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Research article

Analytical study of $\mathcal{ABC}$-fractional pantograph implicit differential equation with respect to another function

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Abstract: This article aims to establish sufficient conditions for qualitative properties of the solutions for a new class of a pantograph implicit system in the framework of Atangana-Baleanu-Caputo ($\mathcal{ABC}$) fractional derivatives with respect to another function under integral boundary conditions. The Schaefer and Banach fixed point theorems (FPTs) are utilized to investigate the existence and uniqueness results for this pantograph implicit system. Moreover, some stability types such as the Ulam-Hyers ($\mathbb{UH}$), generalized $\mathbb{UH}$, Ulam-Hyers-Rassias ($\mathbb{UHR}$) and generalized $\mathbb{UHR}$ are discussed. Finally, interpretation mathematical examples are given in order to guarantee the validity of the main findings. Moreover, the fractional operator used in this study is more generalized and supports our results to be more extensive and covers several new and existing problems in the literature.

Keywords: fractional differential equations; fractional calculus with respect to another function; non-singular fractional operators; existence and uniqueness results
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1. Introduction

Fractional differential equations (FDEs) are a type of mathematical equation that involves non-integer orders of derivatives. These equations have become increasingly important in recent years, as they provide a powerful tool for modeling systems with long-range dependence and memory. The solutions to FDEs can exhibit unique behaviors, including power-law decay and non-locality, which distinguish them from solutions to traditional differential equations, for more detailed information, see these works [1–11]. Furthermore, $\mathbb{FPT}$ is a mathematical discipline that explores the existence and behavior of fixed points of the mapping equation. These points are those that remain invariant under the action of the mapping. The study of fixed points has many applications in diverse fields such as analysis, geometry, topology, physics, economics, and engineering. For instance, in physics, $\mathbb{FPT}$ can be used to assess the stability of dynamic systems. In economics, it can help identify equilibrium points of economic models. In general, $\mathbb{FPT}$ provides a powerful tool for finding solutions to problems involving nonlinear equations and systems, we refer the researchers to some works applied $\mathbb{FPT}$ in the literature studies [12–21]. In fact, pantograph problems are a type of differential equations that arises in the study of systems with a delayed feedback, particularly in mechanical systems. These equations are characterized by fractional derivatives and a time delay term, making them difficult to solve. Pantograph problems have a wide range of applications in fields such as physics, engineering, and biology [22–27].

In fractional calculus field, Atangana and Baleanu ($\mathcal{AB}$) [28] introduced a new non-singular fractional operator via a function of Mittag-Leffler in the sense of Caputo and Riemann-Liouville. These two operators attracted the attention of many researchers to study various problems and applications [29–37]. Afterwards, Fernandez and Baleanu [38] generalized the $\mathcal{AB}$ operator to differintegration with respect to other functions and so called ϱ-$\mathcal{ABC}$ in the Caputo sense and ϱ-$\mathcal{ABR}$ in the Riemann-Liouville sense, where ϱ is a positive and increasing function of its domains. Very recently, Abdeljawad et al. [39] extended these operators to higher-order and derived the Gronwall-type inequality in the framework of an ϱ-$\mathcal{AB}$ factional integral.

In 2020, Ali et al. [40] studied some essential conditions of qualitative results for a delay problem in the sense of $\mathcal{ABC}$ fractional derivatives, given by:

$$\begin{cases} {}^{\mathcal{ABC}}\mathfrak{D}_0^{\vartheta}\varpi(\upsilon) = \mathfrak{g}\left(\upsilon, \varpi(\upsilon), \varpi(\delta\upsilon)\right), \upsilon \in \mathrm{J} = [0, \mathfrak{z}], \vartheta \in (1, 2], \\ \varpi(0) = \gamma_1, \quad \varpi(\mathfrak{z}) = \gamma_2, \ \gamma_1, \gamma_2 \in \mathbb{R}, \ \delta \in (0, 1). \end{cases} \tag{1.1}$$

After that, in 2021 [41], Banach and Krasnoselskii used $\mathbb{FPT}s$ to establish the existence and uniqueness theorems for a $\mathcal{ABC}$-fractional implicit problem, given as follows:

$$\begin{cases} {}^{\mathcal{ABC}}\mathfrak{D}_0^{\vartheta}\varpi(\upsilon) = \mathfrak{g}\left(\upsilon, \varpi(\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_0^{\vartheta}\varpi(\upsilon)\right), \upsilon \in \mathrm{J} = [0, \mathfrak{z}], \vartheta \in (1, 2], \\ \varpi(0) = \gamma_1, \quad \varpi(\mathfrak{z}) = \gamma_2, \ \gamma_1, \gamma_2 \in \mathbb{R}. \end{cases} \tag{1.2}$$

Furthermore, the authors of [42] investigated an implicit $\mathcal{ABC}$-fractional differential system under integral conditions:

$$\begin{cases} {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta}\varpi(\upsilon) = \mathfrak{g}\left(\upsilon, \varpi(\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta}\varpi(\upsilon)\right), \upsilon \in \mathrm{J} = [\iota, \mathfrak{z}], \vartheta \in (1, 2], \\ \varpi(\iota) = 0, \quad \varpi(\mathfrak{z}) = \gamma_2 \displaystyle\int_\iota^{\mathfrak{z}} \mathfrak{f}(\kappa, \varpi(\kappa)) d\kappa. \end{cases} \tag{1.3}$$

Motivated by the aforementioned research works [40–42], and as an application for the work in [39], in this article, we aim to study the existence, uniqueness and verity types of $\mathbb{UH}$ stability of the solution for a new class of ϱ-$\mathcal{ABC}$ fractional pantograph implicit systems supplemented by the integral boundary conditions of the form:

$$\begin{cases} {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon) = \mathfrak{g}\left(\upsilon, \varpi(\upsilon), \varpi(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon)\right), \upsilon \in \mathrm{J} = [\iota, \mathfrak{z}], \\ \varpi(\iota) = \gamma_1, \quad \varpi(\mathfrak{z}) = \gamma_2 \displaystyle\int_{\iota}^{\mathfrak{z}} \mathfrak{f}(\kappa, \varpi(\kappa))d\kappa, \end{cases} \tag{1.4}$$

where $0 < \delta < 1, \gamma_1, \gamma_2 \in \mathbb{R}$, ${}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}$ denotes the ϱ-$\mathcal{ABC}$ fractional derivative of the arbitrary order $\vartheta \in (1, 2]$, and the functions $\mathfrak{g} : \mathrm{J}\times\mathbb{R}^3 \to \mathbb{R}, \mathfrak{f} : \mathrm{J}\times\mathbb{R} \to \mathbb{R}$, are continuous. Moreover, let $\varrho : [\iota, \mathfrak{z}] \to \mathbb{R}^+$ be an increasing and positive function with $\varrho(u) \in C^1((\iota, \mathfrak{z}), \mathbb{R}^+)$ and $\varrho'(\upsilon) \neq 0, \ \forall\, \upsilon \in \mathrm{J}$.

Here, we ensure that this research work displays a novelty by considering a new class of pantograph implicit problems in the framework of ϱ-$\mathcal{ABC}$ fractional derivatives. Furthermore, the contributions of this work are interesting and more generalized of a new and existing problems in the literature, for instance:

- **i)** If $\varrho(\upsilon) = \upsilon$, then the system (1.4) returns to a system in the sense of $\mathcal{ABC}$-fractional derivatives;
- **ii)** If $\varrho(\upsilon) = \upsilon$, and $\int_{\iota}^{\mathfrak{z}} \mathfrak{f}(\kappa, \varpi(\kappa))d\kappa = 1$, then the system (1.4), reduces to the problem (1.1) when the implicit term finishes and reduces to problem (1.2) when the delay term finishes;
- **iii)** The system (1.4), returns to Eq (1.3) in the case where $\varrho(\upsilon) = \upsilon$ and delay term finishes;
- **iv)** Our system (1.4) covers numerous problems by taking any positive and increasing the function ϱ on the interval J.

The paper's remaining sections are organized as follows. In Section 2, some fundamental preliminaries are presented. In Section 3, a corresponding fractional integral equation of the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4) is derived. In Section 4, the existence and uniqueness results are proven by applying the Schaefer and Banach $\mathbb{FPT}s$. In Section 5, $\mathbb{UH}$ and $\mathbb{UHR}$ stability are discussed. In Section 6, some examples are provided to illustrate the main findings.

2. Preliminaries

In this section, we recall several major preliminaries and definitions. Let $\Omega := C(\mathrm{J} = [\iota, \mathfrak{z}], \mathbb{R})$ denote the Banach space of all the continuous functions ϖ endowed with the supremum norm $\|\varpi\| = \sup_{\upsilon\in[\iota,\mathfrak{z}]} |\varpi(\upsilon)|$.

Definition 2.1. *([43]) The ϱ-Riemann-Liouville fractional integral of order $\vartheta > 0$ for an integrable function $\varpi : [\iota, \mathfrak{z}] \to \mathbb{R}$ is given by the following:*

$$ {}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon) = \frac{1}{\Gamma(\vartheta)} \int_{\iota}^{\upsilon} (\varrho(\upsilon) - \varrho(\kappa))^{\vartheta-1}\varrho'(\kappa)\, \varpi(\kappa)\, \mathrm{d}\kappa, $$

where $\Gamma(\vartheta) = \int_0^{+\infty} e^{-\upsilon}\, \upsilon^{\vartheta-1}\, \mathrm{d}\upsilon$, $\vartheta > 0$.

Lemma 2.2. *([43]) Let $\vartheta, \mu > 0$ and $\varpi : [\iota, \mathfrak{z}] \to \mathbb{R}$. Then,*

(i) ${}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}[\varrho(\upsilon) - \varrho(\iota)]^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\vartheta+\mu)}[\varrho(\upsilon) - \varrho(\iota)]^{\vartheta+\mu-1}$;
(ii) ${}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\, {}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{\mu,\varrho}\varpi(\upsilon) = {}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{\vartheta+\mu,\varrho}\varpi(\upsilon)$;
(iii) $\left(\left(\frac{1}{\varrho(\upsilon)}\frac{d}{d\upsilon}\right)^n \, {}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{n,\varrho}\varpi\right)(\upsilon) = \varpi(\upsilon), n \in \mathbb{N}$.

Definition 2.3. *([38, 44]) The ϱ-$\mathcal{ABC}$ fractional derivative of order $\vartheta \in (0,1]$ of a function $\varpi \in \mathcal{H}^1(\iota, \mathfrak{z})$ is given by the following:*

$$({}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi)(\upsilon) = \frac{\Lambda(\vartheta)}{1-\vartheta}\int_{\iota}^{\upsilon} \varrho'(\kappa)\mathbb{E}_{\vartheta}\left(\frac{-\vartheta}{1-\vartheta}(\varrho(\upsilon)-\varrho(\kappa))^{\vartheta}\right)\varpi'_{\varrho}(\kappa)d\kappa, \quad \upsilon \in [\iota, \mathfrak{z}],$$

where $\varpi'_{\varrho}(\upsilon) = \dfrac{\varpi'(\upsilon)}{\varrho'(\upsilon)}$, $\Lambda(\vartheta)$ *is the normalization function with* $\Lambda(0) = \Lambda(1) = 1$*, and* $\mathbb{E}_{\vartheta}$ *is called the Mittag-Leffler function defined by the following:*

$$\mathbb{E}_{\vartheta}(\mathrm{z}) = \sum_{i=0}^{\infty} \frac{\mathrm{z}^i}{\Gamma(\vartheta i + 1)}, \quad Re(\vartheta) > 0, \mathrm{z} \in \mathbb{C}.$$

Definition 2.4. *([44]) The ϱ-$\mathcal{AB}$ fractional integral of order $\vartheta \in (0,1]$ of an integrable function ϖ is given by the following:*

$$({}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\varpi)(\upsilon) = \frac{1-\vartheta}{\Lambda(\vartheta)}\varpi(\upsilon) + \frac{\vartheta}{\Lambda(\vartheta)}\,{}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon), \quad \upsilon \in [\iota, \mathfrak{z}].$$

Definition 2.5. *([39]) The ϱ-$\mathcal{ABC}$ fractional derivative of order $\vartheta \in (n, n+1], \nu = \vartheta - n, n = 0, 1, 2, \ldots,$ of a function $\varpi^{(n+1)} \in \mathcal{H}^1(\iota, \mathfrak{z})$ is given by the following:*

$$\begin{aligned}({}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\,\varpi)(\upsilon) &= ({}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\nu,\varrho}\,\varpi_{\varrho}^{(n)})(\upsilon)\\ &= \frac{\Lambda(\vartheta-n)}{n+1-\vartheta}\int_{\iota}^{\upsilon} \varrho'(\kappa)\mathbb{E}_{\vartheta-n}\left(\frac{-(\vartheta-n)}{n+1-\vartheta}(\varrho(\upsilon)-\varrho(\kappa))^{\vartheta-n}\right)\varpi_{\varrho}^{(n+1)}(\kappa)d\kappa,\end{aligned}$$

where $\varpi_{\varrho}^{(n)}(\upsilon) = \left(\dfrac{1}{\varrho'(\upsilon)}\dfrac{d}{d\upsilon}\right)^n \varpi(\upsilon)$ *and* $\varpi_{\varrho}^{(0)}(\upsilon) = \varpi(\upsilon)$*. If* $\vartheta = k \in \mathbb{N}$*, then* $({}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi)(\upsilon) = \varpi_{\varrho}^{(k)}(\upsilon)$.

Definition 2.6. *([39]) The ϱ-$\mathcal{AB}$ fractional integral of order $\vartheta \in (n, n+1], \nu = \vartheta - n, n = 0, 1, 2, \ldots,$ of a function ϖ is given by the following:*

$$\begin{aligned}({}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\varpi)(\upsilon) &= ({}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{n,\varrho}\,{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\nu,\varrho}\,\varpi)(\upsilon) = ({}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\nu,\varrho}\,{}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{n,\varrho}\,\varpi)(\upsilon)\\ &= \frac{n+1-\vartheta}{\Lambda(\vartheta-n)}\,{}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{n,\varrho}\varpi(\upsilon) + \frac{(\vartheta-n)}{\Lambda(\vartheta-n)}\,{}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon),\end{aligned}$$

where ${}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{n,\varrho}$ *is defined as:*

$$({}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{n,\varrho}\varpi)(\upsilon) = \frac{1}{\Gamma(n)}\int_{\iota}^{\upsilon} \varrho'(\kappa)(\varrho(\upsilon)-\varrho(\kappa))^{n-1}\varpi(\kappa)d\kappa, \ \upsilon > \iota.$$

Lemma 2.7. *([39]) Let $\varpi \in C(\mathrm{J}, \mathbb{R})$ and $\varrho \in C^n(\mathrm{J}, \mathbb{R})$. For $\vartheta \in (n, n+1], \nu = \vartheta - n, n = 0, 1, 2, \ldots,$ the following relation holds:*

$$({}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\,{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi)(\upsilon) = \varpi(\upsilon) - \sum_{k=0}^{n} e_k\left(\varrho(\upsilon)-\varrho(\iota)\right)^k, \quad e_k \in \mathbb{R}.$$

Lemma 2.8. *([39]) Let $\varrho \in C^n(\mathrm{J}, \mathbb{R}^+)$ with $\varrho'(\upsilon) \neq 0$. For $\vartheta \in (n, n+1], \nu = \vartheta - n, n = 0, 1, 2, \ldots, \beta \geq n+1$. Then, the following equations hold:*

(i) ${}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}[\varrho(\upsilon)-\varrho(\iota)]^{\beta}=\dfrac{\Lambda(\vartheta-n)}{(n+1-\vartheta)}\sum_{i=0}^{\infty}\left(\dfrac{-(\vartheta-n)}{n+1-\vartheta}\right)^{i}\dfrac{\Gamma(\beta+1)[\varrho(\upsilon)-\varrho(\iota)]^{i(\vartheta-n)+\beta-n}}{\Gamma(i(\vartheta-n)+\beta-n+1)}$;

(ii) ${}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}[\varrho(\upsilon)-\varrho(\iota)]^{\zeta}=0,\ \ \zeta=0,1,\ldots,n$;

(iii) $({}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\,1)(\upsilon)=\dfrac{(n+1-\vartheta)[\varrho(\upsilon)-\varrho(\iota)]^{n}}{\Lambda(\vartheta-n)\Gamma(n+1)}+\dfrac{(\vartheta-n)[\varrho(\upsilon)-\varrho(\iota)]^{\vartheta}}{\Lambda(\vartheta-n)\Gamma(\vartheta+1)}$.

Theorem 2.9. *([45], Banach's $\mathbb{FPT}$) Suppose that K be a Banach space. If $\aleph: K\to K$ is a contraction, then $\aleph$ has one and only one fixed point in K.*

Theorem 2.10. *([46], Schaefer's $\mathbb{FPT}$) Let K be a Banach space and $\aleph: K\to K$ be a completely continuous mapping. If $\mathrm{U}=\{\zeta\in\mathrm{K}:\varpi=\eta\aleph\varpi,$ for some $\eta\in(0,1)\}$ is bounded set. Then, $\aleph$ possesses fixed points.*

3. The corresponding fractional integral equation

We start this section by deriving the corresponding fractional integral equation of the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4). For this, we need to introduce the next lemma.

Lemma 3.1. *Consider $\varpi\in\Omega$. Then, the corresponding fractional integral equation of the following ϱ-$\mathcal{ABC}$ fractional system,*

$$\begin{cases}{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon)=\mathfrak{q}\,(\upsilon)\,,\upsilon\in\mathrm{J}=[\iota,\mathfrak{z}],\vartheta\in(1,2],\\ \varpi(\iota)=\gamma_1,\quad \varpi(\mathfrak{z})=\gamma_2\displaystyle\int_{\iota}^{\mathfrak{z}}\mathfrak{f}(\kappa,\varpi(\kappa))d\kappa,\end{cases}\tag{3.1}$$

is given by

$$\varpi(\upsilon)=\gamma_1\left[1-\frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\right]+\frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\left[\gamma_2\int_{\iota}^{\mathfrak{z}}\mathfrak{f}(\kappa,\varpi(\kappa))d\kappa-{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{q}\,(\mathfrak{z})\right]+{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{q}\,(\upsilon)\,.$$

Proof. We start our proof by applying ${}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}$ on both sides of Eq (3.1). According to the Definition 2.6 with using Lemma 2.7, we get

$$\begin{aligned}\varpi(\upsilon)&=e_0+e_1\,(\varrho(\upsilon)-\varrho(\iota))+{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{q}\,(\upsilon)\\&=e_0+e_1\,(\varrho(\upsilon)-\varrho(\iota))+\frac{2-\vartheta}{\Lambda(\vartheta-1)}\,{}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{1,\varrho}\mathfrak{q}\,(\upsilon)+\frac{(\vartheta-1)}{\Lambda(\vartheta-1)}\,{}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{q}\,(\upsilon)\,.\end{aligned}\tag{3.2}$$

Then, by the boundary condition $\varpi(\iota)=\gamma_1$, we deduce that $e_0=\gamma_1$, which implies that

$$\varpi(\upsilon)=\gamma_1+e_1\,(\varrho(\upsilon)-\varrho(\iota))+\frac{2-\vartheta}{\Lambda(\vartheta-1)}\,{}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{1,\varrho}\mathfrak{q}\,(\upsilon)+\frac{(\vartheta-1)}{\Lambda(\vartheta-1)}\,{}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{q}\,(\upsilon)\,.$$

Again, due to the boundary condition $\varpi(\mathfrak{z})=\gamma_2\int_{\iota}^{\mathfrak{z}}\mathfrak{f}(\kappa,\varpi(\kappa))d\kappa$, we conclude that

$$e_1=\frac{1}{(\varrho(\mathfrak{z})-\varrho(\iota))}\left[\gamma_2\int_{\iota}^{\mathfrak{z}}\mathfrak{f}(\kappa,\varpi(\kappa))d\kappa-\gamma_1-\frac{2-\vartheta}{\Lambda(\vartheta-1)}{}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{1,\varrho}\mathfrak{q}\,(\mathfrak{z})-\frac{(\vartheta-1)}{\Lambda(\vartheta-1)}{}^{\mathcal{RL}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{q}\,(\mathfrak{z})\right].$$

By substituting the values of e_0 and e_1 into Eq (3.2), we obtain

$$\begin{aligned}\varpi(\upsilon) &= \gamma_1\left[1-\frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\right]+\frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\Bigg[\gamma_2\int_\iota^{\mathfrak{z}}\mathfrak{f}(\kappa,\varpi(\kappa))d\kappa-\frac{2-\vartheta}{\Lambda(\vartheta-1)}\,{}^{\mathcal{RL}}\mathfrak{I}_\iota^{1,\varrho}\mathfrak{q}\,(\mathfrak{z})\\ &\quad-\frac{(\vartheta-1)}{\Lambda(\vartheta-1)}\,{}^{\mathcal{RL}}\mathfrak{I}_\iota^{\vartheta,\varrho}\mathfrak{q}\,(\mathfrak{z})\Bigg]+\frac{2-\vartheta}{\Lambda(\vartheta-1)}\,{}^{\mathcal{RL}}\mathfrak{I}_\iota^{1,\varrho}\mathfrak{q}\,(\upsilon)+\frac{(\vartheta-1)}{\Lambda(\vartheta-1)}\,{}^{\mathcal{RL}}\mathfrak{I}_\iota^{\vartheta,\varrho}\mathfrak{q}\,(\upsilon)\,.\end{aligned}$$

Hence, the proof is finished. □

As a consequence of the above lemma, we conclude the following interesting lemma.

Lemma 3.2. *Consider $\varpi\in\Omega$. Then, the corresponding fractional integral equation of the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system* (1.4)*, is given by*

$$\begin{aligned}\varpi(\upsilon) &= \gamma_1\left[1-\frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\right]+\frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\Bigg[\gamma_2\int_\iota^{\mathfrak{z}}\mathfrak{f}(\kappa,\varpi(\kappa))d\kappa \qquad (3.3)\\ &\quad-{}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho}\mathfrak{g}\left(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(\delta\mathfrak{z}),{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\mathfrak{z})\right)\Bigg]+{}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho}\mathfrak{g}\left(\upsilon,\varpi(\upsilon),\varpi(\delta\upsilon),{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\upsilon)\right).\end{aligned}$$

4. Existence and uniqueness

We start this section by investigating the uniqueness criteria of the solution for the system (1.4) via Banach $\mathbb{FPT}$s. Thus, the next assumption is required:
H$_1$) There are constants $\ell_1,\ell_2,\ell_3>0$ such that for all $\upsilon\in[\iota,\mathfrak{z}]$ and $\varpi_i,\hat{\varpi}_i\in\Omega,(i=1,2,3)$ satisfy

$$|\mathfrak{g}\,(\upsilon,\varpi_1,\varpi_2,\varpi_3)-\mathfrak{g}\,(\upsilon,\hat{\varpi}_1,\hat{\varpi}_2,\hat{\varpi}_3)|\le\ell_1\,(|\varpi_1-\hat{\varpi}_1|+|\varpi_2-\hat{\varpi}_2|)+\ell_2|\varpi_3-\hat{\varpi}_3|,$$

and

$$|\mathfrak{f}(\upsilon,\varpi_1)-\mathfrak{f}(\upsilon,\hat{\varpi}_1)|\le\ell_3|\varpi_1-\hat{\varpi}_1|.$$

Theorem 4.1. *Let* H$_1$ *hold, and if*

$$\beta_1:=\left\{|\gamma_2|(\mathfrak{z}-\iota)\ell_3+\frac{4\ell_1}{1-\ell_2}\left[\frac{(2-\vartheta)[\varrho(\mathfrak{z})-\varrho(\iota)]}{\Lambda(\vartheta-1)}+\frac{(\vartheta-1)[\varrho(\mathfrak{z})-\varrho(\iota)]^\vartheta}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)}\right]\right\}<1, \qquad (4.1)$$

then there exists a exactly one solution for the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4) on J.

Proof. At the beginning, we define the operator $\aleph:\Omega\to\Omega$ as follows:

$$\begin{aligned}(\aleph\varpi)(\upsilon) &= \gamma_1\left(1-\frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\right)+\frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\Bigg[\gamma_2\int_\iota^{\mathfrak{z}}\mathfrak{f}(\kappa,\varpi(\kappa))d\kappa \qquad (4.2)\\ &\quad-{}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho}\mathfrak{g}\left(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(\delta\mathfrak{z}),{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\mathfrak{z})\right)\Bigg]+{}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho}\mathfrak{g}\left(\upsilon,\varpi(\upsilon),\varpi(\delta\upsilon),{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\upsilon)\right).\end{aligned}$$

In order to investigate the uniqueness solution of the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4), we will use the Banach $\mathbb{FPT}$. Regarding this, let $\varpi,\hat{\varpi}\in\Omega$, $\upsilon\in$ J, and by using H$_1$, we have

$$|(\aleph\varpi)(\upsilon)-(\aleph\hat{\varpi})(\upsilon)|$$

$$
\begin{aligned}
&\leq \frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\Bigg[|\gamma_2|\int_\iota^{\mathfrak{z}} |\mathfrak{f}(\kappa,\varpi(\kappa))-\mathfrak{f}(\kappa,\hat{\varpi}(\kappa))|\,d\kappa \\
&\quad + {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho}\left|\mathfrak{g}\left(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(\delta\mathfrak{z}),{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\mathfrak{z})\right)-\mathfrak{g}\left(\mathfrak{z},\hat{\varpi}(\mathfrak{z}),\hat{\varpi}(\delta\mathfrak{z}),{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\hat{\varpi}(\mathfrak{z})\right)\right|\Bigg] \\
&\quad + {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho}\left|\mathfrak{g}\left(\upsilon,\varpi(\upsilon),\varpi(\delta\upsilon),{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\upsilon)\right)-\mathfrak{g}\left(\upsilon,\hat{\varpi}(\upsilon),\hat{\varpi}(\delta\upsilon),{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right)\right| \\
&\leq \frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\Bigg[|\gamma_2|\int_\iota^{\mathfrak{z}} \ell_3\,|\varpi(\kappa)-\hat{\varpi}(\kappa)|\,d\kappa \\
&\quad + {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho}\left[\ell_1\left(|\varpi(\mathfrak{z})-\hat{\varpi}(\mathfrak{z})|+|\varpi(\delta\mathfrak{z})-\hat{\varpi}(\delta\mathfrak{z})|\right)+\ell_2|{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\mathfrak{z})-{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\hat{\varpi}(\mathfrak{z})|\right]\Bigg] \\
&\quad + {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho}\left[\ell_1\left(|\varpi(\upsilon)-\hat{\varpi}(\upsilon)|+|\varpi(\delta\upsilon)-\hat{\varpi}(\delta\upsilon)|\right)+\ell_2|{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\upsilon)-{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\hat{\varpi}(\upsilon)|\right].
\end{aligned}
\tag{4.3}
$$

In fact, we have ${}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\upsilon)=\mathfrak{g}\left(\upsilon,\varpi(\upsilon),\varpi(\delta\upsilon),{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\upsilon)\right)$, thus

$$
\begin{aligned}
&|{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\upsilon)-{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\hat{\varpi}(\upsilon)| \\
&=|\mathfrak{g}\left(\upsilon,\varpi(\upsilon),\varpi(\delta\upsilon),{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\upsilon)\right)-\mathfrak{g}\left(\upsilon,\hat{\varpi}(\upsilon),\hat{\varpi}(\delta\upsilon),{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right)| \\
&\leq \ell_1\left(|\varpi(\upsilon)-\hat{\varpi}(\upsilon)|+|\varpi(\delta\upsilon)-\hat{\varpi}(\delta\upsilon)|\right)+\ell_2|{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\upsilon)-{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\hat{\varpi}(\upsilon)|.
\end{aligned}
$$

Therefore,

$$
|{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\upsilon)-{}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\hat{\varpi}(\upsilon)| \leq \frac{\ell_1}{1-\ell_2}\left(|\varpi(\upsilon)-\hat{\varpi}(\upsilon)|+|\varpi(\delta\upsilon)-\hat{\varpi}(\delta\upsilon)|\right). \tag{4.4}
$$

In view of Eqs (4.3), (4.4), and Lemma 2.8 part (*iii*), we obtain

$$
\begin{aligned}
&|(\aleph\varpi)(\upsilon)-(\aleph\hat{\varpi})(\upsilon)| \\
&\leq \frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\Bigg[|\gamma_2|\int_\iota^{\mathfrak{z}} \ell_3\,|\varpi(\kappa)-\hat{\varpi}(\kappa)|\,d\kappa \\
&\quad + {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho}\left[\frac{\ell_1}{1-\ell_2}\left(|\varpi(\mathfrak{z})-\hat{\varpi}(\mathfrak{z})|+|\varpi(\delta\mathfrak{z})-\hat{\varpi}(\delta\mathfrak{z})|\right)\right]\Bigg] \\
&\quad + {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho}\left[\frac{\ell_1}{1-\ell_2}\left(|\varpi(\upsilon)-\hat{\varpi}(\upsilon)|+|\varpi(\delta\upsilon)-\hat{\varpi}(\delta\upsilon)|\right)\right] \\
&\leq |\gamma_2|(\mathfrak{z}-\iota)\ell_3\,\|\varpi-\hat{\varpi}\| \\
&\quad + \frac{4\ell_1}{1-\ell_2}\left[\frac{(2-\vartheta)[\varrho(\mathfrak{z})-\varrho(\iota)]}{\Lambda(\vartheta-1)}+\frac{(\vartheta-1)[\varrho(\mathfrak{z})-\varrho(\iota)]^\vartheta}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)}\right]\|\varpi-\hat{\varpi}\| \\
&\leq \left\{|\gamma_2|(\mathfrak{z}-\iota)\ell_3+\frac{4\ell_1}{1-\ell_2}\left[\frac{(2-\vartheta)[\varrho(\mathfrak{z})-\varrho(\iota)]}{\Lambda(\vartheta-1)}+\frac{(\vartheta-1)[\varrho(\mathfrak{z})-\varrho(\iota)]^\vartheta}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)}\right]\right\}\|\varpi-\hat{\varpi}\|.
\end{aligned}
$$

Hence, $\|\aleph\varpi-\aleph\hat{\varpi}\|\leq\beta_1\,\|\varpi-\hat{\varpi}\|$, and by condition (4.1), the operator $\aleph$ is a contraction. Based on the Banach $\mathbb{FPT}$ 2.9, the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4) possesses exactly one solution on J. □

Next, we discuss the existence result of the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4) via Schaefer's $\mathbb{FPT}$. Thus, we need to present the following hypothesis:
H$_2$) There are constants $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5 > 0$, such that for all $\upsilon \in [\iota, \mathfrak{z}]$ and $\varpi_i \in \Omega, (i = 1, 2, 3)$ satisfy

$$|\mathfrak{g}(\upsilon, \varpi_1, \varpi_2, \varpi_3)| \leq \xi_1 + \xi_2(|\varpi_1| + |\varpi_2|) + \xi_3|\varpi_3|, \tag{4.5}$$

and

$$|\mathfrak{f}(\upsilon, \varpi_1)| \leq \xi_4 + \xi_5|\varpi_1|.$$

Theorem 4.2. *Let* H$_2$ *hold. Then, there exists a solution for the* ϱ-$\mathcal{ABC}$ *fractional pantograph implicit system (1.4) provided that*

$$\beta_3 := \xi_5|\gamma_2|(\mathfrak{z} - \iota) + \frac{4\xi_2}{1 - \xi_3}\left[\frac{(2 - \vartheta)[\varrho(\mathfrak{z}) - \varrho(\iota)]}{\Lambda(\vartheta - 1)} + \frac{(\vartheta - 1)[\varrho(\mathfrak{z}) - \varrho(\iota)]^{\vartheta}}{\Lambda(\vartheta - 1)\Gamma(\vartheta + 1)}\right] < 1. \tag{4.6}$$

Proof. In order to investigate the conditions of Schaefer's $\mathbb{FPT}$, we define the operator $\aleph : \Omega \to \Omega$ as given in (4.2). Additionally, we set the bounded closed and convex ball $\mho_\varsigma := \{\varpi \in \Omega : \|\varpi\| \leq \varsigma\}$, with radius $\varsigma \geq \dfrac{\beta_2}{1 - \beta_3}$ and $\beta_3 < 1$, such that

$$\beta_2 := 2|\gamma_1| + \xi_4|\gamma_2|(\mathfrak{z} - \iota) + \frac{2\xi_1}{1 - \xi_3}\left[\frac{(2 - \vartheta)[\varrho(\mathfrak{z}) - \varrho(\iota)]}{\Lambda(\vartheta - 1)} + \frac{(\vartheta - 1)[\varrho(\mathfrak{z}) - \varrho(\iota)]^{\vartheta}}{\Lambda(\vartheta - 1)\Gamma(\vartheta + 1)}\right],$$

and β_3 given in Eq (4.6). Our proof will be divided into two of the procedures.
First: We show that the operator $\aleph$ is completely continuous. For this, let the sequence $\{\varpi_n\}_{n\in\mathbb{N}}$ is convergence to ϖ in the ball $\mho_\varsigma$ as $n \to \infty$. Therefore, based on the continuity of the functions $\mathfrak{g}$, $\mathfrak{f}$, and by using the dominated convergence theorem in the sense of Lebesgue, we find that

$$\begin{aligned}
\lim_{n\to\infty}(\aleph\varpi_n)(\upsilon) &= \gamma_1\left[1 - \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\right] + \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\Bigg[\gamma_2\int_\iota^{\mathfrak{z}} \lim_{n\to\infty}\mathfrak{f}(\kappa, \varpi_n(\kappa))d\kappa \\
&\quad - {}^{\mathcal{AB}}\mathfrak{J}_\iota^{\vartheta,\varrho}\lim_{n\to\infty}\mathfrak{g}\left(\mathfrak{z}, \varpi_n(\mathfrak{z}), \varpi_n(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi_n(\mathfrak{z})\right)\Bigg] \\
&\quad + {}^{\mathcal{AB}}\mathfrak{J}_\iota^{\vartheta,\varrho}\lim_{n\to\infty}\mathfrak{g}\left(\upsilon, \varpi_n(\upsilon), \varpi_n(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi_n(\upsilon)\right) \\
&= (\aleph\varpi)(\upsilon).
\end{aligned}$$

Thus, the operator $\aleph$ is continuous.
Next, we show that $\aleph\mho_\varsigma \subset \mho_\varsigma$. By using H$_2$, for $\varpi \in \mho_\varsigma$ and $\upsilon \in$ J, we obtain

$$\begin{aligned}
|(\aleph\varpi)(\upsilon)| &\leq |\gamma_1|\left|1 - \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\right| + \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\Bigg[|\gamma_2|\int_\iota^{\mathfrak{z}} |\mathfrak{f}(\kappa, \varpi(\kappa))|d\kappa \\
&\quad + {}^{\mathcal{AB}}\mathfrak{J}_\iota^{\vartheta,\varrho}|\mathfrak{g}\left(\mathfrak{z}, \varpi(\mathfrak{z}), \varpi(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\mathfrak{z})\right)|\Bigg] \\
&\quad + {}^{\mathcal{AB}}\mathfrak{J}_\iota^{\vartheta,\varrho}|\mathfrak{g}\left(\upsilon, \varpi(\upsilon), \varpi(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho}\varpi(\upsilon)\right)| \\
&\leq |\gamma_1|\left|1 - \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\right| + \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\Bigg[|\gamma_2|\int_\iota^{\mathfrak{z}} (\xi_4 + \xi_5|\varpi(\upsilon)|)d\kappa
\end{aligned}$$

$$
\begin{aligned}
&+{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\left[\xi_1+\xi_2(|\varpi(\mathfrak{z})|+|\varpi(\delta\mathfrak{z})|)+\xi_3|{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\mathfrak{z})|\right]\Bigg]\\
&+{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\left[\xi_1+\xi_2(|\varpi(\upsilon)|+|\varpi(\delta\upsilon)|)+\xi_3|{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon)|\right]. \qquad (4.7)
\end{aligned}
$$

By the relation 4.5, we find that

$$
\begin{aligned}
|{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon)| &= |\mathfrak{g}\left(\upsilon,\varpi(\upsilon),\varpi(\delta\upsilon),{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon)\right)|\\
&\le \xi_1+\xi_2(|\varpi(\upsilon)|+|\varpi(\delta\upsilon)|)+\xi_3|{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon)|.
\end{aligned}
$$

Thus,

$$
|{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon)| \le \frac{\xi_1}{1-\xi_3}+\frac{\xi_2}{1-\xi_3}(|\varpi(\upsilon)|+|\varpi(\delta\upsilon)|). \qquad (4.8)
$$

Now, by substituting Eq (4.8) into Eq (4.7) and by taking the supremum to both sides, and based on Lemma 2.8, part (*iii*), with $\frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))}<1$, we obtain

$$
\begin{aligned}
\|\aleph\varpi\| &\le 2|\gamma_1|+|\gamma_2|(\mathfrak{z}-\iota)(\xi_4+\xi_5\|\varpi\|)\\
&\quad+\left[\frac{2\xi_1}{1-\xi_3}+\frac{4\xi_2}{1-\xi_3}\|\varpi\|\right]\left[\frac{(2-\vartheta)[\varrho(\mathfrak{z})-\varrho(\iota)]}{\Lambda(\vartheta-1)}+\frac{(\vartheta-1)[\varrho(\mathfrak{z})-\varrho(\iota)]^{\vartheta}}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)}\right]\\
&\le 2|\gamma_1|+\xi_4|\gamma_2|(\mathfrak{z}-\iota)\\
&\quad+\frac{2\xi_1}{1-\xi_3}\left[\frac{(2-\vartheta)[\varrho(\mathfrak{z})-\varrho(\iota)]}{\Lambda(\vartheta-1)}+\frac{(\vartheta-1)[\varrho(\mathfrak{z})-\varrho(\iota)]^{\vartheta}}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)}\right]\\
&\quad+\left\{\xi_5|\gamma_2|(\mathfrak{z}-\iota)+\frac{4\xi_2}{1-\xi_3}\left[\frac{(2-\vartheta)[\varrho(\mathfrak{z})-\varrho(\iota)]}{\Lambda(\vartheta-1)}+\frac{(\vartheta-1)[\varrho(\mathfrak{z})-\varrho(\iota)]^{\vartheta}}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)}\right]\right\}\|\varpi\|\\
&\le \beta_2+\beta_3\|\varpi\| \qquad (4.9)\\
&\le \beta_2+\beta_3\varsigma\le\varsigma.
\end{aligned}
$$

Hence, $\aleph\mho_{\varsigma}\subset\mho_{\varsigma}$, concluding that $\aleph:\mho_{\varsigma}\to\mho_{\varsigma}$.
In the following, we prove that the operator $\aleph:\mho_{\varsigma}\to\mho_{\varsigma}$ is equicontinuos on $\mho_{\varsigma}$. For each $\varpi\in\mho_{\varsigma}$ and $\iota\le\upsilon_1\le\upsilon_2\le\mathfrak{z}$, we have the following:

$$
\begin{aligned}
&|(\aleph\varpi)(\upsilon_2)-(\aleph\varpi)(\upsilon_1)|\\
&\le|\gamma_1|\left|\frac{(\varrho(\upsilon_2)-\varrho(\upsilon_1))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\right|+\frac{|(\varrho(\upsilon_2)-\varrho(\upsilon_1))|}{(\varrho(\mathfrak{z})-\varrho(\iota))}\Bigg[|\gamma_2|\int_{\iota}^{\mathfrak{z}}|\mathfrak{f}(\kappa,\varpi(\kappa))|d\kappa\\
&\quad+{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}|\mathfrak{g}\left(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(\delta\mathfrak{z}),{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\mathfrak{z})\right)|\Bigg]\\
&\quad+|{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\upsilon_2,\varpi(\upsilon_2),\varpi(\delta\upsilon_2),{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon_2)\right)-{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\upsilon_1,\varpi(\upsilon_1),\varpi(\delta\upsilon_1),{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon_1)\right)|\\
&\le|\gamma_1|\left|\frac{(\varrho(\upsilon_2)-\varrho(\upsilon_1))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\right|+\frac{|(\varrho(\upsilon_2)-\varrho(\upsilon_1))|}{(\varrho(\mathfrak{z})-\varrho(\iota))}\Bigg[|\gamma_2|\int_{\iota}^{\mathfrak{z}}|\mathfrak{f}(\kappa,\varpi(\kappa))|d\kappa\\
&\quad+{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}|\mathfrak{g}\left(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(\delta\mathfrak{z}),{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\mathfrak{z})\right)|\Bigg]
\end{aligned}
$$

$$+\left|\frac{2-\vartheta}{\Lambda(\vartheta-1)}\int_{\iota}^{\upsilon_2}\varrho'(\kappa)\mathfrak{g}\left(\kappa,\varpi(\kappa),\varpi(\delta\kappa),{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\kappa)\right)d\kappa\right.$$
$$\left.-\frac{2-\vartheta}{\Lambda(\vartheta-1)}\int_{\iota}^{\upsilon_1}\varrho'(\kappa)\mathfrak{g}\left(\kappa,\varpi(\kappa),\varpi(\delta\kappa),{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\kappa)\right)d\kappa\right|$$
$$+\left|\frac{(\vartheta-1)}{\Lambda(\vartheta-1)\Gamma(\vartheta)}\int_{\iota}^{\upsilon_2}\varrho'(\kappa)(\varrho(\upsilon_2)-\varrho(\kappa))^{\vartheta-1}\mathfrak{g}\left(\kappa,\varpi(\kappa),\varpi(\delta\kappa),{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\kappa)\right)d\kappa\right.$$
$$\left.-\frac{(\vartheta-1)}{\Lambda(\vartheta-1)\Gamma(\vartheta)}\int_{\iota}^{\upsilon_1}\varrho'(\kappa)(\varrho(\upsilon_1)-\varrho(\kappa))^{\vartheta-1}\mathfrak{g}\left(\kappa,\varpi(\kappa),\varpi(\delta\kappa),{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\kappa)\right)d\kappa\right|$$
$$\leq|\gamma_1|\left|\frac{(\varrho(\upsilon_2)-\varrho(\upsilon_1))}{(\varrho(\mathfrak{z})-\varrho(\iota))}\right|+\frac{|(\varrho(\upsilon_2)-\varrho(\upsilon_1))|}{(\varrho(\mathfrak{z})-\varrho(\iota))}\Big[|\gamma_2|(\mathfrak{z}-\iota)(\xi_4+\varsigma\,\xi_5)$$
$$+\left[\frac{\xi_1}{1-\xi_3}+\frac{2\,\varsigma\xi_2}{1-\xi_3}\right]\frac{1}{\Gamma(\vartheta+1)}[\varrho(\mathfrak{z})-\varrho(\iota)]^{\vartheta}\Big]$$
$$+\left|\frac{2-\vartheta}{\Lambda(\vartheta-1)}\left[\frac{\xi_1}{1-\xi_3}+\frac{2\,\varsigma\xi_2}{1-\xi_3}\right][\varrho(\upsilon_2)-\varrho(\iota)]-\frac{2-\vartheta}{\Lambda(\vartheta-1)}\left[\frac{\xi_1}{1-\xi_3}+\frac{2\,\varsigma\xi_2}{1-\xi_3}\right][\varrho(\upsilon_1)-\varrho(\iota)]\right|$$
$$+\left|\frac{(\vartheta-1)}{\Lambda(\vartheta-1)\Gamma(\vartheta)}\left[\frac{\xi_1}{1-\xi_3}+\frac{2\,\varsigma\xi_2}{1-\xi_3}\right]\frac{1}{\Gamma(\vartheta+1)}[\varrho(\upsilon_1)-\varrho(\iota)]^{\vartheta}\right.$$
$$\left.-\frac{(\vartheta-1)}{\Lambda(\vartheta-1)\Gamma(\vartheta)}\left[\frac{\xi_1}{1-\xi_3}+\frac{2\,\varsigma\xi_2}{1-\xi_3}\right]\frac{1}{\Gamma(\vartheta+1)}[\varrho(\upsilon_1)-\varrho(\iota)]^{\vartheta}\right|.$$

Clearly, $|(\aleph\varpi)(\upsilon_2)-(\aleph\varpi)(\upsilon_1)|\to 0$ as $\upsilon_1\to\upsilon_2$ and is independent of $\varpi\in\mho_{\varsigma}$. Therefore, $\aleph$ is equicontinuos on $\mho_{\varsigma}$. Thus, according to the Arzela Ascoli theorem, the mapping $\aleph:\Omega\to\Omega$ is completely continuous.

Second: We investigate that the set $\mathrm{U}=\{\varpi\in\Omega:\varpi=\eta\aleph\varpi, 0<\eta<1\}$ is bounded. Then, for $\varpi\in\mathrm{U}$, $\upsilon\in[\iota,\mathfrak{z}]$, and by direct computation as in (4.9), we obtain

$$\|\varpi\|=\sup_{z\in[\iota,\mathfrak{z}]}|\eta\aleph\varpi(\upsilon)|\leq\beta_2+\beta_3\|\varpi\|.$$

Hence, $\|\varpi\|=\dfrac{\beta_2}{1-\beta_3}$, then U is bounded set. Therefore, based on Theorem 2.10, the mapping $\aleph$ possesses at least fixed point in Ω, which is a solution of the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4). □

5. Stability

The $\mathbb{UH}$ stability are a type of stability for functional equations. This stability was first introduced by Ulam in 1940 [48]; Hyers [49] gave a partial affirmative answer to Ulam's question in 1941, after which, Rassias extended Hyers's theorem in 1978 [50]. The $\mathbb{UH}$ and $\mathbb{UHR}$ stabilities have been applied to a variety of problems in mathematics, physics, and economics and are also useful for studying the behavior of solutions to functional equations when the equations are only approximately satisfied.

In this section, we are going to discuss the $\mathbb{UH}$ and $\mathbb{UHR}$ stabilities by using nonlinear analysis themes. Thus, we present the following concepts that are related to $\mathbb{UH}$ stability:

Definition 5.1. *([51]) The ϱ-$\mathcal{ABC}$ fractional pantograph implicit problem (1.4) is defined as $\mathbb{UH}$ stable if there is a constant $\chi_{\mathfrak{g}} > 0$ such that for each $\varepsilon > 0$, when $\hat{\varpi} \in \Omega$ is any solution of the inequality*

$$|{}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon) - \mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right)| \leq \varepsilon, \quad \upsilon \in [\iota, \mathfrak{z}], \tag{5.1}$$

then there is one solution $\varpi \in \Omega$ of the Eq (1.4) satisfied

$$|\varpi(\upsilon) - \hat{\varpi}(\upsilon)| \leq \chi_{\mathfrak{g}}\varepsilon, \quad \upsilon \in [\iota, \mathfrak{z}]. \tag{5.2}$$

Definition 5.2. *([51]) The ϱ-$\mathcal{ABC}$ fractional pantograph implicit problem (1.4) is defined as generalized $\mathbb{UH}$ stable if there is a function $\alpha \in C(\mathbb{R}^+, \mathbb{R}^+), \alpha(0) = 0$, such that for when $\hat{\varpi} \in \Omega$ is any solution of the identity* (5.1)*, then there is one solution $\varpi \in \Omega$, of the Eq (1.4) satisfied by*

$$|\hat{\varpi}(\upsilon) - \varpi(\upsilon)| \leq \chi_{\mathfrak{g}}\alpha(\varepsilon), \quad \upsilon \in [\iota, \mathfrak{z}]. \tag{5.3}$$

Remark 5.3. *For $\varepsilon > 0$, a function $\hat{\varpi} \in \Omega$ is a solution of the inequality* (5.1)*, for all $\upsilon \in [\iota, \mathfrak{z}]$ iff there is a function $\rho \in \Omega$ such that*
i) $|\rho(\upsilon)| \leq \varepsilon, \quad \upsilon \in [\iota, \mathfrak{z}]$;
ii) ${}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon) = \mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right) + \rho(\upsilon)$.

Theorem 5.4. *Suppose that* H_1 *is satisfied and $\beta_1 < 1$. Then, the unique solution of the ϱ-$\mathcal{ABC}$ fractional pantograph system (1.4) is* $\mathbb{UH}$ *stable and generalized* $\mathbb{UH}$ *stable.*

Proof. Consider $\hat{\varpi} \in \Omega$ which satisfies the inequality (5.1), hence by using the Remark 5.3, we obtain

$${}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon) = \mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right) + \rho(\upsilon), \ \forall \upsilon \in [\iota, \mathfrak{z}].$$

In view of Eq (3.3), we have

$$\begin{aligned}
\hat{\varpi}(\upsilon) = {} & \gamma_1\left[1 - \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\right] + \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\Bigg[\gamma_2\int_{\iota}^{\mathfrak{z}} \mathfrak{f}(\kappa, \hat{\varpi}(\kappa))d\kappa \\
& - {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\mathfrak{z}, \hat{\varpi}(\mathfrak{z}), \hat{\varpi}(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\mathfrak{z})\right) - {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\rho(\mathfrak{z})\Bigg] \\
& + {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right) + {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\rho(\upsilon),
\end{aligned} \tag{5.4}$$

which gives

$$\begin{aligned}
& \Bigg|\hat{\varpi}(\upsilon) - \gamma_1\left[1 - \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\right] - \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\Bigg[\gamma_2\int_{\iota}^{\mathfrak{z}} \mathfrak{f}(\kappa, \hat{\varpi}(\kappa))d\kappa \\
& \quad - {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\mathfrak{z}, \hat{\varpi}(\mathfrak{z}), \hat{\varpi}(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\mathfrak{z})\right)\Bigg] - {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right)\Bigg| \\
& \leq \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}|\rho(\mathfrak{z})| + {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}|\rho(\upsilon)| \\
& \leq 2\,\varepsilon\left[\frac{(2 - \vartheta)[\varrho(\mathfrak{z}) - \varrho(\iota)]}{\Lambda(\vartheta - 1)} + \frac{(\vartheta - 1)[\varrho(\mathfrak{z}) - \varrho(\iota)]^{\vartheta}}{\Lambda(\vartheta - 1)\Gamma(\vartheta + 1)}\right].
\end{aligned} \tag{5.5}$$

Now, for $\varpi, \hat{\varpi} \in \Omega$, by using Eqs (3.3), (5.5) and H_1, we obtain

$$|\hat{\varpi}(\upsilon) - \varpi(\upsilon)|$$

$$
\begin{aligned}
&= \bigg| \hat{\varpi}(\upsilon) - \gamma_1 \left[1 - \frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))} \right] - \frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))} \bigg[\gamma_2 \int_\iota^{\mathfrak{z}} \mathfrak{f}(\kappa, \varpi(\kappa)) d\kappa \\
&\quad - {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho} \mathfrak{g}\left(\mathfrak{z}, \varpi(\mathfrak{z}), \varpi(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho} \varpi(\mathfrak{z}) \right) \bigg] - {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho} \mathfrak{g}\left(\upsilon, \varpi(\upsilon), \varpi(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho} \varpi(\upsilon) \right) \bigg| \\
&\leq \bigg| \hat{\varpi}(\upsilon) - \gamma_1 \left[1 - \frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))} \right] - \frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))} \bigg[\gamma_2 \int_\iota^{\mathfrak{z}} \mathfrak{f}(\kappa, \hat{\varpi}(\kappa)) d\kappa \\
&\quad - {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho} \mathfrak{g}\left(\mathfrak{z}, \hat{\varpi}(\mathfrak{z}), \hat{\varpi}(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho} \hat{\varpi}(\mathfrak{z}) \right) \bigg] - {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho} \mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho} \hat{\varpi}(\upsilon) \right) \bigg| \\
&\quad + \frac{(\varrho(\upsilon)-\varrho(\iota))}{(\varrho(\mathfrak{z})-\varrho(\iota))} \bigg[|\gamma_2| \int_\iota^{\mathfrak{z}} \left| \mathfrak{f}(\kappa, \hat{\varpi}(\kappa)) - \mathfrak{f}(\kappa, \varpi(\kappa)) \right| d\kappa \\
&\quad + {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho} \left| \mathfrak{g}\left(\mathfrak{z}, \hat{\varpi}(\mathfrak{z}), \hat{\varpi}(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho} \hat{\varpi}(\mathfrak{z}) \right) - \mathfrak{g}\left(\mathfrak{z}, \varpi(\mathfrak{z}), \varpi(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho} \varpi(\mathfrak{z}) \right) \right| \bigg] \\
&\quad + {}^{\mathcal{AB}}\mathfrak{I}_\iota^{\vartheta,\varrho} \left| \mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho} \hat{\varpi}(\upsilon) \right) - \mathfrak{g}\left(\upsilon, \varpi(\upsilon), \varpi(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho} \varpi(\upsilon) \right) \right| \\
&\leq 2\,\varepsilon \left[\frac{(2-\vartheta)[\varrho(\mathfrak{z})-\varrho(\iota)]}{\Lambda(\vartheta-1)} + \frac{(\vartheta-1)[\varrho(\mathfrak{z})-\varrho(\iota)]^\vartheta}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)} \right] \\
&\quad + \left\{ |\gamma_2|(\mathfrak{z}-\iota)\ell_3 + \frac{4\ell_1}{1-\ell_2} \left[\frac{(2-\vartheta)[\varrho(\mathfrak{z})-\varrho(\iota)]}{\Lambda(\vartheta-1)} + \frac{(\vartheta-1)[\varrho(\mathfrak{z})-\varrho(\iota)]^\vartheta}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)} \right] \right\} \|\hat{\varpi}-\varpi\| \\
&\leq 2\,\varepsilon \left[\frac{(2-\vartheta)[\varrho(\mathfrak{z})-\varrho(\iota)]}{\Lambda(\vartheta-1)} + \frac{(\vartheta-1)[\varrho(\mathfrak{z})-\varrho(\iota)]^\vartheta}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)} \right] + \beta_1 \|\hat{\varpi}-\varpi\|,
\end{aligned}
$$

which further implies

$$
\|\hat{\varpi}-\varpi\| \leq \frac{\beta_4}{1-\beta_1}\varepsilon, \tag{5.6}
$$

where

$$
\beta_4 := 2 \left[\frac{(2-\vartheta)[\varrho(\mathfrak{z})-\varrho(\iota)]}{\Lambda(\vartheta-1)} + \frac{(\vartheta-1)[\varrho(\mathfrak{z})-\varrho(\iota)]^\vartheta}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)} \right].
$$

Thus, yields that

$$
\|\hat{\varpi}-\varpi\| \leq \chi_{\mathfrak{g}}\,\varepsilon; \quad \chi_{\mathfrak{g}} := \frac{\beta_4}{1-\beta_1}.
$$

Therefore, the solution of the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4) is $\mathbb{UH}$ stable. Moreover, there is an increasing function $\alpha : (0,\infty) \to (0,\infty)$, where $\alpha(\varepsilon) = \varepsilon$ with $\alpha(0) = 0$; thus from (5.6), we obtain

$$
\|\hat{\varpi}-\varpi\| \leq \chi_{\mathfrak{g}}\alpha(\varepsilon).
$$

Hence, the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4) is generalized $\mathbb{UH}$ stable. □

Next, before discussing the $\mathbb{UHR}$ stability, we present the following concepts:

Definition 5.5. *([52]) The ϱ-$\mathcal{ABC}$ fractional pantograph implicit problem (1.4) is called $\mathbb{UHR}$ stable with respect to $\theta \in C(\mathrm{J},\mathbb{R})$, if there is a constant $\chi_{\mathfrak{g},\theta} > 0$, $\forall\, \varepsilon > 0$, for all $\hat{\varpi} \in \Omega$ which satisfies*

$$
\left| {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho} \hat{\varpi}(\upsilon) - \mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_\iota^{\vartheta,\varrho} \hat{\varpi}(\upsilon) \right) \right| \leq \varepsilon\theta(\upsilon), \quad \upsilon \in [\iota, \mathfrak{z}], \tag{5.7}
$$

then there is one solution $\varpi \in \Omega$ of the Eq (1.4) which satisfies

$$|\varpi(\upsilon) - \hat{\varpi}(\upsilon)| \leq \chi_{\mathfrak{g},\theta}\varepsilon\theta(\upsilon), \quad \upsilon \in [\iota, \mathfrak{z}]. \tag{5.8}$$

Definition 5.6. *([52]) The ϱ-$\mathcal{ABC}$ fractional pantograph implicit problem (1.4) is called generalized $\mathbb{UHR}$ stable with respect to $\theta \in C(\mathrm{J}, \mathbb{R})$, if there is a constant $\chi_{\mathfrak{g},\theta} > 0$, for all $\hat{\varpi} \in \Omega$ satisfy (5.7), then there is one solution $\varpi \in \Omega$ of the Eq (1.4) which satisfies*

$$|\hat{\varpi}(\upsilon) - \varpi(\upsilon)| \leq \chi_{\mathfrak{g},\theta}\theta(\upsilon), \quad \upsilon \in [\iota, \mathfrak{z}]. \tag{5.9}$$

Remark 5.7. *For $\varepsilon > 0$, a function $\hat{\varpi} \in \Omega$ is a solution of the inequality (5.7), for all $\upsilon \in [\iota, \mathfrak{z}]$ iff there is a function $\zeta \in \Omega$ such that*
i) $|\zeta(\upsilon)| \leq \varepsilon\theta(\upsilon), \quad \upsilon \in [\iota, \mathfrak{z}]$;
ii) ${}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon) = \mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right) + \zeta(\upsilon)$.

Remark 5.8. *There exists a real number $\lambda_\theta > 0$ and non-decreasing function $\theta(\upsilon) \in \Omega$ such that ${}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\theta(\upsilon) \leq \lambda_\theta\theta(\upsilon), \forall \upsilon \in [\iota, \mathfrak{z}]$.*

Theorem 5.9. *If H_1 holds, subject to $\beta_1 < 1$, then the unique solution of the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4) is $\mathbb{UHR}$ stable and is consequently generalized $\mathbb{UHR}$ stable.*

Proof. Suppose that $\hat{\varpi} \in \Omega$ verifies the inequality (5.7). Thus, due to the Remark 5.7, we find

$${}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon) = \mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right) + \zeta(\upsilon), \ \forall \upsilon \in [\iota, \mathfrak{z}].$$

Based on the Eq (3.3), we have

$$\begin{aligned}
\hat{\varpi}(\upsilon) = {} & \gamma_1\left[1 - \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\right] + \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\Bigg[\gamma_2\int_{\iota}^{\mathfrak{z}} \mathfrak{f}(\kappa, \hat{\varpi}(\kappa))d\kappa \\
& - {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\mathfrak{z}, \hat{\varpi}(\mathfrak{z}), \hat{\varpi}(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\mathfrak{z})\right) - {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\rho(\mathfrak{z})\Bigg] \\
& + {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right) + {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\zeta(\upsilon).
\end{aligned} \tag{5.10}$$

In view of the Remarks 5.7 and 5.8, it is implied that

$$\begin{aligned}
& \Bigg|\hat{\varpi}(\upsilon) - \gamma_1\left[1 - \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\right] - \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\Bigg[\gamma_2\int_{\iota}^{\mathfrak{z}} \mathfrak{f}(\kappa, \hat{\varpi}(\kappa))d\kappa \\
& \quad - {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\mathfrak{z}, \hat{\varpi}(\mathfrak{z}), \hat{\varpi}(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\mathfrak{z})\right)\Bigg] - {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right)\Bigg| \\
& \leq \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}{}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}|\zeta(\mathfrak{z})| + {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}|\zeta(\upsilon)| \\
& \leq \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\varepsilon\, {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\theta(\mathfrak{z}) + \varepsilon\, {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\theta(\upsilon) \\
& \leq 2\varepsilon\, \lambda_\theta\theta(\upsilon).
\end{aligned} \tag{5.11}$$

Now, for $\varpi, \hat{\varpi} \in \Omega$, by using Eqs (3.3), (5.11) and H_1, we obtain

$$|\hat{\varpi}(\upsilon) - \varpi(\upsilon)|$$

$$
\begin{aligned}
&\leq \bigg|\hat{\varpi}(\upsilon) - \gamma_1\left[1 - \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\right] - \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\bigg[\gamma_2 \int_{\iota}^{\mathfrak{z}} \mathfrak{f}(\kappa, \hat{\varpi}(\kappa))d\kappa \\
&\quad - {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\mathfrak{z}, \hat{\varpi}(\mathfrak{z}), \hat{\varpi}(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\mathfrak{z})\right)\bigg] - {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right)\bigg| \\
&\quad + \frac{(\varrho(\upsilon) - \varrho(\iota))}{(\varrho(\mathfrak{z}) - \varrho(\iota))}\bigg[|\gamma_2| \int_{\iota}^{\mathfrak{z}} \Big|\mathfrak{f}(\kappa, \hat{\varpi}(\kappa)) - \mathfrak{f}(\kappa, \varpi(\kappa))\Big|d\kappa \\
&\quad + {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\Big|\mathfrak{g}\left(\mathfrak{z}, \hat{\varpi}(\mathfrak{z}), \hat{\varpi}(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\mathfrak{z})\right) - \mathfrak{g}\left(\mathfrak{z}, \varpi(\mathfrak{z}), \varpi(\delta\mathfrak{z}), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\mathfrak{z})\right)\Big|\bigg] \\
&\quad + {}^{\mathcal{AB}}\mathfrak{I}_{\iota}^{\vartheta,\varrho}\Big|\mathfrak{g}\left(\upsilon, \hat{\varpi}(\upsilon), \hat{\varpi}(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\hat{\varpi}(\upsilon)\right) - \mathfrak{g}\left(\upsilon, \varpi(\upsilon), \varpi(\delta\upsilon), {}^{\mathcal{ABC}}\mathfrak{D}_{\iota}^{\vartheta,\varrho}\varpi(\upsilon)\right)\Big| \\
&\leq 2\varepsilon\, \lambda_\theta\theta(\upsilon) + \beta_1 \left\|\hat{\varpi} - \varpi\right\|,
\end{aligned}
$$

which yields

$$
\|\hat{\varpi} - \varpi\| \leq \frac{2\, \lambda_\theta\theta(\upsilon)}{1 - \beta_1}\varepsilon. \tag{5.12}
$$

Hence, we have

$$
\|\hat{\varpi} - \varpi\| \leq \chi_{\mathfrak{g},\theta}\, \varepsilon\theta(\upsilon), \text{ such that } \chi_{\mathfrak{g},\theta} := \frac{2\, \lambda_\theta}{1 - \beta_1}.
$$

Then, the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4) is $\mathbb{UHR}$ stable. Additionally, if $\varepsilon = 1$, the solution of the ϱ-$\mathcal{ABC}$ fractional pantograph implicit system (1.4) is generalized $\mathbb{UHR}$ stable. □

6. Examples

Throughout this section, we present two examples for testing the effectiveness of our findings.

Example 6.1. *Consider the following ϱ-$\mathcal{ABC}$ fractional pantograph implicit system:*

$$
\begin{cases}
{}^{\mathcal{ABC}}\mathfrak{D}_{1}^{1.5,\varrho}\varpi(\upsilon) = \dfrac{\sqrt{\upsilon}}{1 + \upsilon^3} + \dfrac{\sin(\varpi(\upsilon))}{\frac{1}{9} + \varpi(\upsilon)} + \dfrac{\varpi(\frac{1}{4}\upsilon)}{\frac{1}{9} + \varpi(\frac{1}{4}\upsilon)} + \dfrac{1}{4}{}^{\mathcal{ABC}}\mathfrak{D}_{1}^{1.5,\varrho}\varpi(\upsilon), \upsilon \in [1, e], \\
\varpi(1) = 1, \quad \varpi(e) = 0.5 \displaystyle\int_1^e \frac{\varpi(\kappa)}{\log(\sqrt{\kappa}) + \varpi(\kappa)}d\kappa.
\end{cases} \tag{6.1}
$$

Here, $\vartheta = 1.5, \iota = 1, \mathfrak{z} = e, \gamma_1 = 1, \gamma_2 = 0.5, \varrho(\upsilon) = \log(\upsilon), \mathfrak{f}(\upsilon, \varpi) = \dfrac{\varpi(\upsilon)}{\log(\sqrt{\upsilon}) + \varpi(\upsilon)}$, *and*

$$
\mathfrak{g}\left(\upsilon, \varpi, \varpi, \varpi\right) = \frac{\sqrt{\upsilon}}{1 + \upsilon^3} + \frac{\sin(\varpi(\upsilon))}{\frac{1}{9} + \varpi(\upsilon)} + \frac{\varpi(\frac{1}{4}\upsilon)}{\frac{1}{9} + \varpi(\frac{1}{4}\upsilon)} + \frac{1}{4}{}^{\mathcal{ABC}}\mathfrak{D}_{1}^{1.5,\varrho}\varpi(\upsilon).
$$

Hence,

$$
|\mathfrak{f}(\upsilon, \varpi) - \mathfrak{f}(\upsilon, \hat{\varpi})| \leq \left|\frac{\varpi}{\log(\sqrt{\upsilon}) + \varpi} - \frac{\hat{\varpi}}{\log(\sqrt{\upsilon}) + \hat{\varpi}}\right| \leq \frac{1}{2}|\varpi - \hat{\varpi}|,
$$

and

$$
|\mathfrak{g}\left(\upsilon, \varpi_1, \varpi_2, \varpi_3\right) - \mathfrak{g}\left(\upsilon, \hat{\varpi}_1, \hat{\varpi}_2, \hat{\varpi}_3\right)|
$$

$$\leq \left|\frac{\sin(\varpi_1)}{\frac{1}{9}+\varpi_1} - \frac{\sin(\hat{\varpi}_1)}{\frac{1}{9}+\hat{\varpi}_1}\right| + \left|\frac{\varpi_2}{\frac{1}{9}+\varpi_2} - \frac{\hat{\varpi}_2}{\frac{1}{9}+\hat{\varpi}_2}\right| + \frac{1}{4}\left|\varpi_3 - \hat{\varpi}_3\right|$$
$$\leq \frac{1}{9}\left|\varpi_1 - \hat{\varpi}_1\right| + \frac{1}{9}\left|\varpi_2 - \hat{\varpi}_2\right| + \frac{1}{4}\left|\varpi_3 - \hat{\varpi}_3\right|.$$

Thus, we have $\ell_1 = \frac{1}{9}, \ell_2 = \frac{1}{4}$, *and* $\ell_3 = \frac{1}{2}$. *Moreover, for* $\Lambda(\vartheta - 1) = 1$, *by the Mathematica software, we can calculate the following:*

$$\beta_1 := \left\{|\gamma_2|(\mathfrak{z} - \iota)\ell_3 + \frac{4\ell_1}{1-\ell_2}\left[\frac{(2-\vartheta)[\varrho(\mathfrak{z}) - \varrho(\iota)]}{\Lambda(\vartheta-1)} + \frac{(\vartheta-1)[\varrho(\mathfrak{z}) - \varrho(\iota)]^{\vartheta}}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)}\right]\right\} \approx 0.948756 < 1.$$

Therefore, based on the Theorem 4.1, the ϱ-$\mathcal{ABC}$ *fractional pantograph implicit system* (6.1) *possesses exactly one solution on* $[1, e]$. *Furthermore, in view of Theorem 5.4, the solution is* $\mathbb{UH}$ *stable with* $\chi_{\mathfrak{g}} \approx 20.263474$, *and is consequently generalized* $\mathbb{UH}$ *stable. Analogously, we can easily draw a conclusion that* $\mathbb{UHR}$ *and generalized* $\mathbb{UHR}$ *stability conditions by obtaining a non-decreasing function* $\theta(\upsilon) = \upsilon, \forall \upsilon \in [1, e]$.

Example 6.2. *Consider the following* ϱ-$\mathcal{ABC}$ *fractional pantograph implicit system:*

$$\begin{cases} {}^{\mathcal{ABC}}\mathfrak{D}_0^{1.8,\varrho}\varpi(\upsilon) = \dfrac{1}{10e^{\upsilon^3}} + \dfrac{1}{15}\tan^{-1}(\varpi(\upsilon)) + \dfrac{\varpi(\frac{1}{2}\upsilon)}{36+12\varpi(\frac{1}{2}\upsilon)} + \dfrac{{}^{\mathcal{ABC}}\mathfrak{D}_0^{1.8,\varrho}\varpi(\upsilon)}{4+2\,{}^{\mathcal{ABC}}\mathfrak{D}_0^{1.8,\varrho}\varpi(\upsilon)}, \upsilon \in [0,1], \\ \varpi(0) = 0, \quad \varpi(1) = 0.25\displaystyle\int_0^1 \left(\frac{\kappa}{3} + \frac{\cos(\varpi(\kappa))}{1+\varpi(\kappa)}\right) d\kappa. \end{cases} \tag{6.2}$$

Here, $\vartheta = 1.8, \iota = 0, \mathfrak{z} = 1, \gamma_1 = 0, \gamma_2 = 0.25, \varrho(\upsilon) = \upsilon^2, \mathfrak{f}(\upsilon, \varpi) = \dfrac{\upsilon}{3} + \dfrac{\cos(\varpi(\upsilon))}{1+\varpi(\upsilon)}$, *and*

$$\mathfrak{g}(\upsilon, \varpi, \varpi, \varpi) = \frac{1}{10e^{\upsilon^3}} + \frac{1}{15}\tan^{-1}(\varpi(\upsilon)) + \frac{\varpi(\frac{1}{2}\upsilon)}{36+12\varpi(\frac{1}{2}\upsilon)} + \frac{{}^{\mathcal{ABC}}\mathfrak{D}_0^{1.8,\varrho}\varpi(\upsilon)}{4+2\,{}^{\mathcal{ABC}}\mathfrak{D}_0^{1.8,\varrho}\varpi(\upsilon)}.$$

Hence,

$$|\mathfrak{f}(\upsilon, \varpi)| \leq \left|\frac{\upsilon}{3} + \frac{\cos(\varpi)}{1+\varpi}\right| \leq \frac{1}{3} + |\varpi|,$$

and

$$\begin{aligned} &|\mathfrak{g}(\upsilon, \varpi_1, \varpi_2, \varpi_3)| \\ &= \left|\frac{1}{10e^{\upsilon^3}} + \frac{1}{15}\tan^{-1}(\varpi_1) + \frac{\varpi_2}{36+12\varpi_2} + \frac{\varpi_3}{4+2\varpi_3}\right| \\ &\leq \frac{1}{10} + \frac{1}{12}(|\varpi_1| + |\varpi_2|) + \frac{1}{2}|\varpi_3|. \end{aligned}$$

Then, we obtain $\xi_1 = \frac{1}{10}, \xi_2 = \frac{1}{12}, \xi_3 = \frac{1}{2}, \xi_4 = \frac{1}{3}$, *and* $\xi_5 = 1$. *Furthermore, by putting* $\Lambda(\vartheta - 1) = 1$, *and using the Mathematica software, we find the following:*

$$\beta_3 := \xi_5|\gamma_2|(\mathfrak{z} - \iota) + \frac{4\xi_2}{1-\xi_3}\left[\frac{(2-\vartheta)[\varrho(\mathfrak{z}) - \varrho(\iota)]}{\Lambda(\vartheta-1)} + \frac{(\vartheta-1)[\varrho(\mathfrak{z}) - \varrho(\iota)]^{\vartheta}}{\Lambda(\vartheta-1)\Gamma(\vartheta+1)}\right] \approx 0.701456 < 1.$$

Hence, in view of the Theorem 4.2, the ϱ-$\mathcal{ABC}$ *fractional pantograph implicit system* (6.2) *possesses a solution on* $[0, 1]$.

7. Conclusions

The current study was devoted to investigate the existence, uniqueness and several types of $\mathbb{UH}$ stability of the solution for a new class of ϱ-$\mathcal{ABC}$ factional pantograph implicit systems under integral boundary conditions (1.4). The Scheafer and Banach $\mathbb{FPT}s$ were employed to establish sufficient conditions of the existence and uniqueness results of such system (1.4). The $\mathbb{UH}$, generalized $\mathbb{UH}$, $\mathbb{UHR}$ and generalized $\mathbb{UHR}$ stabilities were discussed by non-linear analysis themes and fractional calculus. Finally, two mathematical examples were enhanced to examine the validity of the main outcomes. In fact, with respect to another function which was used in this study, the $\mathcal{ABC}$ fractional operator is more generalized and supports our system to be more extensive and covers several of new and existing problems in the literature. As a future direction, the studied problem would be interesting if it was studied with an integral term and under impulsive boundary conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. M. I. Abbas, M. Ghaderi, S. Rezapour, S. T. M. Thabet, On a coupled system of fractional differential equations via the generalized proportional fractional derivatives, *J. Funct. Space.*, **2022** (2022), 4779213. https://doi.org/10.1155/2022/4779213
2. M. S. Abdo, T. Abdeljawad, K. D. Kucche, M. A. Alqudah, S. M. Ali, M. B. Jeelani, On nonlinear pantograph fractional differential equations with Atangana-Baleanu-Caputo derivative, *Adv. Diff. Equ.*, **2021** (2021), 65. https://doi.org/10.1186/s13662-021-03229-8
3. S. T. M. Thabet, M. B. Dhakne, On boundary value problems of higher order abstract fractional integro-differential equations, *Int. J. Nonlinear Anal. Appl.*, **7** (2016), 165–184.
4. M. S. Abdo, T. Abdeljawad, K. Shah, F. Jarad, Study of impulsive problems under Mittag-Leffler power law, *Heliyon*, 2020, 6e05109. https://doi.org/10.1016/j.heliyon.2020.e05109

5. R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, Singapore, 2000.

6. S. T. M. Thabet, M. B. Dhakne, M. A. Salman, R. Gubran, Generalized fractional Sturm-Liouville and Langevin equations involving Caputo derivative with nonlocal conditions, *Progr. Fract. Differ. Appl.*, **6** (2020), 225–237. https://doi.org/10.18576/pfda/060306

7. O. Nikan, S. M. Molavi-Arabshai, H. Jafari, Numerical simulation of the nonlinear fractional regularized long-wave model arising in ion acoustic plasma waves, *Discrete Cont. Dyn.-S*, **14** (2021), 3685–3701. https://doi.org/10.3934/dcdss.2020466

8. N. H. Can, O. Nikan, M. N. Rasoulizadeh, H. Jafari, Y. S. Gasimov, Numerical computation of the time non-linear fractional generalized equal width model arising in shallow water channel, *Therm. Sci.*, **24** (2020), 49–58. https://doi.org/10.2298/TSCI20S1049C

9. W. Qiu, D. Xu, H. Chen, J. Guo, An alternating direction implicit Galerkin finite element method for the distributed-order time-fractional mobile-immobile equation in two dimensions, *Comput. Math. Appl.*, **80** (2020), 3156–3172. https://doi.org/10.1016/j.camwa.2020.11.003

10. X. Yang, W. Qiu, H. Zhang, L. Tang, An efficient alternating direction implicit finite difference scheme for the three-dimensional time-fractional telegraph equation, *Comput. Math. Appl.*, **102** (2021), 233–247. https://doi.org/10.1016/j.camwa.2021.10.021

11. W. Qiu, D. Xu, J. Guo, J. Zhou, A time two-grid algorithm based on finite difference method for the two-dimensional nonlinear time-fractional mobile/immobile transport model, *Numer. Algorithms*, **85** (2020), 39–58. https://doi.org/10.1007/s11075-019-00801-y

12. S. T. M. Thabet, M. M. Matar, M. A. Salman, M. E. Samei, M. Vivas-Cortez, I. Kedim, On coupled snap system with integral boundary conditions in the $\mathbb{G}$-Caputo sense, *AIMS Math.*, **8** (2023), 12576–12605. https://doi.org/10.3934/math.2023632

13. F. Jarad, T. Abdeljawad, Z. Hammouch, On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative, *Chaos Soliton. Fract.*, **117** (2018), 16–20.

14. S. T. M. Thabet, B. Ahmad, R. P. Agarwal, On abstract Hilfer fractional integrodifferential equations with boundary conditions, *Arab J. Math. Sci.*, **26** (2020), 107–125. https://doi.org/10.1016/j.ajmsc.2019.03.001

15. S. Rezapour, S. T. M. Thabet, M. M. Matar, J. Alzabut, S. Etemad, Some existence and stability criteria to a generalized FBVP having fractional composite p-Laplacian operator, *J. Funct. Space.*, **2021** (2021), 9554076. https://doi.org/10.1155/2021/9554076

16. S. T. M. Thabet, M. B. Dhakne, On positive solutions of higher order nonlinear fractional integro-differential equations with boundary conditions, *Malaya J. Mat.*, **7** (2019), 20–26. https://doi.org/10.26637/MJM0701/0005

17. M. S. Abdo, Boundary value problem for fractional neutral differential equations with infinite delay, *Abh. J. Basic Appl. Sci.*, **1** (2022), 1–18. https://doi.org/10.59846/ajbas.v1i1.357

18. A. S. Rafeeq, Periodic solution of Caputo-Fabrizio fractional integro-differential equation with periodic and integral boundary conditions, *Eur. J. Pure Apppl. Math.*, **15** (2022), 144–157. https://doi.org/10.29020/nybg.ejpam.v15i1.4247

19. S. T. M. Thabet, I. Kedim, Study of nonlocal multiorder implicit differential equation involving Hilfer fractional derivative on unbounded domains, *J. Math.*, **2023** (2023), 1–14. https://doi.org/10.1155/2023/8668325

20. A. Boutiara, S. Etemad, S. T. M. Thabet, S. K. Ntouyas, S. Rezapour, J. Tariboon, A mathematical theoretical study of a coupled fully hybrid (κ, ϕ)-fractional order system of BVPs in generalized Banach spaces, *Symmetry*, **15** (2023), 1041. https://doi.org/10.3390/sym15051041

21. S. T. M. Thabet, M. B. Dhakne, On nonlinear fractional integro-differential equations with two boundary conditions, *Adv. Stud. Contemp. Math.*, **26** (2016), 513–526.

22. A. Ali, I. Mahariq, K. Shah, T. Abdeljawad, B. Al-Sheikh, Stability analysis of initial value problem of pantograph-type implicit fractional differential equations with impulsive conditions, *Adv. Diff. Equ.*, **2021** (2021), 55. https://doi.org/10.1186/s13662-021-03218-x

23. S. T. M. Thabet, S. Al-Sádi, I. Kedim, A. S. Rafeeq, S. Rezapour, Analysis study on multi-order ϱ-Hilfer fractional pantograph implicit differential equation on unbounded domains, *AIMS Math.*, **8** (2023), 18455–18473. https://doi.org/10.3934/math.2023938

24. M. Houas, K. Kaushik, A. Kumar, A. Khan, T. Abdeljawad, Existence and stability results of pantograph equation with three sequential fractional derivatives, *AIMS Math.*, **8** (2022), 5216–5232. https://doi.org/10.3934/math.2023262

25. Z. H. Yu, Variational iteration method for solving the multi-pantograph delay equation, *Phys. Lett. A*, **372** (2008), 6475–6479. https://doi.org/10.1016/j.physleta.2008.09.013

26. K. Balachandran, S. Kiruthika, J. J. Trujillo, Existence of solution of nonlinear fractional pantograph equations, *Acta Math. Sci.*, **33** (2013), 712–720. https://doi.org/10.1016/S0252-9602(13)60032-6

27. E. Tohidi, A. H. Bhrawy, K. Erfani, A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation, *Appl. Math. Model.*, **37** (2013), 4283–4294. https://doi.org/10.1016/j.apm.2012.09.032

28. A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769. https://doi.org/10.2298/TSCI160111018A

29. T. Abdeljawad, A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel, *J. Inequal. Appl.*, **2017** (2017), 130. https://doi.org/10.1186/s13660-017-1400-5

30. T. Abdeljawad, D. Baleanu, Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel, *J. Nonlinear Sci. Appl.*, **9** (2017), 1098–1107.

31. M. I. Ayari, S. T. M. Thabet, Qualitative properties and approximate solutions of thermostat fractional dynamics system via a nonsingular kernel operator, *Arab J. Math. Sci.*, 2023. https://doi.org/10.1108/AJMS-06-2022-0147

32. M. Khan, Z. Ahmad, F. Ali, N. Khan, I. Khan, K. S. Nisar, Dynamics of two-step reversible enzymatic reaction under fractional derivative with Mittag-Leffler Kernel, *Plos One*, **18** (2023), e0277806. https://doi.org/10.1371/journal.pone.0277806

33. S. Rashid, Z. Hammouch, R. Ashraf, Y. M. Chu, New computation of unified bounds via a more general fractional operator using generalized Mittag-Leffler function in the kernel, *Comput. Model. Eng. Sci.*, **126** (2021), 359–378. https://doi.org/10.32604/cmes.2021.011782

34. Y. M. Chu, M. F. Khan, S. Ullah, S. A. Shah, M. Farooq, M. B. Mamat, Mathematical assessment of a fractional-order vector-host disease model with the Caputo-Fabrizio derivative, *Math. Method. Appl. Sci.*, **46** (2023), 232–247. https://doi.org/10.1002/mma.8507

35. S. Rashid, Z. Hammouch, D. Baleanu, Y. M. Chu, New generalizations in the sense of the weighted non-singular fractional integral operator, *Fractals*, **28** (2020), 2040003. https://doi.org/10.1142/S0218348X20400034

36. F. Jin, Z. S. Qian, Y. M. Chu, M. Rahman, On nonlinear evolution model for drinking behavior under Caputo-Fabrizio derivative, *J. Appl. Anal. Comput.*, **12** (2022), 790–806. https://doi.org/10.11948/20210357

37. M. D. Ikram, M. A. Imran, Y. M. Chu, A. Akgül, MHD flow of a Newtonian fluid in symmetric channel with ABC fractional model containing hybrid nanoparticles, *Comb. Chem. High T. Scr.*, **25** (2022), 1087–1102. https://doi.org/10.2174/1386207324666210412122544

38. A. Fernandez, D. Baleanu, Differintegration with respect to functions in fractional models involving Mittag-Leffler functions, *SSRN Electron. J.*, 2018. https://doi.org/10.2139/ssrn.3275746

39. T. Abdeljawad, S. T. M. Thabet, I. Kedim, M. I. Ayari, A. Khan, A higher-order extension of Atangana-Baleanu fractional operators with respect to another function and a Gronwall-type inequality, *Bound. Value Probl.*, **2023** (2023), 49. https://doi.org/10.1186/s13661-023-01736-z

40. G. Ali, K. Shah, T. Abdeljawad, H. Khan, G. U. Rahman, A. Khan, On existence and stability results to a class of boundary value problems under Mittag-Leffler power law, *Adv. Diff. Equ.*, **2020** (2020), 407. https://doi.org/10.1186/s13662-020-02866-9

41. S. Asma, K. Shabbir, T. Shah, T. Abdeljawad, Stability analysis for a class of implicit fractional differential equations involving Atangana-Baleanu fractional derivative, *Adv. Diff. Equ.*, **2021** (2021), 395. https://doi.org/10.1186/s13662-021-03551-1

42. M. S. Abdo, T. Abdeljawad, S. M. Ali, K. Shah, On fractional boundary value problems involving fractional derivatives with Mittag-Leffler kernel and nonlinear integral conditions, *Adv. Diff. Equ.*, **2021** (2021), 37. https://doi.org/10.1186/s13662-020-03196-6

43. R. Almeida, A. B. Malinowska, M. T. T. Monteiro, Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, *Math. Method. Appl. Sci.*, **41** (2018), 336–352. https://doi.org/10.1002/mma.4617

44. P. O. Mohammed, T. Abdeljawad, Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel, *Adv. Diff. Equ.*, **2020** (2020), 363. https://doi.org/10.1186/s13662-020-02825-4

45. A. Granas, J. Dugundji, *Fixed point theory*, New York, Springer, 2003. https://doi.org/10.1007/978-0-387-21593-8

46. Y. Zhou, *Basic theory of fractional differential equations*, World Scientific, Singapore, 2014.

47. Z. Ahmad, F. Ali, N. Khan, I. Khan, Dynamics of fractal-fractional model of a new chaotic system of integrated circuit with Mittag-Leffler kernel, *Chaos Soliton. Fract.*, **153** (2021), 111602. https://doi.org/10.1016/j.chaos.2021.111602

48. S. M. Ulam, A problem of stability in functional equations, *Bull. Am. Math. Soc.*, **46** (1940), 1–14.

49. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA*, **27** (1941), 222–224. https://doi.org/10.1073/pnas.27.4.222

50. T. M. Rassias, On the stability of functional equations in Banach spaces, *J. Funct. Anal.*, **32** (1978), 168–175.

51. C. Wang, T. Xu, Hyers-Ulam stability of fractional linear differential equations involving Caputo fractional derivatives, *Appl. Math.*, **60** (2015), 383–393. https://doi.org/10.1007/s10492-015-0102-x

52. T. M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta Appl. Math.*, **62** (2000), 23–130. https://doi.org/10.1023/A:1006499223572

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