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Research article

Numerical scheme for estimating all roots of non-linear equations with applications

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Abstract: The roots of non-linear equations are a major challenge in many scientific and professional fields. This problem has been approached in a number of ways, including use of the sequential Newton's method and the traditional Weierstrass simultaneous iterative scheme. To approximate all of the roots of a given nonlinear equation, sequential iterative algorithms must use a deflation strategy because rounding errors can produce inaccurate results. This study aims to develop an efficient numerical simultaneous scheme for approximating all nonlinear equations' roots of convergence order 12. The numerical outcomes of the considered engineering problems show that, in terms of accuracy, validations, error, computational CPU time, and residual error, recently developed simultaneous methods perform better than existing methods in the literature.

Keywords: simultaneous methods; error graph; computational efficiency; computer algorithm
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1. Introduction

Numerical iterative methods for solving nonlinear equations have a long and rich history, dating back to ancient times. They have played a significant role in the development of mathematics and have a wide range of applications in science and engineering. Today, these methods continue to be an important area of research and development, as mathematicians and scientists seek to develop new and more efficient methods for solving nonlinear equations. These methods are particularly useful in situations in which it is difficult or impossible to find the exact roots of a polynomial by using analytical methods. Iterative methods are used in a wide range of applications, including engineering, science, finance, and computer science. One of the main areas where iterative methods for finding polynomial roots play a significant role is in signal processing. Signals, such as sound waves, images and videos, can be represented mathematically as polynomials. The roots of these polynomials correspond to the frequencies present in the signal [1,2]. Therefore, finding the roots of a polynomial is an essential task in signal processing, and iterative methods are commonly used to estimate the roots of the polynomial representing a signal. Another area where iterative methods are used for finding polynomial roots is in control systems. Control systems are used to regulate the behavior of physical systems such as machines, vehicles, and robots. The roots of the polynomial describing the behavior of a system can be used to design control strategies to stabilize or improve the system's performance. Iterative methods for finding polynomial roots are used to estimate the roots of the polynomial model of the system, enabling the design of control strategies that work in real-time.

Iterative methods are also used in finance, where they are used to estimate the roots of polynomial equations used in to price financial derivatives, such as options and futures. These financial instruments can be modeled mathematically as polynomials, and finding the roots of these polynomials is essential to pricing them accurately.

In an effort to simultaneously find all polynomial roots Cordero et al [3] used the Ehrlich method to develop a simultaneous method of convergence order 3p; Chinesta et al. [4] used a simultaneous method to solve a vectorial problem, Proinov and Vasileva [5] accelerated the convergence order of the Simultaneous-Weierstrass method; Zhang et al. [6] presented a fifth order simultaneous method with derivatives; Iliev and Semerdzhiev [7] generalized the Chebyshev method into the Chebyshev-simultaneous method, and many others. Iterative methods for approximating all roots of nonlinear equations have grown in prominence in recent years, due to their global convergence and parallel computer application (see, e.g., Proinov and Vasileva [8], Kanno et al. [9], Proinov and Cholakov [10], Weidner [11], Mir et al. [12], Farmer [13], Nourein [14], Aberth [15], Cholakov and Vasileva [16] and the references cited there in [17–20]).

Motivated by the aforementioned work, we develop a higher-order simultaneous method for solving nonlinear equations in this article. In the future, this article will assist other researchers in the further development of this topic.

The main contributions of this research works are

- A simultaneous numerical technique for locating all of the roots of scalar nonlinear equations is developed.
- The proposed numerical scheme for solving polynomial equations was subjected to a local convergence analysis.
- For some random initial guess values, we execute numerical simultaneous methods to show the behavior of global convergence.
- The efficiency, stability, and application of the suggested technique are evaluated using mathematical computational tools.

The method's applicability to various nonlinear engineering applications [21–23] is considered. In last few years, a lot of work is done on numerical iterative methods which approximate single at one time of nonlinear equation. Besides these single root estimating methods in literature, we found another class of derivative free iterative schemes which approximates all roots of nonlinear equations simultaneously.

Among the derivative free simultaneous methods, the Weierstrass-Dochive [24] method is the most attractive method, and it is given by

$$u_i^{[\varsigma]} = r_i^{[\varsigma]} - w(r_i^{[\varsigma]}), \tag{1.1}$$

where

$$w(r_i^{[\varsigma]}) = \frac{f(r_i^{[\varsigma]})}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(r_i^{[\varsigma]} - r_j^{[\varsigma]})}, \quad (i, j = 1, 2, 3, ..., n), \tag{1.2}$$

is Weierstrass' correction. Method (1.2) has a local quadratic convergence.

In 1977, Ehrlich presented the following convergent simultaneous method [25] of third order as:

$$u_i^{[\varsigma]} = r_i^{[\varsigma]} - \frac{1}{\frac{1}{N_i(r_i^{[\varsigma]})} - \sum\limits_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{1}{\left(r_i^{[\varsigma]} - r_j^{[\varsigma]}\right)}\right)}; \tag{1.3}$$

using $r_j^{[\varsigma]} = \overset{*}{u}_j^{[\varsigma]}$ as a correction in (1.3), Petkovic et al. [26] accelerated the convergence order of (1.3) from three to 6:

$$u_i^{[\varsigma]} = r_i^{[\varsigma]} - \frac{1}{\frac{1}{N_i(r_i^{[\varsigma]})} - \sum\limits_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{1}{\left(r_i^{[\varsigma]} - \overset{*}{u}_j^{[\varsigma]}\right)}\right)}, \tag{1.4}$$

where $\overset{*}{u}_j^{[\varsigma]} = r_j^{[\varsigma]} - \frac{f(s_j^{[\varsigma]}) - f(r_j^{[\varsigma]})}{2 * f(s_j^{[\varsigma]}) - f(r_j^{[\varsigma]})} \frac{f(r_j^{[\varsigma]})}{f'(r_j^{[\varsigma]})}$ and $s_j^{[\varsigma]} = r_j^{[\varsigma]} - \frac{f(r_j^{[\varsigma]})}{f'(r_j^{[\varsigma]})}$.

Petkovic et al. [27] accelerated the convergence order of (1.3) from three to 10 as (abbreviated as $\mathrm{MM}_{10\alpha}$):

$$x_i^{[\varsigma+1]} = x_i^{[\varsigma]} - \frac{1}{\frac{1}{N_i(x_i^{[\varsigma]})} - \sum\limits_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{1}{\left(x_i^{[\varsigma]} - Z_j^{[\varsigma]}\right)}\right)}, \tag{1.5}$$

where $Z_j^{[\varsigma]} = u_j^{[\varsigma]} - \frac{\left(y_j^{[\varsigma]} - u_j^{[\varsigma]}\right) f\left(u_j^{[\varsigma]}\right)\left(\frac{f\left(x_j^{[\varsigma]}\right)}{f'\left(x_j^{[\varsigma]}\right)}\right)}{\left(f\left(x_j^{[\varsigma]}\right) - f\left(u_j^{[\varsigma]}\right)\right)^2}\left[f\left(y_j^{[\varsigma]}\right) - \frac{f\left(x_j^{[\varsigma]}\right)}{f\left(y_j^{[\varsigma]}\right) - f\left(u_j^{[\varsigma]}\right)}\right]$, $u_j^{[\varsigma]} = y_j^{[\varsigma]} - \frac{f\left(x_j^{[\varsigma]}\right) f\left(y_j^{[\varsigma]}\right)\left(\frac{f\left(x_j^{[\varsigma]}\right)}{f'\left(x_j^{[\varsigma]}\right)}\right)}{\left(f\left(x_j^{[\varsigma]}\right) - f\left(y_j^{[\varsigma]}\right)\right)^2}$, $y_j^{[\varsigma]} = x_j^{[\varsigma]} - \frac{f\left(x_j^{[\varsigma]}\right)}{f'\left(x_j^{[\varsigma]}\right)}$.

Shams et al. [28] proposed the following three-step simultaneous scheme for finding all polynomial roots (abbreviated as $\mathrm{MM}_{12\alpha}$):

$$x_i^{[\varsigma+1]} = z_i^{[\varsigma]} - \frac{f(z_i^{[\varsigma]})}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(z_i^{[\varsigma]} - z_j^{[\varsigma]})} - \frac{f(z_i^{[\varsigma]})}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(z_i^{[\varsigma]} - z_j^{[\varsigma]})}, \tag{1.6}$$

where $z_i^{[\varsigma]} = y_i^{[\varsigma]} - \frac{f(y_i^{[\varsigma]})}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}(y_i^{[\varsigma]}-y_j^{[\varsigma]})}$, $y_i^{[\varsigma]} = x_i^{[\varsigma]} - \frac{f(x_i^{[\varsigma]})}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}(x_i^{[\varsigma]}-x_j^{*[\varsigma]})}$ and $x_j^{*[\varsigma]} = x_j^{[\varsigma]} - \frac{\alpha\left(f(x_i^{[\varsigma]})\right)^2}{f(x_i^{[\varsigma]}+\alpha f(x_i^{[\varsigma]}))-f(x_i^{[\varsigma]})}$; $\alpha \in R$. The order of convergence of the numerical scheme (1.6) is 12.

2. Construction of family of simultaneous methods for distinct roots

Here, we propose the following family of methods as (abbreviated as MM_{12}):

$$\left\{\begin{aligned} y^{[\varsigma]} &= x^{[\varsigma]} - \frac{f\left(x^{[\varsigma]}\right)}{f'(x^{[\varsigma]})},\\ z^{[\varsigma]} &= y^{[\varsigma]} - \frac{f\left(y^{[\varsigma]}\right)}{f'(y^{[\varsigma]})},\\ x^{[\varsigma+1]} &= z^{[\varsigma]} - \frac{f\left(z^{[\varsigma]}\right)}{f'(y^{[\varsigma]})+f\left(y^{[\varsigma]}\right)+f\left(z^{[\varsigma]}\right)}. \end{aligned}\right. \tag{2.1}$$

The following theorem proves the convergence order of the numerical scheme MM_{12}.

Theorem 1. *Assume that $\zeta \in I$ is the simple root of a sufficiently differential function $f : I \subseteq R \longrightarrow R$. The convergence order of (2.1) is six if x_0 is sufficiently close to ζ, and the error equation is given by*

$$e^{[\varsigma+1]} = (2\sigma_2^5 - \sigma_2^4)\left(e^{[\varsigma]}\right)^6 + O(e^{[\varsigma]}), \tag{2.2}$$

where $\sigma_m = \frac{f^m(\zeta)}{m!f'(\zeta)}, m \geq 2$.

Proof. Let ζ be a simple root of f and $x^{[\varsigma]} = \zeta + e^{[\varsigma]}, x^{[\varsigma+1]} = \zeta + e^{[\varsigma+1]}$. By Taylor series expansion about ζ, taking $f(\zeta) = 0$, we get

$$f(x^{[\varsigma]}) = f'(\zeta)(e^{[\varsigma]} + \sigma_2\left(e^{[k]}\right)^2 + \sigma_3\left(e^{[\varsigma]}\right)^3 + \sigma_4\left(e^{[k]}\right)^4 + \sigma_5\left(e^{[\varsigma]}\right)^5 + ...) \tag{2.3}$$

and

$$f'(x^{[\varsigma]}) = f'(\zeta)(1 + 2c_2\left(e^{[\varsigma]}\right) + 3\sigma_3\left(e^{[k]}\right)^2 + 4\sigma_4\left(e^{[\varsigma]}\right)^3 + ...). \tag{2.4}$$

Dividing (2.3) by (2.4), we have

$$\frac{f(x^{[\varsigma]})}{f'(x^{[\varsigma]})} = e^{[\varsigma]} + \sigma_2\left(e^{[\varsigma]}\right)^2 + (2\sigma_2^2 - 2\sigma_3)\left(e^{[\varsigma]}\right)^3 + (7\sigma_2\sigma_3 - 3\sigma_4 - 4\sigma_2^3)\left(e^{[\varsigma]}\right)^4 + ... \tag{2.5}$$

Using (2.5) in the first-step of (2.2), we have

$$y^{[\varsigma]} = \zeta + \sigma_2\left(e^{[\varsigma]}\right)^2 + (2\sigma_3 - 2\sigma_2^2)\left(e^{[\varsigma]}\right)^3 + ... \tag{2.6}$$

Thus, using a Taylor series, we have

$$f(y^{[\varsigma]}) = f'(\zeta)(\sigma_2\left(e^{[k]}\right)^2 + 2(\sigma_3 - \sigma_2^2)\left(e^{[\varsigma]}\right)^3 + (3\sigma_4 - 7\sigma_2\sigma_3 + 5\sigma_2^3)\left(e^{[\varsigma]}\right)^4 + ... \tag{2.7}$$

$$f'(y^{[\varsigma]}) = 1 + 2\sigma_2^2\left(e^{[\varsigma]}\right)^2 + 2(-2\sigma_2^2 + 2\sigma_3)\left(e^{[\varsigma]}\right)^3 \tag{2.8}$$

$$+(2\sigma_2(4c_2^3 - 7\sigma_2\sigma_3 + 3\sigma_4) + 3\sigma_2^2\sigma_3)\left(e^{[\varsigma]}\right)^4 + ...$$

This gives

$$\frac{f(y^{[\varsigma]})}{f'(y^{[\varsigma]})} = \sigma_2\left(e^{[\varsigma]}\right)^2 + (2\sigma_3 - 2\sigma_2^2)\left(e^{[\varsigma]}\right)^3 + (3\sigma_2^3 - 7\sigma_2\sigma_3 + 3\sigma_4)\left(e^{[\varsigma]}\right)^4 \tag{2.9}$$

$$+6\sigma_2\sigma_3(-2\sigma_2^2 + 2\sigma_3)\left(e^{[\varsigma]}\right)^5 + O\left(\left(e^{[\varsigma]}\right)^6\right)$$

$$z^{[\varsigma]} = \zeta + (\sigma_2^3)\left(e^{[\varsigma]}\right)^4 + (-4\sigma_2^4 + 4\sigma_2^2\sigma_3)\left(e^{[\varsigma]}\right)^5 + ... \tag{2.10}$$

Adding (2.3), (2.7) and a function of (2.10), we have

$$f'(y^{[\varsigma]}) + f(y^{[\varsigma]}) + f(z^{[\varsigma]}) = 1 + (2\sigma_2^2 + \sigma_2)\left(e^{[\varsigma]}\right)^2 + (-4\sigma_2^3 - 2\sigma_2^2 + 4\sigma_2\sigma_3 + 2\sigma_3)\left(e^{[\varsigma]}\right)^3 + ... \tag{2.11}$$

$$\left(\frac{f(z^{[\varsigma]})}{f'(y^{[\varsigma]}) + f(y^{[\varsigma]}) + f(z^{[\varsigma]})}\right) = \sigma_2^3\left(e^{[\varsigma]}\right)^4 + (-4\sigma_2^4 + 4\sigma_2^2\sigma_3)\left(e^{[\varsigma]}\right)^5 + ... \tag{2.12}$$

This implies that

$$x^{[\varsigma+1]} = \zeta + (2\sigma_2^5 - \sigma_2^4)\left(e^{[\varsigma]}\right)^6 + O\left(e^{[\varsigma]}\right)^7. \tag{2.13}$$

Hence we arrive at the desired result. □

Using (1.2), we convert (2.1) into a simultaneous iterative method for approximating all nonlinear equation roots as follows:

$$w_i^{[\varsigma]} = z_i^{[\varsigma]} - \frac{f\left(z_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(z_i^{[\varsigma]} - z_j^{[\varsigma]}\right)}\left[\frac{1}{\Psi^{[\varsigma]} * \left(1 + \frac{f\left(y_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(y_i^{[\varsigma]} - y_j^{[\varsigma]}\right)} + \frac{f\left(z_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(z_i^{[\varsigma]} - z_j^{[\varsigma]}\right)}\right)}\right], \tag{2.14}$$

where $y_i^{[\varsigma]} = x_i^{[\varsigma]} - \frac{f\left(x_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(x_i^{[\varsigma]} - x_j^{[\varsigma]} + \frac{f\left(x_i^{[\varsigma]}\right)}{f'\left(x_i^{[\varsigma]}\right)}\frac{1}{\left[1-\alpha\left[\frac{f\left(x_i^{[\varsigma]}\right)}{1+f\left(x_i^{[\varsigma]}\right)}\right]\right]}\right)}$, $z_i^{[\varsigma]} = y_i^{[\varsigma]} - \frac{f\left(y_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(y_i^{[\varsigma]} - y_j^{[\varsigma]}\right)}$. Therefore

$$\begin{cases} y_i^{[\varsigma]} = x_i^{[\varsigma]} - w_i^*\left(x_i^{[\varsigma]}\right), \\ z_i^{[\varsigma]} = y_i^{[\varsigma]} - w_i\left(y_i^{[\varsigma]}\right), \\ w_i^{[\varsigma]} = z_i^{[\varsigma]} - w_i\left(z_i^{[\varsigma]}\right)\left[\frac{1}{\Psi^{[\varsigma]}*\left(1+w\left(y_i^{[\varsigma]}\right)+w_i\left(z_i^{[\varsigma]}\right)\right)}\right], \end{cases} \tag{2.15}$$

where $w_i^*\left(x_i^{[\varsigma]}\right) = \frac{f\left(x_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(x_i^{[\varsigma]} - x_j^{[\varsigma]} + \frac{f\left(x_i^{[\varsigma]}\right)}{f'\left(x_i^{[\varsigma]}\right)}\frac{1}{\left[1-\alpha\left[\frac{f\left(x_i^{[\varsigma]}\right)}{1+f\left(x_i^{[\varsigma]}\right)}\right]\right]}\right)}$, $w_i\left(y_i^{[\varsigma]}\right) = \frac{f(r_i^{(\sigma)})}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}(r_i^{(\sigma)} - r_j^{(\sigma)})}$,

$w_i\left(z_i^{[\varsigma]}\right) = \frac{f\left(z_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(z_i^{[\varsigma]} - z_j^{[\varsigma]}\right)}$ and $\Psi^{[\varsigma]} = \frac{f'(y^{[\varsigma]})}{f'(z^{[\varsigma]})}$.

In the following theorem, we prove the convergence order of MM_{12}.

Theorem 2. *Let $\zeta_1, ..., \zeta_\sigma$ be a simple zero of the nonlinear equation and for sufficiently close initial distinct estimation $x_1^{[0]}, ..., x_n^{[0]}$ of the roots respectively; then, MM_{12} has a convergence of order 12.*

Proof. Let $\epsilon_i = x_i^{[\sigma]} - \zeta_i, \epsilon_i' = y_i^{[\sigma]} - \zeta_i, \epsilon_i'' = z_i^{[\sigma]} - \zeta_i$ and $\epsilon_i''' = w_i^{[\sigma]} - \zeta_i$ be the errors in $x_i^{[\sigma]}, y_i^{[\sigma]}, z_i^{[\sigma]}$, and $w_i^{[\sigma]}$, respectively. From the first-step of MM_{12}, we have

$$y_i^{[\varsigma]} - \zeta_i = x_i^{[\varsigma]} - \zeta_i - \frac{f\left(x_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(x_i^{[\varsigma]} - x_j^{[\varsigma]} + \frac{f\left(x_i^{[\varsigma]}\right)}{f'\left(x_i^{[k\varsigma]}\right)}\frac{x1}{\left[1-\alpha\left[\frac{f\left(x_i^{[\varsigma]}\right)}{1+f\left(x_i^{[\varsigma]}\right)}\right]\right]}\right)}. \tag{2.16}$$

$$\epsilon_i' = \epsilon_i - \vartheta_i^*\left(x_i^{[\varsigma]}\right) = \epsilon_i - \epsilon_i\frac{\vartheta_i^*\left(x_i^{[\varsigma]}\right)}{\epsilon_i}, \tag{2.17}$$

$$\epsilon_i' = \epsilon_i\left(1 - Q_i^{[1]}\right), \tag{2.18}$$

where

$$Q_i^{[1]} = \frac{\vartheta_i^*\left(x_i^{[\varsigma]}\right)}{\epsilon_i} = \prod_{\substack{j=1\\j\neq i}}^{n}\left[\frac{x_i^{[\varsigma]} - \zeta_j}{x_i^{[\varsigma]} - \overset{*}{x}_j^{[\varsigma]}}\right], \tag{2.19}$$

and $\overset{*}{x}_j^{[\varsigma]} = x_j^{[\varsigma]} - \frac{f\left(x_i^{[\varsigma]}\right)}{f'\left(x_i^{[\varsigma]}\right)}\frac{1}{\left[1-\alpha\left[\frac{f\left(x_i^{[\varsigma]}\right)}{1+f\left(x_i^{[\varsigma]}\right)}\right]\right]}$.

$$\frac{x_i^{[\varsigma]} - \zeta_j}{x_i^{[\varsigma]} - \overset{*}{x}_j^{[\varsigma]}} = 1 + \frac{x_j^{[\varsigma]} - \zeta_j}{x_i^{[\varsigma]} - \overset{*}{x}_j^{[\varsigma]}}1 + O\left(\left|\epsilon^2\right|\right), \tag{2.20}$$

$$\begin{aligned} Q_i^{[1]} = \prod_{\substack{j=1\\j\neq i}}^{n}\left[\frac{x_i^{[\varsigma]} - \zeta_j}{x_i^{[\varsigma]} - \overset{*}{x}_j^{[\varsigma]}}\right] &= \left(1 + O\left(\left|\epsilon^2\right|\right)\right)^{n-1}, \\ &= 1 + (n-1)\,O\left(\left|\epsilon^2\right|\right) = 1 + O\left(\left|\epsilon^2\right|\right), \end{aligned} \tag{2.21}$$

$$Q_i^{[1]} - 1 = O\left(\left|\epsilon^2\right|\right). \tag{2.22}$$

Thus, we get

$$\epsilon_i' = \epsilon_i\left(O\left(\left|\epsilon^2\right|\right)\right) = O\left(\left|\epsilon^3\right|\right), \tag{2.23}$$

$$z_i^{[\varsigma]} - \zeta_i = y_i^{[\varsigma]} - \zeta_i - \frac{f\left(y_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(y_i^{[\varsigma]} - y_j^{[\varsigma]}\right)}. \tag{2.24}$$

$$\epsilon_i'' = \epsilon_i' - \vartheta_i^*\left(y_i^{[\varsigma]}\right) = \epsilon_i' - \epsilon_i'\frac{\vartheta_i^*\left(y_i^{[\varsigma]}\right)}{\epsilon_i'} = \epsilon_i'\left(1 - Q_i^{[2]}\right), \tag{2.25}$$

where

$$Q_i^{[2]} = \frac{\vartheta_i^*\left(y_i^{[\varsigma]}\right)}{\epsilon_i'} = \prod_{\substack{j=1\\ j\neq i}}^{n}\left[\frac{y_i^{[\varsigma]}-\zeta_j}{y_i^{[\varsigma]}-\overset{*}{y}_j^{[\varsigma]}}\right], \tag{2.26}$$

$$\frac{y_i^{[\varsigma]}-\zeta_j}{y_i^{[\varsigma]}-y_j^{[\varsigma]}} = 1+\frac{y_j^{[\varsigma]}-\zeta_j}{y_i^{[\varsigma]}-y_j^{[\varsigma]}} = 1+O\left(\left|\epsilon^2\right|\right), \tag{2.27}$$

$$\begin{aligned} Q_i^{[2]} = \prod_{\substack{j=1\\ j\neq i}}^{n}\left[\frac{y_i^{[\varsigma]}-\zeta_j}{y_i^{[\varsigma]}-y_j^{[\varsigma]}}\right] &= \left(1+O\left(|\epsilon'|\right)\right)^{n-1}, \\ &= 1+(n-1)\,O\left(|\epsilon'|\right) = 1+O\left(|\epsilon'|\right), \end{aligned} \tag{2.28}$$

$$Q_i^{[2]}-1 = O\left(|\epsilon'|\right). \tag{2.29}$$

Assume that $|\epsilon_i| = \left|\epsilon_j\right| = |\epsilon|$, then, we get

$$\epsilon_i'' = \epsilon_i'\left(O\left(|\epsilon'|\right)\right) = O\left(|\epsilon'|^2\right) = O\left(\left|\epsilon^6\right|\right). \tag{2.30}$$

Also

$$z_i^{[\varsigma]}-\zeta_i = z_i^{[\varsigma]}-\zeta_i-\frac{f\left(z_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}\left(z_i^{[\varsigma]}-z_j^{[\varsigma]}\right)}\left[\frac{1}{\Psi^{[\varsigma]}*\left(1+\frac{f\left(y_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}\left(y_i^{[\varsigma]}-y_j^{[\varsigma]}\right)}+\frac{f\left(z_i^{[\varsigma]}\right)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}\left(z_i^{[\varsigma]}-z_j^{[\varsigma]}\right)}\right)}\right], \tag{2.31}$$

$$\epsilon_i''' = \epsilon_i''-\epsilon_i''\frac{\vartheta_i^*\left(z_i^{[\varsigma]}\right)}{\epsilon_i''}\left[\frac{1}{\Psi^{[\varsigma]}*\left(1+\vartheta_i^*\left(y_i^{[\varsigma]}\right)+\vartheta_i^*\left(z_i^{[\varsigma]}\right)\right)}\right], \tag{2.32}$$

$$\epsilon_i''' = \epsilon_i''-\epsilon_i''\frac{\vartheta_i^*\left(z_i^{[\varsigma]}\right)}{\epsilon_i''}\left[\frac{1}{\Psi^{[\varsigma]}*\left(1+\vartheta_i^*\left(y_i^{[\varsigma]}\right)+\vartheta_i^*\left(z_i^{[\varsigma]}\right)\right)}\right] \tag{2.33}$$

as $\Psi^{[\varsigma]} = \frac{f'\left(y_i^{[\varsigma]}\right)}{f'\left(z_i^{[\varsigma]}\right)}$.

$$f\left(x_i^{[\varsigma]}\right) = \left(x_1^{[\varsigma]}-\zeta_1\right)\ldots\left(x_i^{[\varsigma]}-\zeta_i\right) = \epsilon_i\prod_{\substack{j=1\\ j\neq i}}^{n}\left(x_i^{[\varsigma]}-\zeta_j\right), \tag{2.34}$$

$$f\left(y_i^{[\varsigma]}\right) = \left(y_1^{[\varsigma]}-\zeta_1\right)\ldots\left(y_i^{[\varsigma]}-\zeta_i\right) = \epsilon_i'\prod_{\substack{j=1\\ j\neq i}}^{n}\left(y_i^{[\varsigma]}-\zeta_j\right), \tag{2.35}$$

$$
\begin{aligned}
&= \epsilon_i' \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(x_i^{[\varsigma]} - x_j^{[\varsigma]} - \frac{f\left(x_i^{[\varsigma]}\right)}{f'\left(x_i^{[\varsigma]}\right)} \frac{1}{\left[1 - \alpha \left[\frac{f\left(x_i^{[\varsigma]}\right)}{1+f\left(x_i^{[\varsigma]}\right)}\right]\right]} - \zeta_j \right), \\
&= \epsilon_i' \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(x_i^{[\varsigma]} - \vartheta_i^* \left(x_i^{[\varsigma]}\right) - \zeta_j \right),
\end{aligned}
$$

$$
f\left(z_i^{[\varsigma]}\right) = \left(z_1^{[\varsigma]} - \zeta_1\right) \ldots \left(z_i^{[\varsigma]} - \zeta_i\right) \tag{2.36}
$$

$$
= \epsilon_i'' \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(z_i^{[\varsigma]} - \zeta_j\right) = \epsilon_i'' \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(y_i^{[\varsigma]} - \vartheta_i^*\left(y_i^{[\varsigma]}\right) - \zeta_j\right), \tag{2.37}
$$

$$
= \epsilon_i'' \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(y_i^{[\varsigma]} - \vartheta_i^*\left(y_i^{[\varsigma]}\right) - \zeta_j\right).
$$

$$
\frac{f\left(y_i^{[\varsigma]}\right)}{f\left(x_i^{[\varsigma]}\right)} = \frac{\epsilon_i'}{\epsilon_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left[\frac{y_i^{[\varsigma]} - \zeta_j}{x_i^{[\varsigma]} - \zeta_j}\right] = \frac{\epsilon_i \left(1 - Q_i^{[1]}\right)}{\epsilon_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left[\frac{x_i^{[\varsigma]} - \vartheta_i^*\left(x_i^{[\varsigma]}\right) - \zeta_j}{x_i^{[\varsigma]} - \zeta_j}\right], \tag{2.38}
$$

$$
= \left(1 - Q_i^{[1]}\right) \prod_{\substack{j=1 \\ j\neq i}}^{n} \left[1 - \frac{\vartheta_i^*\left(x_i^{[\varsigma]}\right)}{x_i^{[\varsigma]} - \zeta_j}\right] = 1 - O(\epsilon),
$$

$$
\frac{f\left(z_i^{[\varsigma]}\right)}{f\left(y_i^{[\varsigma]}\right)} = \frac{\epsilon_i''}{\epsilon_i'} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left[\frac{z_i^{[\varsigma]} - \zeta_j}{y_i^{[\varsigma]} - \zeta_j}\right] = \frac{\epsilon_i''}{\epsilon_i'} \left(1 - O(\epsilon')\right), \tag{2.39}
$$

$$
\frac{f'\left(y_i^{[\varsigma]}\right)}{f'\left(z_i^{[\varsigma]}\right)} = \frac{f\left(y_i^{[\varsigma]}\right) f\left(z_i^{[\varsigma]}\right)}{f\left(y_i^{[\varsigma]}\right) f\left(z_i^{[\varsigma]}\right)} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left[\frac{y_i^{[\varsigma]} - \zeta_j}{z_i^{[\varsigma]} - \zeta_j}\right] = 1; \tag{2.40}
$$

therefore

$$
\Psi^{[\varsigma]} * \left(1 + \frac{f\left(y_i^{[\varsigma]}\right)}{\prod_{\substack{j=1 \\ j\neq i}}^{n} \left(y_i^{[\varsigma]} - y_j^{[\varsigma]}\right)} + \frac{f\left(z_i^{[\varsigma]}\right)}{\prod_{\substack{j=1 \\ j\neq i}}^{n} \left(z_i^{[\varsigma]} - z_j^{[\varsigma]}\right)}\right), \tag{2.41}
$$

$$
\begin{aligned}
&= 1 + \frac{f\left(z_i^{[\varsigma]}\right)}{f'\left(z_i^{[\varsigma]}\right)} \left[\frac{\frac{f\left(z_i^{[\varsigma]}\right)}{f\left(y_i^{[\varsigma]}\right)} + 1}{\frac{f\left(z_i^{[\varsigma]}\right)}{f\left(y_i^{[\varsigma]}\right)}}\right] = 1 + \epsilon_i'' \prod_{\substack{j=1 \\ j\neq i}}^{n} \left[\frac{z_i^{[\varsigma]} - \zeta_j}{z_i^{[\varsigma]} - z_j^{[\varsigma]}}\right] \left[\frac{\frac{\epsilon_i''}{\epsilon_i'} \left(1 - O(\epsilon')\right) + 1}{\frac{\epsilon_i''}{\epsilon_i'} \left(1 - O(\epsilon')\right)}\right], \\
&= 1 + \epsilon_i'' \prod_{\substack{j=1 \\ j\neq i}}^{n} \left[\frac{z_i^{[\varsigma]} - \zeta_j}{z_i^{[\varsigma]} - z_j^{[\varsigma]}}\right] \left[\frac{\epsilon_i'' \left(1 - O(\epsilon')\right) + \epsilon'}{\epsilon_i'' \left(1 - O(\epsilon')\right)}\right],
\end{aligned}
$$

$$= 1+\prod_{\substack{j=1\\ j\neq i}}^{n}\left[\frac{z_i^{[\varsigma]}-\zeta_j}{z_i^{[\varsigma]}-z_j^{[\varsigma]}}\right]\left[\frac{\epsilon_i''\left(1-O\left(\epsilon'\right)\right)+\epsilon'}{\left(1-O\left(\epsilon'\right)\right)}\right],$$
$$= 1+O\left(\epsilon''\right).$$

Thus

$$\epsilon_i''' = \epsilon_i'' - \epsilon_i''\frac{\vartheta_i^*\left(z_i^{[\varsigma]}\right)}{\epsilon_i''}\left[1-O\left(\epsilon''\right).\right] = \epsilon_i''\left(1-Q_i^{[3]}\right), \tag{2.42}$$

where

$$Q_i^{[3]} = \frac{\vartheta_i^*\left(z_i^{[\varsigma]}\right)}{\epsilon_i''} = \prod_{\substack{j=1\\ j\neq i}}^{n}\left[\frac{z_i^{[\varsigma]}-\zeta_j}{z_i^{[\varsigma]}-z_j^{[\varsigma]}}\right], \tag{2.43}$$

$$\frac{y_i^{[\varsigma]}-\zeta_j}{y_i^{[\varsigma]}-y_j^{[\varsigma]}} = 1+\frac{z_j^{[\varsigma]}-\zeta_j}{z_i^{[\varsigma]}-z_j^{[\varsigma]}}1+O\left(\left|\epsilon^2\right|\right), \tag{2.44}$$

$$Q_i^{[3]} = \prod_{\substack{j=1\\ j\neq i}}^{n}\left[\frac{z_i^{[\varsigma]}-\zeta_j}{z_i^{[\varsigma]}-z_j^{[\varsigma]}}\right] = \left(1+O\left(\left|\epsilon''\right|\right)\right)^{n-1}, \tag{2.45}$$

$$Q_i^{[3]} = 1+(n-1)\,O\left(\left|\epsilon''\right|\right) = 1+O\left(\left|\epsilon''\right|\right),$$

$$Q_i^{[1]}-1 = O\left(\left|\epsilon''\right|\right). \tag{2.46}$$

Thus, we get

$$\epsilon_i''' = \epsilon_i''\left(O\left(\left|\epsilon''\right|\right)\right) = O\left(\left|\epsilon''\right|^2\right) = O\left(\left|\epsilon^6\right|^2\right) = O\left(|\epsilon|\right)^{12}. \tag{2.47}$$

Hence, we have completed the proof of this theorem. □

3. Computational analysis of simultaneous methods

The computational analysis of the numerical approach comprises analyzing its computational complexity and convergence characteristics. In general, the approach converges more quickly when the initial guess is nearer to the exact roots. The computational complexity of the simultaneous technique is dominated by global convergence behavior unlike a single root-finding algorithm. This indicates that the overall complexity of the simultaneous technique, where n is the degree of the polynomial, is $\mathrm{O}\left[m^2\right]$. Here, we contrast the computing effectiveness of the recently introduced methods MM_{10} with the MM_{10} and $\mathrm{MM}_{12\alpha}$. As presented in [29], the computational efficiency of an iterative method can be estimated by using the efficiency index given by

$$\Lambda^{[*]}[m] = \frac{\log[\mathbf{r}]}{\theta_{11}^{[*]}+\theta_{12}^{[*]}+\theta_{13}^{[*]}}, \tag{3.1}$$

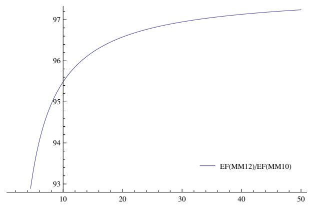
where $\theta_{11}^{[*]} = w_{as} * AS_m, \theta_{12}^{[*]} = w_m M_m$, and $\theta_{13}^{[*]} = w_d D_m$ [30]. Applying the data given in Table 1, we have

$$\Lambda_1^{[*]}[m,n] = \left(\frac{\Lambda^{[*]}[m]-\Lambda^{[*]}[n]}{\Lambda^{[*]}[n]}\times 100\right). \tag{3.2}$$

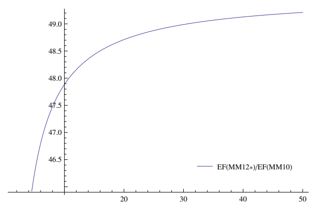
Table 1. Basic operations for simultaneous schemes.

Methods	MM_{10}	MM_{12}	$MM_{12\alpha}$
AA_m	$22*\left(m^2\right)+O[m]$	$17*\left(m^2\right)+O[m]$	$19*\left(m^2\right)+O[m]$
MM_m	$12*\left(m^2\right)+O[m]$	$6*\left(m^2\right)+O[m]$	$8*\left(m^2\right)+O[m]$
DD_m	$2*\left(m^2\right)+O[m]$	$2*\left(m^2\right)+O[m]$	$2*\left(m^2\right)+O[m]$

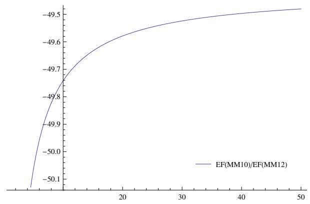
Remark 1. *Figure 1 graphically illustrates the computational efficiency ratios. It is evident from Figure 1 that the newly constructed simultaneous methods MM_{12} is more efficient than MM_{10} and $MM_{12\alpha}$*



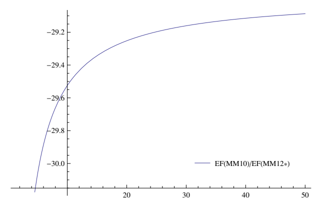
(**a**) Computational efficiency of $MM_{12\alpha}$ w.r.t MM_{10}.



(**b**) Computational efficiency of MM_{12} w.r.t MM_{10}.



(**c**) Computational efficiency of MM_{10} w.r.t $MM_{12\alpha}$.



(**d**) Computational efficiency of MM_{10} w.r.t MM_{12}.

Figure 1. (a–d) Computational efficiency ratios for simultaneous methods.

4. Numerical outcomes

4.1. Real world application

In this section, we discuss some real world applications whose solutions are approximated by our newly constructed methods MM_{10},MM_{12}, and $MM_{12\alpha}$.

Example 1: Quarter car suspension model

One component of the suspension system i.e., the shock absorbed, is also utilized to regulate the transient behavior of the vehicle mass and the suspension mass (see Pulvirenti and Faria [31], Konieczny [32]). Due to its nonlinear behavior, it is one of the most complicated components of the suspension system. The damping force of the dampers is, however, described by an asymmetric nonlinear hysteresis loop [33]. A two-degrees-of-freedom quarter-car model is used to simulate the vehicle characteristics in this situation, and linear and nonlinear damping characteristics are used to analyze the damper effect. Construction of a damper model that describes the damper's nonlinear hysteresis features, such as a polynomial model, is crucial because simpler models, such as linear and piece-wise linear models, fail to characteristic the damper's behavior. What follows are the equations of mass motion:

$$\begin{cases} m_s x_s'' + \varsigma_s(x_s - x_u) + F = 0, \\ m_u x_u'' - \varsigma_s(x_s - x_u) - \varsigma_\sigma(x_r - x_u) - F = 0, \end{cases} \tag{4.1}$$

where m_s is masses that are over-sprung, m_u denotes masses that are under sprung, x_s displacement over masses, x_u displacement under masses, x_r is the disturbance from road bumps, ς_s denotes coefficients relating to the spring, and ς_σ coefficients relating to the spring stiffness in the Tyre stiffness. The following polynomial is used to fit the damper force F in (4.1):

$$f(x) = -77.14 * x^4 + 23.14 * x^3 + 342.7 * x^2 + 956.7x + 124.5. \tag{4.2}$$

The quarter-car model's suspension is centered between the sprung mass and the unsprung mass. The force-velocity relation of the shock absorbed is based on the models covered in the section before, whereas the suspension system's spring has a stiffness of ς_s. The stiffness of the tyre is represented by a second spring attached to the unsprung mass with a coefficient of ς_σ The two springs in the system are considered to be linear in nature and to have constant spring coefficients. It should be mentioned that system damping is thought to be more important than tyre damping. The response of the system can be analyzed by determining the displacement, velocity, and acceleration of the mass over time. The model can be used to design and optimize suspension systems for various driving conditions, such as ride comfort, handling, and stability.

The exact roots of (4.2) are

$$\begin{aligned} \zeta_1 &= 3.090556803, \zeta_2 = -1.326919946 + 1.434668028 * i, \zeta_3 = -0.1367428388, \\ \zeta_4 &= -1.326919946 - 1.434668028 * i. \end{aligned}$$

We have determined the convergence history and computational order for the numerical schemes MM10, MM_{12}, and $MM_{12\alpha}$. Using the standard "rand()" function in CAS-Matlab, a random initial guess value of v1, v2 and v3 of Appendix A1.1 was generated in order to observe the overall convergence behavior of the simultaneous scheme. With a random initial estimate value, MM_{10}, MM_{12}, $MM_{12\alpha}$ converges to exact zeros after 10, 8, 8; 7,6,6 and 8,7,6 iterations respectively. Table 2 reveals the CPU time and the local convergence order of the computational algorithm for MM_{10}, MM_{12}, $MM_{12\alpha}$. Table 2, clearly shows that the rate of convergence of MM_{12} is better than those of MM_{10} and $MM_{12}\alpha$. Error analysis from Table 3, shows that in terms of maximum error and in maximum iteration MM_{12} is better as compared to MM_{10} and $MM_{12}\alpha$. Our newly developed inverse numerical scheme converges to exact roots at different random initial guesses values, which indicates the better global convergence behavior as compared to MM_{10} and $MM_{12\alpha}$. The numerical out of the iterative schemes on random initial guesses presented in Table 3 shows that the simultaneous scheme MM_{12} is more stable than MM_{10} and $MM_{12}\alpha$.

The rate of convergence of simultaneous schemes MM_{10}, MM_{12}, $MM_{12\alpha}$ increases as the initial guesses values are chosen to be close to the exact roots. Choosing initial guesses values close to the exact roots results in significant improvements in the CPU time, computational order of convergence and error iterations, as shown in Table 4.

Table 2. Finding all polynomial roots simultaneously.

Methods	$e_1^{[\varsigma]}$	$e_2^{[\varsigma]}$	$e_3^{[\varsigma]}$	$e_4^{[\varsigma]}$	$\rho_{\varsigma i}^{[\varsigma-1]}$	C-Time
Random initial guess vector v1 taken from Appendix A1.1						
MM_{10}	0.214E-10	2.7E-15	8.17E-15	0.251E-14	6.12015	2.4533
MM_{12}	4.87E-20	5.66E-25	9.52E-30	7.221E-30	9.78914	005733
$MM_{12\alpha}$	2.98E-21	0.561E-20	0.33E-20	0.221E-25	8.01241	0.9342
Random initial guess vector v2 taken from Appendix A1.1						
MM_{10}	0.146E-03	0.451E-05	3.252E-10	2.114E-05	5.41534	1.0124
MM_{12}	0.186E-10	6.145E-15	0.241E-20	0.141E-25	9.41534	0.00124
$MM_{12\alpha}$	0.168E-09	7.1548E-10	1.122E-12	1.251E-15	9.41534	0.00124
Random initial guess vector v3 taken from Appendix A1.1						
MM_{10}	8.116E-18	0.1352E-15	2.214E-09	0.178E-25	7.41534	0.00124
MM_{12}	9.165E-29	8.1012E-30	2.3562E-35	0.198E-35	11.41534	0.00124
$MM_{12\alpha}$	3.126E-21	0.114E-25	0.2541E-26	0.581E-27	9.41534	0.00124

Table 3. Error analysis on random initial values.

Methods	MM_{10}	MM_{12}	$MM_{12\alpha}$
Maximum error on random initial vector v1			
Error it	10.0	8.00	8.00
Max-Err	2.14E-10	5.66E-25	0.561E-20
Maximum error on random initial vector v2			
Error it	7.0	6.00	6.00
Max-Err	3.252E-10	0.186E-10	0.168E-09
Maximum error on random initial vector v3			
Error it	8.00	7.00	6.00
Max-Err	2.14E-09	8.1012E-30	3.126E-21

Table 4. Determination of all polynomial roots.

Methods	MM_{10}	MM_{12}	$MM_{12\alpha}$
Error it	10	08	08
CPU	0.014145	0.01642585	0.05442551
$e_1^{[\varsigma]}$	3.3120e-45	0.7226e-65	3.5874e-65
$e_2^{[\varsigma]}$	1.6220e-43	5.2526e-64	0.2398e-56
$e_3^{[\varsigma]}$	3.3150e-53	8.8583e-75	7.2325e-70
$e_4^{[\varsigma]}$	1.6254e-53	9.4258e-85	0.2544e-65
$\rho_{\varsigma i}^{[\sigma-1]}$	9.1221512	12.031452	11.914556

Convergence rates rise when the following initial guess value set is used

$$x_1^{[0]} = 3,\ x_2^{[0]} = -1 + 1i,\ x_3^{[0]} = -0.1, x_4^{[0]} = -1 - 1i.$$

Example 2: Blood rheology model [30]

The "Casson fluid," a non-Newtonian fluid, is used to represent blood. The Casson fluid model predicts that a simple fluid, such as water or blood, will flow through a tube such by its center core moves as a plug with minimal deformation and that there is a velocity gradient towards the tube wall. The plug flow of Casson fluids can be described by using the non-linear polynomial equation as follows [34, 35]:

$$G = 1 - \frac{16}{7}\sqrt{x} + \frac{4}{3}x - \frac{1}{21}x^4, \tag{4.3}$$

Using flow rate reduction $G = 0.40$ in Eq (4.3), we have:

$$f_1(x) = \frac{1}{441}x^8 - \frac{8}{63}x^5 - 0.05714285714x^4 + \frac{16}{9}x^2 - 3.624489796x + 0.36. \tag{4.4}$$

The exact roots of (4.4) are

$$\begin{aligned}
\zeta_1 &= 0.1046986515, \zeta_2 = 3.822389235, \zeta_3 = 1.553919850 + .9404149899i,\\
\zeta_4 &= -1.238769105 + 3.408523568i, \zeta_5 = -2.278694688 + 1.987476450i,\\
\zeta_6 &= -2.278694688 - 1.987476450i, \zeta_7 = -1.238769105 - 3.408523568,\\
\zeta_8 &= 1.553919850 - .9404149899.
\end{aligned}$$

We determined the convergence history and computational order for the numerical schemes MM10, MM_{12}, and $MM_{12\alpha}$. Using the standard "rand()" function in CAS-Matlab, a random initial guess value of v1, v2 and v3 Appendix A1.2 was generated in order to observe the overall convergence behavior of the inverse simultaneous scheme. With a random initial estimate value, MM_{10},MM_{12},$MM_{12\alpha}$ converges to exact zeros after 10, 8, 8; 7,6,6 and 8,7,6 iterations respectively. The CPU time and local computational order of convergence are represented in Table 5. Table 5, clearly shows that the rate of convergence of MM_{12} is better than those of MM_{10} and $MM_{12}\alpha$. Error analysis from Table 6 shows that in terms of maximum error and in maximum iteration MM_{12} is better as compared to MM_{10} and $MM_{12}\alpha$. Our newly developed inverse numerical scheme converges to exact roots at different random initial guesses values, which indicates the better global convergence behavior as compared to MM_{10} and $MM_{12\alpha}$. The numerical out of the iterative schemes on random initial guesses presented in Table 6 shows that the simultaneous scheme MM_{12} is more stable than MM_{10} and $MM_{12}\alpha$.

The rate of convergence of simultaneous schemes MM_{10}, MM_{12}, $MM_{12\alpha}$ increases as the initial guesses values are chosen to be close to the exact roots. Choosing initial guesses values close to the exact roots results in significant improvements in CPU time, computational order of convergence, and error iterations, as shown in Table 7.

Convergence rates rise when the following initial guess value set is used

$$\begin{aligned}
x_1^{[0]} &= 0.1,\ x_2^{[0]} = 3.8,\ x_3^{[0]} = 1.5 + 0.9i,\ x_4^{[0]} = 1.2 + 3.4i,\\
x_5^{[0]} &= -2.2 + 1.9i,\ x_6^{[0]} = -2.2 - 1.9i\ x_7^{[0]} = -1.2 - 3.4i,\ x_8^{[0]} = 1.5 + 0.9i.
\end{aligned}$$

Table 5. Simultaneous determination of all polynomial roots using the random initial approximations in A1.2.

Methods	MM_{10}	MM_{12}	$MM_{12\alpha}$	MM_{10}	MM_{12}	$MM_{12\alpha}$	MM_{10}	MM_{12}	$MM_{12\alpha}$
Error it	10	8	8	7	6	6	8	7	6
	v1 from Appendix A1.2			**v**2 from Appendix A1.2			**v3** from Appendix A1.2		
CPU	2.514114	0.0165847	0.9541451	3.0678541	1.365521	1.76554	2.0141	1.016145	1.052544
$e_1^{[\varsigma]}$	3.30E-06	0.76E-26	5.5E-20	0.012E-06	3.67E-47	3.67E-37	8.30E-16	3.76E-16	3.5E-26
$e_2^{[\varsigma]}$	1.62E-04	0.26E-30	4.23E-20	6.725E-14	6.26E-34	6.26E-34	1.62E-14	1.26E-14	1.23E-24
$e_3^{[\varsigma]}$	3.013E-03	0.83E-25	3.23E-20	4.356E-13	1.77E-49	1.77E-39	5.35E-13	1.83E-13	1.23E-23
$e_4^{[\varsigma]}$	1.6104E-03	9.48E-20	3.44E-18	1.84E-13	1.64E-48	1.64E-38	6.64E-13	9.48E-13	1.44E-23
$e_5^{[\varsigma]}$	8.551E-03	0.05E-15	0.51E-10	7.525E-22	8.5E-45	8.554E-35	9.55E-13	0.05E-10	3.5E-10
$e_6^{[\varsigma]}$	0.65E-04	0.56E-20	0.76E-11	9.526E-21	6.66E-45	6.66E-45	9.65E-14	0.56E-11	8.76E-21
$e_7^{[\varsigma]}$	0.26E-03	0.52E-12	5.62E-20	15.24E-13	1.27E-46	1.27E-26	0.26E-18	0.52E-12	5.62E-23
$e_8^{[\varsigma]}$	0.34E-03	0.33E-20	7.43E-20	12.43E-13	4.37E-45	4.37E-35	4.34E-18	4.33E-03	7.43E-23
$\rho_{\varsigma i}^{[\varsigma-1]}$	5.145215	7.01214	7.5145	4.01445	9.12405	7.31445	6.1215	0.2314	1.5145

Table 6. Error analysis on random initial values.

Methods	MM_{10}	MM_{12}	$MM_{12\alpha}$
Maximum error on random initial vector v1			
Error it	10.0	8.00	8.00
Max-Err	0.34E-03	0.05E-15	0.51E-10
Maximum error on random initial vector v2			
Error it	7.0	6.00	6.00
Max-Err	0.012E-06	8.58E-45	8.554E-45
Maximum error on random initial vector v3			
Error it	8.00	7.00	6.00
Max-Err	9.55E-13	0.56E-11	3.5E-10

Table 7. Determination of all polynomial roots.

Methods	MM_{10}	MM_{12}	$MM_{12\alpha}$
Error it	10	06	07
CPU	0.014114	0.016145	0.0545656
$e_1^{[\varsigma]}$	3.30E-26	0.0	0.0
$e_2^{[\varsigma]}$	1.62E-34	0.0	1.23E-64
$e_3^{[\varsigma]}$	3.3E-33	1.83E-75	1.23E-53
$e_4^{[\varsigma]}$	1.64E-43	9.48E-95	1.44E-69
$e_5^{[\varsigma]}$	8.55E-33	0.05E-101	0.0
$e_6^{[\varsigma]}$	2.65E-34	0.0	0.0
$e_7^{[\varsigma]}$	1.26E-38	0.0	0.62E-63
$e_8^{[\varsigma]}$	4.34E-45	0.0	4.43E-73
$\rho_{\varsigma i}^{[\varsigma-1]}$	9.121556	12.01412	11.514525

5. Conclusions

A new family of simultaneous methods with a convergence order of 12 is introduced to approximate the roots of all nonlinear equations. To demonstrate the global convergence of the newly developed schemes MM_{12}, engineering applications for various random initial approximations have been solved. The rate of convergence increases when some close initial approximations are used, as shown in Tables 4 and 7. Similar higher order simultaneous iterative techniques for finding polynomial roots are required to solve more complex engineering applications.
Future research will focus on the same methodologies described in this article, and we will use fractional calculus and the weight function technique to develop optimal higher-order efficient fractional simultaneous iterative methods with and without derivatives for finding all roots of nonlinear equations.

Furthermore, we will investigate the dynamical analysis, computational convergent history, and root trajectories under the condition of random initial guesses values in the future in order to better understand the global convergence of simultaneous schemes.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest regarding the publication of this paper.

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Supplementary

Abbreviations

In this study's article, the following abbreviations are used

MM_{12}	Proposed Simultaneous Scheme
Error it	Number of Error Iterations
Ex-Time	Computational Time in Seconds
Max-Err	Maximum Error
e-	$10^{-()}$
$\rho_{\varsigma i}^{(\sigma-1)}$	Computational local order of convergence

Appendix

Table 8. Appendix A1.1: Matlab was used to generate a random set of initial guess values for engineering application 1.

$x_{i*}^{[0]}$	$[x_1^{[0]}, x_2^{[0]}, x_3^{[0]}, x_4^{[0]}]$
v1	[-0.160,0.643,0.967,0.085]
v2	[0.743,0.392,0.655,0.171]
v3	[-0..145,0.874,0.475,0.876]
⋮	[⋮, ⋮, ⋮, ⋮]

Table 9. Appendix A1.2: Matlab was used to generate a random set of initial guess values for engineering application 2.

$x_{i*}^{[0]}$	$[x_1^{[0]}, x_2^{[0]}, x_3^{[0]}, x_4^{[0]}, x_5^{[0]}, x_6^{[0]}, x_7^{[0]}, x_8^{[0]}]$
v1	[-0.760,0.643,0.967,0.881,0.760,0.643,0.967,0.085]
v2	[-0.153,0.392,0.615,0.171,0.743,0.392,0.855,0.071]
v3	[-0.905,0.874,0.473,0.076,0.145,0.874,0.775,0.076]
⋮	[⋮, ⋮ ,⋮ ,⋮ ,⋮ ,⋮ ,⋮ ,⋮]