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Research article

On the vectorial multifractal analysis in a metric space

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Abstract: Multifractal analysis is typically used to describe objects possessing some type of scale invariance. During the last few decades, multifractal analysis has shown results of outstanding significance in theory and applications. In particular, it is widely used to characterize the geometry of the singularity of a measure μ or to study the time series, which has become an important tool for the study of several natural phenomena. In this paper, we investigate a more general level set studied in multifractal analysis. We use functions defined on balls in a metric space and that are Banach valued which is more general than measures used in the classical multifractal analysis. This is done by investigating Peyrière's multifractal Hausdorff and packing measures to study a relative vectorial multifractal formalism. This leads to results on the simultaneous behavior of possibly many branching random walks or many local Hölder exponents. As an application, we study the relative multifractal binomial measure in symbolic space $\partial\mathcal{A}$.

Keywords: multifractal Hausdorff measure; multifractal packing measure; relative vectorial formalism
Mathematics Subject Classification: 28A78, 28A80

1. Introduction

The concept of multifractal analysis was developed around 1980, following the work of B. Mandelbrot, when he studied the multiplicative cascades for energy dissipation in the context of turbulence [24, 25]. Since then, it has been developed rapidly and discussed by several authors, emphasizing its importance in the study of local properties of functions and measures. In particular, the multifractal spectrum provides a characterization in terms of the geometric properties of the singularities of a distribution. More precisely, let $X : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a signal; the multifractal analysis is a processing method that allows the examination of X by using the characteristics of its pointwise regularity, which are measured by $\alpha_X(x)$, i.e., the exponent of pointwise regularity. This is done by using the multifractal spectrum, which is the Hausdorff dimension of the set of locations where the function $\alpha_X(x)$ is distributed, to characterize the set of x such that $\alpha_X(x) = \alpha$. Specifically, consider the set

$$E(\alpha) = \Big\{x \in \mathbb{R}^{\mathrm{d}};\ \ \alpha_X(x) = \alpha\Big\}, \tag{1.1}$$

which gives a geometric and global account of the variations in X's regularity along x. Usually, we use the Hurst exponent H as a quantification of the degree of self-similarity of the time series which is directly correlated with the fractal dimension D and describes the complexity of the signals. A higher value of D indicates a higher irregularity of the signals: $D = 2 - H$ [11, 18]. In the last few decades, multifractal analysis has become a powerful tool to study the time series which has become an important tool for the study of several natural phenomena. In fact, such series present complex statistical fluctuations that are associated with long-range correlations between the dynamical variables present in these series, and which obey the behavior usually described by the decay of the fractal power law. This theory in time series was first introduced by B. Mandelbrot in [21–23] including early approaches by Hurst and colleagues [18,19]. Since then, fractal and multifractal scaling behavior has been reported in many natural time series generated by complex systems, including medical and physiological time series especially recordings of the heartbeat, respiration, blood pressure wind speed, seismic events, etc.

Recall the set $E(\alpha)$ given in (1.1) and consider, for $n \geq 1$, the dyadic interval $I_n(k) = [(k-1)2^{-n}, k2^{-n}]$ with $1 \leq k \leq 2^n$ and with length $|I_n(k)| = 2^{-n}$. In fact, there are various definitions of the exponent α:

$$\alpha = \lim_{n\to\infty} \frac{\log A_X(I_n(k))}{\log |I_n(k)|},$$

where $A_X(I_n(k))$ may be chosen to be the wavelet-leaders $L_X(I_n(k))$ or the oscillation $Osc_X(I_n(k))$ of X over the interval $I_n(k)$ [20]. Therefore, it is interesting to introduce the local dimension of a probability measure μ at a point x:

$$\dim_{\mathrm{loc}}(x,\mu) = \lim_{r\to 0} \frac{\log(\mu(\mathsf{B}(x,r))}{\log r},$$

as well as the set $E_\mu(\alpha) = \Big\{x \in \mathbb{R}^{\mathrm{d}};\ \ \dim_{\mathrm{loc}}(x,\mu) = \alpha\Big\}$, where $\mathsf{B}(x,r)$ stands for the closed ball of center x and radius r and $\alpha \geq 0$. In the beginning, the multifractal formalism used "boxes", or in other terms took place in a totally disconnected metric space. To get rid of these boxes and have a formalism meaningful in geometric measure theory, Olsen [27] introduced a formalism which is now commonly used. Especially, we compute the Hausdorff multifractal spectrum function f_μ defined as

$$f_\mu(\alpha) = \dim\left(E_\mu(\alpha)\right),$$

where dim denotes the Hausdorff dimension. To this end, multifractal analysis can be considered as another way to describe the local properties of time series. Since then, numerous writers have looked at these measures, stressing their significance for the study of local fractal properties and fractal products [5–7, 13–16, 26].

Moreover, the developments of this field showed that getting a valid variant of the multifractal formalism does not require the application of radius power-laws equivalent measures. This leads one to think about a general framework wherein the restriction of the vector-valued function on balls may be any vector-valued function $\xi(\mathsf{B}(x, r))$ which is not equivalent to power-laws r^α and develops a general multifractal analysis. In particular, and in another context, to overcome the problem of being a non-doubling, non-Hölderian measure, Cole, in [10] proposed to control the analyzed measure μ by another suitable measure ν via a relative multifractal analysis of the relative singularity sets. More specifically, he calculated, for $\alpha \geq 0$, the size of the set

$$E(\alpha) = \left\{ x \in \operatorname{supp} \mu \cap \operatorname{supp} \nu; \;\; \lim_{r \to 0} \frac{\log \mu(\mathsf{B}(x, r))}{\log \nu(\mathsf{B}(x, r))} = \alpha \right\},$$

where $\operatorname{supp}\mu$ is the topological support of the measure μ. For this, he introduced a generalized Hausdorff and packing measures denoted by $\mathcal{H}^{q,s}_{\mu,\nu}$ and $\mathcal{P}^{q,s}_{\mu,\nu}$ respectively. One can emphasize the duality by replacing $\mathbb{R}^{\mathrm{d}}$ by a general metric space $(\mathbb{X}, d)$ and then replacing the diameter by a more general function defined on balls in $\mathbb{X}$ and analyzing functions defined on balls which are more general than measures. More precisely, let $\mathbb{E}$ be a separable real Banach space, whose dual is denoted by $\mathbb{E}'$ and the form of the duality $\langle \, , \, \rangle$. We denote by $\mathcal{B}(\mathbb{X})$ the set of closed balls on $\mathbb{X}$. We consider the functions

$$\begin{cases} \xi : \mathcal{B}(\mathbb{X}) & \to \mathbb{R}, \\ \varkappa : \mathbb{X} \times \mathbb{R}_+ & \to \mathbb{E}', \end{cases} \tag{1.2}$$

such that, for all $x \in \mathbb{X}$, one has that $\lim_{r \to 0} \xi(\mathsf{B}(x, r)) = +\infty$. For $\alpha \in \mathbb{E}'$, we consider the set

$$\mathrm{X}_\chi(\alpha) = \left\{ x \in \mathbb{X}; \;\; \lim_{r \to 0} \frac{\langle w, \varkappa(x, r) \rangle}{\xi(x, r)} = \langle w, \alpha \rangle, \quad \forall\, w \in \mathbb{E} \right\},$$

where $\chi = (\varkappa, \xi)$. The set $\mathrm{X}_\chi(\alpha)$ may be thought of as the set of points x such that $\frac{\varkappa(x,r)}{\xi(x,r)}$ tends to α in the sense of topology $\sigma(\mathbb{E}, \mathbb{E}')$ when r tends to 0. In [28], Peyrière introduced vectorial Hausdorff and packing measures denoted by $\mathcal{H}^{q,t}_\chi$ and $\mathcal{P}^{q,t}_\chi$ respectively. He defined, in a natural way, the Hausdorff and packing dimensions denoted respectively as $\dim^q_\chi$ and Dim^q_χ. In particular, if $\varkappa = 0$ then $\dim^q_\chi$ will be denoted by $\dim_\xi$ and Dim^q_χ will be denoted by Dim_ξ. In fact, such measures are appropriate for the study of a general formalism by relating

$$\dim_\xi\big(\mathrm{X}_\chi(\alpha)\big) \quad \text{and} \quad \mathrm{Dim}_\xi\big(\mathrm{X}_\chi(\alpha)\big)$$

to the Legendre transform of the multifractal Hausdorff and packing functions denoted respectively by b_χ and B_χ (see Section 2 for the definition).

The purpose of this paper is to study the Hausdorff and packing dimensions of the set $\mathrm{X}_\chi(\alpha)$. In fact, it is difficult to compute these dimensions in general, but we can compute a lower bound of the Hausdorff and packing dimensions of this level set. Indeed, we can decompose the set $\mathrm{X}_\chi(\alpha)$ and calculate the size of the subset of $\mathrm{X}_\chi(\alpha)$ whose points satisfy that $\lim_{r \to 0} \frac{\xi(x, r)}{-\log r} = \beta$. Inspired by [4,10,29], we define $\alpha \in \mathbb{E}'$ and $\beta \geq 0$; then the set is given as

$$\mathrm{X}_\chi(\alpha, \beta) = \left\{ x \in \mathbb{X}; \;\; \lim_{r \to 0} \frac{\langle w, \varkappa(x, r) \rangle}{\xi(x, r)} = \langle w, \alpha \rangle \;\; \text{and} \;\; \lim_{r \to 0} \frac{\xi(x, r)}{-\log r} = \beta, \;\; \forall\, w \in \mathbb{E} \right\}.$$

This article is organized as follows. The next section is devoted to recalling the definitions of the various multifractal dimensions and measures investigated in the paper. In Section 3, we will state and prove our main results concerning the study of Hausdorff and packing dimensions of the set $\mathrm{X}_\chi(\alpha,\beta)$. In general settings, we have that $\dim \mathrm{X}_\chi(\alpha,\beta) \neq \mathrm{Dim}\, \mathrm{X}_\chi(\alpha,\beta)$; for this, we give in Section 4 a sufficient condition so that we have the equality. In this case, we say that the relative multifractal formalism holds. As an application, we study the validity of the relative multifractal formalism for the binomial measure in symbolic space $\partial\mathcal{A}$.

2. Preliminaries

2.1. Vectorial multifractal measures and dimensions

In this section, we recall the multifractal Hausdorff and packing measures introduced in [28]. We assume throughout this paper that $\mathbb{X}$ is a separable metric space verifying the Besicovitch covering property [8, 9]. We define

$$\mathsf{B}(x,r) := \{y \in \mathbb{X};\ \ d(x,y) \le r\},$$

i.e., the closed ball with center $x \in \mathbb{X}$ and radius $r > 0$. We denote by $\mathcal{B}(\mathbb{X})$ the set of closed balls on $\mathbb{X}$. Let $\xi : \mathcal{B}(\mathbb{X}) \longrightarrow \mathbb{R}$ be an application such that, for all $x \in \mathbb{X}$, one has that $\lim\limits_{r\to 0} \xi(\mathsf{B}(x,r)) = +\infty$. Such a function will be called a valuation on $\mathbb{X}$ and we will write that $\xi(x,r) = \xi(\mathsf{B}(x,r))$, for simplicity. When such a valuation is given, one sets

$$\mathbb{X}_n = \Big\{x \in \mathbb{X};\ \ \xi(x,r) > 1 \ \text{ for } r \le 1/n\Big\}.$$

We consider the function $\varkappa : \mathbb{X} \times \mathbb{R}_+ \longrightarrow \mathbb{E}'$. We denote by $\langle\ ,\ \rangle$ the duality bracket between $\mathbb{E}$ and $\mathbb{E}'$. Let $A \subseteq \mathbb{X}$, $t \in \mathbb{R}$, $q \in \mathbb{E}$, $\chi = (\varkappa, \xi)$ and $\delta > 0$; we write

$$\overline{\mathcal{H}}^{q,t}_{\chi,\delta}(A) = \inf \sum_i e^{-\left(\langle q, \varkappa(x_i,r_i)\rangle + t\,\xi(x_i,r_i)\right)},$$

where the infimum is taken over all families $\{(x_i, r_i)\}_i$ satisfying that $\{\mathsf{B}(x_i, r_i)\}_i$ is a centered δ-cover of A, that is, $A \subseteq \bigcup\limits_i \mathsf{B}(x_i, r_i)$, $r_i \le \delta$ and $x_i \in A$. Let

$$\overline{\mathcal{H}}^{q,t}_{\chi}(A) = \lim_{\delta\to 0} \overline{\mathcal{H}}^{q,t}_{\chi,\delta}(A) \quad \text{and} \quad \widetilde{\mathcal{H}}^{q,t}_{\chi}(A) = \sup_{F\subseteq A} \overline{\mathcal{H}}^{q,t}_{\chi}(F).$$

Now $\widetilde{\mathcal{H}}^{q,t}_{\chi}$ is a metric outer measure. In addition, the function $t \longmapsto \widetilde{\mathcal{H}}^{q,t}_{\chi}(A)$ is non-decreasing; nevertheless, it is so if A is a subset of one of the $\mathbb{X}_n$. This is why one more step is needed in the construction. We write

$$\mathcal{H}^{q,t}_{\chi}(A) = \lim_{n\to\infty} \widetilde{\mathcal{H}}^{q,t}_{\chi}(A \cap \mathbb{X}_n).$$

Similarly, multifractal packing measures are defined as

$$\overline{\mathcal{P}}^{q,t}_{\chi,\delta}(A) = \sup \sum_i e^{-\left(\langle q, \varkappa(x_i,r_i)\rangle + t\,\xi(x_i,r_i)\right)},$$

where the supremum is taken over all families $\{(x_i, r_i)\}_i$ such that $\big(\mathsf{B}(x_i, r_i)\big)_i$ is a δ-packing of A, that is, $r_i \leq \delta$, $x_i \in A$ and $\mathsf{B}(x_i, r_i) \cap \mathsf{B}(x_j, r_j) = \emptyset$, for $i \neq j$. Then, we define

$$\begin{aligned}\overline{\mathcal{P}}^{q,t}_{\chi}(A) &= \lim_{\delta\to 0} \overline{\mathcal{P}}^{q,t}_{\chi,\delta}(A),\\ \widetilde{\mathcal{P}}^{q,t}_{\chi}(A) &= \inf\Big\{ \sum_i \overline{\mathcal{P}}^{q,t}_{\chi}(A_i) \;\big|\; A \subseteq \bigcup_i A_i \Big\},\end{aligned}$$

and

$$\mathcal{P}^{q,t}_{\chi}(A) = \lim_{n\to\infty} \widetilde{\mathcal{P}}^{q,t}_{\chi}(A \cap \mathbb{X}_n).$$

The functions $\widetilde{\mathcal{P}}^{q,t}_{\chi}$ and $\mathcal{P}^{q,t}_{\chi}$ are metric outer measures. Furthermore, we may prove using the well known Besicovitch covering theorem that there exists an integer $\theta \in \mathbb{N}$ such that

$$\mathcal{H}^{q,t}_{\chi} \leq \theta \mathcal{P}^{q,t}_{\chi}. \tag{2.1}$$

The measures $\mathcal{H}^{q,t}_{\chi}$ and $\mathcal{P}^{q,t}_{\chi}$ assign in the usual way a multifractal dimension to each subset A of $\mathbb{X}$. They are respectively denoted by $\dim^q_{\chi}(A)$ and $\mathrm{Dim}^q_{\chi}(A)$. More precisely, we have

$$\begin{aligned}\dim^q_{\chi}(A) &= \inf\Big\{t \in \mathbb{R} \;\big|\; \mathcal{H}^{q,t}_{\chi}(A) = 0\Big\} = \sup\Big\{t \in \mathbb{R} \;\big|\; \mathcal{H}^{q,t}_{\chi}(A) = \infty\Big\},\\ \mathrm{Dim}^q_{\chi}(A) &= \inf\Big\{t \in \mathbb{R} \;\big|\; \mathcal{P}^{q,t}_{\chi}(A) = 0\Big\} = \sup\Big\{t \in \mathbb{R} \;\big|\; \mathcal{P}^{q,t}_{\chi}(A) = \infty\Big\}.\end{aligned}$$

One also defines Δ^q_{χ}, which generalizes the Minkowski-Bouligand dimension; for a bounded set A, one sets

$$\Delta^q_{\chi}(A) = \inf\Big\{t \geq 0 \;\Big|\; \lim_{n\to+\infty} \overline{\mathcal{P}}^{q,t}_{\chi}(A \cap \mathbb{X}_n) = 0\Big\}.$$

If A is unbounded, one chooses $x_0 \in \mathbb{X}$ and can set

$$\Delta^q_{\chi}(A) = \lim_{n\to+\infty} \Delta^q_{\chi}(A \cap B(x_0, n)).$$

As a direct consequence of the definition, the dimensions defined above satisfy that $\dim^q_{\chi}(A) \leq \mathrm{Dim}^q_{\chi}(A) \leq \Delta^q_{\chi}(A)$. Moreover, for $\varkappa = 0$, the functions $\mathcal{H}^{q,t}_{\chi}$, $\mathcal{P}^{q,t}_{\chi}$ and $\overline{\mathcal{P}}^{q,t}_{\chi}$ will be denoted respectively by $\mathcal{H}^t_{\xi}$, $\mathcal{P}^t_{\xi}$ and $\overline{\mathcal{P}}^t_{\xi}$; then, we will write

$$\dim_{\xi}(A) = \dim^q_{\chi}(A), \qquad \mathrm{Dim}_{\xi}(A) = \mathrm{Dim}^q_{\chi}(A) \quad \text{and} \quad \Delta_{\xi}(A) = \Delta^q_{\chi}(A).$$

Remark 1. *In the special case where $\varkappa = 0$ and $\xi(x, r) = -\log r$, we come back to the classical definitions of the Hausdorff and packing measures and dimensions in their original forms [27]. In particular, we get*

$$\mathcal{H}^{q,t}_{\chi} = \mathcal{H}^t, \qquad \mathcal{P}^{q,t}_{\chi} = \mathcal{P}^t,$$

and

$$\dim^q_{\chi}(A) = \dim(A), \qquad \mathrm{Dim}^q_{\chi}(A) = \mathrm{Dim}(A).$$

Finally, we respectively define the multifractal functions b_{χ}, B_{χ} and $\Lambda_{\chi} : \mathbb{E} \longrightarrow [-\infty, +\infty]$ by

$$b_{\chi}(q) = \dim^q_{\chi}(\mathbb{X}), \quad B_{\chi}(q) = \mathrm{Dim}^q_{\chi}(\mathbb{X}) \quad \text{and} \quad \Lambda_{\chi}(q) = \Delta^q_{\chi}(\mathbb{X}). \tag{2.2}$$

Moreover, it is well known [28] that Λ_{χ} and B_{χ} are convex and $b_{\chi} \leq B_{\chi} \leq \Lambda_{\chi}$.

2.2. Example: Homogeneous tree

Let $b \geq 2$ and consider the set $\mathcal{A}^* = \bigcup_{k\geq 0} \mathcal{A}^k$ as a free monoid consisting of words on $\mathcal{A} = \{0, 1, 2, \dots, b-1\}$. The empty word ε is the identity element and it is convenient to set $\mathcal{A}^0 = \{\epsilon\}$. The concatenation of two words u and v will be simply denoted by a juxtaposition, that is the word. The length of the word u is denoted by $|u|$. Moreover, we may define an order " $\prec$ " on $\mathcal{A}^*$: if a word v is a prefix of the word u, we write $v \prec u$. The set of infinite sequences of elements of $\mathcal{A}$ will be denoted by $\partial\mathcal{A}$. We identify $u \in \mathcal{A}^*$ with the cylinder $[u] := \{x \in \partial\mathcal{A},\ u \prec x\}$. We define an ultrametric distance on $\partial\mathcal{A}$ by

$$d(u, v) = b^{-|u\wedge v|}, \tag{2.3}$$

where $u \wedge v$ stands for their largest common prefix. In this example, we consider $\mathbb{X}$ to be the space $\partial\mathcal{A}$ and $\chi = (\varkappa, \xi)$ defined in (1.2) such that χ constitutes functions defined on the cylinder. Let $\delta > 0$; A is a bounded subset of $\mathbb{X}$. We set

$$\mathcal{P}^{*q,\,t}_{\chi,\delta}(A) = \sup \sum_j e^{-\langle q,\,\varkappa(x_j,\,r_j)\rangle - t\,\xi(x_i,r_i)},$$

where the supremum is taken over by the collection of δ-packings $\{B(x_j, r_j)\}$ of A such that $\delta/b < r_j \leq \delta$. We define

$$\mathcal{P}^{*q,\,t}_{\chi}(A) = \sup_{n\geq 1} \limsup_{\delta\to 0} \mathcal{P}^{*q,\,t}_{\chi,\delta}(A \cap \mathbb{X}_n)$$

and

$$\Delta^{*q}_{\chi}(A) = \inf\Big\{t \geq 0 \ \mid\ \mathcal{P}^{*q,\,t}_{\chi}(A) = 0\Big\}.$$

Definition 1. *For $b \geq 2$, the valuation ξ is said to be normal if, for all $n \geq 1$ and all $\epsilon > 0$, there exists $\rho > 0$, such that $\sum_{j\geq 0} e^{-t\tilde{\xi}_n(\rho b^{-j})} < \infty$, where*

$$\tilde{\xi}_n(t) = \inf_{x\in\mathbb{X}_n} \inf_{t/b\leq r<t} \xi(x, r).$$

Lemma 1. *Let $q \in \mathbb{E}$, $t \in \mathbb{R}$ and $k \geq 1$. If ξ is normal, then we have the following*

(1) $\mathcal{P}^{*q,\,t}_{\chi,\,b^{-k}}(\partial\mathcal{A}) = \sum_{u\in\mathcal{A}^k} e^{-\langle q,\,\varkappa([u])\rangle - t\xi([u])}$.

(2) $\Delta^{q}_{\chi} = \Delta^{*q}_{\chi}$.

Proof. (1) Let $\{B(x_j, r_j)\}_j$ be a packing of $\partial\mathcal{A}$ such that $b^{-k-1} < r_j \leq b^{-k}$; then,

$$\sum_j e^{-\langle q,\chi(B(x_j,r_j))\rangle - t\xi(x_i,r_i)} \quad \leq \quad \sum_{u\in\mathcal{A}^k} e^{-\langle q,\,\varkappa([u]))\rangle - \xi([u])}.$$

It follows that $\mathcal{P}^{*q,\,t}_{\chi,\,b^{-k}}(\partial\mathcal{A}) \leq \sum_{u\in\mathcal{A}^k} e^{\langle q,\,\varkappa([u]))\rangle - \xi([u])}$. On the other hand, since $\Big\{[u], u \in \mathcal{A}^k\Big\}$ is a b^{-k}-packing of $\partial\mathcal{A}$, we have

$$\sum_{u\in\mathcal{A}^k} e^{\langle q,\,\varkappa([u]))\rangle - \xi([u])} \leq \mathcal{P}^{*q,t}_{\varkappa,b^{-k}}(\partial\mathcal{A})$$

as required.

(2) Since, for all $n \geq 1$, $\mathcal{P}_{\chi}^{*q,t}(A) \leq \overline{\mathcal{P}}_{\chi}^{q,t}(A \cap \mathbb{X}_n)$, one has that $\Delta_{\chi}^{*q}(A) \leq \Delta_{\chi}^{q}(A)$. Now, suppose that $\Delta_{\chi}^{q}(A) > 0$. Let t and ϵ be two positive numbers such that $0 < t - \epsilon < t < \Delta_{\chi}^{q}(E)$. Therefore, $\overline{\mathcal{P}}_{\chi}^{q,t}(A) = +\infty$. We define recursively a sequence $\{\eta_m\}_{m\geq 0}$. First, $\eta_0 = b^{-k_0}\rho$, where ρ given by the normality of ξ and k_0 is chosen so that $\eta_0 \leq 1/n$. Suppose that η_m has been defined. Then, there exists an (η_m/b)-packing of $A \cap \mathbb{X}_n$ with

$$\sum e^{-\left(\left\langle q, \varkappa(x_j, r_j)\right\rangle + t\,\xi(x_i, r_j)\right)} \geq 1.$$

There exists a positive integer $k \geq 1$ such that

$$\sum_{j:\, \eta_m/b < b^k r_j \leq \eta_m} e^{-\left(\left\langle q, \varkappa(x_j, r_j)\right\rangle + t\,\xi(x_i, r_j)\right)} \geq e^{-\epsilon\tilde{\xi}(b^{-k}\eta_m)} \Big/ \sum_{k\geq 1} e^{-\epsilon\tilde{\xi}(b^{-k}\eta_m)}.$$

Then we set $\eta_{m+1} = b^{-k}\eta_m$. It follows that

$$\overline{\mathcal{P}}_{\chi,\eta_{m+1}}^{*q,t}(A \cap \mathbb{X}_n) \geq \sum_{j:\, \eta_m/b < b^k r_j \leq \eta_m} e^{-\left(\left\langle q, \varkappa(x_j, r_j)\right\rangle + t\,\xi(x_i, r_j)\right)} e^{\epsilon\xi(x_j r_j)} > 1 \Big/ \sum_{k\geq 1} e^{-\epsilon\tilde{\xi}(b^{-k}\eta_m)}.$$

Therefore $\mathcal{P}_{\chi}^{*q,t-\epsilon}(A) = +\infty$ and $\Delta_{\chi}^{*q}(A) \geq t - \epsilon$.

□

Proposition 1. *Let $q \in \mathbb{E}$, $t \in \mathbb{R}$ and $k \geq 1$. If the valuation ξ is normal, then we have*

$$\Lambda_{\chi}(q) = \inf\Big\{t \in \mathbb{R},\ \ \limsup_{k\to\infty} \frac{1}{k} \log \sum_{u\in\mathcal{A}^k} e^{-\langle q, \varkappa([u])\rangle - t\xi([u])} \leq 0\Big\}.$$

Proof. Let $t > f(q) := \inf\Big\{t \in \mathbb{R},\ \ \limsup_{k\to\infty} \frac{1}{k} \log \sum_{u\in\mathcal{A}^k} e^{-\langle q, \varkappa([u])\rangle - t\xi([u])} \leq 0\Big\}$. Then, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{u\in\mathcal{A}^k} e^{-\langle q, \varkappa([u])\rangle - t\xi([u])} \leq 1, \qquad k \geq k_0.$$

It follows that

$$\mathcal{P}_{\chi, b^{-k}}^{*q,t}(\partial\mathcal{A}) \;=\; \sum_{u\in\mathcal{A}^k} e^{-\langle q, \varkappa([u])\rangle - t\xi([u])} \leq 1,$$

and, then $\mathcal{P}_{\chi}^{*q,t} < \infty$. This implies that $\Lambda_{\chi}(q) \leq f(q)$. On the other hand, assume that $t < f(q)$; then, there exists a sequence $(k_m)_{m\geq 1}$ such that

$$\sum_{u\in\mathcal{A}^{k_m}} e^{-\langle q, \chi([u])\rangle - t\xi([u])} > 1.$$

It follows that

$$\mathcal{P}_{\chi, b^{-k_m}}^{*q,t}(\partial\mathcal{A}) \;=\; \sum_{u\in\mathcal{A}^{k_m}} e^{-\langle q, \varkappa([u])\rangle - t\xi([u])} > 0$$

and then $\mathcal{P}_{\chi}^{*q,t} > 0$. This implies that $\Lambda_{\chi}(q) \geq f(q)$ as required. □

Remark 2. *If* $\chi = (\varkappa, -\log r)$ *then*

$$\mathcal{P}^{*q,t}_{\chi, b^{-k}}(\partial\mathcal{A}) = b^{-kt}\sum_{u\in\mathcal{A}^k} e^{-\langle q, \varkappa([u])\rangle}$$

and

$$\Lambda_\chi(q) = \limsup_{k\to\infty}\frac{1}{k}\log_b \sum_{u\in\mathcal{A}^k} e^{-\langle q, \varkappa([u])\rangle}.$$

3. Main results

Multifractal analysis is typically used to describe objects possessing some type of scale invariance. The investigation has focused on structures produced by one mechanism which were analyzed with respect to the ordinary volume or metric. The most imported examples include branching random walk and self-similar measures [1,2,27]. In particular, the multifractal spectrum provides a characterization of the singularities of a distribution in terms of the geometrical properties. Unfortunately, we may obtain identical spectra despite having strikingly different measures. For this, we will study a more general level set. More precisely, let $(\mathbb{X}, d)$ be a separable metric space verifying the Besicovitch covering property; $\mathbb{E}'$ is the dual of a separable real Banach space $\mathbb{E}$ and $\chi = (\varkappa, \xi)$ such that $\varkappa$ and ξ satisfy (1.2). For $\alpha \in \mathbb{E}'$ and $\beta \geq 0$, we recall the set

$$\mathrm{X}_\chi(\alpha,\beta) = \left\{x\in\mathbb{X};\ \lim_{r\to 0}\frac{\langle w, \varkappa(x,r)\rangle}{\xi(x,r)} = \langle w, \alpha\rangle \ \text{ and }\ \lim_{r\to 0}\frac{\xi(x,r)}{-\log r} = \beta,\ \ \forall\, w\in\mathbb{E}\right\}.$$

In this section, we will state our main results concerning the estimation of the Hausdorff and packing dimensions of the set $\mathrm{X}_\chi(\alpha)$ by using the Legendre transform of the multifractal Hausdorff and packing functions, where the Legendre transform of a real valued function $f : \mathbb{E}\to\overline{\mathbb{R}}$ is a function $f^* : \mathbb{E}' \longrightarrow \overline{\mathbb{R}}$ defined by

$$f^*(\alpha) = \inf_{q\in\mathbb{E}}\langle q,\ \alpha\rangle + f(q).$$

More precisely, we have the following results.

Theorem A. *(1) Let $q \in \mathbb{E}$ and $\beta \geq 0$. Assume that, at some point q, the multifractal function b_χ is convex and differentiable and set $\alpha = -b_\chi'(q)$. Then, provided that $b^*_\chi(\alpha) \geq 0$ and $\mathcal{H}^{q,b_\chi(q)}_\chi(\mathrm{X}_\chi(\alpha,\beta)) > 0$, one has*

$$\dim \mathrm{X}_\chi(\alpha,\beta) = \beta\, b^*_\chi(\alpha).$$

(2) Let $q \in \mathbb{E}$ and $\beta \geq 0$. Assume that, at some point q, the multifractal function B_χ is differentiable and set $\alpha = -B_\chi'(q)$. Then, provided that $B^*_\chi(\alpha) \geq 0$ and $\mathcal{P}^{q,B_\chi(q)}_\chi(\mathrm{X}_\chi(\alpha,\beta)) > 0$, one has

$$\mathrm{Dim}\ \mathrm{X}_\chi(\alpha,\beta) = \beta\, B^*_\chi(\alpha).$$

The most common example in this context is considered when we study the multifractal measure μ with respect to arbitrary measure ν. More precisely, take

$$\varkappa(x,r) = -\log\mu(B(x,r)) \quad \text{and} \quad \xi(x,r) = -\log\nu(B(x,r)),$$

where μ and ν are two Borel measures defined in the metric space $\mathbb{X}$. The major interest of this is to use a partition of the space in sets of equal ν measures instead of equal size (when considering the diameter). In [10] the author formalizes the idea of performing multifractal analysis with respect to an arbitrary reference measure by developing a formalism for the multifractal analysis of one measure with respect to another. This formalism is based on the ideas of the 'multifractal formalism' as first introduced by Halsey et al. [17], and closely parallels Olsen's formal treatment of this formalism in [27]. The Hausdorff and packing dimensions of $\mathrm{X}_\chi(\alpha)$ are fully carried by some subset $\mathrm{X}_\chi(\alpha,\beta)$. The following corollary provides us with a sufficient condition that gives the lower bound for the Hausdorff and packing dimensions of $\mathrm{X}_\chi(\alpha)$.

Corollary B. *(1) Assume that, at some point q, the multifractal function b_χ is convex and differentiable. Set $\alpha = -b'_\chi(q)$ and*

$$I = \Big\{\beta \geq 0 \mid \mathcal{H}_\chi^{q,b_\chi(q)}(\mathrm{X}_\chi(\alpha,\beta)) > 0\Big\}.$$

Suppose that $b^*_\chi(\alpha) \geq 0$; then,

$$\dim \mathrm{X}_\chi(\alpha) \;\geq\; \sup_{\beta\in I} \beta\, b^*_\chi(\alpha).$$

(2) Assume that, at some point q, the multifractal function B_χ is differentiable. Set $\alpha = -B'_\chi(q)$ and

$$J = \Big\{\beta \geq 0 \mid \mathcal{P}_\chi^{q,B_\chi(q)}(\mathrm{X}_\chi(\alpha,\beta)) > 0\Big\}.$$

Suppose that $B^*_\chi(\alpha) \geq 0$; then,

$$\mathrm{Dim}\; \mathrm{X}_\chi(\alpha) \;\geq\; \sup_{\beta\in J} \beta\, B^*_\chi(\alpha).$$

Remark 3. *It is not difficult to observe that the second assertion of the preview corollary remains true when we consider Λ_χ instead of B_χ. In particular, let $\alpha = -\Lambda'_\chi(q)$ and*

$$\widetilde{I} = \Big\{\beta \geq 0 \mid \mathcal{H}_\chi^{q,\Lambda_\chi(q)}(\mathrm{X}_\chi(\alpha,\beta)) > 0\Big\}.$$

Then, provided that $\Lambda^*_\chi(\alpha) \geq 0$, we have that $\dim \mathrm{X}_\chi(\alpha) = \mathrm{Dim}\, \mathrm{X}_\chi(\alpha) \;\geq\; \sup_{\beta\in\widetilde{I}} \beta\, \Lambda^*_\chi(\alpha)$.

In the following example, we will consider a special case when the function Λ_χ is differentiable. This fact will be used in Section 4.

Example 1. *In this example, we will use the same notation as in Section 2.2. Let $\mathbb{X} = \partial\mathcal{A}$, $\mathbb{E}$ be the Euclidean space $\mathbb{R}^N$ and $\{(p_{i,j})_{0\leq j<b}\}_{1\leq i\leq N}$ be a family of positive numbers. Define the recurrence $p_{i,u}$ for given i and $u \in \mathcal{A}^*$:*

$$p_{i,\epsilon} = 1 \qquad \text{and} \qquad p_{i,uj} = p_{i,u}\, p_{i,j}.$$

Then, when $\sum_{j=0}^{b-1} p_{i,j} = 1$, the function $[u] \longmapsto p_{i,u}$ extends to a probability measure on $\partial\mathcal{A}$. We set the function $\varkappa([u]) = (-\log p_{i,u})_{1\leq i\leq N}$ and $\xi([u]) = -\log r$. For $q = (q_1, q_2, \ldots, q_N) \in \mathbb{R}^N$, we have

$$\sum_{u\in\mathcal{A}^{k+1}} e^{-\langle q,\varkappa([u])\rangle} = \sum_{u\in\mathcal{A}^{k+1}} \prod_{i=1}^{N} p_{i,u}^{q_i} = \sum_{u\in\mathcal{A}^{k}} \sum_{j=0}^{b-1} \prod_{i=1}^{N} p_{i,u}^{q_i} p_{i,j}^{q_i} = \left(\sum_{u\in\mathcal{A}^{k}} e^{-\langle q,\varkappa([u])\rangle}\right)\left(\sum_{j=0}^{b-1} \prod_{i=1}^{N} p_{i,j}^{q_i}\right).$$

It follows that the sequence $\left(\sum_{u\in\mathcal{A}^k} e^{-\langle q,\varkappa([u])\rangle}\right)_k$ *is geometric; then, using Remark 2,*

$$\Lambda_\chi(q)=\limsup_{n\to\infty}\frac{1}{k}\log_b\sum_{u\in\mathcal{A}^k}e^{-\langle q,\varkappa([u])\rangle}=\limsup_{k\to\infty}\frac{1}{k}\log_b\left(\sum_{j=0}^{b-1}\prod_{i=1}^{N}p_{i,j}^{q_i}\right)^k=\log_b\sum_{j=0}^{b-1}\prod_{i=1}^{N}p_{i,j}^{q_i},$$

which is clearly differentiable.

3.1. Upper bound of Hausdorff and packing dimensions

Let $A\subseteq\mathbb{E}$, $\alpha\in\mathbb{E}'$ and $\beta\geq 0$; we define

$$\mathrm{X}_\chi(\underline{\alpha},\underline{\beta};A):=\left\{x\;\Big|\;\varliminf_{r\to0}\frac{\langle w,\varkappa(x,r)\rangle}{\xi(x,r)}\geq\langle w,\alpha\rangle\;\text{ and }\;\varliminf_{r\to0}\frac{\xi(x,r)}{-\log r}\geq\beta,\quad\forall\,w\in A\right\},$$

$$\mathrm{X}_\chi(\overline{\alpha},\overline{\beta};A):=\left\{x\;\Big|\;\varlimsup_{r\to0}\frac{\langle w,\varkappa(x,r)\rangle}{\xi(x,r)}\leq\langle w,\alpha\rangle\;\text{ and }\;\varlimsup_{r\to0}\frac{\xi(x,r)}{-\log r}\leq\beta,\quad\forall\,w\in A\right\}.$$

The sets $\mathrm{X}_\chi(\underline{\alpha},\underline{\beta};\mathbb{E})$ and $\mathrm{X}_\chi(\overline{\alpha},\overline{\beta};\mathbb{E})$ will simply be denoted by $\mathrm{X}_\chi(\underline{\alpha},\underline{\beta})$ and $\mathrm{X}_\chi(\overline{\alpha},\overline{\beta})$ respectively. We will be interested in the set

$$\mathrm{X}_\chi(\alpha,\beta)\quad:=\quad\mathrm{X}_\chi(\overline{\alpha},\overline{\beta})\cap\mathrm{X}_\chi(\underline{\alpha},\underline{\beta}).$$

Theorem 1. *For* $\alpha\in\mathbb{E}'$ *and* $\beta\geq0$*, we have the following:*

(1) $\dim\left(\mathrm{X}_\chi(\alpha,\beta)\right)\;\leq\;\beta\,{b^*}_\chi(\alpha)$.

(2) $\mathrm{Dim}\left(\mathrm{X}_\chi(\alpha,\beta)\right)\;\leq\;\beta\,{B^*}_\chi(\alpha)$.

A negative dimension means that $\mathrm{X}_\chi(\alpha,\beta)$ *is empty.*

Proof. This theorem follows immediately from the following lemma. □

Lemma 2. *Let* $\alpha\in\mathbb{E}'$, $q\in\mathbb{E}$, $A\subseteq\mathbb{E}$ *and* $\beta\geq0$.

(1) If $\langle q,\alpha\rangle+b_\chi(q)\geq0$, *then*

$$\dim\left(\mathrm{X}_\chi(\overline{\alpha},\overline{\beta};A)\right)\leq\beta(\langle q,\alpha\rangle+b_\chi(q)).$$

(2) If $\langle q,\alpha\rangle+B_\chi(q)\geq0$, *then*

$$\mathrm{Dim}\left(\mathrm{X}_\chi(\overline{\alpha},\overline{\beta};A)\right)\leq\beta(\langle q,\alpha\rangle+B_\chi(q)).$$

Proof. It is clear that we only have to consider the case when the set $A=\{q\}$. Let n and m be two positive integers such that $m\geq n$, $q\in\mathbb{E}$, $t\in\mathbb{R}$ and ε_1 and ε_2 are two positive numbers such that

$$\varepsilon_1\leq\langle q,\alpha\rangle+t\quad\text{ and }\quad\varepsilon_2\leq\beta(\langle q,\alpha\rangle+t-\varepsilon_1).$$

We consider the set

$$A_{m,n}(\varepsilon_1,\varepsilon_2)\;=\;\left\{x\in\mathbb{X}_n\;\Big|\frac{\langle q,\varkappa(x,r)\rangle}{\xi(x,r)}\;\leq\;\langle q,\alpha\rangle+\varepsilon_1\;\text{ and }\;\frac{\xi(x,r)}{-\log r}\leq\beta+\frac{\varepsilon_2}{\langle q,\alpha\rangle+t+\varepsilon_1}\;\text{ for }\;r\leq\frac{1}{m}\right\}.$$

Then, we have

$$\mathrm{X}_\chi(\overline{\alpha},\overline{\beta};\{q\})\subseteq\bigcup_{n\geq1}\;\bigcap_{p_1,p_2\geq1}\;\bigcup_{m\geq n}A_{m,n}(1/p_1,1/p_2).$$

(1) Let $\left(\mathsf{B}(x_i, r_i)\right)_i$ be a centered δ-covering of a subset $F \subseteq A_{m,n}(\varepsilon_1, \varepsilon_2)$ with $0 < \delta \leq \frac{1}{m}$. Then one has $e^{-(\langle q,\alpha\rangle+\varepsilon_1)\xi(x_i,r_i)} \leq e^{-\langle q,\varkappa(x_i,r_i)\rangle}$ and $r_i^{\beta(\langle q,\alpha\rangle+t+\varepsilon_1)+\varepsilon_2} \leq e^{-(\langle q,\alpha\rangle+t+\varepsilon_1)\xi(x_i,r_i)}$. It follows that, for $t = b_\chi(q)+\eta$

$$r_i^{\beta\left(\langle q,\alpha\rangle+b_\chi(q)+\eta+\varepsilon_1\right)+\varepsilon_2} \leq e^{-\left(\langle q,\alpha\rangle+b_\chi(q)+\eta+\varepsilon_1\right)\xi(x_i,r_i)} \leq e^{-\left(\langle q,\varkappa(x_i,r_i)\rangle+(b_\chi(q)+\eta)\,\xi(x_i,r_i)\right)}.$$

Therefore, we have

$$\overline{\mathcal{H}}_\delta^{\beta(\langle q,\alpha\rangle+b_\chi(q)+\eta+\varepsilon_1)+\varepsilon_2}(F) \leq \sum_i r_i^{\beta\left(\langle q,\alpha\rangle+b_\chi(q)+\eta+\varepsilon_1\right)+\varepsilon_2} \leq \sum_i e^{-\left(\langle q,\varkappa(x_i,r_i)\rangle+(b_\chi(q)+\eta)\xi(x_i,r_i)\right)}.$$

From this, we can deduce that for $0 < \delta \leq \frac{1}{m}$, $\overline{\mathcal{H}}_\delta^{\beta(\langle q,\alpha\rangle+b_\chi(q)+\eta+\varepsilon_1)+\varepsilon_2}(F) \leq \overline{\mathcal{H}}_{\chi,\delta}^{q,b_\chi(q)+\eta}(F)$. Now, letting $\delta \to 0$, we obtain, for all $F \subseteq A_{m,n}(\varepsilon_1, \varepsilon_2)$,

$$\mathcal{H}^{\beta(\langle q,\alpha\rangle+b_\chi(q)+\eta+\varepsilon_1)+\varepsilon_2}(A_{m,n}(\varepsilon_1,\varepsilon_2)) \leq \mathcal{H}_\chi^{q,b_\chi(q)+\eta}(A_{m,n}(\varepsilon_1,\varepsilon_2)).$$

Then it is easy to conclude that $\mathcal{H}^{\beta(\langle q,\alpha\rangle+b_\chi(q)+\eta+\varepsilon_1)+\varepsilon_2}(A_{m,n}(\varepsilon_1,\varepsilon_2)) = 0$. This implies that

$$\dim\left(A_{m,n}(\varepsilon_1,\varepsilon_2)\right) \leq \beta(\langle q,\alpha\rangle + b_\chi(q) + \varepsilon_1) + \varepsilon_2;$$

then by the countable stability and monotony of the Hausdorff dimension, we have

$$\dim\left(\mathrm{X}_\chi(\overline{\alpha},\overline{\beta};A\right) \leq \beta(\langle q,\alpha\rangle + b_\chi(q)).$$

(2) Let $\left(\mathsf{B}(x_i, r_i)\right)_i$ be a δ-packing of $F \subseteq A_{m,n}(\varepsilon_1, \varepsilon_2)$ with $0 < \delta \leq \frac{1}{m}$. Then, for $t = B_\chi(q) + \eta$, we have that $e^{-(\langle q,\alpha\rangle+\varepsilon_1)\xi(x_i,r_i)} \leq e^{-\langle q,\varkappa(x_i,r_i)\rangle}$ and $r_i^{\beta\left(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1\right)+\varepsilon_2} \leq e^{-\left(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1\right)\xi(x_i,r_i)}$. Putting these together we see that

$$r_i^{\beta\left(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1\right)+\varepsilon_2} \leq r_i^{\beta\left(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1\right)+\varepsilon_2} \leq e^{-\left(\langle q,\varkappa(x_i,r_i)\rangle+(B_\chi(q)+\eta)\,\xi(x_i,r_i)\right)}.$$

Hence $\sum_i r_i^{\beta\left(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1)\right)+\varepsilon_2} \leq \sum_i e^{-\left(\langle q,\varkappa(x_i,r_i)\rangle+(B_\chi(q)+\eta)\,\xi(x_i,r_i)\right)}$. Then, we can deduce that, for $0 < \delta \leq \frac{1}{m}$

$$\overline{\mathcal{P}}_\delta^{\beta(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1)+\varepsilon_2}(F) \leq \overline{\mathcal{P}}_{\chi,\delta}^{q,B_\chi(q)+\eta}(F).$$

Letting $\delta \to 0$, we obtain that $\overline{\mathcal{P}}^{\beta(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1)+\varepsilon_2}(F) \leq \overline{\mathcal{P}}_{\xi,\varkappa}^{q,B_\chi(q)+\eta}(F)$. Now, let $(A_i)_{i\in\mathbb{N}}$ be a covering of $A_{m,n}(\varepsilon_1, \varepsilon_2)$. We have

$$\begin{aligned}
\mathcal{P}^{\beta(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1)+\varepsilon_2}(A_{m,n}(\varepsilon_1,\varepsilon_2)) &\leq \sum_i \overline{\mathcal{P}}^{\beta(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1)+\varepsilon_2}(A \cap A_i) \\
&\leq \sum_i \overline{\mathcal{P}}_\chi^{q,B_\chi(q)+\eta}(A \cap A_i) \\
&\leq \sum_i \overline{\mathcal{P}}_\chi^{q,B_\chi(q)+\eta}(A_i).
\end{aligned}$$

It results that $\mathcal{P}^{\beta(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1)+\varepsilon_2}(A_{m,n}(\varepsilon_1,\varepsilon_2)) \le \widetilde{\mathcal{P}}_\chi^{q,B_\chi(q)+\eta}(A_{m,n}(\varepsilon_1,\varepsilon_2))$. Since $\mathcal{P}_\chi^{q,B_\chi(q)+\eta}(\mathbb{X}) = 0$, it follows that, for all n, $\widetilde{\mathcal{P}}_\chi^{q,\,B_\chi(q)+\eta}(\mathbb{X}_n) = 0$. Therefore,

$$\mathcal{P}^{\beta(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1)+\varepsilon_2}(A_{m,n}(\varepsilon_1,\varepsilon_2)) = \widetilde{\mathcal{P}}^{\beta(\langle q,\alpha\rangle+B_\chi(q)+\eta+\varepsilon_1)+\varepsilon_2}(A_{m,n}(\varepsilon_1,\varepsilon_2)) = 0.$$

So, we have that $\mathrm{Dim}\,(A_{m,n}(\varepsilon_1,\varepsilon_2)) \le \beta(\langle q,\alpha\rangle + B_\chi(q) + \varepsilon_1) + \varepsilon_2$; then,

$$\mathrm{Dim}\left(\mathrm{X}_\chi(\overline{\alpha},\overline{\beta};A\right) \le \beta(\langle q,\alpha\rangle + B_\chi(q)).$$

□

3.2. Lower bound of Hausdorff and packing dimensions

Let $v,q \in \mathbb{E}$ and assume that $\left|B_{\xi,\varkappa}(q)\right| < \infty$. We define

$$\partial_v B_\chi(q) = \lim_{t\to 0} \frac{B_\chi(q+tv) - B_\chi(q)}{t}.$$

We will denote by $B_\chi'(q)$ (as an element of $\mathbb{E}'$) the derivative of B_χ at q when it exists. When B_χ has a partial derivative at point q along the direction v, one has that $\partial_{-v}B_\chi(q) = -\partial_v B_\chi(q)$. In this case, we have

$$\partial_v B_\chi(q) = \langle v,\ B_\chi'(q)\rangle.$$

Assume that the function $v \longmapsto \partial_v B_\chi(q)$ is lower semi-continuous; then, from [28, Proposition 10] and (2.1), one gets that $\mathcal{P}_\chi^{q,B_\chi(q)}(\mathrm{X}_\chi(\alpha)) > 0$, which implies that there exists β such that $\mathcal{P}_\chi^{q,B_\chi(q)}(\mathrm{X}_\chi(\alpha,\beta)) > 0$. Similarly, if the function b_χ is convex and differentiable and $v \longmapsto \partial_v b_\chi(q)$ is lower semi-continuous, then

$$\mathcal{H}_\chi^{q,b_\chi(q)}(\mathrm{X}_\chi(\alpha)) > 0 \qquad \text{or} \qquad \mathcal{H}_\chi^{q,b_\chi(q)}(\mathbb{X}\setminus \mathrm{X}_\chi(\alpha)) = 0,$$

which implies that there exists β such that $\mathcal{H}_\chi^{q,b_\chi(q)}(\mathrm{X}_\chi(\alpha,\beta)) > 0$.

Theorem 2. *(1) If, for some q, $\mathcal{H}_\chi^{q,b_\chi(q)}\big(\mathrm{X}_\chi(\alpha,\beta)\big) > 0$ and if $v \longmapsto \partial_v b_\chi(q)$ is lower semi-continuous, then, if $b_\chi(q)$ is convex and differentiable at q, one has*

$$\dim\Big(\mathrm{X}_\chi(-b_\chi'(q),\beta)\Big) \ge \beta\Big(b_\chi(q) - \partial_q b_\chi(q)\Big).$$

(2) If, for some q, $\mathcal{P}_\chi^{q,B_\chi(q)}(\mathrm{X}_\chi(\alpha,\beta)) > 0$ and if $v \longmapsto \partial_v \mathsf{B}_\chi(q)$ is lower semi-continuous, then one has

$$\mathrm{Dim}\Big(\mathrm{X}_\chi(-B_\chi'(q),\beta)\Big) \ge \beta\Big(B_\chi(q) - \partial_q B_\chi(q)\Big).$$

Proof. This theorem follows immediately from the following Lemma. □

Lemma 3. *(1) If $b_\chi(q)$ is convex and differentiable at q and we set $\alpha = -b'_\chi(q)$, then for each Borel set $E \subseteq \mathrm{X}_\chi(\underline{\alpha},\underline{\beta}) \cap \mathbb{X}_n$, we have*

$$\mathcal{H}_\chi^{q,b_\chi(q)}(E) \le \mathcal{H}^{\beta\left(b_\chi(q)-\partial_q b_\chi(q)-\varepsilon_1\right)-\varepsilon_2}(E).$$

(2) Set $\alpha = -B'_\chi(q)$; then, for each Borel set $E \subseteq \mathrm{X}_\chi(\underline{\alpha}, \underline{\beta}) \cap \mathbb{X}_n$, we have

$$\mathcal{P}_\chi^{q,B_\chi(q)}(E) \leq \mathcal{P}^{\beta\left(B_\chi(q)-\partial_q B_\chi(q)-\varepsilon_1\right)-\varepsilon_2}(E).$$

Proof. (1) For $m \geq n$, we consider the set

$$A_m = \Big\{ x \in \mathrm{X}_\chi(\underline{\alpha}, \underline{\beta}) \cap \mathbb{X}_n \ \Big|\ \langle q, \varkappa(x,r)\rangle + (\partial_q b_\chi(q) + \varepsilon_1)\xi(x,r) \geq 0$$

and

$$\frac{\xi(x,r)}{-\log r} \geq \beta + \frac{\varepsilon_2}{b_\chi(q) - \partial_q b_\chi(q) - \varepsilon_1} \quad \text{for} \quad r \leq \frac{1}{m} \Big\}.$$

Given n and a subset F of A_m, let $(\mathsf{B}(x_i, r_i))_i$ a centered δ-covering of F with $0 < \delta < \min\{1/n, 1/m\}$. We have that $e^{-\left(b_\chi(q)-\partial_q b_\chi(q)-\varepsilon_1\right)\xi(x_i,r_i)} \geq e^{-\left(\langle q,\varkappa(x_i,r_i)\rangle + b_\chi(q)\,\xi(x_i,r)\right)}$ and $r_i^{\left(\beta\left(b_\chi(q)-\partial_q b_\chi(q)-\varepsilon_1\right)-\varepsilon_2\right)} \geq e^{-\left(b_\chi(q)-\partial_q b_\chi(q)-\varepsilon_1\right)\xi(x_i,r_i)}$. Therefore, we have

$$\overline{\mathcal{H}}_{\chi,\delta}^{q,b_\chi(q)}(F) \leq \sum e^{-\left(\langle q,\varkappa(x_i,r_i)\rangle + b_\chi(q)\,\xi(x_i,r)\right)} \leq \sum r_i^{-\left(\beta\left(b_\chi(q)-\partial_q b_\chi(q)-\varepsilon_1\right)-\varepsilon_2\right)}.$$

Then, for $\delta \leq \min\{1/n, 1/m\}$, we have that $\overline{\mathcal{H}}_{\chi,\delta}^{q,b_\chi(q)}(F) \leq \overline{\mathcal{H}}_{\delta}^{\beta\left(b_\chi(q)-\partial_q b_\chi(q)-\varepsilon_1\right)-\varepsilon_2}(F)$, and letting $\delta \to 0$ gives that for all $F \subseteq A_m$

$$\overline{\mathcal{H}}_{\chi}^{q,b_\chi(q)}(F) \leq \overline{\mathcal{H}}^{\beta\left(b_\chi(q)-\partial_q b_\chi(q)-\varepsilon_1\right)-\varepsilon_2}(F) \leq \mathcal{H}^{\beta\left(b_\chi(q)-\partial_q b_\chi(q)-\varepsilon_1\right)-\varepsilon_2}(A_m),$$

which gives that t$\mathcal{H}_{\chi}^{q,b_\chi(q)}(A_m) \leq \mathcal{H}^{\beta\left(b_\chi(q)-\partial_q b_\chi(q)-\varepsilon_1\right)-\varepsilon_2}(A_m)$. Finally, since $E = \bigcup_m A_m$, we obtain

$$\mathcal{H}_{\chi}^{q,b_\chi(q)}(E) \leq \mathcal{H}^{\beta\left(b_\chi(q)-\partial_q b_\chi(q)-\varepsilon_1\right)-\varepsilon_2}(E).$$

(2) For $m \geq n$, consider

$$A_m = \Big\{ x \in \mathrm{X}_\chi(\underline{\alpha}, \underline{\beta}) \cap \mathbb{X}_n \ \Big|\ \langle q, \varkappa(x,r)\rangle + (\partial_q B_\chi(q) + \varepsilon_1)\xi(x,r) \geq 0$$

and

$$\frac{\xi(x,r)}{-\log r} \geq \beta + \frac{\varepsilon_2}{B_\chi(q) - \partial_q B_\chi(q) - \varepsilon_1} \quad \text{for} \quad r \leq \frac{1}{m} \Big\}.$$

Given n and a subset F of A_m, $0 < \delta < \frac{1}{m}$ and let $(\mathsf{B}(x_i, r_i))_i$ be a δ-packing of F. Then, we have that $e^{-\left(B_\chi(q)-\partial_q B_\chi(q)-\varepsilon_1\right)\xi(x_i,r_i)} \geq e^{-\left(\langle q,\varkappa(x_i,r_i)\rangle + B_\chi(q)\,\xi(x_i,r)\right)}$ and $r_i^{-\left(\beta\left(B_\chi(q)-\partial_q B_\chi(q)-\varepsilon_1\right)-\varepsilon_2\right)} \geq e^{-\left(B_\chi(q)-\partial_q B_\chi(q)-\varepsilon_1\right)\xi(x_i,r_i)}$. Putting these together we see that

$$\sum_i e^{-\left(\langle q,\varkappa(x_i,r_i)\rangle + B_\chi(q)\,\xi(x_i,r)\right)} \leq \sum_i r_i^{-\left(\beta\left(B_\chi(q)-\partial_q B_\chi(q)-\varepsilon_1\right)-\varepsilon_2\right)} \leq \overline{\mathcal{P}}_{\delta}^{q,B_\chi(q)}(F);$$

then, $\overline{\mathcal{P}}_{\chi,\delta}^{q,B_\chi(q)}(F) \leq \overline{\mathcal{P}}_{\delta}^{q,B_\chi(q)}(F)$. Thus, letting $\delta \to 0$ gives that for all $F \subseteq A_m$, $\overline{\mathcal{P}}_{\chi}^{q,B_\chi(q)}(F) \leq \overline{\mathcal{P}}^{q,B_\chi(q)}(F)$. Now, let $(A_i)_i$ be a covering of A_m. Therefore, we have

$$\mathcal{P}_\chi^{q,t}(A_m) \leq \mathcal{P}_\chi^{q,B_\chi(q)}(\cup_i (A_m \cap A_i)) \leq \sum_i \mathcal{P}_\chi^{q,B_\chi(q)}(A_m \cap A_i) \leq \sum_i \overline{\mathcal{P}}_\chi^{q,B_\chi(q)}(A_m \cap A_i).$$

It follows that

$$\mathcal{P}_{\chi}^{q,B_\chi(q)}(A_m) \le \sum_i \overline{\mathcal{P}}^{\beta\left(B_\chi(q)-\partial_q B_\chi(q)-\varepsilon_1\right)-\varepsilon_2}(A_m \cap A_i). \le \sum_i \overline{\mathcal{P}}^{\beta\left(B_\chi(q)-\partial_q B_\chi(q)-\varepsilon_1\right)-\varepsilon_2}(A_i).$$

We can deduce now that $\mathcal{P}_{\chi}^{q,B_\chi(q)}(E) \le \mathcal{P}^{\beta\left(B_\chi(q)-\partial_q B_\chi(q)-\varepsilon_1\right)-\varepsilon_2}(E)$.

□

As mentioned above, in the last decay, there has been a great interest in the validity and non-validity of the multifractal formalism. Many positive results have been written in various situations. What follows, we state a sufficient condition so that we obtain the validity of the multifractal formalism. This result will be used to study the binomial measure in symbolic space $\partial\mathcal{A}$.

Proposition 2. *Let $q \in \mathbb{E}$ and $\beta \ge 0$. Assume that, at some point q, the function Λ_χ is differentiable and set $\alpha = -\Lambda'_\chi(q)$. Then, provided that $\mathcal{H}_{\chi}^{q,\Lambda_\chi(q)}(\mathrm{X}_\chi(\alpha,\beta)) > 0$, one has*

$$\dim(\mathrm{X}_\chi(\alpha,\beta)) = \mathrm{Dim}(\mathrm{X}_\chi(\alpha,\beta)) = \beta\, b_\chi^*(\alpha) = \beta\, B_\chi^*(\alpha) = \beta\, \Lambda_\chi^*(\alpha).$$

Proof. It is known from Theorem 1, that for all $\beta \ge 0$ and $\alpha \in \mathbb{E}$, one has

$$\mathrm{Dim}\ (\mathrm{X}_\chi(\alpha,\beta)) \ \le\ \beta\, B_\chi^*(\alpha)\ \le\ \beta\, \Lambda_\chi^*(\alpha).$$

It is clear that $\mathrm{X}_\chi(\alpha,\beta) \subseteq \mathrm{X}_\chi(\alpha)$. Then the assumption $\mathcal{H}_{\chi}^{q,\Lambda_\chi(q)}(\mathrm{X}_\chi(\alpha,\beta)) > 0$ implies that

$$\mathcal{H}_{\chi}^{q,\Lambda_\chi(q)}(\mathrm{X}_\chi(\alpha)) > 0.$$

Therefore from [28, Theorem 12] we obtain that $b_\chi(q) = B_\chi(q) = \Lambda_\chi(q)$. Hence, using Lemma 4 and the fact that Λ_χ is differentiable at q, we get

$$0 < \mathcal{H}_{\chi}^{q,\Lambda_\chi(q)}\left(\mathrm{X}_\chi(\alpha,\beta)\right) \le \mathcal{H}^{\beta\left(\partial_q\Lambda_\chi(q)+\Lambda_\chi(q)-\varepsilon_1\right)-\varepsilon_2}\left(\mathrm{X}_\chi(\alpha,\beta)\right)$$

and then

$$\dim\left(\mathrm{X}_\chi(\alpha,\beta)\right) \ge \beta(\partial_q\Lambda_\chi(q) + \Lambda_\chi(q) - \varepsilon_1) - \varepsilon_2.$$

Letting $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ yields that $\dim \mathrm{X}_\chi(\alpha,\beta) \ge \beta(\partial_q\Lambda_\chi(q) + \Lambda_\chi(q))$, which achieves the proof. □

Usually, it is difficult to check the hypothesis that $\mathcal{H}_{\chi}^{q,\Lambda_\chi(q)}(\mathrm{X}_\chi(\alpha,\beta)) > 0$. For this, we use the Frostman lemma, which is a useful tool to verify this hypothesis.

Lemma 4. (*Frostman lemma [28]*) *For $\beta \ge 0$, if there exists a Borel measure μ_q, and two positive numbers η and C such that $\mu_q\left(\mathrm{X}_\chi(\alpha,\beta)\right) > 0$ and such that, for all $x \in \mathrm{X}_\chi(\alpha,\beta)$ and all $r \le \eta$, one has*

$$\mu_q(\mathsf{B}(x,r)) \le C\, e^{-\left(\langle q,\varkappa(x,r)\rangle + \Lambda_\chi(q)\,\xi(x,r)\right)},$$

then $\mathcal{H}_{\chi}^{q,\Lambda_\chi(q)}\left(\mathrm{X}_\chi(\alpha,\beta)\right) > 0$.

4. Application

In this section, we will consider a special case when $\varkappa$ and ξ are two functions defined by using binomial measures. In this situation, we are able to construct an auxiliary measure μ_q so that we obtain the validity of the relative multifractal formalism, that is

$$\dim\left(\mathrm{X}_\chi(\alpha,\beta)\right) = \mathrm{Dim}\left(\mathrm{X}_\chi(\alpha,\beta)\right).$$

Moreover, we can compute explicitly the Hausdorff and packing dimensions in this case. Take the space $\mathbb{E}$ to be the Euclidean space $\mathbb{R}$ and we denote by $\mathbb{X}$ the space $\partial\mathcal{A}$ with $b = 2$, that is, $\mathbb{X} = \{0,1\}^{\mathbb{N}}$. Let (p_0, p_1) and (ω_0, ω_1) be two probability vectors, that is $p_0, p_1, \omega_0, \omega_1 \geq 0$ and $\sum p_i = \sum \omega_i = 1$. We define on $\partial\mathcal{A}$ two binomial probability measures μ_p, ν_ω by $\mu_p([\epsilon]) = \nu_\omega([\epsilon]) = 1$ and, for all $u \in \mathcal{A}^*$ and $i \in \{0,1\}$,

$$\mu([ui]) = p_u\, p_i \quad \text{ and } \quad \nu([ui]) = \omega_u\, \omega_i.$$

Now, we consider the functions $\varkappa$ and ξ to be defined on the cylinder such that, for all $u \in \mathcal{A}^k$, we have that $\varkappa([u]) = -\log\mu([u])$ and

$$\nu([u])^{1+h(k)} \leq e^{-\xi([u])} \leq \nu([u])^{1-h(k)},$$

where $h : \mathbb{N} \longrightarrow \mathbb{R}^*$ is a non-increasing function with $\lim\limits_{k\to\infty} h(k) = 0$. It is clear that a special example of the function ξ is when it is defined using the measure ν by $\xi([u]) = -\log\nu([u])$. For $q \in \mathbb{R}$, we define $\tau(q)$ as the unique number satisfying

$$p_0^q \omega_0^{\tau(q)} + p_1^q \omega_1^{\tau(q)} = 1. \tag{4.1}$$

Choose $h(k)$ small enough so that $1/2 \leq \inf_{u\in\mathcal{A}^k} \nu([u])^{-\tau(q)h(k)} \leq \sup_{u\in\mathcal{A}^k} \nu([u])^{-\tau(q)h(k)} \leq 3/2$ (take for instance $h(k) = \circ\left(\inf\{\ln\nu(u), u \in \mathcal{A}^k\}\right)$). Finally, we define

$$\beta(q) := -p_0\omega_0^{\tau(q)} \log_2 \omega_0 - p_1\omega_1^{\tau(q)} \log_2 \omega_1.$$

Theorem 3. *Let $(\alpha,\beta) \in \mathbb{R}^2$ such that $\alpha = -\tau'(q)$ and $\beta = \beta(q)$ for some $q \in \mathbb{R}$. Then,*

$$\dim\left(\mathrm{X}_\chi(\alpha,\beta)\right) = \mathrm{Dim}\left(\mathrm{X}_\chi(\alpha,\beta)\right) = \beta\,\tau^*(\alpha).$$

Observe that, for all $k \geq 1$, we have

$$\begin{aligned}
\sum_{u\in\mathcal{A}^{k+1}} e^{-\langle q,\varkappa([u])\rangle - \tau(q)\xi([u])} &= \sum_{u\in\mathcal{A}^{k+1}} \mu([u])^q \nu([u])^{\tau(q)(1-h(k))} \\
&\leq 3/2 \sum_{u\in\mathcal{A}^{k+1}} \mu([u])^q \nu([u])^{\tau(q)} \\
&\leq 3/2 \sum_{u\in\mathcal{A}^{k}} \mu([u])^q \nu([u])^{\tau(q)} \underbrace{\left(p_0^q\omega_0^{\tau(q)} + p_1^q\omega_1^{\tau(q)}\right)}_{=1} \leq 3/2.
\end{aligned}$$

Similarly, we have that $\sum\limits_{u\in\mathcal{A}^{k+1}} e^{-\langle q,\varkappa([u])\rangle - \tau(q)\xi([u])} \geq 1/2$. It is clear that ξ is normal; therefore, according to Lemma 1, we have

$$0 < \mathcal{P}_\chi^{*q,\tau(q)}(\partial\mathcal{A}) < \infty \quad \text{ and then } \quad \Lambda_\chi(q) = \tau(q).$$

We define, for each $q \in \mathbb{R}$, the measure μ_q on $\partial\mathcal{A}$ by

$$\mu_q([\epsilon]) = \emptyset \qquad \text{and} \qquad \mu_q([u]) = p_u^q\, \omega_u^{\tau(q)} \tag{4.2}$$

for all $u \in \mathcal{A}^*$.

Lemma 5. *Let μ_l be a binomial probability with the parameter $l \in (0,1)$; then, for μ_q-almost every x*

$$\lim_{k\to\infty} \frac{\log_2 \mu_l([x_{|n}])}{-n} = -p_0\omega_0^{\tau(q)} \log_2 l - p_1\omega_1^{\tau(q)} \log_2(1-l),$$

where $x_{|k} = x_1 \ldots x_k \in \mathcal{A}^k$.

Proof. The proof follows immediately from the law of large numbers see the details in [29], or [3] in a more general case. □

In particular, using the Lemma 5, for μ_q-almost every $x \in \partial\mathcal{A}$, we have

$$\lim_{k\to\infty} \frac{\xi([x_{|k}])}{k \log 2} = \lim_{k\to\infty} \frac{(1-h(k))\log_2 \nu([x_{|k}])}{-k} = -p_0\omega_0^{\tau(q)} \log_2 \omega_0 - p_1\omega_1^{\tau(q)} \log_2 \omega_1 = \beta(q)$$

and

$$\lim_{k\to\infty} \frac{\varkappa([x_{|k}])}{\xi([x_{|k}])} = \lim_{k\to\infty} \frac{\log_2 \mu([x_{|k}])}{-k} \frac{-k(1-h(k))^{-1}}{\log_2 \nu([x_{|k}])} = \frac{p_0\omega_0^{\tau(q)} \log_2 p_0 + p_1\omega_1^{\tau(q)} \log_2 p_1}{p_0\omega_0^{\tau(q)} \log_2 \omega_0 + p_1\omega_1^{\tau(q)} \log_2 \omega_1} = -\tau'(q) = \alpha.$$

Hence, $\mu_q(\mathrm{X}_\chi(\alpha,\beta)) = 1$. Moreover, for any $u \in \partial\mathcal{A}$, we have

$$\begin{aligned}
\frac{\mu_q([u])}{e^{-\langle q, \varkappa([u])\rangle - \tau(q)\xi([u])}} &\leq \frac{\mu_q([u])}{\mu([u])^q \nu([u])^{\tau(q)(1+h(k))}} \leq C_k \frac{\mu_q([u])}{\mu([u])^q \nu([u])^{\tau(q)}} \\
&\leq \frac{3}{2} \frac{\mu_q([u])}{\mu([u])^q \nu([u])^{\tau(q)}} \leq \frac{3}{2} \frac{p_u^q\, \omega_u^{\tau(q)}}{p_u^q\, \omega_u^{\tau(q)}} \leq \frac{3}{2}.
\end{aligned}$$

Therefore, from Lemma 4, we have that $\mathcal{H}_\chi^{q,\tau(q)}(\mathrm{X}_\chi(\alpha,\beta)) > 0$, which implies that

$$b_\chi(q) = B_\chi(q) = \Lambda_\chi(q) = \tau(q).$$

Since τ is differentiable at q, Theorem 2 gives that

$$\dim(\mathrm{X}_\chi(\alpha,\beta)) \geq \beta(q\alpha + \tau(q)).$$

On the other hand, by Theorem 1, we have that $\dim\left(\mathrm{X}_\chi(\alpha,\beta)\right) \leq \beta\ b_\chi^*(\alpha) = \beta\ \tau^*(\alpha)$. Finally, we obtain

$$\dim(\mathrm{X}_\chi(\alpha,\beta)) = \mathrm{Dim}(\mathrm{X}_\chi(\alpha,\beta)) = \beta\, b_\chi^*(\alpha) = \beta\, B_\chi^*(\alpha) = \beta\, \tau^*(\alpha).$$

Remark 4. *In fact, we can use the mass distribution principle [12] to compute the validity of the multifractal analysis. Indeed, for μ_q-almost every $x \in \partial\mathcal{A}$, we have*

$$\lim_{k\to\infty} \frac{\log_2 \mu_q([x_{|k}])}{-k} = \lim_{k\to\infty} \frac{\log \mu_q([x_{|k}])}{-\xi([x_{|k}]} \frac{\xi([x_{|k}])}{k \log 2} = \beta\Big(q \lim_{k\to\infty} (1-h(k))^{-1} \frac{\log p_{x_{|k}}}{\log \omega_{x_{|k}}} + \tau(q)\Big) = \beta(q\alpha + \tau(q)).$$

Therefore, the Hausdorff dimension of the measure μ_q is $\beta\, \tau^*(\alpha)$, where $\beta = \beta(q)$.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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