Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 23534–23547.
DOI: 10.3934/math.20231196
Received: 08 May 2023
Revised: 18 July 2023
Accepted: 24 July 2023
Published: 28 July 2023

Research article

A new subclass of analytic and bi-univalent functions associated with Legendre polynomials

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Abstract: In this paper, we introduce a new subclass of analytic and bi-univalent functions in the open unit disc U. For this subclass of functions, estimates of the initial coefficients $|A_2|$ and $|A_3|$ of the Taylor-Maclaurin series are given. An application of Legendre polynomials to this subclass of functions is presented. Furthermore, our study discusses several special cases.

Keywords: bi-univalent functions; analytic functions; starlike and convex functions; Legendre polynomials
Mathematics Subject Classification: 30C45, 30C50, 30C55

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with the following Taylor series representation

$$\xi(z) = z + \sum_{n=2}^{\infty} A_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in U. The Koebe function

$$\kappa(z) = z(1-z)^{-2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right], \quad (z \in U),$$

is one of the most important members of class $\mathcal{S}$. The range of this function is the entire complex plane except for a slit along the negative real axis from $w = -\infty$ to $w = -\frac{1}{4}$. Bieberbach [8] in 1916 proved that if $\xi \in \mathcal{S}$, and is given by (1.1), then $|a_2| \leq 2$, equality holds if and only if ξ is the Koebe function or one of its rotations. This theorem was the main basis for the famous Bieberbach's conjecture below.

Conjecture 1. *(Bieberbach's conjecture [8]) If $\xi \in \mathcal{S}$, and is given by (1.1), then $|a_n| \leq n$ for any integer $n \geq 2$, equality holds if and only if ξ is the Koebe function or one of its rotations.*

The Bieberbach conjecture was unproven until de Branges found a proof in 1984, the difficulty in solving Bieberbach's conjecture led many mathematicians to investigate subclasses of $\mathcal{S}$, for example, starlike, convex, and close-to-convex functions for which sharp coefficient bounds can be obtained.

A function $\xi \in \mathcal{A}$ is said to be strongly starlike of order γ if it satisfies the following inequality

$$\left|\arg\left(\frac{z\xi'(z)}{\xi(z)}\right)\right| < \frac{\gamma\pi}{2}, \quad (0 < \gamma \leq 1, \ \ z \in U). \tag{1.2}$$

We denote the class of all strongly starlike of order γ by $S(\gamma)$. We note that $S(1) = S^*$ is the familiar class of starlike functions. Also, a function $\xi \in \mathcal{A}$ is said to be strongly convex of order γ if it satisfies the following inequality

$$\left|\arg\left(1 + \frac{z\xi''(z)}{\xi'(z)}\right)\right| < \frac{\gamma\pi}{2}, \quad (0 < \gamma \leq 1, \ \ z \in U). \tag{1.3}$$

We denote the class of all strongly convex of order γ by $\tilde{K}(\gamma)$. We note that $\tilde{K}(1) = C$ is the well-known class of convex functions. In 1976, Miller [24] introduced the class $S(\alpha, \beta)$ of functions $\xi \in \mathcal{A}$ of the form

$$\Re\left(\left(\frac{z\xi'(z)}{\xi(z)}\right)^{\alpha}\left(1 + \frac{z\xi''(z)}{\xi'(z)}\right)^{\beta}\right) > 0, \quad (z \in U), \tag{1.4}$$

where α and β are fixed real numbers, and he proved that all functions in this class are univalent and starlike. This class contains many subclasses of univalent functions.

In fact

(1) $S(1,0) = S^*, \ S(0,1) = C, \ S(\frac{1}{\gamma}, 0) = S(\gamma)$ with $0 < \gamma \leq 1$;

(2) $S(0, \frac{1}{\gamma}) = \tilde{K}(\gamma)$ with $0 < \gamma \leq 1$;

(3) $S(1-\gamma, \gamma)$ is the class of gamma-starlike functions introduced by Lewandowski et al. in [19];

(4) $S(1, 1)$ is the subclass of starlike function of the form

$$\Re\left(\frac{z\xi'(z)}{\xi(z)} + \frac{z^2\xi''(z)}{\xi(z)}\right) > 0, \quad (z \in U), \tag{1.5}$$

which was studied by Ramesha el al. in [33], Obraddovic and Joshi in [28] and Padmanabhan in [32];

(5) $S(-1, 1)$ is the subclass of starlike function of the form

$$\Re\left(\frac{1 + z\xi''(z)/\xi'(z)}{z\xi'(z)/\xi(z)}\right) > 0, \quad (z \in U), \tag{1.6}$$

which was introduced by Nunokawa [26, 27], Silverman [35] and Obradovic and Owa [29].

Let Ω be the class of all analytic Schwarz functions ω, normalized by $\omega(0) = 0$, and satisfying the condition $|\omega(z)| < 1$ for all $z \in U$, and let ξ and ϕ be two analytic functions in U. Then we say that the function ξ is subordinate to ϕ (denoted by $\xi(z) \prec \phi(z)$) if there exists a function $\omega \in \Omega$, such that $\xi(z) = \phi(\omega(z))$. The subordination is identical to $\xi(0) = \phi(0)$ and $\xi(U) \subset \phi(U)$ if the function ϕ is univalent in U.

An application of the Bieberbach theorem is the Kobe one-quarter theorem [8] which states that any univalent function $\xi \in \mathcal{S}$ contains the disc $U^* = \{w : |w| < \frac{1}{4}\}$. Therefore, each univalent function $\xi \in \mathcal{S}$ has an inverse function

$$\xi^{-1} := G$$

given by

$$G(\xi(z)) = z\,, \quad (z \in U),$$

and

$$\xi(G(w)) = w, \quad (w \in U^*),$$

where the inverse function $\xi^{-1} = G$ has a series expansion of the form

$$G(w) = \xi^{-1}(w) = w - A_2w^2 + (2A_2^2 - A_3)w^3 - (5A_2^3 - 5A_2A_3 + A_4)w^4 + \dots .$$

The function $\xi \in \mathcal{S}$ is said to be in the class σ of all bi-univalent functions in U if its inverse ξ^{-1} is also univalent in U. Lewin [18] is the first author to introduce analytic bi-univalent functions and estimate the second coefficient $|A_2|$. The bounds for the first two coefficients $|A_2|$ and $|A_3|$ have been estimated by many authors for analytic bi-univalent functions (see for example [1, 2, 5, 7, 10, 12, 14–16, 20, 21, 23, 31, 34, 37]).

Let $\mathcal{P}$ be the Caratheodory class analytic functions ϕ in U, defined by

$$\phi(z) = 1 + \varkappa_1 z + \varkappa_2 z^2 + \varkappa_3 z^3 + \dots, \tag{1.7}$$

such that

$$\Re\,(\phi(z)) > 0\,, \quad (z \in U).$$

In Definition 1 below, we define a new class of analytic and bi-univalent functions in U that generalizes several subclasses of bi-univalent functions given by many authors.

Definition 1. *Let the function $\phi \in \mathcal{P}$ of the form (1.7) such that $\phi(U)$ is symmetric about the real axis. A function $\xi \in \sigma$ given by (1.1) is said to be in the class $L_\sigma(\alpha, \beta, \phi)$ if the following subordinations hold*

$$\left(\frac{z\xi'(z)}{\xi(z)}\right)^{\alpha}\left(1 + \frac{z\xi''(z)}{\xi'(z)}\right)^{\beta} \prec \phi(z), \quad (z \in U),$$

and

$$\left(\frac{wG'(w)}{G(w)}\right)^{\alpha}\left(1 + \frac{wG''(w)}{G'(w)}\right)^{\beta} \prec \phi(w)\,, \quad (w \in U),$$

where $G(w) = \xi^{-1}(w)$ and $\alpha, \beta \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers).

Remark 1. *It is obvious that*

(1) The class $L_\sigma(-1, 1, \phi) = K_\sigma(\phi)$ has been studied by Lashin in [15];

(2) The classes $L_\sigma(\frac{1}{\alpha}, 0, \frac{1+z}{1-z}) = S^*_\sigma(\alpha)$, $(0 < \alpha \leq 1)$ and $L_\sigma(0, \frac{1}{\alpha}, \frac{1+z}{1-z}) = C_\sigma(\alpha)$, $(0 < \alpha \leq 1)$ were introduced and studied by Brannan and Taha in [4] and Taha in [36];

(3) The class $L_\sigma(1, 1, \phi) = ST_\alpha(1, \phi)$ and $L_\sigma(1-\alpha, \alpha, \phi) = L_\sigma(\alpha, \phi)$ were introduced and studied by Ali et al. in [3], see also Peng and Han in [30] and Hamidi and Jahangiri in [11].

This paper presents estimates for the initial coefficients $|A_2|$ and $|A_3|$ of the Taylor-Maclaurin series of functions in the class $L_\sigma(\alpha, \beta, \phi)$. It also gives applications for the Legendre polynomials to functions in the class $L_\sigma(\alpha, \beta, \phi)$. Many subclasses associated with the Legendre polynomials are also discussed.

2. Main results

In this section, we give estimates for the initial coefficients $|A_2|$ and $|A_3|$ of the Taylor-Maclaurin series of functions in the class $L_\sigma(\alpha, \beta, \phi)$.

Theorem 1. *If* $\xi \in L_\sigma(\alpha, \beta, \phi)$, *then*

$$|A_2| \leq \frac{|\varkappa_1| \sqrt{|\varkappa_1|}}{\sqrt{\left|[2\beta(\beta+\alpha) + \frac{\alpha(\alpha+1)}{2}]\varkappa_1^2 - \varkappa_2(\alpha+2\beta)^2\right| + (\alpha+2\beta)^2 |\varkappa_1|}}, \tag{2.1}$$

and

$$|A_3| \leq \begin{cases} \frac{|\varkappa_1|}{2|\alpha+3\beta|}, & |\varkappa_1| \leq \frac{(\alpha+2\beta)^2}{2|\alpha+3\beta|}, \\ \frac{\left[2|\alpha+3\beta||\varkappa_1| - (\alpha+2\beta)^2\right]|\varkappa_1|^2}{2|\alpha+3\beta|\left(\left|[2\beta(\beta+\alpha)+\frac{\alpha(\alpha+1)}{2}]\varkappa_1^2 - \varkappa_2(\alpha+2\beta)^2\right| + (\alpha+2\beta)^2|\varkappa_1|\right)} \\ + \frac{|\varkappa_1|}{2|\alpha+3\beta|}, & |\varkappa_1| > \frac{(\alpha+2\beta)^2}{2|\alpha+3\beta|}. \end{cases} \tag{2.2}$$

Proof. Let $u, v \in \Omega$ have the series expansion of the form

$$u(z) = \sum_{n=1}^{\infty} b_n z^n, \ v(z) = \sum_{n=1}^{\infty} c_n z^n, \ (z \in U). \tag{2.3}$$

Then, it is well-known that

$$|b_1| < 1, \ |b_2| < 1 - |b_1|^2, \ \ |c_1| < 1 \text{ and } |c_2| < 1 - |c_1|^2, \tag{2.4}$$

(see [25], Page 172). As a result of a simple calculation, we can conclude that

$$\phi(u(z)) = 1 + \varkappa_1 b_1 z + (\varkappa_1 b_2 + \varkappa_2 b_1^2) z^2 + ..., \ \ (z \in U), \tag{2.5}$$

and

$$\phi(v(w)) = 1 + \varkappa_1 b_1 w + (\varkappa_1 b_2 + \varkappa_2 b_1^2) w^2 + ..., \ \ (w \in U). \tag{2.6}$$

Since $\xi \in L_{\sigma}(\alpha, \beta, \phi)$, then Definition 1 gives

$$\left(\frac{z\xi'(z)}{\xi(z)}\right)^{\alpha}\left(1+\frac{z\xi''(z)}{\xi'(z)}\right)^{\beta}=\phi(u(z)), \quad z \in U, \tag{2.7}$$

and

$$\left(\frac{wG'(w)}{G(w)}\right)^{\alpha}\left(1+\frac{wG''(w)}{G'(w)}\right)^{\beta}=\phi(v(w)), \ w \in U. \tag{2.8}$$

Now,

$$\begin{aligned}
&\left(\frac{z\xi'(z)}{\xi(z)}\right)^{\alpha}\left(1+\frac{z\xi''(z)}{\xi'(z)}\right)^{\beta}\\
=\ &\left(1+\alpha A_2 z+\left(2\alpha A_3+\frac{\alpha(\alpha-3)}{2}A_2^2\right)z^2+...\right)\\
&\times\left(1+2\beta A_2 z+\left(6\beta A_3+2\beta(\beta-3)A_2^2\right)z^2+...\right)\\
=\ &1+\varkappa_1 b_1 z+\left(\varkappa_1 b_2+\varkappa_2 b_1^2\right)z^2+...,
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&\left(\frac{wG'(w)}{G(w)}\right)^{\alpha}\left(1+\frac{wG''(w)}{G'(w)}\right)^{\beta}\\
=\ &\left(1-\alpha A_2 w+\left(-2\alpha A_3+\frac{\alpha(\alpha+5)}{2}A_2^2\right)w^2+...\right)\\
&\times(1-2\beta A_2 w+\left(-6\beta A_3+2\beta(\beta+3)A_2^2\right)w^2+...)\\
=\ &1+\varkappa_1 c_1 w+\left(\varkappa_1 c_2+\varkappa_2 c_1^2\right)w^2+... .
\end{aligned} \tag{2.10}$$

Equating the corresponding coefficients in (2.9) and (2.10) , we get

$$(\alpha+2\beta)A_2=\varkappa_1 b_1, \tag{2.11}$$

$$2(\alpha+3\beta)A_3+\left(2\beta(\alpha+\beta-3)+\frac{\alpha(\alpha-3)}{2}\right)A_2^2=\varkappa_1 b_2+\varkappa_2 b_1^2, \tag{2.12}$$

$$-(\alpha+2\beta)A_2=\varkappa_1 c_1, \tag{2.13}$$

$$-2(\alpha+3\beta)A_3+\left(2\beta(\alpha+\beta+3)+\frac{\alpha(\alpha+5)}{2}\right)A_2^2=\varkappa_1 c_2+\varkappa_2 c_1^2. \tag{2.14}$$

From (2.11) and (2.13), we get

$$b_1=-c_1, \tag{2.15}$$

$$b_1^2 + c_1^2 = \frac{2(\alpha+2\beta)^2}{\varkappa_1^2} A_2^2. \tag{2.16}$$

Using (2.12), (2.14) and (2.16), we have

$$\left[(4\beta(\beta+\alpha)+\alpha(\alpha+1))\,\varkappa_1^2 - 2\varkappa_2(\alpha+2\beta)^2\right] A_2^2 = \varkappa_1^3(b_2+c_2). \tag{2.17}$$

By using (2.4) and (2.15), we get

$$\left|(4\beta(\beta+\alpha)+\alpha(\alpha+1))\,\varkappa_1^2 - 2\varkappa_2(\alpha+2\beta)^2\right| |A_2|^2 \leq 2\,|\varkappa_1|^3\,(1-|b_1|^2). \tag{2.18}$$

If we apply (2.11) again, we obtain

$$\left[\left|\left(2\beta(\beta+\alpha)+\frac{\alpha(\alpha+1)}{2}\right)\varkappa_1^2 - \varkappa_2(\alpha+2\beta)^2\right| + (\alpha+2\beta)^2\,|\varkappa_1|\right] |A_2|^2 \leq |\varkappa_1|^3\,, \tag{2.19}$$

which is equivalent to

$$|A_2| \leq \frac{|\varkappa_1|\,\sqrt{|\varkappa_1|}}{\sqrt{\left|\left(2\beta(\beta+\alpha)+\frac{\alpha(\alpha+1)}{2}\right)\varkappa_1^2 - \varkappa_2(\alpha+2\beta)^2\right| + (\alpha+2\beta)^2\,|\varkappa_1|}}. \tag{2.20}$$

To give an estimation to $|A_3|$, subtracting (2.14) from (2.12), we get

$$A_3 = A_2^2 + \frac{\varkappa_1(b_2-c_2)}{4\,|\alpha+3\beta|}.$$

On using (2.4) and (2.11), we get

$$|A_3| \leq (1 - \frac{(\alpha+2\beta)^2}{2\,|\alpha+3\beta|\,|\varkappa_1|})\left|A_2^2\right| + \frac{|\varkappa_1|}{2\,|\alpha+3\beta|}. \tag{2.21}$$

Case 1: If $|\varkappa_1| \leq \frac{(\alpha+2\beta)^2}{2|\alpha+3\beta|}$, then we have

$$|A_3| \leq \frac{|\varkappa_1|}{2\,|\alpha+3\beta|}. \tag{2.22}$$

Case 2: If $|\varkappa_1| > \frac{(\alpha+2\beta)^2}{2|\alpha+3\beta|}$, then

$$\begin{aligned} |A_3| \quad \leq \quad & \frac{\left(2\,|\alpha+3\beta|\,|\varkappa_1| - (\alpha+2\beta)^2\right)|\varkappa_1|^2}{|\alpha+3\beta|\left(\left|\left(2\beta(\beta+\alpha)+\frac{\alpha(\alpha+1)}{2}\right)\varkappa_1^2 - \varkappa_2(\alpha+2\beta)^2\right| + (\alpha+2\beta)^2\,|\varkappa_1|\right)} \\ & + \frac{|\varkappa_1|}{2\,|\alpha+3\beta|}, \end{aligned} \tag{2.23}$$

which completes the proof. □

Remark 2. *In Theorem 1, if we put*

(1) $\alpha = -1$ and $\beta = 1$, we get the results obtained by Lashin in [15];

(2) $\alpha = \frac{1}{\alpha}$ and $\beta = 0$ $(0 < \alpha \leq 1)$, we get the results obtained by Brannan and Taha in [4] and Taha [36];

(3) $\alpha = 1$ and $\beta = 1$, we get the results obtained by Ali et al. in [3], Peng and Han in [30], and Hamidi and Jahangiri in [11];

(4) $\alpha = 1 - \alpha$ and $\beta = \alpha$, Ali et al. in [3], Peng and Han in [30], and Hamidi and Jahangiri in [11].

We get the new class $L_\sigma(\alpha, \beta, \gamma)$ described by Definition 2 below if we insert

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + ..., \ \ (0 < \gamma \leq 1, \ \ z \in U),$$

in Definition 1 of the bi-univalent function class $L_\sigma(\alpha, \beta, \phi)$.

Definition 2. *Let $L_\sigma(\alpha, \beta, \gamma)$ be the class of bi-univalent function $\xi \in \sigma$ such that:*

$$\left|\arg\left(\frac{z\xi'(z)}{\xi(z)}\right)^\alpha \left(1 + \frac{z\xi''(z)}{\xi'(z)}\right)^\beta\right| < \frac{\pi\gamma}{2}, \ \ (0 < \gamma \leq 1, \quad z \in U), \tag{2.24}$$

and

$$\left|\arg\left(\frac{wG'(w)}{G(w)}\right)^\alpha \left(1 + \frac{wG''(w)}{G'(w)}\right)^\beta\right| < \frac{\pi\gamma}{2}, \ \ (0 < \gamma \leq 1, \ \ w \in U), \tag{2.25}$$

where $G(w) = \xi^{-1}(w)$ and $\alpha, \beta \in \mathbb{R}$.

The following Corollary is produced using the parameter setting of Definition 2 in Theorem 1.

Corollary 1. *Let $\alpha, \beta \in \mathbb{R}$ and $0 < \gamma \leq 1$. If $\xi \in L_\sigma(\alpha, \beta, \gamma)$, then*

$$|A_2| \leq \frac{2\gamma}{\sqrt{|\alpha|\,\gamma + (\alpha + 2\beta)^2}}, \tag{2.26}$$

and

$$|A_3| \leq \begin{cases} \frac{\gamma}{|\alpha+3\beta|}, & \gamma \leq \frac{(\alpha+2\beta)^2}{4|\alpha+3\beta|}, \\ \frac{\left(4\gamma|\alpha+3\beta|-(\alpha+2\beta)^2\right)\gamma}{|\alpha+3\beta|\left(|\alpha|\gamma+(\alpha+2\beta)^2\right)} + \frac{\gamma}{|\alpha+3\beta|}, & \gamma > \frac{(\alpha+2\beta)^2}{4|\alpha+3\beta|}. \end{cases}$$

If we set

$$\phi(z) = \frac{1 + (1 - 2v)z}{1 - z} = 1 + 2(1 - v)z + 2(1 - v)z^2 + ..., \ (0 \leq v < 1, \ \ z \in U),$$

in Definition 1 of the bi-univalent function class $L_\sigma(\alpha, \beta, \phi)$, we obtain a new class $L^v_\sigma(\alpha, \beta)$ given by Definition 3 below.

Definition 3. *Let $L^v_\sigma(\alpha, \beta)$ be the class of bi-univalent function $\xi \in \sigma$ such that:*

$$\Re\left(\frac{z\xi'(z)}{\xi(z)}\right)^\alpha \left(1 + \frac{z\xi''(z)}{\xi'(z)}\right)^\beta > \upsilon, \ \ (0 \leq \upsilon < 1, \ \ z \in U), \tag{2.27}$$

and

$$\Re\left(\frac{wG'(w)}{G(w)}\right)^{\alpha}\left(1+\frac{wG''(w)}{G'(w)}\right)^{\beta}>\upsilon,\ \ (0\le\upsilon<1,\ \ w\in U), \tag{2.28}$$

where $G(w)=\xi^{-1}(w)$ *and* $\alpha,\beta\in\mathbb{R}$.

The following corollary is produced using the parameter setting of Definition 3 in the Theorem 1.

Corollary 2. *Let* $\alpha,\beta\in\mathbb{R}$, *and* $0\le\upsilon<1$. *If* $\xi\in L_{\sigma}^{\upsilon}(\alpha,\beta)$, *then*

$$|A_2|\le\frac{2(1-\upsilon)}{\sqrt{\left|(4\beta(\beta+\alpha)+\alpha(\alpha+1))(1-\upsilon)-(\alpha+2\beta)^2\right|+(\alpha+2\beta)^2}},$$

and

$$|A_3|\le\begin{cases}\frac{(1-\upsilon)}{|\alpha+3\beta|}, & \upsilon\ge 1-\frac{(\alpha+2\beta)^2}{4|\alpha+3\beta|},\\ \frac{\left(4|\alpha+3\beta|(1-\upsilon)-(\alpha+2\beta)^2\right)(1-\upsilon)}{|\alpha+3\beta|\left(\left|(4\beta(\beta+\alpha)+\alpha(\alpha+1))(1-\upsilon)-(\alpha+2\beta)^2\right|+(\alpha+2\beta)^2\right)} & \\ \quad+\frac{(1-\upsilon)}{|\alpha+3\beta|}, & \upsilon<1-\frac{(\alpha+2\beta)^2}{4|\alpha+3\beta|}.\end{cases}$$

The following section introduces applications some of the Legendre polynomials to a certain subclass of the bi-univalent class σ. Many subclasses associated of σ with the Legendre polynomials are also discussed.

3. Applications of Legendre functions

Legendre polynomials have a wide range of applications, particularly in mathematics, physics, and chemistry. Among the applications of Legendre polynomials are the determination of electron wave functions in the orbits of atoms [22] and in the determination of potential functions in spherically symmetric geometry [6]. Also, in developing the mathematical models for flow and heat analysis of fluid [13]. The particular solutions to the Legendre differential equation

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0,\ n\in Z^{+},|x|<1,$$

are the Legendre functions of the first kind $P_n(x)$, these functions are given by the following Rodrigues' formula

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n.$$

The functions P_n are also defined as the coefficients in a formal expansion in powers of t of the generating function

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n, \tag{3.1}$$

which is convergent if $|x|\le 1$ and $|t|<1$. The first few Legendre polynomials are

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{1}{2}\left(3x^2-1\right),\ P_3(x)=\frac{1}{2}\left(5x^3-3x\right).$$

The function

$$\varphi(z) = \frac{1-z}{\sqrt{1-2z\cos\delta+z^2}},$$

is in the class $\mathcal{P}$ for every $\delta \in \mathbb{R}$ (see [9, Page 102]). In [17], Lashin et al. proved that the function φ maps the unit disc U onto the right half plane $\Re(w) > 0$ except for the slit along the positive real axis from $\frac{1}{|\cos\frac{\delta}{2}|}$ to ∞, this means that φ is starlike with respect to 1. By using (3.1), it is easy to check that

$$\begin{aligned}\phi(z) &= 1 + \sum_{n=1}^{\infty} \left[P_n(\cos\delta) - P_{n-1}(\cos\delta)\right] z^n, \\ &= 1 + \sum_{n=1}^{\infty} B_n z^n, \quad z \in U.\end{aligned} \tag{3.2}$$

We get the new class $R_\sigma(\alpha, \beta, \delta)$ described by Definition 4 below if we set

$$\phi(z) = \frac{1-z}{\sqrt{1-2z\cos\delta+z^2}} = 1 + (\cos\delta - 1)z + \frac{1}{2}(\cos\delta - 1)(1+3\cos\delta)z^2 + ..., \ (z \in U),$$

in Definition 1 of the bi-univalent function class $L_\sigma(\alpha, \beta, \phi)$.

Definition 4. *Let $R_\sigma(\alpha, \beta, \delta)$ be the class of bi-univalent function $\xi \in \sigma$ such that:*

$$\left(\frac{z\xi'(z)}{\xi(z)}\right)^{\alpha}\left(1+\frac{z\xi''(z)}{\xi'(z)}\right)^{\beta} \prec \frac{1-z}{\sqrt{1-2z\cos\delta+z^2}}, \quad (z \in U), \tag{3.3}$$

and

$$\left(\frac{wG'(w)}{G(w)}\right)^{\alpha}\left(1+\frac{wG''(w)}{G'(w)}\right)^{\beta} \prec \frac{1-w}{\sqrt{1-2w\cos\delta+w^2}}, \quad (w \in U), \tag{3.4}$$

where $G(w) = \xi^{-1}(w)$ and $\alpha, \beta, \delta \in \mathbb{R}$.

In the limit case when $\delta \to \pi$, the class $R_\sigma(\alpha, \beta, \delta)$ extends the classes given by Brannan and Taha [4], Taha [36], Ali et al. [3], Peng and Han [30] and Hamidi and Jahangiri [11].

The following corollary is produced using the parameter setting of Definition 4 in Theorem 1.

Corollary 3. *Let $\alpha, \beta, \delta \in \mathbb{R}$. If $\xi \in R_\sigma(\alpha, \beta, \delta)$, then*

$$|A_2| \leq \frac{1-\cos\delta}{\sqrt{\left|\left(2\beta(\beta+\alpha)+\frac{\alpha(\alpha+1)}{2}\right)(1-\cos\delta)+\frac{1}{2}(1+3\cos\delta)(\alpha+2\beta)^2\right|+(\alpha+2\beta)^2}},$$

and

$$|A_3| \leq \begin{cases} \frac{1-\cos\delta}{2|\alpha+3\beta|}, & \cos\delta \geq 1 - \frac{(\alpha+2\beta)^2}{2|\alpha+3\beta|}, \\ \frac{\left(2|\alpha+3\beta|(1-\cos\delta)-(\alpha+2\beta)^2\right)(1-\cos\delta)}{2|\alpha+3\beta|\left(\left|\left([2\beta(\beta+\alpha)+\frac{\alpha(\alpha+1)}{2}\right)(1-\cos\delta)+\frac{1}{2}(1+3\cos\delta)(\alpha+2\beta)^2\right|+(\alpha+2\beta)^2\right)} \\ \quad +\frac{1-\cos\delta}{2|\alpha+3\beta|}, & \cos\delta < 1 - \frac{(\alpha+2\beta)^2}{2|\alpha+3\beta|}. \end{cases}$$

Putting $\alpha = 1$ and $\beta = 1$ in Corollary 3, we get the following corollary.

Corollary 4. *If $\xi \in \sigma$ given by (1.1) satisfies the following conditions*

$$\left(\frac{z\xi'(z)}{\xi(z)} + \frac{z^2\xi''(z)}{\xi(z)}\right) \prec \frac{1-z}{\sqrt{1-2z\cos\delta + z^2}}, \quad (z \in U),$$

and

$$\frac{wG'(w)}{G(w)} + \frac{w^2G''(w)}{G(w)} \prec \frac{1-w}{\sqrt{1-2w\cos\delta + w^2}}, \quad (w \in U),$$

where $G(w) = \xi^{-1}(w)$, then we have

$$|A_2| \leq \frac{1-\cos\delta}{\sqrt{\left|5\,(1-\cos\delta) + \frac{9}{2}(1+3\cos\delta)\right| + 9}},$$

and

$$|A_3| \leq \begin{cases} \frac{1-\cos\delta}{8}, & \cos\delta \geq -\frac{1}{8}, \\ \frac{[8(1-\cos\delta)-9](1-\cos\delta)}{8\left(\left|5(1-\cos\delta)+\frac{9}{2}(1+3\cos\delta)\right|+9\right)} + \frac{1-\cos\delta}{8}, & \cos\delta < -\frac{1}{8}. \end{cases}$$

Putting $\alpha = -1$ and $\beta = 1$ in Corollary 3, we get the following corollary.

Corollary 5. *If $\xi \in \sigma$ given by (1.1) satisfies the following conditions*

$$\frac{1 + \frac{z\xi''(z)}{\xi'(z)}}{\frac{z\xi'(z)}{\xi(z)}} \prec \frac{1-z}{\sqrt{1-2z\cos\delta + z^2}}, \quad (z \in U),$$

and

$$\frac{1 + \frac{wG''(w)}{G'(w)}}{\frac{wG'(w)}{G(w)}} \prec \frac{1-w}{\sqrt{1-2w\cos\delta + w^2}}, \quad (w \in U),$$

where $G(w) = \xi^{-1}(w)$ and $\delta \in \mathbb{R}$, then we have

$$|A_2| \leq \frac{\sqrt{2}(1-\cos\delta)}{\sqrt{|(1+3\cos\delta)| + 2}},$$

and

$$|A_3| \leq \begin{cases} \frac{1-\cos\delta}{4}, & \cos\delta \geq \frac{3}{4}, \\ \frac{(4(1-\cos\delta)-1)(1-\cos\delta)}{4\left(\left|\frac{1}{2}(1+3\cos\delta)\right|+1\right)} + \frac{1-\cos\delta}{2|\alpha+3\beta|}, & \cos\delta < \frac{3}{4}. \end{cases}$$

Putting $\alpha = 1-\gamma$ and $\beta = \gamma$ in Corollary 3, we get the following corollary.

Corollary 6. *If $\xi \in \sigma$ given by (1.1) satisfies the following conditions*

$$\left(\frac{z\xi'(z)}{\xi(z)}\right)^{1-\gamma}\left(1 + \frac{z\xi''(z)}{\xi'(z)}\right)^{\gamma} \prec \frac{1-z}{\sqrt{1-2z\cos\delta + z^2}}, \quad (z \in U),$$

and

$$\left(\frac{wG'(w)}{Gg(w)}\right)^{1-\gamma}\left(1 + \frac{wG''(w)}{G'(w)}\right)^{\gamma} \prec \frac{1-w}{\sqrt{1-2w\cos\delta + w^2}}, \quad (w \in U),$$

where $G(w)=\xi^{-1}(w)$ *and* $0\leq\gamma\leq 1$, *then we have*

$$|A_2|\leq \frac{1-\cos\delta}{\sqrt{\left|\left(2\gamma+\frac{(1-\gamma)(2-\gamma))}{2}\right)(1-\cos\delta)+\frac{1}{2}(1+3\cos\delta)(1+\gamma)^2\right|+(1+\gamma)^2}},$$

and

$$|A_3|\leq\begin{cases}\frac{1-\cos\delta}{2(1+2\gamma)}, & \cos\delta\geq 1-\frac{(1+\gamma)^2}{2(1+2\gamma)},\\ \frac{\left[2(1+2\gamma)(1-\cos\delta)-(1+\gamma)^2\right](1-\cos\delta)}{2(1+2\gamma)\left(\left|\left(2\gamma+\frac{(1-\gamma)(2-\gamma))}{2}\right)(1-\cos\delta)+\frac{1}{2}(1+3\cos\delta)(\alpha+2\beta)^2\right|+(1+\gamma)^2\right)} \\ +\frac{1-\cos\delta}{2(1+2\gamma)}, & \cos\delta<1-\frac{(1+\gamma)^2}{2(1+2\gamma)}.\end{cases}$$

Putting $\alpha=\frac{1}{\gamma}$ and $\beta=0,\ (0<\alpha\leq 1)$ in Corollary 3, we get the following corollary.

Corollary 7. *If* $\xi\in\sigma$ *given by (1.1) satisfies the following conditions*

$$\frac{z\xi'(z)}{\xi(z)}\prec\left(\frac{1-z}{\sqrt{1-2z\cos\delta+z^2}}\right)^{\gamma},\quad (z\in U),$$

and

$$\frac{wG'(w)}{G(w)}\prec\left(\frac{1-w}{\sqrt{1-2w\cos\delta+w^2}}\right)^{\gamma},\quad (w\in U),$$

where $G(w)=\xi^{-1}(w)$ *and* $0<\alpha\leq 1$, *then we have*

$$|A_2|\leq\frac{\gamma\sqrt{2}(1-\cos\delta)}{\sqrt{\gamma(1-\cos\delta)+2(2+\cos\delta)}},$$

and

$$|A_3|\leq\begin{cases}\frac{\gamma(1-\cos\delta)}{2}, & \cos\delta\geq 1-\frac{1}{2\gamma},\\ \frac{\gamma(2\gamma(1-\cos\delta)-1)(1-\cos\delta)}{\gamma(1-\cos\delta)+2(1+\cos\delta)+1}+\frac{\gamma(1-\cos\delta)}{2}, & \cos\delta<1-\frac{1}{2\gamma}.\end{cases}$$

Putting $\alpha=0$ and $\beta=\frac{1}{\gamma},\ (0<\gamma\leq 1)$ in Corollary 3, we get the following corollary.

Corollary 8. *If* $\xi\in\sigma$ *given by(1.1) satisfies the following conditions*

$$1+\frac{z\xi''(z)}{\xi'(z)}\prec\left(\frac{1-z}{\sqrt{1-2z\cos\delta+z^2}}\right)^{\gamma},\quad (z\in U),$$

and

$$1+\frac{wG''(w)}{G'(w)}\prec\left(\frac{1-w}{\sqrt{1-2w\cos\delta+w^2}}\right)^{\gamma},\quad (w\in U),$$

where $G(w)=\xi^{-1}(w)$. *Then we have*

$$|A_2|\leq\frac{\gamma(1-\cos\delta)}{2\sqrt{2+\cos\delta}},$$

and

$$|A_3|\leq\begin{cases}\frac{\gamma(1-\cos\delta)}{6}, & \cos\delta\geq 1-\frac{2}{3\gamma},\\ \frac{\gamma(3\gamma(1-\cos\delta)-2)(1-\cos\delta)}{6(|2(1-\cos\delta)+(1+3\cos\delta)|+2)}+\frac{\gamma(1-\cos\delta)}{6}, & \cos\delta<1-\frac{2}{3\gamma}.\end{cases}$$

4. Conclusions

The bounds for the first two coefficients $|A_2|$ and $|A_3|$ have been estimated by many authors for analytic bi-univalent functions class σ. This paper defines a new subclass of σ associated with the Legendre polynomials. For this class, we find estimations for the two initial coefficients $|A_2|$ and $|A_3|$. Furthermore, it presents several subclasses of class σ and generalizes many previous works of various authors.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to express their thanks to the referees for their helpful comments and suggestions, which improved the presentation of the paper.

Conflict of interest

The authors declare no competing interests.

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