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Research article

Riccati equation and metric geometric means of positive semidefinite matrices involving semi-tensor products

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Abstract: We investigate the Riccati matrix equation $XA^{-1}X = B$ in which the conventional matrix products are generalized to the semi-tensor products $\ltimes$. When A and B are positive definite matrices satisfying the factor-dimension condition, this equation has a unique positive definite solution, which is defined to be the metric geometric mean of A and B. We show that this geometric mean is the maximum solution of the Riccati inequality. We then extend the notion of the metric geometric mean to positive semidefinite matrices by a continuity argument and investigate its algebraic properties, order properties and analytic properties. Moreover, we establish some equations and inequalities of metric geometric means for matrices involving cancellability, positive linear map and concavity. Our results generalize the conventional metric geometric means of matrices.

Keywords: metric geometric mean; semi-tensor product; positive definite matrix; Riccati equation
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1. Introduction

In classical matrix theory, the conventional matrix multiplication is a fundamental operation for processing of one/two-dimensional data. However, in modern data science, the conventional product is difficult to work with big or multidimensional data in order to extract information. In the early 2000s, Cheng [1] proposed the semi-tensor product (STP) of matrices as a tool for dealing with higher-dimensional data. The STP is a generalization of the conventional matrix multiplication, so that the multiplied matrices do not need to satisfy the matching-dimension condition. The symbol for this operation is $\ltimes$. The STP keeps all fundamental properties of the conventional matrix multiplication. In addition, it possesses some incomparable advantages over the latter, such as interchangeability and complete compatibility. Due to these advantages, the STP is widely used in various fields, such as engineering [2], image encryption [3, 4], Boolean networks [5, 6], networked games [7, 8], classical logic and fuzzy mathematics [9, 10], finite state machines [11, 12], finite systems [13] and others.

Matrix equations are fundamental tools in mathematics and they are applied in diverse fields. Recently, the theory of linear matrix equations with respect to the STP were investigated by many authors. Such theory includes necessary/sufficient conditions for existence and uniqueness of solutions (concerning ranks and linear independence) and methods to solve the matrix equations. The solutions of the matrix linear equation $A \ltimes X = B$ were studied by Yao et al. [14]. Li et al. [15] investigated a system of two matrix equations $A \ltimes X = B$ and $X \ltimes C = D$. Ji et al. [16] discussed the solvability of matrix equation $A \ltimes X \ltimes B = C$. Recently, the theory for the Sylvester equation $A \ltimes X + X \ltimes B = C$, the Lyapunov one $A \ltimes X + X \ltimes A^T = C$ and the Sylvester-transpose one $A \ltimes X + X^T \ltimes B = C$ was investigated in [17] and [18]. For nonlinear matrix equations, higher order algebraic equations can be applied widely in file encoding, file transmission and decoupling of logical networks. Wang et al. [19] investigated a nonlinear equation $A \ltimes X \ltimes X = B$.

Let $\mathbb{H}^{n\times n}, \mathbb{PS}^{n\times n}$ and $\mathbb{P}^{n\times n}$ be the set of $n \times n$ Hermitian matrices, positive semidefinite matrices and positive definite matrices, respectively. This present research focuses on a famous nonlinear equation known as the Riccati equation:

$$XA^{-1}X = B. \tag{1.1}$$

In fact, this equation determines the solution of the linear-quadratic-Gaussian control problem which is one of the most fundamental problems in control theory, e.g., [20, 21]. It is known that the metric geometric mean

$$A \sharp B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2} \tag{1.2}$$

is the unique positive solution of (1.1). This mean was introduced by Pusz and Woronowicz [22] and Ando [23] as the largest Hermitian matrix:

$$A \sharp B = \max\left\{X \in \mathbb{H}^{n\times n} : \begin{bmatrix} A & X \\ X & B \end{bmatrix} \in \mathbb{PS}^{2n\times 2n}\right\},$$

where the maximal element is taken in the sense of the Löwner partial order. A significant property of the metric geometric mean is that $A \sharp B$ is a midpoint of A and B for a natural Finsler metric. Many theoretical and computational research topics on the metric geometric mean have been widely studied, e.g., [24–27].

The metric geometric mean on $\mathbb{PS}^{n\times n}$ is a mean in Kubo-Ando's sense [28]:

(1) joint monotonicity: $A \leqslant C$ and $B \leqslant D$ implies $A \sharp B \leqslant C \sharp D$;
(2) transformer inequality: $T(A \sharp B)T \leqslant (TAT) \sharp (TBT)$;
(3) joint continuity from above: $A_k \downarrow A$ and $B_k \downarrow B$ implies $A_k \sharp B_k \downarrow A \sharp B$;
(4) normalization: $I_n \sharp I_n = I_n$.

Here, $\leqslant$ is the Löwner partial order and $A_k \downarrow A$ indicates that (A_k) is a decreasing sequence converging to A.

There are another axiomatic approaches for means in various frameworks. Lawson and Lim [29, 30] investigated a set of axioms for an algebraic system called a reflection quasigroup. The set $\mathbb{P}^{n\times n}$ with an operation $A \bullet B = AB^{-1}A$ form a reflection quasigroup in the following sense:

(1) idempotency: $A \bullet A = A$;
(2) left distributivity: $A \bullet (B \bullet C) = (A \bullet B) \bullet (A \bullet C)$;
(3) left symmetry: $A \bullet (A \bullet B) = B$;
(4) the equation $X \bullet A = B$ has a unique solution X.

From the axiom 4, we have that $A\sharp B$ is a unique solution of the Riccati equation $XA^{-1}X = B$ and call $A\sharp B$ the mean or the midpoint of A and B.

In this present research, we investigate the Riccati equation with respect to the STP:

$$X \ltimes A^{-1} \ltimes X \;=\; B,$$

where A and B are given positive definite matrices of different sizes, and X is an unknown square matrix. We show that this equation has a unique positive definite solution, which is defined to be the metric geometric mean of A and B. Then, we extend this notion to the case of positive semidefinite matrices by a continuity argument. We establish fundamental properties of this mean. Moreover, we investigate certain equations and inequalities involving metric geometric means.

The paper is organized as follows. In Section 2, we setup basic notation and give basic results on STP and Kronecker products. Positive (semi) definiteness of matrices concerning semi-tensor products is also presented in this section. In Section 3, we define the metric geometric mean for positive definite matrices from the Riccati equation. In Section 4, we extend the notion of metric geometric mean to positive semidefinite matrices and provide fundamental properties of geometric means. In Section 5, we present matrix equations and inequalities of metric geometric mean involving cancellability, concavity and positive linear maps. We conclude the whole work in Section 6.

2. Preliminaries

Throughout, let $\mathbb{C}^{m\times n}$ be the set of $m \times n$ complex matrices. We consider the following subsets of $\mathbb{C}^{m\times n}$: $\mathbb{H}^{n\times n}$ the $n \times n$ Hermitian matrices, $\mathbb{GL}^{n\times n}$ the $n \times n$ invertible matrices, $\mathbb{PS}^{n\times n}$ the $n \times n$ positive semidefinite matrices and $\mathbb{P}^{n\times n}$ the $n \times n$ positive definite matrices. Define $\mathbb{C}^n = \mathbb{C}^{n\times 1}$, the set of n-dimensional complex vectors. For any $A, B \in \mathbb{H}^{n\times n}$, the Löwner partial order $A \geqslant B$ means that $A-B \in \mathbb{PS}^{n\times n}$, while the strict order $A > B$ indicates that $A-B \in \mathbb{P}^{n\times n}$. A matrix pair $(A, B) \in \mathbb{C}^{m\times n}\times\mathbb{C}^{p\times q}$ is said to satisfy factor-dimension condition if $n|p$ or $p|n$. In this case, we write $A \succ_k B$ when $n = kp$ and $A \prec_k B$ when $p = kn$. Denote A^T and A^* the transpose and conjugate transpose of A, respectively. We denote the $n \times n$ identity matrix by I_n.

2.1. *Semi-tensor and Kronecker products of matrices*

This subsection is a brief review on semi-tensor products and Kronecker products of matrices.

Definition 2.1. Let $X \in \mathbb{C}^{1,m}$ and $Y \in \mathbb{C}^n$. If $X \succ_k Y$, we split X into $X_1, X_2, \ldots, X_n \in \mathbb{C}^{1,k}$ and define the STP of X and Y as

$$X \ltimes Y \;=\; \sum_{i=1}^{n} y_i X_i \;\in \mathbb{C}^{1,k}.$$

If $X \prec_k Y$, we split Y into $Y^1, Y^2, \ldots, Y^m \in \mathbb{C}^k$ and define the STP of X and Y as

$$X \ltimes Y \ = \ \sum_{i=1}^{m} x_i Y^i \ \in \mathbb{C}^k.$$

From the STP between vectors, we define the STP between matrices as follows.

Definition 2.2. Let a pair $(A, B) \in \mathbb{C}^{m\times n} \times \mathbb{C}^{p\times q}$ satisfy the factor-dimensional condition. Then, we define the STP of A and B to be an $m \times q$ block matrix

$$A \ltimes B \ = \ \left[A_i \ltimes B^j\right]_{i,j=1}^{m,q},$$

where A_i is i-th row of A and B^j is the j-th column of B.

Lemma 2.1. *(e.g., [31, 32]) Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}, P \in \mathbb{C}^{m\times m}, Q \in \mathbb{C}^{n\times n}$. Provided that all matrix operations are well-defined, we have*

(1) the operation $(A, B) \mapsto A \ltimes B$ is bilinear and associative;
(2) $(A \ltimes B)^* = B^* \ltimes A^*$;
(3) if $P \in \mathbb{GL}^{m\times m}$ and $Q \in \mathbb{GL}^{n\times n}$, then $(P \ltimes Q)^{-1} = Q^{-1} \ltimes P^{-1}$;
(4) if $P \prec_k Q$, then $\det(P \ltimes Q) = (\det P)^k(\det Q)$.

Recall that for any matrices $A = [a_{ij}] \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$, their Kronecker product is defined by

$$A \otimes B \ = \ [a_{ij}B] \ \in \ \mathbb{C}^{mp,nq}.$$

Lemma 2.2. *(e.g., [31, 32]) Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$.*

(1) If $A \succ_k B$ then $A \ltimes B = A(B \otimes I_k)$.
(2) If $A \prec_k B$ then $A \ltimes B = (A \otimes I_k)B$.

Lemma 2.3. *(e.g., [33]) Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}, P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$. Then, we have*

(1) the operation $(A, B) \mapsto A \otimes B$ is bilinear and associative;
(2) $(A \otimes B)^* = A^* \otimes B^*$;
(3) $\operatorname{rank}(A \otimes B) = \operatorname{rank}(A)\operatorname{rank}(B)$;
(4) $A \otimes B = 0$ if and only if either $A = 0$ or $B = 0$;
(5) if $P \in \mathbb{GL}^{m\times m}$ and $Q \in \mathbb{GL}^{n\times n}$, then $(P \otimes Q)^{-1} = P^{-1} \otimes Q^{-1}$;
(6) if $P \geqslant 0$ and $Q \geqslant 0$, then $P \otimes Q \geqslant 0$ and $(P \otimes Q)^{1/2} = P^{1/2} \otimes Q^{1/2}$;
(7) if $P > 0$ and $Q > 0$, then $P \otimes Q > 0$;
(8) $\det(P \otimes Q) = (\det P)^m(\det Q)^n$.

2.2. Positive definiteness of matrices involving semi-tensor products

In this subsection, we provide positive (semi) definiteness of matrices involving semi-tensor products.

Proposition 2.1. *Let $A \in \mathbb{PS}^{n\times n}, B \in \mathbb{PS}^{m\times m}, X \in \mathbb{C}^{m\times m}$ and $S, T \in \mathbb{H}^{n\times n}$. Provided that all matrix operations are well-defined, we have*

(1) $X^* \ltimes A \ltimes X \geqslant 0$;
(2) $A \ltimes B \geqslant 0$ *if and only if* $A \ltimes B = B \ltimes A$;
(3) *if* $S \geqslant T$ *then* $X^* \ltimes S \ltimes X \geqslant X^* \ltimes T \ltimes X$.

Proof. 1) Assume that $A \prec_k X$. Since $(X^* \ltimes A \ltimes X)^* = X^* \ltimes A \ltimes X$, we have that $X^* \ltimes A \ltimes X$ is Hermitian. Let $u \in \mathbb{C}^m$ and set $v = X \ltimes u$. Using positive semidefiniteness of Kronecker products (Lemma 2.3), we obtain that $A \otimes I_k \geqslant 0$. Then, by Lemmas 2.1 and 2.2,

$$u^*(X^* \ltimes A \ltimes X)u \;=\; (X \ltimes u)^* \ltimes A \ltimes (X \ltimes u) \;=\; v^*(A \otimes I_k)v \geqslant 0.$$

This implies that $X^* \ltimes A \ltimes X \geqslant 0$. For the case $A \succ_k X$, the proof is similar to the case $A \prec_k X$.
2) Suppose that $A \prec_k B$. ($\Rightarrow$) Since A, B and $A \ltimes B$ are Hermitian, we have by Lemma 2.1 that $A \ltimes B = (A \ltimes B)^* = B^* \ltimes A^* = B \ltimes A$.
($\Leftarrow$) We know that B and $B^{1/2}$ are commuting matrices. Since $A \ltimes B = B \ltimes A$, we get $A \ltimes B^{1/2} = B^{1/2} \ltimes A$. Thus, $A \ltimes B = A \ltimes B^{1/2} \ltimes B^{1/2} = B^{1/2} \ltimes A \ltimes B^{1/2}$. Using the assertion 1, $A \ltimes B = B^{1/2} \ltimes A \ltimes B^{1/2} \geqslant 0$. For the case $A \succ_k B$, the proof is similar to the case $A \prec_k B$.
3) Since $S \geqslant T$, we have $S - T \geqslant 0$. Applying the assertion 1, we get $X^* \ltimes (S - T) \ltimes X \geqslant 0$, i.e., $X^* \ltimes S \ltimes X \geqslant X^* \ltimes T \ltimes X$.

Proposition 2.2. *Let* $A \in \mathbb{P}^{n\times n}, B \in \mathbb{P}^{m\times m}, X \in \mathbb{C}^{p\times q}, Y \in \mathbb{GL}^{p\times p}$ *and* $S, T \in \mathbb{H}^{n\times n}$. *Provided that all matrix operations are well-defined, we have:*

(1) *If* $\operatorname{rank} X = q$ *then* $X^* \ltimes A \ltimes X > 0$.
(2) $Y^* \ltimes A \ltimes Y > 0$.
(3) $A \ltimes B > 0$ *if and only if* $A \ltimes B = B \ltimes A$.
(4) *If* $S > T$ *then* $Y^* \ltimes S \ltimes Y > Y^* \ltimes T \ltimes Y$.

Proof. 1) Suppose $A \prec_k X$ and $\operatorname{rank} X = q$. Applying Lemma 2.1, $X^* \ltimes A \ltimes X \in \mathbb{H}^{q\times q}$. Let $u \in \mathbb{C}^q - \{0\}$. Set $v = X \ltimes u$. Since $\operatorname{rank} X = q$, we have $v \neq 0$. Since $A \otimes I_k > 0$ (Lemma 2.1), we obtain

$$u^*(X^* \ltimes A \ltimes X)u \;=\; v^* \ltimes A \ltimes v \;=\; v^*(A \otimes I_k)v \;>\; 0.$$

Thus, $X^* \ltimes A \ltimes X > 0$. For the case $A \succ_k X$, we have by Lemma 2.1 that $X^* \ltimes A \ltimes X \in \mathbb{H}^{kq\times kq}$. Since $\operatorname{rank} X = q$, we get by Lemma 2.3 that $\operatorname{rank}(X \otimes I_k) = kq$. Thus, $v = (X \otimes I_{kq})u \neq 0$. Since $A > 0$ and $v \neq 0$, we obtain $u^*(X^* \ltimes A \ltimes X)u = v^*Av > 0$, i.e., $X^* \ltimes A \ltimes X$ is positive definite.
2) Since Y is invertible, we have $\operatorname{rank} Y = p$. Using the assertion 1, $Y^* \ltimes A \ltimes Y > 0$.
3)–4). The proof is similar to Proposition 2.1.

3. Metric geometric means of matrices induced from the Riccati equation

In this section, we define the metric geometric mean of two positive definite matrices when the two matrices satisfy factor-dimension condition, as a solution of the Riccati equation. Our results include the conventional metric geometric means of matrices as special case.

Definition 3.1. Let $m, n, k \in \mathbb{N}$ be such that $m = nk$. We define a binary operation

$$\bullet : \mathbb{P}^{m\times m} \times \mathbb{P}^{m\times m} \to \mathbb{P}^{m\times m}, \quad (X, Y) \;\mapsto\; XY^{-1}X,$$

and define an external binary operation

$$* : \mathbb{P}^{m\times m} \times \mathbb{P}^{n\times n} \to \mathbb{P}^{m\times m}, \quad (X, Y) \;\mapsto\; X \ltimes Y^{-1} \ltimes X.$$

For convenience, we write $\bullet$ and $*$ to the same notation $\bullet$.

Proposition 3.1. *Let $m, n, k \in \mathbb{N}$ be such that $m = nk$. Then,*

(1) $A \bullet A = A$;
(2) $\alpha(A \bullet B) = (\alpha A) \bullet (\alpha B)$ for all $\alpha > 0$;
(3) $A \bullet (A \bullet B) = B \otimes I_k$;
(4) $(A \bullet B)^{-1} = A^{-1} \bullet B^{-1}$;
(5) $A \bullet (B \bullet C) = (A \bullet B) \bullet (A \bullet C)$;
(6) if $A \leqslant B$ then $T \bullet A \geqslant T \bullet B$ for all $T \in \mathbb{P}^{m\times m}$.

Proof. The proof is immediate.

Theorem 3.2. *Let $A \in \mathbb{P}^{n\times n}$ and $B \in \mathbb{P}^{m\times m}$ be such that $A \prec_k B$. Then, the Riccati equation $X \bullet A = B$ has a unique solution $X \in \mathbb{P}^{m\times m}$.*

Proof. Set $X = A^{1/2} \ltimes (A^{-1/2} \ltimes B \ltimes A^{-1/2})^{1/2} \ltimes A^{1/2}$. Since $B \geqslant 0$ and $A \in \mathbb{GL}^{n\times n}$, we have by Proposition 2.2 that $A^{-1/2} \ltimes B \ltimes A^{-1/2} > 0$. Thus, $(A^{-1/2} \ltimes B \ltimes A^{-1/2})^{1/2} > 0$. Using Proposition 2.2 again, we obtain

$$X \;=\; A^{1/2} \ltimes \left(A^{-1/2} \ltimes B \ltimes A^{-1/2}\right)^{1/2} \ltimes A^{1/2} \;>\; 0.$$

Applying Lemma 2.1, we get

$$X \bullet A \;=\; A^{1/2} \ltimes \left(A^{-1/2} \ltimes B \ltimes A^{-1/2}\right)^{1/2} \ltimes I_n \ltimes \left(A^{-1/2} \ltimes B \ltimes A^{-1/2}\right)^{1/2} \ltimes A^{1/2} \;=\; B.$$

Thus, $A \sharp B := A^{1/2} \ltimes (A^{-1/2} \ltimes B \ltimes A^{-1/2})^{1/2} \ltimes A^{1/2}$ is a solution of $X \bullet A = B$. Suppose that $Y \in \mathbb{P}^{m\times m}$ satisfying $X \bullet X = B = Y \bullet A$. Consider

$$\begin{aligned}\left(A^{-1/2} \ltimes X \ltimes A^{-1/2}\right)^2 \;&=\; A^{-1/2} \ltimes (X \bullet A) \ltimes A^{-1/2} \;=\; A^{-1/2} \ltimes (Y \bullet A) \ltimes A^{-1/2} \\ &=\; \left(A^{-1/2} \ltimes Y \ltimes A^{-1/2}\right)^2.\end{aligned}$$

From the uniqueness of positive-definite square root, we get $A^{-1/2} \ltimes X \ltimes A^{-1/2} = A^{-1/2} \ltimes Y \ltimes A^{-1/2}$. Thus,

$$X \;=\; A^{1/2} \ltimes \left(A^{-1/2} \ltimes X \ltimes A^{-1/2}\right) \ltimes A^{1/2} \;=\; A^{1/2} \ltimes \left(A^{-1/2} \ltimes Y \ltimes A^{-1/2}\right) \ltimes A^{1/2} \;=\; Y.$$

For the special case $m = n$ of Theorem 3.2, the Riccati equation $A \prec_k B$ is reduced to $XA^{-1}X = B$ which has been already studied by many authors. e.g., [22], [25], [34].

Definition 3.3. Let $A \in \mathbb{P}^{n\times n}$ and $B \in \mathbb{P}^{m\times m}$ be such that $A \prec_k B$. The metric geometric mean of A and B is defined to be

$$A \sharp B \;=\; A^{1/2} \ltimes \left(A^{-1/2} \ltimes B \ltimes A^{-1/2}\right)^{1/2} \ltimes A^{1/2}. \tag{3.1}$$

Kubo and Ando [28] provided a significant theory of operator means: given an operator monotone function on $(0,\infty)$ such that $f(1) = 1$, the operator mean m_f is defined by

$$A m_f B \;=\; A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}.$$

For the metric geometric mean, we can write

$$A\sharp B \;=\; A^{1/2} \ltimes f\left(A^{-1/2} \ltimes B \ltimes A^{-1/2}\right) \ltimes A^{1/2},$$

where $f = \sqrt{x}$, $A \in \mathbb{P}^{n\times n}$ and $B \in \mathbb{P}^{m\times m}$ with $A \prec_k B$.

Lemma 3.1. *(Löwner-Heinz inequality, e.g., [35]) Let $S,T \in \mathbb{PS}^{n\times n}$. If $S \leqslant T$ then $S^{1/2} \leqslant T^{1/2}$.*

The following theorem gives necessity and sufficiency condition for the Riccati inequality.

Theorem 3.4. *Let $A \in \mathbb{P}^{n\times n}$ and $B \in \mathbb{P}^{m\times m}$ be such that $A \prec_k B$. Let $X \in \mathbb{H}^{m\times m}$. Then, $X \leqslant A\sharp B$ if and only if there exists $Y \in \mathbb{H}^{m\times m}$ such that $X \leqslant Y$ and $Y \bullet A \leqslant B$.*

Proof. Suppose $X \leqslant A\sharp B$. Set $Y = A\sharp B$. We have, by Theorem 3.2, $Y \bullet A = B$ and $Y \geqslant X$. Conversely, suppose that there exists $Y \in \mathbb{H}^{m\times m}$ such that $X \leqslant Y$ and $Y \bullet A \leqslant B$. By Proposition 2.1 and Lemma 3.1, we have

$$A^{-1/2} \ltimes Y \ltimes A^{-1/2} \;=\; \left(A^{-1/2} \ltimes (Y \bullet A) \ltimes A^{-1/2}\right)^{1/2} \;\leqslant\; \left(A^{-1/2} \ltimes B \ltimes A^{-1/2}\right)^{1/2}.$$

Using Proposition 2.1, we obtain

$$X \;\leqslant\; Y \;\leqslant\; A^{1/2} \ltimes \left(A^{-1/2} \ltimes B \ltimes A^{-1/2}\right)^{1/2} \ltimes A^{1/2} \;=\; A\sharp B.$$

From Theorem 3.4, we obtain that $A\sharp B$ is the largest (in the Löwner order) solution of the Riccati inequality $Y \bullet A \leqslant B$.

4. Metric geometric means of positive semidefinite matrices

In this section, the expression $A_k \to A$ means that the matrix sequence $(A_k)_{k\in\mathbb{N}}$ converges to the matrix A. For any sequence $(A_k)_{k\in\mathbb{N}}$ in $\mathbb{H}^{n\times n}$, we write $A_k \downarrow A$ indicates that (A_k) is a decreasing sequence (with respect to the Löwner partial order) and $A_k \to A$.

Lemma 4.1. *Let $m = nk$. Then, the operation $\sharp : \mathbb{P}^{n\times n} \times \mathbb{P}^{m\times m} \to \mathbb{P}^{m\times m}$ is jointly monotone. Moreover, for any sequences $(A_k)_{k\in\mathbb{N}} \in \mathbb{P}^{n\times n}$ and $(B_k)_{k\in\mathbb{N}} \in \mathbb{P}^{m\times m}$ such that $A_k \downarrow A$ and $B_k \downarrow B$, the sequence $(A_k\sharp B_k)_{k\in\mathbb{N}}$ has a common limit, namely, $A\sharp B$.*

Proof. First, let $A_1, A_2 \in \mathbb{P}^{n\times n}$ and $B_1, B_2 \in \mathbb{P}^{m\times m}$. Suppose $A_1 \leqslant A_2$ and $B_1 \leqslant B_2$. By Proposition 3.1 and Theorem 3.2, we have

$$(A_1\sharp B_1) \bullet A_2 \leqslant (A_1\sharp B_1) \bullet A_1 = B_1 \leqslant B_2.$$

Since $A_1\sharp B_1$ satisfies the Riccati inequality $X \bullet A_2 \leqslant B_2$, we obtain $A_1\sharp B_1 \leqslant A_2\sharp B_2$ by Theorem 3.4. Next, let $(A_k)_{k\in\mathbb{N}}$ and $(B_k)_{k\in\mathbb{N}}$ be sequences in $\mathbb{P}^{n\times n}$ and $\mathbb{P}^{m\times m}$, respectively. Assume that $A_k \downarrow A$ and

$B_k \downarrow B$. Using the monotonicity of the metric geometric mean, we conclude that the sequence $(A_k \sharp B_k)$ is decreasing. In addition, it is bounded below by the zero matrix. The order completeness (with respect to the Löwner partial order) of $\mathbb{C}^{n\times n}$ implies that $A_k \sharp B_k$ converges to a positive definite matrix. Recall that the matrix multiplication is continuous. It follows from Lemma 2.2 that $A_k^{-1/2} \ltimes B_k \ltimes A_k^{-1/2}$ converges to $A^{-1/2} \ltimes B \ltimes A^{-1/2}$ in Frobenius norm (or another norm). By Löwner-Heinz inequality (Lemma 3.1), we obtain

$$A_k^{1/2} \ltimes \left(A_k^{-1/2} \ltimes B_k \ltimes A_k^{-1/2}\right)^{1/2} \ltimes A_k^{1/2} \;\to\; A^{1/2} \ltimes \left(A^{-1/2} \ltimes B \ltimes A^{-1/2}\right)^{1/2} \ltimes A^{1/2},$$

i.e., $A_k \sharp B_k \to A \sharp B$.

It is natural to extend the metric geometric mean of positive definite matrices to positive semidefinite matrices by taking limits.

Definition 4.1. Let $A \in \mathbb{PS}^{n\times n}$ and $B \in \mathbb{PS}^{m\times m}$ be such that $A \prec_k B$. We define the metric geometric mean of A and B to be

$$A \sharp B \;=\; \lim_{\varepsilon \to 0^+} (A + \varepsilon I_n) \sharp (B + \varepsilon I_m). \tag{4.1}$$

We see that $A + \varepsilon I_n$ and $B + \varepsilon I_m$ are decreasing sequences where $\varepsilon \downarrow 0^+$. Since $A + \varepsilon I_n \downarrow A$ and $B + \varepsilon I_m \downarrow B$, we obtain by Lemma 4.1 that the limit (4.1) is well-defined. Fundamental properties of metric geometric means are as follows.

Theorem 4.2. *Let $A, C \in \mathbb{PS}^{n\times n}$ and $B, D \in \mathbb{PS}^{m\times m}$ with $A \prec_k B$.*

(1) Positivity: $A \sharp B \geqslant 0$.
(2) Fixed-point property: $A \sharp A = A$.
(3) Positive homogeneity: $\alpha(A \sharp B) = (\alpha A) \sharp (\alpha B)$ for all $\alpha \geqslant 0$.
(4) Congruent invariance: $T^* \ltimes (A \sharp B) \ltimes T = (T^* \ltimes A \ltimes T) \sharp (T^* \ltimes B \ltimes T)$ for all $T \in \mathbb{GL}^{m\times m}$.
(5) Self duality: $(A \sharp B)^{-1} = A^{-1} \sharp B^{-1}$.
(6) Permutation invariance: $A \sharp B = B \sharp (A \otimes I_k)$.
(7) Consistency with scalars: If $A \otimes I_k$ and B commute, then $A \sharp B = A^{1/2} \ltimes B^{1/2}$.
(8) Monotonicity: If $A \leqslant C$ and $B \leqslant D$, then $A \sharp B \leqslant C \sharp D$.
(9) Concavity: the map $(A, B) \mapsto A \sharp B$ is concave.
(10) Continuity from above: If $A_k \downarrow A$ and $B_k \downarrow B$ then $A_k \sharp B_k \downarrow A \sharp B$.
(11) Betweenness: If $A \otimes I_k \leqslant B$, then $A \otimes I_k \leqslant A \sharp B \leqslant B$.
(12) Determinantal identity: $\det(A \sharp B) = \sqrt{(\det A)^k \det B}$.

Proof. By continuity, we may assume that $A, C \in \mathbb{P}^{n\times n}$ and $B, D \in \mathbb{P}^{m\times m}$. It is clear that (1) holds.
(2) Since $A \bullet A = A$, we have by Theorem 3.2 that $A \sharp A = A$.
(3) For $\alpha = 0$, we have $\alpha(A \sharp B) = 0 = (\alpha A) \sharp (\alpha B)$. Let $\alpha > 0$ and $X = A \sharp B$. Since

$$(\alpha X) \bullet (\alpha A) \;=\; \alpha(X \bullet A) \;=\; \alpha B,$$

we have by Theorem 3.2 that $\alpha X = (\alpha A) \sharp (\alpha B)$, i.e., $\alpha(A \sharp B) = (\alpha A) \sharp (\alpha B)$.
(4) Let $T \in \mathbb{GL}^{m\times m}$ and $X = A \sharp B$. Applying Lemma 2.1, we have

$$(T^* \ltimes X \ltimes T) \bullet (T^* \ltimes A \ltimes T) \;=\; T^* \ltimes (X \bullet A) \ltimes T \;=\; T^* \ltimes B \ltimes T.$$

This implies that $T^*\ltimes X\ltimes T = (T^*\ltimes A\ltimes T)\sharp(T^*\ltimes B\ltimes T)$. Hence, $T^*\ltimes(A\sharp B)\ltimes T = (T^*\ltimes A\ltimes T)\sharp(T^*\ltimes B\ltimes T)$.
(5) Let $X = A\sharp B$. Applying Theorem 3.2, we have $X^{-1} \bullet A^{-1} = B^{-1}$. This implies that $X^{-1} = A^{-1}\sharp B^{-1}$, i.e., $(A\sharp B)^{-1} = A^{-1}\sharp B^{-1}$.
(6) Using Lemma 2.2 and Theorem 3.2, we have

$$X^{-1} \bullet B^{-1} \;=\; X^{-1} \bullet (X \bullet A)^{-1} \;=\; X^{-1} \bullet \left(X^{-1} \bullet A^{-1}\right) \;=\; A^{-1} \otimes I_k.$$

It follows that $X^{-1} = B\sharp(A^{-1} \otimes I_k)$, i.e., $(A\sharp B)^{-1} = B^{-1}\sharp(A^{-1} \otimes I_k)$. Using (5), we get $A^{-1}\sharp B^{-1} = B^{-1}\sharp(A^{-1} \otimes I_k)$. By replacing A^{-1} and B^{-1} by A and B, respectively, we obtain $A\sharp B = B\sharp(A \otimes I_k)$.
(7) Since $A \otimes I_k$ and B commute, we have that $A^{1/2} \ltimes B^{1/2} = B^{1/2} \ltimes A^{1/2}$. Then,

$$\left(A^{1/2} \ltimes B^{1/2}\right) \bullet A \;=\; B^{1/2} \ltimes A^{1/2} \ltimes A^{-1} \ltimes A^{1/2} \ltimes B^{1/2} \;=\; B.$$

This implies that $A\sharp B = A^{1/2} \ltimes B^{1/2}$.
(8) Follows from Lemma 4.1.
(9) Let $\lambda \in [0, 1]$. Since $(A\sharp B) \bullet B = A \otimes I_k$ and $(C\sharp D) \bullet D = C \otimes I_k$, we have

$$\begin{bmatrix} A \otimes I_k & A\sharp B \\ A\sharp B & B \end{bmatrix} \geqslant 0 \;\text{ and }\; \begin{bmatrix} C \otimes I_k & C\sharp D \\ C\sharp D & D \end{bmatrix} \geqslant 0.$$

Then,

$$0 \;\leqslant\; \lambda \begin{bmatrix} A \otimes I_k & A\sharp B \\ A\sharp B & B \end{bmatrix} + (1-\lambda) \begin{bmatrix} C \otimes I_k & C\sharp D \\ C\sharp D & D \end{bmatrix} \;=\; \begin{bmatrix} [\lambda A + (1-\lambda)C] \otimes I_k & \lambda A\sharp B + (1-\lambda)C\sharp D \\ \lambda A\sharp B + (1-\lambda)C\sharp D & \lambda B + (1-\lambda)D \end{bmatrix}.$$

We have

$$[\lambda A + (1-\lambda)C] \otimes I_k \;\geqslant\; [\lambda A\sharp B + (1-\lambda)C\sharp D]\,[\lambda B + (1-\lambda)D)]^{-1}\,[\lambda A\sharp B + (1-\lambda)C\sharp D]$$

and then

$$\begin{aligned} &[(\lambda B + (1-\lambda)D)]^{-1/2} \ltimes [\lambda A + (1-\lambda)C] \ltimes [(\lambda B + (1-\lambda)D)]^{-1/2} \\ \geqslant\; &\left\{[(\lambda B + (1-\lambda)D)]^{-1/2} \ltimes [\lambda A\sharp B + (1-\lambda)C\sharp D] \ltimes [(\lambda B + (1-\lambda)D)]^{-1/2}\right\}^2 . \end{aligned}$$

It follows that

$$\begin{aligned} &\left\{[(\lambda B + (1-\lambda)D)]^{-1/2} \ltimes [\lambda A + (1-\lambda)C] \ltimes [(\lambda B + (1-\lambda)D)]^{-1/2}\right\}^{1/2} \\ \geqslant\; &[(\lambda B + (1-\lambda)D)]^{-1/2} \ltimes [\lambda A\sharp B + (1-\lambda)C\sharp D] \ltimes [(\lambda B + (1-\lambda)D)]^{-1/2} . \end{aligned}$$

Thus, $[\lambda A + (1-\lambda)C]\sharp[\lambda B + (1-\lambda)D] \geqslant \lambda(A\sharp C) + (1-\lambda)(C\sharp D)$.
(10) Follows from Lemma 4.1.
(11) Let $A \otimes I_k \leqslant B$. By applying the monotonicity of metric geometric mean, we have

$$A \otimes I_k \;=\; A\sharp(A \otimes I_k) \;\leqslant\; A\sharp B \;\leqslant\; B\sharp B \;=\; B.$$

(12) The determinantal identity follows directly from Lemmas 2.1 and 2.3.

Properties 2, 4, 8 and 10 illustrate that the metric geometric mean (4.1) is a mean in Kubo-Ando's sense. In addition, this mean possesses self-duality and concavity (properties 5 and 9). The following proposition gives another ways of expressing the metric geometric mean.

Proposition 4.1. *Let $A \in \mathbb{PS}^{n\times n}$ and $B \in \mathbb{PS}^{m\times m}$ with $A \prec_k B$.*

(1) There exists a unitary matrix $U \in \mathbb{C}^{m\times m}$ such that

$$A\sharp B \ = \ A^{1/2} \ltimes U \ltimes B^{1/2}.$$

(2) If all eigenvalues of $A^{-1} \ltimes B$ are positive,

$$A\sharp B \ = \ A \ltimes (A^{-1} \ltimes B)^{1/2} \ = \ (A \ltimes B^{-1})^{1/2} \ltimes B.$$

Proof. By continuity, we may assume that $A \in \mathbb{P}^{n\times n}$ and $B \in \mathbb{P}^{m\times m}$.
(1) Set $U = (A^{-1/2} \ltimes B \ltimes A^{-1/2})^{1/2} \ltimes A^{1/2} \ltimes B^{-1/2}$. Since $U^*U = I_m$ and $UU^* = I_m$, we have that U is unitary and

$$A^{1/2} \ltimes U \ltimes B^{1/2} \ = \ A^{1/2} \ltimes (A^{-1/2} \ltimes B \ltimes A^{-1/2})^{1/2} \ltimes A^{1/2} \ = \ A\sharp B.$$

(2) Assume that all eigenvalues of $A^{-1} \ltimes B$ are positive. Recall that if matrix X has positive eigenvalues, it has a unique square root. Since $(A \ltimes (A^{-1} \ltimes B)^{1/2}) \bullet A = B$, we have by Theorem 3.2 that $A\sharp B = A \ltimes (A^{-1} \ltimes B)^{1/2}$. Similarly, $A\sharp B = (A \ltimes B^{-1})^{1/2} \ltimes B$.

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. The Cauchy-Schwarz inequality states that for any $x, y \in V$,

$$|\langle x, y\rangle|^2 \ \leqslant \ \langle x, x\rangle\langle y, y\rangle. \tag{4.2}$$

Corollary 4.1. *Let $A \in \mathbb{PS}^{n\times n}$ and $B \in \mathbb{PS}^{m\times m}$ with $A \prec_k B$. Then, for any $x, y \in \mathbb{C}^m$,*

$$\left|\langle (A\sharp B)x, y\rangle\right| \ \leqslant \ \sqrt{\langle (A \otimes I_k)x, x\rangle\langle By, y\rangle}.$$

Proof. From Proposition 4.1(1), we can write $A\sharp B = A^{1/2} \ltimes U \ltimes B^{1/2}$ for some unitary $U \in \mathbb{C}^{m\times m}$. By applying Cauchy-Schwarz inequality (4.2), we get

$$\begin{aligned}\left|\langle (A\sharp B)x, y\rangle\right|^2 \ &= \ \left|\langle (A\sharp B)y, x\rangle\right|^2 \ = \ \left|\left\langle \left(A^{1/2} \ltimes U \ltimes B^{1/2}\right) y, x\right\rangle\right|^2 \ = \ \left|\left\langle UB^{1/2}y, (A^{1/2} \otimes I_k)x\right\rangle\right|^2 \\ &\leqslant \ \left\langle UB^{1/2}y, UB^{1/2}y\right\rangle \left\langle (A \otimes I_k)^{1/2}x, (A \otimes I_k)^{1/2}x\right\rangle \ = \ \langle (A \otimes I_k)x, x\rangle \langle By, y\rangle .\end{aligned}$$

5. Equations and inequalities involving metric geometric means

In this section, we investigate matrix equations and inequalities concerning metric geometric means.

Theorem 5.1. *Let $A, X_1, X_2 \in \mathbb{PS}^{n\times n}$ and $B, Y_1, Y_2 \in \mathbb{PS}^{m\times m}$ be such that $A \prec_k B$.*

(1) (left cancellability) If $A\sharp Y_1 = A\sharp Y_2$ then $Y_1 = Y_2$.
(2) (right cancellability) If $X_1\sharp B = X_2\sharp B$ then $X_1 = X_2$.

Proof. (1) Assume that $A\sharp Y_1 = A\sharp Y_2$. We have

$$\left(A^{-1/2} \ltimes Y_1 \ltimes A^{-1/2}\right)^{1/2} = \left(A^{-1/2} \ltimes Y_2 \ltimes A^{-1/2}\right)^{1/2}.$$

This implies that $A^{-1/2} \ltimes Y_1 \ltimes A^{-1/2} = A^{-1/2} \ltimes Y_2 \ltimes A^{-1/2}$. Applying Proposition 2.1(3), we get $Y_1 = Y_2$.
(2) Suppose that $X_1\sharp B = X_2\sharp B$. We have by Theorem 4.2(6) that $B\sharp(X_1 \otimes I_k) = B\sharp(X_2 \otimes I_k)$. Using the assertion 1, we get $X \otimes I_k = X_2 \otimes I_k$. Since $(X - Y) \otimes I_k = X \otimes I_k - Y \otimes I_k = 0$, we get by Lemma 2.3 that $X = Y$.

Theorem 5.1 shows that the metric geometric mean is cancellable, i.e., it is both left and right cancellable.

A map $\Psi : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is called a positive linear map if it is linear and $\Psi(X) \in \mathbb{PS}^{m\times m}$ whenever $X \in \mathbb{PS}^{n\times n}$. In addition, it said to be normalized if $\Psi(I_n) = I_m$.

Lemma 5.1. *(e.g., [36]) If $\Psi : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is a normalized positive linear map then for all $X \in \mathbb{PS}^{n\times n}$,*

$$\Psi(X)^2 \leqslant \Psi(X^2).$$

Proposition 5.1. *Let $\Phi : \mathbb{C}^{m\times m} \to \mathbb{C}^{p\times p}$ be a positive linear map. Then, for any $A \in \mathbb{PS}^{n\times n}$ and $B \in \mathbb{PS}^{m\times m}$ such that $A \prec_k B$, we have*

$$\Phi(A\sharp B) \leqslant \Phi(A \otimes I_k)\sharp\Phi(B).$$

Proof. By continuity, we may assume that $A \in \mathbb{P}^{n\times n}$ and $B \in \mathbb{P}^{m\times m}$. Consider the map $\Phi : \mathbb{C}^{m\times m} \to \mathbb{C}^{p\times p}$ defined by

$$\Phi(X) := \Psi(B)^{-1/2} \bullet \Psi(B^{1/2} \bullet X).$$

We see that Φ is a normalized positive linear map. By Lemmas 3.1 and 5.1, we get $\Phi(X^{1/2}) \leqslant \Phi(X)^{1/2}$. Thus,

$$\Phi\left((B^{-1/2} \bullet A)^{1/2}\right) \leqslant \Phi\left(B^{-1/2} \bullet A\right)^{1/2},$$

i.e.,

$$\Psi(B)^{1/2} \bullet \Psi\left(B^{1/2} \bullet (B^{-1/2} \bullet A)^{1/2}\right) \leqslant \left(\Psi(B)^{1/2} \bullet \Psi(A \otimes I_k)\right)^{1/2}.$$

It follows that $\Psi(A\sharp B) = \Psi(B\sharp(A \otimes I_k)) \leqslant \Psi(B)\sharp\Psi(A \otimes I_k) = \Psi(A \otimes I_k)\sharp\Psi(B)$.

For the special map $\Phi_T(X) = T^*XT$, where $T \in \mathbb{C}^{m\times m}$, the result of Proposition 5.1 reduces to the transformer inequality $T^* \ltimes (A\sharp B) \ltimes T \leqslant (T^* \ltimes A \ltimes T)\sharp(T^* \ltimes B \ltimes T)$.

A map $\Psi : \mathbb{C}^{m\times m} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{p\times p}$ is said to be concave if for any $A, C \in \mathbb{C}^{m\times m}, B, D \in \mathbb{C}^{n\times n}$ and $\lambda \in [0, 1]$,

$$\Psi(\lambda A + (1 - \lambda)C), \lambda B + (1 - \lambda)D)) \geqslant \lambda\Psi(A, B) + (1 - \lambda)\Psi(C, D).$$

Proposition 5.2. *Let $\Psi_1 : \mathbb{PS}^{m\times m} \to \mathbb{PS}^{n\times n}$ and $\Psi_2 : \mathbb{PS}^{p\times p} \to \mathbb{PS}^{q\times q}$ be concave maps with $n \mid q$. Then, the map $(A, B) \mapsto \Psi_1(A)\sharp\Psi_2(B)$ is concave.*

Proof. Let $A, C \in \mathbb{PS}^{m\times m}$, $B, D \in \mathbb{PS}^{p\times p}$ and $\lambda \in [0,1]$. Since Ψ_1 and Ψ_2 are concave, we have

$$\Psi_1(\lambda A + (1-\lambda)C) \geqslant \lambda\Psi_1(A) + (1-\lambda)\Psi_2(C) \quad \text{and} \quad \Psi_2(\lambda B + (1-\lambda)D) \geqslant \lambda\Psi_2(B) + (1-\lambda)\Psi_2(D).$$

Applying concavity and monotonicity of the metric geometric mean (Theorem 4.2), we obtain

$$\begin{aligned} \Psi_1(\lambda A + (1-\lambda)C)\sharp\Psi_2(\lambda B + (1-\lambda)D) &\geqslant [\lambda\Psi_1(A) + (1-\lambda)\Phi(C)] \sharp [\lambda\Psi_2(B) + (1-\lambda)\Psi(D)] \\ &\geqslant \lambda\Psi_1(A)\sharp\Psi_2(B) + (1-\lambda)\Psi_1(C)\sharp\Psi_2(D). \end{aligned}$$

This shows the concavity of the map $(A, B) \mapsto \Psi_1(A)\sharp\Psi_2(B)$.

Corollary 5.1. *(Cauchy-Schwarz's inequality) For each $i = 1, 2, \ldots, N$, let $A_i \in \mathbb{PS}^{n\times n}$ and $B_i \in \mathbb{PS}^{m\times m}$ be such that $A_i \prec_k B_i$. Then*

$$\sum_{i=1}^{N}\left(A_i^2\sharp B_i^2\right) \leqslant \left(\sum_{i=1}^{N} A_i^2\right)\sharp\left(\sum_{i=1}^{N} B_i^2\right). \tag{5.1}$$

Proof. By using the concavity of metric geometric mean (Theorem 4.2(9)), we have

$$\sum_{i=1}^{2}(A_i\sharp B_i) \leqslant \left(\sum_{i=1}^{2} A_i\right)\sharp\left(\sum_{i=1}^{2} B_i\right).$$

By mathematical induction, we obtain

$$\sum_{i=1}^{N}(A_i\sharp B_i) \leqslant \left(\sum_{i=1}^{N} A_i\right)\sharp\left(\sum_{i=1}^{N} B_i\right). \tag{5.2}$$

Replacing A_i and B_i by A_i^2 and B_i^2, respectively, in (5.2), we arrive the desire result.

6. Conclusions

We investigate the Riccati matrix equation $X \ltimes A^{-1} \ltimes X = B$, where the operation $\ltimes$ stands for the semi-tensor product. When A and B are positive definite matrices satisfying the factor-dimension condition, this equation has a unique positive solution, which is defined to be the metric geometric mean of A and B. We discuss that the metric geometric mean is the maximum solution of the Riccati inequality. By continuity of the metric geometric mean, we extend the notion of this mean to positive semidefinite matrices. We establish several properties of the metric geometric mean such as positivity, concavity, self-duality, congruence invariance, permutation invariance, betweenness and determinantal identity. In addition, this mean is a mean in the Kubo-Ando sense. Moreover, we investigate several matrix equations and inequalities concerning metric geometric means, concavity, cancellability, positive linear map and Cauchy-Schwarz inequality. Our results include the conventional metric geometric means of matrices as special case.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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