Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 23459–23471.
DOI: 10.3934/math.20231192
Received: 25 April 2023
Revised: 06 June 2023
Accepted: 15 June 2023
Published: 26 July 2023

Research article

Generalizations of some q-integral inequalities of Hölder, Ostrowski and Grüss type

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Abstract: This paper investigates some well-known inequalities for q-h-integrals. These include Hölder, Ostrowski, Grüss and Opial type inequalities. Refinement of the Hadamard inequality for q-h-integrals is also established by applying the definition of strongly convex functions. From main theorems, q-Hölder, q-Ostrowski and q-Grüss inequalities can be obtained in particular cases.

Keywords: q-integral; q-h-integral; Hölder inequality; Ostrowski inequality; Güss inequality; Hadamard inequality
Mathematics Subject Classification: 26A51, 26A33, 33E12

1. Introduction

Integral inequalities are important tools to estimate different kinds of quantities under some constraints. For example, Hölder inequality estimates the cumulative aggregate of product of two functions (vectors) to product of independent aggregates of functions (vectors). The Ostrowski inequality estimates the value of function $f(x)$ to its integral mean provided f is differentiable and its derivative is a bounded function. The well known Grüss inequality estimates the integral mean of product of two functions to the product of their integral means.

Currently, inequalities of q-integrals are studied very frequently by researchers. This article deals with some quantum estimates of integral inequalities involving quantum calculus. In particular, for recent articles closely related to the topic of this paper we refer the readers to [1–5]. We prove Hölder, Ostrowski and Grüss type inequalities for q-h-integrals. These inequalities will provide generalizations of several q-integral and classical integral inequalities.

In the following we recall q-derivative, q-integral and q-derivative/q-integral on a finite interval $[a, b]$.

Definition 1. *[13] The following expression*

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x} \tag{1.1}$$

is called the q-derivative of a continuous function f where $q \in (0, 1)$.

Definition 2. *[4] Assume that $f : J \to \mathbb{R}$ is a continuous function, $x \in J := [a, b]$ and $0 < q < 1$. Then, the expression*

$$_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a \tag{1.2}$$

is called the q-derivative on J of function f at x. Also, $_aD_q f(a) = \lim_{x\to a} {_aD_q f(x)}$.

Definition 3. *[13] Let $J := [a, b]$ and g: $J \to \mathbb{R}$ be a continuous function. The q-definite integral is given by the following formula:*

$$\int_a^x g(t) \; _ad_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n g(q^n x + (1-q^n) a), \tag{1.3}$$

for $x \in J$, $q \in (0, 1)$.

By using definition of q-definite integral it is possible to generalize the theory and concepts based on Riemann integrals. On the other hand, fractional integral operators also generalize Riemann integrals and are used in extending concepts of ordinary calculus. For example, classical integral inequalities have been published for fractional and local fractional integral operators, see [6–8]. In the field of mathematical inequalities, several integral inequalities such as Hadamard, Ostrowski and many other inequalities have been published for q-definite integrals. In [4], authors proved the Hölder, Ostrowski and Grüss type inequalities for these integrals which are stated in the upcoming theorems.

Theorem 1. *Assume that $f, \xi : J \to \mathbb{R}$ are two continuous functions and $x \in J$, $0 < q < 1$, $p_1 > 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then, the upcoming inequality holds for q-integrals:*

$$\int_a^x |f(t)||\xi(t)|_a d_q t \leq \left(\int_a^x |f(t)|^{p_1} \; _ad_q t \right)^{\frac{1}{p_1}} \left(\int_a^x |\xi(t)|^{p_2} \; _ad_q t \right)^{\frac{1}{p_2}}. \tag{1.4}$$

Theorem 2. *Assume that $f : J \to \mathbb{R}$ is q-differentiable function and $_aD_q f$ is continuous on $[a, b]$ where $0 < q < 1$. Then, the upcoming inequality holds for q-integrals:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)_a d_q t\right| \tag{1.5}$$
$$\leq \|_a D_q f\|(b-a)\left(\frac{2q}{1+q}\left(\frac{x-\frac{(3q-1)a+(1+q)b}{4q}}{b-a}\right)^2+\left(\frac{-q^2+6q-1}{8q(1+q)}\right)\right).$$

Theorem 3. *Assume that $g,\xi : J \to \mathbb{R}$ be L_1, L_2 Lipschitzian continuous functions on $[a,b]$ such that*

$$|g(u)-g(v)| \leq L_1|u-v|, \ |\xi(u)-\xi(v)| \leq L_2|u-v| \tag{1.6}$$

for all $u,v\in[a,b]$. Then, the upcoming inequality holds for q-integrals:

$$\left|\frac{1}{b-a}\int_a^b g(t)\xi(t)\ _a d_q t-\left(\frac{1}{b-a}\int_a^b g(t)\ _a d_q t\right)\left(\frac{1}{b-a}\int_a^b \xi(t)\ _a d_q t\right)\right| \tag{1.7}$$
$$\leq \frac{qL_1L_2(b-a)^2}{(1+q+q^2)(1+q)^2}.$$

Next, we recall q-h-derivative and q-h-integral. These combine q-derivative/integral and h-derivative/integral in a single definition.

Definition 4. *[9] Assume that $f\colon I\to\mathbb{R}$ be a continuous function. For $0<q<1$ and $h\in\mathbb{R}$, the q-h derivative of f is given by the formula;*

$$C_h D_q f(x)=\frac{_h d_q f(x)}{_h d_q x}=\frac{f(qx+h)-f(x)}{(q-1)x+qh}. \tag{1.8}$$

For $h=0$, we have $C_0 D_q f(x)=D_q f(x)$.

Definition 5. *[9] Let $0<q<1$ and function $f\colon I:=[a,b]\to\mathbb{R}$ be a continuous function. Then, the left and right q-h integrals on I are defined by*

$$I_{q-h}^{a+}f(x):=\int_a^x f(t)\ _h d_q t \tag{1.9}$$
$$=((1-q)(x-a)+qh)\sum_{n=0}^{\infty}q^n f(q^n a+(1-q^n)x+nq^n h),\ x>a,$$

$$I_{q-h}^{b-}f(x):=\int_x^b f(t)\ _h d_q t \tag{1.10}$$
$$=((1-q)(b-x)+qh)\sum_{n=0}^{\infty}q^n f(q^n x+(1-q^n)b+nq^n h),\ x<b.$$

We will also prove a refinement of the Hadamard q-h-integral inequality for convex functions given in [18]. For this the following definitions will be utilized.

Definition 6. *Suppose a real valued function f be defined on real line's interval I. Then, function f is said to be convex on I if the upcoming inequality holds:*

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b), \tag{1.11}$$

for $t \in [0,1]$, $a, b \in I$.

Definition 7. *[17] Suppose that M is a convex subset of X, $(X, \|.\|)$ be a normed space. A function $f: M \subset X \longrightarrow \mathbb{R}$ will be called strongly convex function with modulus $C \geq 0$ if*

$$f(tu+(1-t)v) \leq tf(u)+(1-t)f(v)-Ct(1-t)\|v-u\|^2 \tag{1.12}$$

holds for all $u, v \in M \subseteq X$, $t \in [0,1]$.

For $C = 0$, (1.12) reduces to (1.11), i.e., the definition of convex function is obtained. In the forthcoming section we prove the Hölder, Ostrowski, Grüss and Hadamard inequalities for q-h-integrals. We also give their special cases for q-integrals and connect them with their classical versions.

2. Main results

First, we give the Hölder inequality for q-h-integrals in the theorem stated below.

Theorem 4. *Let $J=[a,b]$, $q \in (0,1)$, $p_1 > 1$ with $\frac{1}{p_1}+\frac{1}{p_2}=1$. Also, let f and ξ be two continuous functions defined on J. Then, the upcoming inequality holds for q-h-integrals:*

$$I_{q\_h}^{a+}(|f(t)||\xi(t)|) \leq \left(I_{q\_h}^{a+}|f(t)|^{p_1}\right)^{\frac{1}{p_1}} \left(I_{q\_h}^{a+}|\xi(t)|^{p_2}\right)^{\frac{1}{p_2}}. \tag{2.1}$$

Proof. The following equation holds for left q-h-integrals:

$$\begin{aligned} I_{q\_h}^{a+}(|f(t)||\xi(t)|) &= ((1-q)(x-a)+qh)\sum_{n=0}^{\infty} q^n |f(q^n a+(1-q^n)x+nq^n h)| \\ &\times |\xi(q^n a+(1-q^n)x+nq^n h)| \\ &= ((1-q)(x-a)+qh)\sum_{n=0}^{\infty} |f(q^n a+(1-q^n)x+nq^n h)|(q^n)^{\frac{1}{p_1}} \\ &\times |\xi(q^n a+(1-q^n)x+nq^n h)|(q^n)^{\frac{1}{p_2}}. \end{aligned}$$

By applying the Hölder inequality in discrete case on the right hand side of above inequality we have the following inequality:

$$\begin{aligned} I_{q\_h}^{a+}(|f(t)||\xi(t)|) &\leq ((1-q)(x-a)+qh)\left(\sum_{n=0}^{\infty} |f(q^n a+(1-q^n)x+nq^n h)|^{p_1} q^n\right)^{\frac{1}{p_1}} \\ &\times \left(\sum_{n=0}^{\infty} |\xi(q^n a+(1-q^n)x+nq^n h)|^{p_2} q^n\right)^{\frac{1}{p_2}}. \end{aligned}$$

By using the definition of q-h-integral on the right hand side of the above inequality one can have the inequality (2.1). □

Remark 1. *i) If $h = 0$ then we get the q-Hölder inequality stated in Theorem 1.*
ii) If $q \to 1$ and $h = 0$ then inequality (2.1) becomes the Hölder integral inequality as follows:

$$\int_a^b |f(x)\xi(x)| \leq \left[\int_a^b |f(x)|^p \, dx\right]^{\frac{1}{p}} \left[\int_a^b |\xi(x)|^q \, dx\right]^{\frac{1}{q}}. \tag{2.2}$$

In the next theorem we give Ostrowski type inequality for K-Lipschitz function on a finite interval.

Theorem 5. *Suppose that f be a q-h-integerable function. If f is K-Lipschitz function on $J := [a, b]$. Then, we have the following inequality:*

$$\begin{aligned} &\left|\frac{(1-q)(b-a)+qh}{(1-q)(b-a)} f(x) - \frac{1}{b-a}\int_a^b f(t) \;{}_h d_q t\right| \\ &\leq \frac{K((1-q)(b-a)+qh)}{(b-a)(1-q^2)} \Big((|x|+|b|)\,q + |x| + |a| + |h|\left(1-q^2\right)S\Big), \end{aligned} \tag{2.3}$$

where $S = \sum_{n=0}^{\infty} q^{2n} n$.

Proof. From the definition of q-h-integral one can have

$$\begin{aligned} &\left|\frac{(1-q)(b-a)+qh}{(1-q)(b-a)} f(x) - \frac{1}{b-a}\int_a^b f(t) \;{}_h d_q t\right| \\ &= \left|\frac{1}{b-a}\int_a^b (f(x) - f(t)) \;{}_h d_q t\right| \leq \frac{1}{b-a}\int_a^b |f(x) - f(t)| \;{}_h d_q t. \end{aligned} \tag{2.4}$$

It is given that f is K-Lipschitz function, one can have

$$\frac{1}{b-a}\int_a^b |f(x) - f(t)| \;{}_h d_q t \leq \frac{K}{b-a}\int_a^b |x-t| \;{}_h d_q t. \tag{2.5}$$

By using definition of q-h-integral and properties of absolute value function we have

$$\begin{aligned} &\int_a^b |x-t| \;{}_h d_q t \leq \int_a^b (|x| + |t|) \;{}_h d_q t \\ &= ((1-q)(b-a)+qh)\sum_{n=0}^{\infty} q^n \Big(|x| + |q^n a + (1-q^n)\,b + nq^n h|\Big) \\ &\leq \frac{((1-q)(b-a)+qh)}{1-q^2}\Big((|x|+|b|)\,q + |x| + |a| + |h|\left(1-q^2\right)S\Big). \end{aligned} \tag{2.6}$$

Therefore, from (2.5) we obtain

$$\begin{aligned} &\frac{1}{b-a}\int_a^b |f(x) - f(t)| \;{}_h d_q t \leq \frac{K((1-q)(b-a)+qh)}{(b-a)(1-q^2)} \\ &\times\Big((|x|+|b|)\,q + |x| + |a| + |h|\left(1-q^2\right)S\Big). \end{aligned} \tag{2.7}$$

The required inequality is obtained by combining (2.4) and (2.7). □

Here we prove the q-h-Korkine identity as follows:

Lemma 1. *Suppose that f, ξ be two continuous functions on J and $q \in (0,1)$. The following identity for q-integrals holds true:*

$$\begin{aligned}&\frac{1}{2}\int_a^b\int_a^b (f(x)-f(y))(\xi(x)-\xi(y))\ {}_hd_qx\ {}_hd_qy \\ &= \frac{((1-q)(b-a)+qh)}{1-q}\int_a^b f(x)\xi(x)\ {}_hd_qx - \left(\int_a^b f(x)\ {}_hd_qx\right)\left(\int_a^b \xi(x)\ {}_hd_qx\right).\end{aligned} \tag{2.8}$$

Proof. The following equation holds for q-h-integrals:

$$\begin{aligned}&\int_a^b\int_a^b (f(x)-f(y))(\xi(x)-\xi(y))\ {}_hd_qx\ {}_hd_qy \\ &= \int_a^b\int_a^b (f(x)\xi(x)-f(x)\xi(y)-f(y)\xi(x)+f(y)\xi(y))\ {}_hd_qx\ {}_hd_qy \\ &= \Bigg(((1-q)(b-a)+qh)\sum_{n=0}^{\infty} q^n f(q^na+(1-q^n)b+nq^nh) \\ &\times \xi(q^na+(1-q^n)b+nq^nh)\Bigg)\int_a^b\ {}_hd_qy \\ &-((1-q)(b-a)+qh)\sum_{n=0}^{\infty} q^n f(q^na+(1-q^n)b+nq^nh)\int_a^b \xi(y)\ {}_hd_qy \\ &-((1-q)(b-a)+qh)\sum_{n=0}^{\infty} q^n \xi(q^na+(1-q^n)b+nq^nh)\int_a^b f(y)\ {}_hd_qy \\ &+((1-q)(b-a)+qh)\sum_{n=0}^{\infty} q^n f(q^na+(1-q^n)b+nq^nh) \\ &\xi(q^na+(1-q^n)b+nq^nh)\int_a^b\ {}_hd_qx = \frac{2((1-q)(b-a)+qh)}{1-q}\int_a^b f(x)\xi(x)\ {}_hd_qx \\ &-2\left(\int_a^b f(x)\ {}_hd_qx\right)\left(\int_a^b \xi(x)\ {}_hd_qx\right).\end{aligned}$$

From which one deduces (2.8). □

It can be noted that for $h = 0$, the Korkine's identity for q-integrals is obtained. Furthermore, classical Korkine's identity is obtained by setting $q \to 1$ along with $h = 0$. The forthcoming theorem is established by using the Korkine's identity for q-h-integrals.

Theorem 6. *Suppose that $f, \xi : J \to \mathbb{R}$ are L_1, L_2 Lipschitzian continuous functions on $[a,b]$ such that*

$$|f(x)-f(y)| \le L_1\,|x-y|\,,|\xi(x)-\xi(y)| \le L_2\,|x-y| \tag{2.9}$$

for all $x, y \in [a,b]$. Then, we have the inequality

$$\left|\frac{1-q}{(1-q)(b-a)+qh}\int_a^b f(x)\xi(x)\ {}_hd_qx-\left(\frac{1-q}{(1-q)(b-a)+qh}\int_a^b f(x)\ {}_hd_qx\right)\right. \tag{2.10}$$

$$\left.\left(\frac{1-q}{(1-q)(b-a)+qh}\int_a^b \xi(x)_h\, d_qx\right)\right| \leq L_1L_2\Big(\frac{q(b-a)^2}{(1+q+q^2)(1+q)^2}$$

$$+\left(h^2U+2bhS-2hT(b-a)-h^2S^2(1-q)-2hS\frac{(a+qb)}{1+q}\right)(1-q)\Big),$$

where

$$S=\sum_{n=0}^{\infty}nq^2, T=\sum_{n=0}^{\infty}nq^3, U=\sum_{n=0}^{\infty}n^2q^3.$$

Proof. First, we estimate the right hand side of the Korkine's identity under given conditions for functions f and ξ. Then, the following inequality can be obtained:

$$\left|\int_a^b\int_a^b (f(x)-f(y))(\xi(x)-\xi(y))\ {}_hd_qx_hd_qy\right| \leq L_1L_2\int_a^b\int_a^b (x-y)^2\ {}_hd_qx_hd_qy \tag{2.11}$$

$$=L_1L_2\int_a^b\int_a^b \left(x^2-2xy+y^2\right)\ {}_hd_qx\ {}_hd_qy$$

$$=2L_1L_2\left(\frac{((1-q)(b-a)+qh)}{1-q}\int_a^b x^2\ {}_hd_qx-\left(\int_a^b x\ {}_hd_qx\right)^2\right).$$

The following equation holds for q-h-integrals:

$$\int_a^b x^2\ {}_hd_qx=\frac{((1-q)(b-a)+qh)}{1-q}\Big(\frac{a^2(1+q)+b^2q\left(1+q^2\right)+2abq^2}{(1+q)(1+q+q^2)}$$

$$+\left(h^2U+2bhS-2hT(b-a)\right)(1-q)\Big).$$

Hence, we obtain

$$\frac{((1-q)(b-a)+qh)}{1-q}\int_a^b x^2\ {}_hd_qx-\left(\int_a^b x\ {}_hd_qx\right)^2 \tag{2.12}$$

$$=\left(\frac{((1-q)(b-a)+qh)}{1-q}\right)^2\Big(\frac{q(b-a)^2}{(1+q+q^2)(1+q)^2}$$

$$+\left(h^2U+2bhS-2hT(b-a)-h^2S^2(1-q)-2hS\frac{(a+qb)}{1+q}\right)(1-q)\Big).$$

Thus, from (2.11) and (2.12)

$$\frac{1}{2}\int_a^b\int_a^b |(f(x)-f(y))(\xi(x)-\xi(y))|\ {}_hd_qx_hd_qy$$

$$\leq L_1L_2\left(\frac{((1-q)(b-a)+qh)}{1-q}\right)^2\Big(\frac{q(b-a)^2}{(1+q+q^2)(1+q)^2}$$

$$+\left(h^2U+2bhS-2hT(b-a)-h^2S^2(1-q)-2hS\frac{(a+qb)}{1+q}\right)(1-q)\Big).$$

By applying the Korkine's identity we get the required inequality. □

Remark 2. *i) If $q \to 1$ and $h = 0$, then* (2.10) *becomes the classical Grüss-Čebyšev integral inequality as follows [14, 15]:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\xi(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b \xi(x)\,dx\right)\right| \leq \frac{L_1L_2(b-a)^2}{12}. \tag{2.13}$$

ii) If $h = 0$ in (2.10) *then we get Theorem 3.*

Next, we prove an Opial type inequality provided the fundamental theorem of calculus for q-h-integrals holds.

Theorem 7. *Let $p(t)$ be a non-negative and continuous function on $[0,k]$ and $x(t) \in C^{(1)}[0,k]$ be such that $x(0) = x(k) = 0$ and $x(t) > 0$ in $(0,k)$. Then, the following inequality holds:*

$$\int_0^k p(t)\left|\{x(t) + x(qt)\}\ {}_hD_qx(t)\right|\ {}_hd_qt \leq \left(\frac{k}{1-q}\int_0^k p^2(t)\ {}_hd_qt\right)^{\frac{1}{2}} \int_0^k \left|{}_hD_qx(t)\right|^2\ {}_hd_qt. \tag{2.14}$$

Proof. Let us choose $y(t)$ and $z(t)$ functions as follows:

$$y(t) = \int_0^t \left|{}_hD_qx(s)\right|\ {}_hd_qs, \qquad z(t) = \int_t^h \left|{}_hD_qx(s)\right|\ {}_hd_qs. \tag{2.15}$$

For $t \in [0,k]$, it follows that:

$$|x(t)| = \left|\int_0^t {}_hD_qx(s)\ {}_hd_qs\right| \leq \int_0^t \left|{}_hD_qx(s)\right|\ {}_hd_qs = y(t), \qquad |x(t)| = \left|\int_t^k {}_hD_qx(s)\ {}_hd_qs\right| \leq \int_t^k \left|{}_hD_qx(s)\right|\ {}_hd_qs = z(t), \tag{2.16}$$

$$|x(qt)| = \left|\int_0^{qt} {}_hD_qx(s)\ {}_hd_qs\right| \leq \int_0^{qt} \left|{}_hD_qx(s)\right|\ {}_hd_qs = y(t), \qquad |x(qt)| = \left|\int_{qt}^k {}_hD_qx(s)\ {}_hd_qs\right| \leq \int_{qt}^k \left|{}_hD_qx(s)\right|\ {}_hd_qs = z(t). \tag{2.17}$$

From above, we obtain that $|x(t)| \leq y(t)$ and $|x(t)| \leq z(t)$. Thus, we get

$$|x(t)| \leq \frac{y(t) + z(t)}{2} = \frac{1}{2}\int_0^k \left|{}_hD_qx(s)\right|\ {}_hd_qs. \tag{2.18}$$

$$|x(qt)| \leq \frac{y(qt) + z(qt)}{2} = \frac{1}{2}\int_0^k \left|{}_hD_q\ x(s)\right|\ {}_hd_qs. \tag{2.19}$$

By using (2.18), we have

$$\int_0^k p(t)|x(t)|^2 \ {}_h d_q t \le \frac{1}{2}\int_0^k p(t)\left(\left|{}_h D_q x(s)\right|\right)^2 \ {}_h d_q t \tag{2.20}$$
$$\le \frac{1}{4}\left(\int_0^k p(t) \ {}_h d_q t\right)\left(\int_0^k \ {}_h d_q t\right)\left(\int_0^k p(t)\left|{}_h D_q x(s)\right|^2\right)$$
$$\le \frac{k}{4(1-q)}\left(\int_0^k p(t) \ {}_h d_q t\right)\left(\int_0^k \left|{}_h D_q x(t)\right|^2 \ {}_h d_q t\right).$$

Similarly, by using (2.19) we have

$$\int_0^k p(t)|x(qt)|^2 \ {}_h d_q t \le \frac{k}{4(1-q)}\left(\int_0^k p(t) \ {}_h d_q t\right)\left(\int_0^k \left|{}_h D_q x(t)\right|^2 \ {}_h d_q t\right). \tag{2.21}$$

Now, by using Cauchy-Schwarz inequality and (2.20) one can have

$$\int_0^k p(t)\left|x(t) \ {}_h D_q x(t)\right| \ {}_h d_q t$$
$$\le \left(\int_0^k p^2(t)|x(t)|^2 \ {}_h d_q t\right)^{\frac{1}{2}}\left(\int_0^k \left|{}_h D_q x(t)\right|^2 \ {}_h d_q t\right)^{\frac{1}{2}}$$
$$\le \left(\frac{k}{4(1-q)}\left(\int_0^k p^2(t) \ {}_h d_q t\right)\left(\int_0^k \left|{}_h D_q x(t)\right|^2 \ {}_h d_q t\right)\right)^{\frac{1}{2}}\left(\int_0^k \left|{}_h D_q x(t)\right|^2 \ {}_h d_q t\right)^{\frac{1}{2}}$$
$$\le \frac{1}{2}\left(\frac{k}{(1-q)}\int_0^k p^2(t) \ {}_h d_q t\right)^{\frac{1}{2}}\left(\int_0^k \left|{}_h D_q x(t)\right|^2 \ {}_h d_q t\right). \tag{2.22}$$

Similarly,

$$\int_0^k p(t)\left|x(qt) \ {}_h D_q x(t)\right| \ {}_h d_q t \tag{2.23}$$
$$\le \frac{1}{2}\left(\frac{k}{(1-q)}\int_0^k p^2(t) \ {}_h d_q t\right)^{\frac{1}{2}}\left(\int_0^k \left|{}_h D_q x(t)\right|^2 \ {}_h d_q t\right).$$

By adding (2.22) and (2.23), one can get the required inequality (2.14). □

In the last theorem we give the Hadamard type inequality for strongly convex functions. Here, we will frequently use $h_1 = (x-a)h$, $h_2 = (b-x)h$, $S = \sum_{n=0}^{\infty} q^{2n} n$, $T = \sum_{n=0}^{\infty} q^{3n} n$, $U = \sum_{n=0}^{\infty} q^{3n} n^2$.

Theorem 8. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}^+$ be strongly convex function and $q \in (0,1)$. Additionally, let $a, b \in I$, $a < b$. Then, the following inequalities must hold:*

$$f\left(\frac{a+x}{2}\right) \le \frac{1-q}{(1-q)(x-a)+qh_1}\int_a^x f(t) \ {}_{h_1} d_q t \tag{2.24}$$
$$- C(x-a)^2\left(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)} + \left(h^2 U + hS - 2hT\right)(1-q)\right)$$

$$\leq \frac{(1-q)+qh}{1-q}\Big(f(x)\Big(\frac{q}{1+q}+(1-q)\,hS\Big)+f(a)\Big(\frac{q}{1+q}-(1-q)\,hS\Big)$$
$$-C(x-a)^2\Big(\frac{q^2}{(1+q)(1+q+q^2)}+(1-q)\Big(2hT-hS-h^2U\Big)\Big),$$

provided f is left q-h-integrable and symmetric function at $\frac{a+x}{2}$, $x\in(a,b)$ and

$$f\left(\frac{x+b}{2}\right)\leq\frac{1-q}{(1-q)(x-a)+qh_1}\int_x^b f(t)\ {}_{h_2}d_qt \tag{2.25}$$
$$-C(b-x)^2\Big(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)}+\Big(h^2U+hS-2hT\Big)(1-q)\Big)$$
$$\leq\frac{(1-q)+qh}{1-q}\Big(f(b)\Big(\frac{q}{1+q}+(1-q)\,hS\Big)+f(x)\Big(\frac{q}{1+q}-(1-q)\,hS\Big)$$
$$-C(b-x)^2\Big(\frac{q^2}{(1+q)(1+q+q^2)}+(1-q)\Big(2hT-hS-h^2U\Big)\Big),$$

provided f is right q-h-integrable and symmetric function at $\frac{x+b}{2}$, $x\in(a,b)$.

Proof. From definition of strongly convex function the following inequality is yielded:

$$f\left(\frac{a+x}{2}\right)\leq\frac{1}{2}\Big(f(ta+(1-t)x)+f(tx+(1-t)a)\Big)-C(x-a)^2\left(t-\frac{1}{2}\right)^2,$$
$$t\in[0,1].$$

This leads to the following inequality for left q-h-integrals:

$$f\left(\frac{a+x}{2}\right)\leq\frac{1-q}{(1-q)+qh}\Big(\frac{1}{2}\int_0^1 f(ta+(1-t)x)\ {}_hd_qt \tag{2.26}$$
$$+\frac{1}{2}\int_0^1 f(tx+(1-t)a)\ {}_hd_qt-C(x-a)^2\int_0^1\left(t-\frac{1}{2}\right)^2\ {}_hd_qt\Big).$$

Function f is symmetric about $\frac{a+x}{2}$. Therefore,

$$f\left(\frac{a+x}{2}\right)\leq\frac{1-q}{(1-q)+qh}\Big(\int_0^1 f(a+t(x-a))\ {}_hd_qt \tag{2.27}$$
$$-C(x-a)^2\int_0^1\left(t-\frac{1}{2}\right)^2\ {}_hd_qt\Big).$$

By definition, one can have

$$\int_0^1\left(t-\frac{1}{2}\right)^2\ {}_hd_qt=\frac{(1-q)+qh}{1-q}\Big(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)} \tag{2.28}$$
$$+\Big(h^2U+hS-2hT\Big)(1-q)\Big).$$

Then, by using (2.28) in (2.27) it takes the following form:

$$f\left(\frac{a+x}{2}\right) \le \frac{1-q}{(1-q)+qh}\int_0^1 f(a+t(x-a))\ {}_h d_q t \tag{2.29}$$
$$-C(x-a)^2\left(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)}+\left(h^2U+hS-2hT\right)(1-q)\right).$$

The following equation holds for left q-h-integrals:

$$\frac{(1-q)+qh}{(1-q)(x-a)+qh_1}\int_a^x f(t)\ {}_{h_1} d_q t \tag{2.30}$$
$$=((1-q)+qh)\sum_{n=0}^{\infty} q^n f(q^n a+(1-q^n)x+nq^n h_1)=\int_0^1 f(a+(x-a)t)\ {}_h d_q t.$$

The following estimation of the last term in (2.30) holds for strongly convex function f:

$$\int_0^1 f(a+(x-a)t)\ {}_h d_q t \le f(x)\int_0^1 t\ {}_h d_q t+f(a)\int_0^1 (1-t)\ {}_h d_q t \tag{2.31}$$
$$-C(x-a)^2\int_0^1 t(1-t)\ {}_h d_q t.$$

From (2.29)–(2.31) the following inequality can be constituted:

$$f\left(\frac{a+x}{2}\right) \le \frac{(1-q)+qh}{(1-q)(x-a)+qh_1}\int_a^x f(t)\ {}_{h_1} d_q t \tag{2.32}$$
$$-C(x-a)^2\left(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)}+\left(h^2U+hS-2hT\right)(1-q)\right)$$
$$\le f(x)\int_0^1 t\ {}_h d_q t+f(a)\int_0^1 (1-t)\ {}_h d_q t-C(x-a)^2\int_0^1 t(1-t)\ {}_h d_q t.$$

The q-h-integrals on the right hand side of the above inequality can be evaluated as follows:

$$\int_0^1 t\ {}_h d_q t=\frac{(1-q)+qh}{1-q}\left(\frac{q}{1+q}+(1-q)hS\right), \tag{2.33}$$

$$\int_0^1 (1-t)\ {}_h d_q t=\frac{(1-q)+qh}{1-q}\left(\frac{q}{1+q}-(1-q)hS\right), \tag{2.34}$$

and

$$\int_0^1 t(1-t)\ {}_h d_q t=\frac{((1-q)+qh)}{1-q}\left(\frac{q^2}{(1+q)(1+q+q^2)}\right. \tag{2.35}$$
$$\left.+(1-q)\left(2hT-hS-h^2U\right)\right).$$

By using (2.33)–(2.35) in the inequality (2.32), the required inequality (2.24) is obtained.
Similarly, in (2.29) by using definition of right q-h-integral one can obtain the required inequality (2.25). □

Remark 3. *By setting $C=0$, the inequalities* (2.24) *and* (2.25) *hold for left and right q-h-integrals for convex functions.*

3. Conclusions

We proved several well known inequalities for so-called q-h-integrals. These inequalities simultaneously hold for q- and h-integrals implicitly. Results for q-integrals are deduced which have been proven explicitly by different researchers in published articles. Also, we proved a refinement of q-h-Hadamard inequality for convex functions via strongly convex functions. This article may motivates the researchers to utilize q-h-integrals/derivatives in their future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through large group Research Project under grant number RGP2/461/44

Conflict of interest

Authors do not have conflict of interest.

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