Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 23442–23458.
DOI: 10.3934/math.20231191
Received: 19 May 2023
Revised: 10 July 2023
Accepted: 13 July 2023
Published: 26 July 2023

Research article

Further representations and computations of the generalized Moore-Penrose inverse

Kezheng Zuo[1], Yang Chen[2,*] and Li Yuan[2]

[1] School of Mathematics and Statistics, Hubei Normal University, Huangshi, China
[2] Department of Mathematics and Computer Science, Hanjiang Normal University, Shiyan, China

* **Correspondence:** Email: 275477236@qq.com.

Abstract: The aim of this paper is to provide new representations and computations of the generalized Moore-Penrose inverse. Based on the Moore-Penrose inverse, group inverse, Bott-Duffin inverse and certain projections, some representations for the generalized Moore-Penrose inverse are given. An equivalent condition for the continuity of the generalized Moore-Penrose inverse is proposed. Splitting methods and successive matrix squaring algorithm for computing the generalized Moore-Penrose inverse are presented.

Keywords: generalized Moore-Penrose inverse; representation; computation
Mathematics Subject Classification: 15A09

1. Introduction

The sets of all natural number, complex number, n dimensional column vectors and $m \times n$ complex matrices will be denoted by $\mathbb{N}$, $\mathbb{C}$, $\mathbb{C}^n$ and $\mathbb{C}^{m\times n}$, respectively. The identity matrix in $\mathbb{C}^{n\times n}$ and the null matrix in $\mathbb{C}^{m\times n}$ are denoted by I_n and O. For $A \in \mathbb{C}^{m\times n}$, let A^*, $r(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the conjugate transpose, the rank, the range and the null space of A, respectively. For $A \in \mathbb{C}^{n\times n}$, the index of A, denoted by $\mathrm{ind}(A)$, is the smallest nonnegative integer k, such that $r(A^k) = r(A^{k+1})$. The symbol $\mathbb{C}_k^{n\times n}$ stands for the set of all $n \times n$ complex matrices with index k. The symbol $P_{\mathcal{L},\mathcal{M}}$ stands for a projector onto $\mathcal{L}$ along $\mathcal{M}$, when a direct sum of subspaces $\mathcal{L}$ and $\mathcal{M}$ is equal to $\mathbb{C}^m$, and $P_{\mathcal{L}}$ presents the orthogonal projector onto $\mathcal{L}$ along $\mathcal{L}^\perp$, where $\mathcal{L}^\perp$ is the orthogonal complement subspace of $\mathcal{L}$.

The definitions of several helpful generalized inverses are stated now. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies $XAX = X$ is called an outer inverse of $A \in \mathbb{C}^{m\times n}$ and denoted by $A^{(2)}$. Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let $\mathcal{T}$ be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$ and let $\mathcal{S}$ be a subspace of $\mathbb{C}^m$ of dimension $m-s$. There exists a unique outer inverse X of A, such that $\mathcal{R}(X) = \mathcal{T}$ and $\mathcal{N}(X) = \mathcal{S}$ if and only if $A\mathcal{T} \oplus \mathcal{S} = \mathbb{C}^m$.

In this case, X is called an outer inverse with prescribed rang and null space and denoted by $A^{(2)}_{\mathcal{T},\mathcal{S}}$. For main properties please see [1,5,6,25–27].

The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ is the unique matrix $A^\dagger \in \mathbb{C}^{n\times m}$ [1,6,16], such that $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$, $(A^\dagger A)^* = A^\dagger A$.

The Drazin inverse of $A \in \mathbb{C}^{n\times n}_k$ is the unique matrix A^D [1,6,7] satisfying $A^DAA^D = A^D$, $AA^D = A^DA$, $A^DA^{k+1} = A^k$. In a particular case that $\operatorname{ind}(A) = 1$, the Drazin inverse becomes the group inverse $A^D = A^\#$.

The core-EP inverse of $A \in \mathbb{C}^{n\times n}_k$ denoted by $A^{\textcircled{\dagger}}$, is defined in [17] as the unique matrix $X \in \mathbb{C}^{n\times n}$, which satisfies $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$.

The core-EP inverse was recently investigated in numerous studies. Several characterizations and representations for the core-EP inverse were studied in [24,31]. In [12], iterative method and splitting method for finding core-EP inverse were presented. The limit representations were given in [29,30] for core-EP inverse. The core-EP inverse was extended for elements of ring in [9,14], and for tensors in [18]. In [8], the core-EP inverse was generalized to rectangular matrices.

In 2020, the generalized Moore-Penrose inverse (gMP inverse) was introduced by Stojanović and Mosić [22]. More precisely, the gMP inverse of $A \in \mathbb{C}^{n\times n}_k$, defined as $A^{\otimes} = (A^{\textcircled{\dagger}}A)^\dagger A^{\textcircled{\dagger}}$, is the unique solution to the matrix system

$$XAX = X, \quad AX = A(A^{\textcircled{\dagger}}A)^\dagger A^{\textcircled{\dagger}}, \;\; XA = (A^{\textcircled{\dagger}}A)^\dagger A^{\textcircled{\dagger}}A.$$

Especially, if $\operatorname{ind}(A) = 1$, $A^{\otimes}$ becomes $A^\dagger$. Thus, the gMP inverse extends the notion of the Moore-Penrose inverse. Some characterizations, representations and applications of the gMP inverse were proposed in [3,22].

Inspired by recent research about core-EP inverse, our aim is to present new representations and computational procedures of the gMP inverse. The major contributions of the article can be highlighted as follows:

1). Some representations for the gMP inverse are given based on the Moore-Penrose inverse, group inverse, Bott-Duffin inverse and certain projections.

2). Limit and integral representations of the gMP inverse are given.

3). A necessary and sufficient condition for continuity of the gMP inverse is verified.

4). Two splitting methods for computing the gMP inverse are presented.

5). The successive matrix squaring (SMS) algorithm for finding the gMP inverse is proposed.

The paper is organized as follows. In Section 2, some representations of the gMP inverse are presented, continuity of the gMP inverse, as well as maximal classes of matrices, such that the general formula of the gMP inverse is satisfied. Splitting methods for computing the gMP inverse are considered in Section 3. Section 4 gives the SMS algorithm for finding the gMP inverse.

2. Representations of the gMP inverse

The Lemma 2.1 can be got by [3, Lemma 2.2].

Lemma 2.1. [3] *Let $A \in \mathbb{C}^{n\times n}_k$. Then*

(a) $A^{\otimes} = A^{(2)}_{\mathcal{R}(A^*A^k),\mathcal{N}((A^k)^*)}$;

(b) $AA^{\otimes} = P_{\mathcal{R}(AA^*A^k),\mathcal{N}((A^k)^*)}$;

(c) $A^{\otimes}A = P_{\mathcal{R}(A^*A^k)}$.

In [15], the authors gave representations of the weak core inverse by applying the Urquhart formula [1]. Inspired by that, we will apply the Urquhart formula to give the expression of the gMP inverse.

Theorem 2.2. *Let $A \in \mathbb{C}_k^{n\times n}$. Then*

$$A^{\otimes} = ((A^k)^*A)^{\dagger}(A^k)^*. \tag{2.1}$$

Proof. It is easy to verify $r((A^k)^*) = r(A^*A^k) = r((A^k)^*AA^*A^k)$. According to Lemma 2.1 (a) $A^{\otimes} = A^{(2)}_{\mathcal{R}(A^*A^k),\mathcal{N}((A^k)^*)}$ and the Urquhart formula [1],

$$A^{\otimes} = A^*A^k((A^k)^*AA^*A^k)^{\dagger}(A^k)^* = ((A^k)^*A)^{\dagger}(A^k)^*,$$

where the the second identity is obtained by $B^{\dagger} = B^*(BB^*)^{\dagger}$. □

Remark 2.3. *For each nonnegative integer $l \geq k$, the expression* (2.1) *substituting l for k is still valid.*

Lemma 2.4. *Let $A \in \mathbb{C}_k^{n\times n}$. Then $A^{\otimes} = A^{(2)}_{\mathcal{R}(A^*A^k(A^k)^*),\mathcal{N}(A^*A^k(A^k)^*)}$.*

Proof. It follows from $\mathrm{ind}(A) = k$ that $r((A^k)^*) = r(A^*A^k) = r(A^*A^k(A^k)^*)$. Since $\mathcal{R}(A^*A^k(A^k)^*) \subseteq \mathcal{R}(A^*A^k)$ and $\mathcal{N}((A^k)^*) \subseteq \mathcal{N}(A^*A^k(A^k)^*)$, then $\mathcal{R}(A^*A^k) = \mathcal{R}(A^*A^k(A^k)^*)$ and $\mathcal{N}((A^k)^*) = \mathcal{N}(A^*A^k(A^k)^*)$. The rest follows by Lemma 2.1 (a). □

Using Lemma 2.4 and the representation of $A^{(2)}_{\mathcal{T},\mathcal{S}}$ inverse from [25, Theorem 2.1], we get new representations for the gMP inverse.

Theorem 2.5. *Let $A \in \mathbb{C}_k^{n\times n}$. Then*

$$A^{\otimes} = A^*A^k(A^k)^*(AA^*A^k(A^k)^*)^{\#} = (A^*A^k(A^k)^*A)^{\#}A^*A^k(A^k)^*.$$

Mary [13] introduced the inverse along an element, the Lemma 2.4 shows that the gMP is the inverse along $A^*A^k(A^k)^*$. Thus, the Theorem 2.5 also can be got by [13, Theorem 7].

The core-EP decomposition of a square matrix was given in [24] and the corresponding formula of the gMP inverse was verified in [22].

Lemma 2.6. [22, 24] *Let $A \in \mathbb{C}_k^{n\times n}$ and $r(A^k) = t$. Then A is expressed by*

$$A = U\begin{bmatrix} T & S \\ O & N \end{bmatrix}U^*, \tag{2.2}$$

where N is nilpotent with index k, T is $t\times t$ invertible matrix, $U \in \mathbb{C}^{n\times n}$ is unitary. In addition,

$$A^{\otimes} = U\begin{bmatrix} T^*(TT^*+SS^*)^{-1} & O \\ S^*(TT^*+SS^*)^{-1} & O \end{bmatrix}U^*. \tag{2.3}$$

Bott and Duffin [2] defined the Bott-Duffin inverse of $A \in \mathbb{C}^{n\times n}$ by $A^{(-1)}_{\mathcal{L}} = P_{\mathcal{L}}(AP_{\mathcal{L}} + I_n - P_{\mathcal{L}})^{-1}$ when $AP_{\mathcal{L}} + I_n - P_{\mathcal{L}}$ is nonsingular, where $\mathcal{L}$ is a subspace of $\mathbb{C}^n$. In [28], the authors showed the weak group inverse by a special Bott-Duffin inverse. Inspired by that, we use a special Bott-Duffin inverse of $AP_{\mathcal{R}(A^*A^k)}A^*$ to express the gMP inverse of A.

Theorem 2.7. *Let $A \in \mathbb{C}_k^{n\times n}$. Then*

$$\begin{aligned} A^{\otimes} &= P_{\mathcal{R}(A^*A^k)}A^*(AP_{\mathcal{R}(A^*A^k)}A^*)^{(-1)}_{\mathcal{R}(A^k)} \\ &= P_{\mathcal{R}(A^*A^k)}A^*P_{\mathcal{R}(A^k)}(AP_{\mathcal{R}(A^*A^k)}A^*P_{\mathcal{R}(A^k)} + I_n - P_{\mathcal{R}(A^k)})^{-1}. \end{aligned}$$

Proof. Assume that A is given by (2.2) and $L = T^k(T^k)^* + \widetilde{T}(\widetilde{T})^*$, $\widetilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$.

By (2.2), we get $A^k = U\begin{bmatrix} T^k & \widetilde{T} \\ O & O \end{bmatrix}U^*$, then $P_{\mathcal{R}(A^k)} = A^k(A^k)^\dagger = U\begin{bmatrix} I_t & O \\ O & O \end{bmatrix}U^*$ and

$$A^*A^k = U\begin{bmatrix} T^*T^k & T^*\widetilde{T} \\ S^*T^k & S^*\widetilde{T} \end{bmatrix}U^*. \tag{2.4}$$

Applying [5, Ch.3 Corollary 2.3] to (2.4), we get

$$(A^*A^k)^\dagger = U\begin{bmatrix} (T^k)^*L^{-1}(TT^*+SS^*)^{-1}T & (T^k)^*L^{-1}(TT^*+SS^*)^{-1}S \\ (\widetilde{T})^*L^{-1}(TT^*+SS^*)^{-1}T & (\widetilde{T})^*L^{-1}(TT^*+SS^*)^{-1}S \end{bmatrix}U^*,$$

which yields

$$P_{\mathcal{R}(A^*A^k)} = U\begin{bmatrix} T^*(TT^*+SS^*)^{-1}T & T^*(TT^*+SS^*)^{-1}S \\ S^*(TT^*+SS^*)^{-1}T & S^*(TT^*+SS^*)^{-1}S \end{bmatrix}U^*.$$

Let $M = P_{\mathcal{R}(A^*A^k)}A^*P_{\mathcal{R}(A^k)}(AP_{\mathcal{R}(A^*A^k)}A^*P_{\mathcal{R}(A^k)} + I_n - P_{\mathcal{R}(A^k)})^{-1}$. A straightforward calculation gives that

$$\begin{aligned} M &= U\begin{bmatrix} T^* & O \\ S^* & O \end{bmatrix}U^*U\begin{bmatrix} TT^*+SS^* & O \\ NS^* & I_{n-t} \end{bmatrix}^{-1}U^* \\ &= U\begin{bmatrix} T^* & O \\ S^* & O \end{bmatrix}\begin{bmatrix} (TT^*+SS^*)^{-1} & O \\ -NS^*(TT^*+SS^*)^{-1} & I_{n-t} \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^*(TT^*+SS^*)^{-1} & O \\ S^*(TT^*+SS^*)^{-1} & O \end{bmatrix}U^* \\ &= A^{\otimes}. \end{aligned}$$

□

Example 2.8. *Let*

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -a & 0 & -1 \\ -a & 0 & 0 \end{bmatrix}$$

with $\operatorname{ind}(A) = 2$*, where a is a real number. By* (2.1)*, the gMP inverse of A is given by*

$$A^{\otimes} = ((A^2)^*A)^\dagger(A^2)^* = \begin{bmatrix} \frac{12a^2+32}{9a^4+49a^2+64} & \frac{-3a^3-8a}{9a^4+49a^2+64} & \frac{-6a^3-16a}{9a^4+49a^2+64} \\ 0 & 0 & 0 \\ \frac{4a}{9a^4+49a^2+64} & \frac{-a^2}{9a^4+49a^2+64} & \frac{-2a^2}{9a^4+49a^2+64} \end{bmatrix}.$$

By calculation, we get that

$$(A^*A^2)^\dagger = \begin{bmatrix} \frac{3a^2+8}{9a^4+49a^2+64} & 0 & \frac{a}{9a^4+49a^2+64} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P_{\mathcal{R}(A^*A^2)}A^*P_{\mathcal{R}(A^2)} = \begin{bmatrix} \frac{12a^2+32}{5a^2+16} & \frac{-3a^3-8a}{5a^2+16} & \frac{-6a^3-16a}{5a^2+16} \\ 0 & 0 & 0 \\ \frac{4a}{5a^2+16} & \frac{-a^2}{5a^2+16} & \frac{-2a^2}{5a^2+16} \end{bmatrix},$$

$$(AP_{\mathcal{R}(A^*A^2)}A^*P_{\mathcal{R}(A^2)} + I_3 - P_{\mathcal{R}(A^2)})^{-1} = \begin{bmatrix} \frac{9a^4+25a^2+16}{9a^4+49a^2+64} & \frac{6a^3+12a}{9a^4+49a^2+64} & \frac{12a^3+24a}{9a^4+49a^2+64} \\ \frac{12a^3+32a}{9a^4+49a^2+64} & \frac{6a^4+41a^2+64}{9a^4+49a^2+64} & \frac{-6a^4-16a^2}{9a^4+49a^2+64} \\ \frac{12a^3+24a}{9a^4+49a^2+64} & \frac{-3a^4-6a^2}{9a^4+49a^2+64} & \frac{3a^4+37a^2+64}{9a^4+49a^2+64} \end{bmatrix}.$$

Then it can be verified that $P_{\mathcal{R}(A^*A^2)}A^*(AP_{\mathcal{R}(A^*A^2)}A^*)^{(-1)}_{\mathcal{R}(A^2)} = A^{\otimes}$.

The following theorem provides new formulae for the gMP inverse $A^{\otimes}$ based on projections $X = P_{\mathcal{N}((A^k)^*A)}$ and $Y = P_{\mathcal{N}((A^k)^*),\mathcal{R}(AA^*A^k)}$.

Theorem 2.9. *Let* $A \in \mathbb{C}^{n\times n}_k$, $X = P_{\mathcal{N}((A^k)^*A)}$ *and* $Y = P_{\mathcal{N}((A^k)^*),\mathcal{R}(AA^*A^k)}$. *Then for any* $a, b \in \mathbb{C} \setminus \{0\}$, *we have*

$$\begin{aligned} A^{\otimes} &= (A^*A^k(A^k)^*A + aX)^{-1}A^*A^k(A^k)^*(I_n - Y) \\ &= (I_n - X)A^*A^k(A^k)^*(AA^*A^k(A^k)^* + bY)^{-1}. \end{aligned}$$

Proof. By Lemma 2.1, it is not difficult to conclude that

$$(A^*A^k(A^k)^*A + aX)A^{\otimes} = A^*A^k(A^k)^*(I_n - Y).$$

Now we only need to show the invertibility of $A^*A^k(A^k)^*A + aX$. Let $(A^*A^k(A^k)^*A + aX)\xi = 0$ for some $\xi \in \mathbb{C}^n$. Then $A^*A^k(A^k)^*A\xi = -aX\xi$. By Lemma 2.1, we have

$$A^*A^k(A^k)^*A\xi = -aX\xi \in \mathcal{R}(A^*A^k(A^k)^*A) \cap \mathcal{R}(X) = \mathcal{R}(A^*A^k) \cap \mathcal{R}(A^*A^k)^{\perp} = \{0\},$$

which gives $A^*A^k(A^k)^*A\xi = -aX\xi = 0$. Hence,

$$\xi \in \mathcal{N}(A^*A^k(A^k)^*A) \cap \mathcal{N}(X) = \mathcal{N}((A^k)^*A) \cap \mathcal{R}(A^*A^k) = \{0\}.$$

Thus, $\xi = 0$ and $A^*A^k(A^k)^*A + aX$ is invertible.

Analogously, it can be verified that $AA^*A^k(A^k)^* + bY$ is nonsingular and $A^{\otimes} = (I_n - X)A^*A^k(A^k)^*(AA^*A^k(A^k)^* + bY)^{-1}$. □

Example 2.10. *In order to illustrate the representations of Theorem 2.9, let*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -i & 0 & i \\ 1 & 0 & 0 \end{bmatrix}$$

with $\text{ind}(A) = 2$*,* $a = -\frac{1}{5}$ *and* $b = 2i$*, where i stands for the imaginary unit. According to* (2.1)*, exact calculation in Mathematica gives*

$$A^{\otimes} = ((A^2)^*A)^{\dagger}(A^2)^* = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Simple calculation gives

$$X = P_{\mathcal{N}((A^2)^*A)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ Y = P_{\mathcal{N}((A^2)^*),\mathcal{R}(AA^*A^2)} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2}i & 1 & \frac{1}{2}i \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix},$$

$$A^*A^2(A^2)^* = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ (A^*A^2(A^2)^*A - \frac{1}{5}X)^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix},$$

$$(AA^*A^2(A^2)^* + 2iY)^{-1} = \begin{bmatrix} \frac{1}{8} - \frac{1}{4}i & 0 & \frac{1}{8} + \frac{1}{4}i \\ \frac{1}{4} - \frac{1}{8}i & -\frac{1}{2}i & \frac{1}{4} - \frac{1}{8}i \\ \frac{1}{8} + \frac{1}{4}i & 0 & \frac{1}{8} - \frac{1}{4}i \end{bmatrix}.$$

Further, it can be verified that Theorem 2.9 *is valid in this example.*

Inspired by limit and integral expressions of some generalized inverses given in [3, 10, 15, 19, 26, 28–30]. we consider limit and integral representations of the gMP inverse.

Theorem 2.11. *Let* $A \in \mathbb{C}_k^{n\times n}$*. Then*

(a) $A^{\otimes} = \lim\limits_{\lambda\to 0} A^*A^k(\lambda I_n + (A^k)^*AA^*A^k)^{-1}(A^k)^*$;

(b) $A^{\otimes} = \lim\limits_{\lambda\to 0}(\lambda I_n + A^*A^k(A^k)^*A)^{-1}A^*A^k(A^k)^*$.

Proof. (a). According to [19], recall that

$$B^{\dagger} = \lim_{\lambda\to 0} B^*(\lambda I_n + BB^*)^{-1}. \tag{2.5}$$

The result (a) follows by (2.1) and (2.5).

(b). Using Lemma 2.4 and [29, Theorem 2.1], we have

$$A^{\otimes} = \lim_{\lambda\to 0}(\lambda I_n + A^*A^k(A^k)^*A)^{-1}A^*A^k(A^k)^*.$$

□

Example 2.12. *We use the same matrix A as in Example* 2.8*. Simple calculation gives*

$$\begin{aligned} & A^*A^2(\lambda I_3 + (A^2)^*AA^*A^2)^{-1}(A^2)^* \\ = & \begin{bmatrix} \frac{12a^2+32}{9a^4+49a^2+\lambda+64} & \frac{-3a^3-8a}{9a^4+49a^2+\lambda+64} & \frac{-6a^3-16a}{9a^4+49a^2+\lambda+64} \\ 0 & 0 & 0 \\ \frac{4a}{9a^4+49a^2+\lambda+64} & \frac{-a^2}{9a^4+49a^2+\lambda+64} & \frac{-2a^2}{9a^4+49a^2+\lambda+64} \end{bmatrix}. \end{aligned}$$

After simplification, it follows that $\lim\limits_{\lambda\to 0} A^*A^2(\lambda I_3 + (A^2)^*AA^*A^2)^{-1}(A^2)^* = A^{\otimes}$.

Now, we present several integral representations for the gMP inverse.

Theorem 2.13. *Let $A \in \mathbb{C}_k^{n\times n}$. Then*
(a) $A^{\otimes} = \int_0^\infty A^*A^k \exp(-(A^k)^*AA^*A^k u)(A^k)^*\, du$;
(b) *If $D \in \mathbb{C}^{n\times n}$ satisfies $\mathcal{R}(D) = \mathcal{R}(A^*A^k)$ and $\mathcal{N}(D) = \mathcal{N}((A^k)^*)$, then*

$$A^{\otimes} = \int_0^\infty \exp(-D(DAD)^*DAu)D(DAD)^*D\, du.$$

Proof. (a). By [10], it is well-know that

$$B^{\dagger} = \int_0^\infty B^* \exp(-BB^*u)\, du. \tag{2.6}$$

Applying (2.1) and (2.6), we obtain

$$A^{\otimes} = ((A^k)^*A)^{\dagger}(A^k)^* = \int_0^\infty A^*A^k \exp(-(A^k)^*AA^*A^k u)(A^k)^*\, du.$$

(b). Using [26, Theorem 2.2], we conclude that (b) is satisfied. □

Some representations for generalized inverse $A^{(2)}_{\mathcal{T},\mathcal{S}}$ of matrices were given in [4]. For the gMP inverse we have the following results.

Theorem 2.14. *Let $A \in \mathbb{C}_k^{n\times n}$, $a, b, c, d \in \mathbb{C}\setminus\{0\}$. Assume that F and E^* are full column rank matrices, which satisfy $\mathcal{R}(A^*A^k) = \mathcal{R}(F)$ and $\mathcal{N}((A^k)^*) = \mathcal{N}(E)$. Then*

$$A^{\otimes} = b(aP_{\mathcal{R}(A^*A^k)^\perp} + bFEA)^{-1}FE \tag{2.7}$$

$$= dFE(cP_{\mathcal{N}((A^k)^*)} + dAFE)^{-1}. \tag{2.8}$$

Proof. We first show that $aP_{\mathcal{R}(A^*A^k)^\perp} + bFEA$ is nonsingular, let $(aP_{\mathcal{R}(A^*A^k)^\perp} + bFEA)x = 0$ for some $x \in \mathbb{C}^n$. Then $aP_{\mathcal{R}(A^*A^k)^\perp}x = -bFEAx$, we have $-FEAx \in \mathcal{R}(FEA) \subseteq \mathcal{R}(F) = \mathcal{R}(A^*A^k)$ and $aP_{\mathcal{R}(A^*A^k)^\perp}x \in \mathcal{R}(A^*A^k)^\perp$, i.e.,

$$aP_{\mathcal{R}(A^*A^k)^\perp}x = -bFEAx \in \mathcal{R}(A^*A^k)^\perp \cap \mathcal{R}(A^*A^k) = \{0\}.$$

Hence $P_{\mathcal{R}(A^*A^k)^\perp}x = 0$ and $FEAx = 0$. It follows that $x \in \mathcal{R}(A^*A^k)$. Since F is full column rank matrix, we get $EAx = 0$, which gives

$$x \in \mathcal{N}(EA) = \mathcal{R}(A^*E^*)^\perp = \mathcal{R}(A^*A^k)^\perp.$$

Hence $x \in \mathcal{R}(A^*A^k) \cap \mathcal{R}(A^*A^k)^\perp = \{0\}$, so $x = 0$ and $aP_{\mathcal{R}(A^*A^k)^\perp} + bFEA$ is nonsingular. By Lemma 2.1, we obtain $P_{\mathcal{R}(A^*A^k)^\perp}A^{\otimes} = O$ and $EAA^{\otimes} = E$, which imply (2.7).
Similarly, (2.8) can be verified. □

As we know, $A^{\otimes}$ is an outer inverse of A with rang $\mathcal{R}(A^*A^k)$ and null space $\mathcal{N}((A^k)^*)$. The results of Theorems 2.2 and 2.4 in [4] are applicable to the gMP inverse.

Corollary 2.15. *Let $A \in \mathbb{C}_k^{n\times n}$. Let B and C^* be of full column rank matrices and satisfy*

$$\mathcal{N}((A^k)^*) = \mathcal{R}(B), \quad \mathcal{R}(A^*A^k) = \mathcal{N}(C).$$

Let $E_B = I_n - BB^\dagger$, $F_C = I_n - C^\dagger C$. Then,

$$\begin{bmatrix} A^{\otimes} \\ O \end{bmatrix} = \begin{bmatrix} A^*E_BA & C^* \\ C & O \end{bmatrix}^{-1} \begin{bmatrix} A^*E_B \\ O \end{bmatrix},$$

$$\begin{bmatrix} A^{\otimes} & O \end{bmatrix} = \begin{bmatrix} F_CA^* & O \end{bmatrix} \begin{bmatrix} AF_CA^* & B \\ B^* & O \end{bmatrix}^{-1}.$$

Corollary 2.16. *Let $A \in \mathbb{C}_k^{n\times n}$.*

(*a*) *Let E and C be of full row rank matrices and satisfy $\mathcal{N}((A^k)^*) = \mathcal{N}(E)$, $\mathcal{R}(A^*A^k) = \mathcal{N}(C)$. Then*

$$A^{\otimes} = \begin{bmatrix} EA \\ C \end{bmatrix}^{-1} \begin{bmatrix} E \\ O \end{bmatrix}; \tag{2.9}$$

(*b*) *Let F and B be of full column rank matrices and satisfy $\mathcal{N}((A^k)^*) = \mathcal{R}(B)$, $\mathcal{R}(A^*A^k) = \mathcal{R}(F)$. Then*

$$A^{\otimes} = \begin{bmatrix} F & O \end{bmatrix} \begin{bmatrix} AF & B \end{bmatrix}^{-1}. \tag{2.10}$$

Example 2.17. *Let*

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with ind$(A) = 2$. *Using* (2.1), *the gMP inverse of A is given by*

$$A^{\otimes} = ((A^2)^*A)^\dagger (A^2)^* = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0 \end{bmatrix}. \tag{2.11}$$

Let

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ F = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix},\ E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

In order to verify the representations (2.7) *and* (2.8), *let $a = \frac{1}{3}$, $b = 3 + i$, $c = -3i$ and $d = -2$. Simple calculation gives*

$$(\frac{1}{3}P_{\mathcal{R}(A^*A^2)^\perp} + (3+i)FEA)^{-1} = \begin{bmatrix} \frac{31}{30} - \frac{1}{90}i & 0 & -\frac{59}{60} - \frac{1}{180}i & -\frac{59}{60} - \frac{1}{180}i \\ 0 & \frac{3}{20} - \frac{1}{20}i & 0 & 0 \\ -\frac{59}{60} - \frac{1}{180}i & 0 & \frac{301}{120} - \frac{1}{360}i & -\frac{59}{120} - \frac{1}{360}i \\ -\frac{59}{60} - \frac{1}{180}i & 0 & -\frac{59}{120} - \frac{1}{360}i & \frac{301}{120} - \frac{1}{360}i \end{bmatrix},$$

$$(-3iP_{\mathcal{N}((A^2)^*)} - 2AFE)^{-1} = \begin{bmatrix} -\frac{1}{12} & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \\ -\frac{1}{6}i & 0 & \frac{1}{3}i & 0 \\ 0 & 0 & 0 & \frac{1}{3}i \end{bmatrix}.$$

Further calculation gives $(3+i)(\frac{1}{3}P_{\mathcal{R}(A^*A^2)^\perp}+(3+i)FEA)^{-1}FE = A^{\otimes}$ *and* $-2FE(-3iP_{\mathcal{N}((A^2)^*)}-2AFE)^{-1} = A^{\otimes}$.

In order to verify the representations (2.9) *and* (2.10)*, it is necessary to compute*

$$\begin{bmatrix} EA \\ C \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & 0 & -\frac{1}{3} & -\frac{1}{6} \\ \frac{1}{6} & 0 & -\frac{1}{3} & \frac{5}{6} \end{bmatrix}, \begin{bmatrix} AF & B \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Further, $\begin{bmatrix} EA \\ C \end{bmatrix}^{-1} \begin{bmatrix} E \\ O \end{bmatrix} = A^{\otimes}$ *and* $\begin{bmatrix} F & O \end{bmatrix}\begin{bmatrix} AF & B \end{bmatrix}^{-1} = A^{\otimes}$.

For maximal classes of operators and matrices for which the representations of the gMP inverse are valid, see [3, 22]. Now, we present the maximal classes of matrix X such that $X(A^k)^*$ coincides with the gMP inverse of A.

Theorem 2.18. *Let* $A \in \mathbb{C}_k^{n\times n}$ *be given by* (2.2) *and* $r(A^k) = t$. *The following are equivalent:*
(a) $A^{\otimes} = X(A^k)^*$;
(b) $X(A^k)^*A = P_{\mathcal{R}(A^*A^k)}$;
(c) $X = ((A^k)^*A)^\dagger + Y - YP_{\mathcal{R}((A^k)^*A)}$, *where* $Y \in \mathbb{C}^{n\times n}$ *is arbitrary;*
(d) X *is given by*

$$X = U\begin{bmatrix} Y_1 + (T^*\triangle^{-1} - Y_1(T^k)^* - Y_2(\widetilde{T})^*)L^{-1}T^k & Y_2 + (T^*\triangle^{-1} - Y_1(T^k)^* - Y_2(\widetilde{T})^*)L^{-1}\widetilde{T} \\ Y_3 + (S^*\triangle^{-1} - Y_3(T^k)^* - Y_4(\widetilde{T})^*)L^{-1}T^k & Y_4 + (S^*\triangle^{-1} - Y_3(T^k)^* - Y_4(\widetilde{T})^*)L^{-1}\widetilde{T} \end{bmatrix}U^*,$$

where $\triangle = TT^*+SS^*$, $L = T^k(T^k)^*+\widetilde{T}(\widetilde{T})^*$, $\widetilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$ *and* $Y_1 \in \mathbb{C}^{t\times t}$, $Y_2 \in \mathbb{C}^{t\times(n-t)}$, $Y_3 \in \mathbb{C}^{(n-t)\times t}$, $Y_4 \in \mathbb{C}^{(n-t)\times(n-t)}$ *are arbitrary.*

Proof. (a) $\Rightarrow$ (b). It follows from (2.1) that $X(A^k)^*A = P_{\mathcal{R}(A^*A^k)}$.

(b) $\Rightarrow$ (c). Obviously, $((A^k)^*A)^\dagger$ satisfies the equation $X(A^k)^*A = P_{\mathcal{R}(A^*A^k)}$. Applying [1, Ch.2 Theorem 1] to the above equation, we get the general solution $X = ((A^k)^*A)^\dagger + Y - YP_{\mathcal{R}((A^k)^*A)}$, where $Y \in \mathbb{C}^{n\times n}$ is arbitrary.

(c) $\Rightarrow$ (d). Using (2.2), we have

$$(A^k)^*A = U\begin{bmatrix} (T^k)^*T & (T^k)^*S \\ (\widetilde{T})^*T & (\widetilde{T})^*S \end{bmatrix}U^*. \quad (2.12)$$

Applying [5, Ch.3 Corollary 2.3] to (2.12), we get

$$((A^k)^*A)^\dagger = U\begin{bmatrix} T^*\triangle^{-1}L^{-1}T^k & T^*\triangle^{-1}L^{-1}\widetilde{T} \\ S^*\triangle^{-1}L^{-1}T^k & S^*\triangle^{-1}L^{-1}\widetilde{T} \end{bmatrix}U^*, \quad (2.13)$$

where $\triangle = TT^* + SS^*$, $L = T^k(T^k)^* + \widetilde{T}(\widetilde{T})^*$, $\widetilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$. Next,

$$Y = U \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} U^*,$$

where $Y_1 \in \mathbb{C}^{t\times t}$, $Y_2 \in \mathbb{C}^{t\times(n-t)}$, $Y_3 \in \mathbb{C}^{(n-t)\times t}$ and $Y_4 \in \mathbb{C}^{(n-t)\times(n-t)}$ are arbitrary. By direct calculation, we get that

$$X = U \begin{bmatrix} Y_1 + (T^*\triangle^{-1} - Y_1(T^k)^* - Y_2(\widetilde{T})^*)L^{-1}T^k & Y_2 + (T^*\triangle^{-1} - Y_1(T^k)^* - Y_2(\widetilde{T})^*)L^{-1}\widetilde{T} \\ Y_3 + (S^*\triangle^{-1} - Y_3(T^k)^* - Y_4(\widetilde{T})^*)L^{-1}T^k & Y_4 + (S^*\triangle^{-1} - Y_3(T^k)^* - Y_4(\widetilde{T})^*)L^{-1}\widetilde{T} \end{bmatrix} U^*,$$

where $\triangle = TT^* + SS^*$, $L = T^k(T^k)^* + \widetilde{T}(\widetilde{T})^*$, $\widetilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$ and Y_1, Y_2, Y_3, Y_4 are arbitrary.

$(d) \Rightarrow (a)$. By computation, we get $X(A^k)^* = A^{\otimes}$ as in (2.3). □

Remark 2.19. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.2)*. Using* (2.13)*, further calculations confirm $((A^k)^*A)^\dagger(A^k)^* = A^{\otimes}$ as in* (2.3)*, which gives another proof of Theorem* 2.2*.*

Using the continuity of the Moore-Penrose inverse given in [21], we develop a necessary and sufficient condition for the continuity of the gMP inverse.

Lemma 2.20. [21] *Let $A \in \mathbb{C}^{m\times n}$ and $A_p \in \mathbb{C}^{m\times n}$, $p \in \mathbb{N}$. If $A_p \to A$ as $p \to \infty$, then $A_p^\dagger \to A^\dagger$ as $p \to \infty$ iff there is $p_0 \in \mathbb{N}$ such that $r(A_p) = r(A)$ for $p \geq p_0$.*

Theorem 2.21. *Let $A \in \mathbb{C}^{n\times n}$ and $A_p \in \mathbb{C}^{n\times n}$, $p \in \mathbb{N}$ be a sequence satisfying $A_p \to A$ as $p \to \infty$. Then $A_p^{\otimes} \to A^{\otimes}$ as $p \to \infty$ iff there is $p_0 \in \mathbb{N}$ such that $r(A_p^l) = r(A^l)$ for $p \geq p_0$ and $l = \max\{\mathrm{ind}(A), \mathrm{ind}(A_p), \mathrm{ind}(A_{p+1}), \ldots\}$.*

Proof. Suppose that there is $p_0 \in \mathbb{N}$ such that $r(A_p^l) = r(A^l)$, for $p \geq p_0$ and $l = \max\{\mathrm{ind}(A), \mathrm{ind}(A_p), \mathrm{ind}(A_{p+1}), \ldots\}$. Then

$$r((A_p^l)^*A_p) = r((A_p^l)^*) = r(A_p^l) = r(A^l) = r((A^l)^*A),$$

for $p \geq p_0$. By Lemma 2.20 and $(A_p^l)^*A_p \to (A^l)^*A$, it follows that $((A_p^l)^*A_p)^\dagger \to ((A^l)^*A)^\dagger$. Therefore, using $(A_p^l)^* \to (A^l)^*$ and Theorem 2.2,

$$A_p^{\otimes} = ((A_p^l)^*A_p)^\dagger(A_p^l)^* \to ((A^l)^*A)^\dagger(A^l)^* = A^{\otimes}.$$

Conversely, from $A_p^{\otimes} \to A^{\otimes}$ and $A_p \to A$, we have $A_p^{\otimes}A_p \to A^{\otimes}A$. Applying [23, Lemma 9.2.2] for projectors $A_p^{\otimes}A_p$ and $A^{\otimes}A$, there exists $p_0 \in \mathbb{N}$ such that $r(A_p^{\otimes}A_p) = r(A^{\otimes}A)$, for $p \geq p_0$. Let $l = \max\{\mathrm{ind}(A), \mathrm{ind}(A_p), \mathrm{ind}(A_{p+1}), \ldots\}$. Applying Lemma 2.1, we have $\mathcal{R}(A^{\otimes}A) = \mathcal{R}(A^*A^{\mathrm{ind}(A)}) = \mathcal{R}(A^*A^l)$ and $\mathcal{R}(A_p^{\otimes}A_p) = \mathcal{R}(A_p^*A_p^{\mathrm{ind}(A_p)}) = \mathcal{R}(A_p^*A_p^l)$, which yield

$$r(A^l) = r(A^*A^l) = r(A^{\otimes}A) = r(A_p^{\otimes}A_p) = r(A_p^*A_p^l) = r(A_p^l),$$

for $p \geq p_0$. □

3. Splitting methods for computing the gMP inverse

Many characterizations of several generalized inverses were investigated in terms of splitting methods [11, 12, 15, 27]. Corresponding splitting methods for finding the gMP inverse are verified in this section.

Theorem 3.1. *Let $A \in \mathbb{C}_k^{n\times n}$. Suppose that $(A^k)^*AA^*A^k = H - K$, $\mathcal{N}(A^k) = \mathcal{N}(H)$ and $\mathcal{R}((A^k)^*) = \mathcal{R}(H)$. Then*
(*a*) *$H^\#$ exists;*
(*b*) *$I_n - H^\# K$ is invertible;*
(*c*) *$A^{\otimes} = A^*A^k(I_n - H^\# K)^{-1}H^\#(A^k)^*$.*

Proof. (*a*). Notice that $\mathrm{ind}(H) = 1$ by $\mathbb{C}^n = \mathcal{R}(H) \oplus \mathcal{N}(H) = \mathcal{R}((A^k)^*) \oplus \mathcal{N}(A^k)$.
(*b*). In order to check that $I_n - H^\# K$ is nonsingular, let $(I_n - H^\# K)x = 0$ for some $x \in \mathbb{C}^n$. Then

$$x = H^\# Kx \in \mathcal{R}(H^\#) = \mathcal{R}(H) = \mathcal{R}((A^k)^*) = \mathcal{R}((A^k)^*A)$$

and

$$x = H^\# Kx \;=\; H^\#(H - (A^k)^*AA^*A^k)x = H^\# Hx - H^\#(A^k)^*AA^*A^k x = x - H^\#(A^k)^*AA^*A^k x.$$

Hence, we get $H^\#(A^k)^*AA^*A^k x = 0$, which gives

$$(A^k)^*AA^*A^k x \in \mathcal{N}(H^\#) = \mathcal{N}(H) = \mathcal{N}(A^*A^k) = \mathcal{N}(((A^k)^*A)^*) = \mathcal{N}(((A^k)^*A)^\dagger),$$

i.e., $((A^k)^*A)^\dagger(A^k)^*AA^*A^k x = A^*A^k x = 0$. Thus,

$$x \in \mathcal{R}((A^k)^*A) \cap \mathcal{N}(A^*A^k) = \mathcal{R}((A^k)^*A) \cap \mathcal{R}((A^k)^*A)^\perp = \{0\}.$$

Therefore, $I_n - H^\# K$ is invertible.
(*c*). Since

$$\mathcal{R}(H) = \mathcal{R}((A^k)^*A) = \mathcal{R}(((A^k)^*AA^*A^k)^*) = \mathcal{R}(((A^k)^*AA^*A^k)^\dagger),$$

we get

$$H^\# H((A^k)^*AA^*A^k)^\dagger = P_{\mathcal{R}(H),\mathcal{N}(H)}((A^k)^*AA^*A^k)^\dagger = ((A^k)^*AA^*A^k)^\dagger. \tag{3.1}$$

Since

$$\mathcal{N}((A^k)^*A((A^k)^*A)^\dagger) = \mathcal{N}(((A^k)^*A)^*) = \mathcal{N}(A^*A^k) = \mathcal{N}(H) = \mathcal{N}(H^\#),$$

we have

$$H^\#(A^k)^*A((A^k)^*A)^\dagger = H^\#. \tag{3.2}$$

The equalities $K = H - (A^k)^*AA^*A^k$, (3.1) and (3.2) give

$$\begin{aligned}
&(I_n - H^\# K)((A^k)^*AA^*A^k)^\dagger \\
=\;& ((A^k)^*AA^*A^k)^\dagger - H^\# H((A^k)^*AA^*A^k)^\dagger + H^\#(A^k)^*AA^*A^k((A^k)^*AA^*A^k)^\dagger \\
=\;& H^\#(A^k)^*AA^*A^k((A^k)^*AA^*A^k)^\dagger \\
=\;& H^\#(A^k)^*A((A^k)^*A)^\dagger = H^\#.
\end{aligned}$$

Hence, $(I_n - H^\# K)^{-1} H^\# = ((A^k)^* A A^* A^k)^\dagger$. Therefore, by Theorem 2.2, we have

$$A^* A^k (I_n - H^\# K)^{-1} H^\# (A^k)^* = A^* A^k ((A^k)^* A A^* A^k)^\dagger (A^k)^* = A^\otimes.$$

□

Example 3.2. *Let A and $A^\otimes$ as in Example* 2.17*. To verify Theorem* 3.1*, let*

$$H = \begin{bmatrix} 4 & 0 & 2 & 5 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & \frac{5}{2} \\ 5 & 0 & \frac{5}{2} & \frac{25}{4} \end{bmatrix}.$$

Further calculation gives that the matrices

$$H^\# = H(H^3)^\dagger H = \begin{bmatrix} \frac{64}{2025} & 0 & \frac{32}{2025} & \frac{16}{405} \\ 0 & 1 & 0 & 0 \\ \frac{32}{2025} & 0 & \frac{16}{2025} & \frac{8}{405} \\ \frac{16}{405} & 0 & \frac{8}{405} & \frac{4}{81} \end{bmatrix}, \; A^* A^2 = \begin{bmatrix} 8 & 0 & 4 & 10 \\ 0 & 4 & 0 & 0 \\ 4 & 0 & 2 & 5 \\ 4 & 0 & 2 & 5 \end{bmatrix},$$

$$K = \begin{bmatrix} -92 & 0 & -46 & -115 \\ 0 & -63 & 0 & 0 \\ -46 & 0 & -23 & -\frac{115}{2} \\ -115 & 0 & -\frac{115}{2} & -\frac{575}{4} \end{bmatrix}, \; (I_4 - H^\# K)^{-1} = \begin{bmatrix} \frac{89}{135} & 0 & -\frac{23}{135} & -\frac{23}{54} \\ 0 & \frac{1}{64} & 0 & 0 \\ -\frac{23}{135} & 0 & \frac{247}{270} & -\frac{23}{108} \\ -\frac{23}{54} & 0 & -\frac{23}{108} & \frac{101}{216} \end{bmatrix}.$$

Final verification of Theorem 3.1 *confirms that the expression $A^* A^2 (I_4 - H^\# K)^{-1} H^\# (A^2)^*$ coincides with $A^\otimes$ given in* (2.11)*.*

By Lemma 2.4, we know that $A^\otimes$ is an outer inverse of A with rang $\mathcal{R}(A^* A^k (A^k)^*)$ and null space $\mathcal{N}(A^* A^k (A^k)^*)$. Let $G = A^* A^k (A^k)^*$. The following result follows from Corollary 2.4 given in [11].

Corollary 3.3. *Let $A \in \mathbb{C}_k^{n\times n}$. Suppose that $A^* A^k (A^k)^* A = H - K$, $\mathcal{N}((A^k)^* A) = \mathcal{N}(H)$ and $\mathcal{R}(A^* A^k) = \mathcal{R}(H)$. Then*
(a) $H^\#$ exists;
(b) $I_n - H^\# K$ is invertible;
(c) $A^\otimes = (I_n - H^\# K)^{-1} H^\# A^* A^k (A^k)^*$.

Example 3.4. *Consider A and $A^\otimes$ as in Example* 2.17*. Let*

$$H = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

We calculate the matrices

$$K = \begin{bmatrix} -178 & 0 & -89 & -89 \\ 0 & -63 & 0 & 0 \\ -89 & 0 & -\frac{89}{2} & -\frac{89}{2} \\ -89 & 0 & -\frac{89}{2} & -\frac{89}{2} \end{bmatrix}, \; (I_4 - H^\# K)^{-1} = \begin{bmatrix} \frac{46}{135} & 0 & -\frac{89}{270} & -\frac{89}{270} \\ 0 & \frac{1}{64} & 0 & 0 \\ -\frac{89}{270} & 0 & \frac{451}{540} & -\frac{89}{540} \\ -\frac{89}{270} & 0 & -\frac{89}{540} & \frac{451}{540} \end{bmatrix}.$$

Further verification gives $(I_4 - H^\# K)^{-1} H^\# A^* A^2 (A^2)^* = A^\otimes$.

4. SMS algorithm for finding the gMP inverse

In this section, we modify successive matrix squaring algorithm from [20] and define an efficient algorithm for computing the gMP inverse.

By Theorem 2.2 and $B^* = B^*BB^\dagger$, we have $A^*A^k(A^k)^*AA^\otimes = A^*A^k(A^k)^*A((A^k)^*A)^\dagger(A^k)^* = A^*A^k(A^k)^*$ which implies

$$A^\otimes = A^\otimes - \beta(A^*A^k(A^k)^*AA^\otimes - A^*A^k(A^k)^*) = (I_n - \beta A^*A^k(A^k)^*A)A^\otimes + \beta A^*A^k(A^k)^*.$$

Let

$$Q = \beta A^*A^k(A^k)^*, \quad P = I_n - \beta A^*A^k(A^k)^*A, \quad \beta > 0.$$

The iterative scheme for computing the gMP inverse $A^\otimes$ will be given by [20]

$$X_1 = Q = \beta A^*A^k(A^k)^*, \quad X_{m+1} = PX_m + Q, \quad m \in \mathbb{N}.$$

Considering

$$T = \begin{bmatrix} P & Q \\ O & I_n \end{bmatrix}, \quad T^m = \begin{bmatrix} P^m & \sum_{i=0}^{m-1} P^iQ \\ O & I_n \end{bmatrix},$$

we have X_m is top right block of T^m, i.e., $X_m = \sum_{i=0}^{m-1} P^iQ$. Notice that

$$T_m = T^{2^m} = \begin{bmatrix} P^{2^m} & \sum_{i=0}^{2^m-1} P^iQ \\ O & I_n \end{bmatrix}.$$

Applying [20], we obtain the next result. The norm used in Theorem 4.1 is an arbitrary matrix norm.

Theorem 4.1. *Let $A \in \mathbb{C}_k^{n\times n}$ and $\epsilon > 0$. The sequence of approximations*

$$X_{2^m} = \sum_{i=0}^{2^m-1}(I_n - \beta A^*A^k(A^k)^*A)^i\beta A^*A^k(A^k)^*,$$

determined by SMS algorithm

$$T_0 = T, \quad T_{i+1} = T_i^2, \quad i = 0, 1, ..., m-1,$$

converges in the matrix norm $\|\cdot\|$ to the gMP inverse $A^\otimes$, if β is a fixed real number such that

$$\max_{1\le i\le s} | 1 - \beta\lambda_i | < 1,$$

where $r(AA^*A^k(A^k)^*) = s$, $\lambda_i (i = 1, 2, ..., s)$ are the nonzero eigenvalues of $AA^*A^k(A^k)^*$ and

$$\rho(I_n - \beta AA^*A^k(A^k)^*) \le \|I_n - \beta AA^*A^k(A^k)^*\| \le \rho(I_n - \beta AA^*A^k(A^k)^*) + \epsilon,$$

where $\rho(I_n - \beta AA^*A^k(A^k)^*)$ is the spectral radius of $I_n - \beta AA^*A^k(A^k)^*$.

Example 4.2. *In this example, we reuse the following matrix from* [20]*:*

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

Simple verification gives $\mathrm{ind}(A) = 2$*. According to* (2.1)*, exact calculation in Mathematica gives*

$$A^{\otimes} = ((A^2)^* A)^{\dagger}(A^2)^* = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & 0 & 0 & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{4} & \frac{1}{4} & 0 & 0 & -\frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} \\ 0 & 0 & \frac{1}{12} & -\frac{1}{12} & \frac{1}{4} & -\frac{1}{12} \\ 0 & 0 & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{4} \end{bmatrix}.$$

The eigenvalues of $AA^*A^2(A^2)^*$ *are nonnegative and contained in the set* $\{12, 40.1, 64, 1163.9\}$*, which ensures the applicability of successive matrix squaring algorithm. Using the rule* (2.22) *from* [20]*, we get* $\beta = 1.7716 \times 10^{-5}$*. According to Theorem* 4.1 *, the matrices Q and P are given as follows:*

$$Q = \beta A^* A^2 (A^2)^*, \quad P = I_6 - \beta A^* A^2 (A^2)^* A, \quad \beta = 1.7716 \times 10^{-5}.$$

Notice that $A^{\otimes}$ *can be approximated by the upper right block of the* 16*th approximation* $(T^2)^{16}$ *of the SMS method, which is given by*

$$(T^2)^{16} = \left[\begin{array}{rrrrrrrrrrrr} 0.1667 & 0.1667 & -0.1667 & -0.1667 & 0.1667 & 0.1667 & 0.2500 & -0.2500 & -0.0000 & 0.0000 & -0.1667 & -0.1667 \\ 0.1667 & 0.1667 & -0.1667 & -0.1667 & 0.1667 & 0.1667 & -0.2500 & 0.2500 & -0.0000 & 0.0000 & -0.1667 & -0.1667 \\ -0.1667 & -0.1667 & 0.4167 & 0.4167 & 0.0833 & 0.0833 & 0.0000 & -0.0000 & 0.2500 & -0.2500 & -0.0833 & -0.0833 \\ -0.1667 & -0.1667 & 0.4167 & 0.4167 & 0.0833 & 0.0833 & 0.0000 & -0.0000 & -0.2500 & 0.2500 & -0.0833 & -0.0833 \\ 0.1667 & 0.1667 & 0.0833 & 0.0833 & 0.4167 & 0.4167 & -0.0000 & 0.0000 & 0.0833 & -0.0833 & 0.2500 & -0.0833 \\ 0.1667 & 0.1667 & 0.0833 & 0.0833 & 0.4167 & 0.4167 & -0.0000 & 0.0000 & -0.0833 & 0.0833 & -0.0833 & 0.2500 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \end{array}\right],$$

which gives

$$A^{\otimes} \approx X = \begin{bmatrix} 0.2500 & -0.2500 & -0.0000 & 0.0000 & -0.1667 & -0.1667 \\ -0.2500 & 0.2500 & -0.0000 & 0.0000 & -0.1667 & -0.1667 \\ 0.0000 & -0.0000 & 0.2500 & -0.2500 & -0.0833 & -0.0833 \\ 0.0000 & -0.0000 & -0.2500 & 0.2500 & -0.0833 & -0.0833 \\ -0.0000 & 0.0000 & 0.0833 & -0.0833 & 0.2500 & -0.0833 \\ -0.0000 & 0.0000 & -0.0833 & 0.0833 & -0.0833 & 0.2500 \end{bmatrix}.$$

Also, the SMS method yields $r(A^{\otimes}) = 4$ *and* $\|A^{\otimes} - X\| = 1.5127 \times 10^{-12}$ *is the norm of the error matrix.*

5. Conclusions

Our main goal is to present representations and computations of the gMP inverse. Limit and integral representations, as well as representations using the Moore-Penrose inverse, Bott-Duffin inverse and projectors are given for the gMP inverse. Continuity of the gMP inverse is studied. Splitting methods and the SMS algorithm for calculating the gMP inverse are obtained. Some numerical examples are provided to illustrate the results obtained.

We believe that investigation related to the gMP inverse will attract attention, and we describe perspectives for further research:

(1). Considering the gMP inverse of tensors.

(2). Extending the gMP inverse to rectangular matrices and rings.

(3). Studying relation of the gMP inverse and some partial order.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research is supported by the Natural Science Foundation of China under Grants 11961076, the Science and Technology Research Program for the Education Department of Hubei province of China under Grant No.D20163101.

Conflict of interest

No potential conflict of interest was reported by the authors.

References

1. A. Ben-Israel, T. Greville, *Generalized Inverses: Theory and Applications*, New York: Springer, 2003.
2. R. Bott, R. J. Duffin, On the algebra of networks, *Trans. Amer. Math. Soc.*, **74** (1953), 99–109.

3. Y. Chen , K. Z. Zuo, Z. M. Fu, New characterizations of the generalized Moore-Penrose inverse of matrices, *AIMS Math.*, **7** (2022), 4359–4375. http://dx.doi.org/10.3934/math.2022242
4. Y. L. Chen, Expressions and determinantal formulas for the generalized inverse $A^{(2)}_{\mathcal{T},\mathcal{S}}$ and their applications, *J. Natural Sci. Nanjing Normal University*, **16** (1993), 3–16.
5. Y. L. Chen, *The Theory and Method of Generakized Inverse Matrix*, in Chinese, Nangjing: Nanjing Normal University press, 2005.
6. D. S. Cvetković-Ilić, Y. M. Wei, *Algebraic Properties of Generalized Inverses*, Singapore: Springer, 2017.
7. M. P. Drazin, Pseudo-inverses in associative rings and semigroups, *Amer. Math. Mon.*, **65** (1958), 506–514.
8. D. Ferreyra, F. Levis, N. Thome, Revisiting the core-EP inverse and its extension to rectangular matrices, *Quaest. Math.*, **41** (2018), 265–281. https://doi.org/10.2989/16073606.2017.1377779
9. Y. F. Gao, J. L. Chen, Pseudo core inverses in rings with involution, *Commun. Algebra*, **46** (2018), 38–50. https://doi.org/10.1080/00927872.2016.1260729
10. C. W. Groetsch, Generalized inverses of linear operators: representation and approximation, *Monographs Textbooks Pure Appl. Math.*, **37** (1977).
11. X. J. Liu, S. W. Huang, Proper splitting for the generalized inverse $A^{(2)}_{\mathcal{T},\mathcal{S}}$ and its application on Banach spaces, *Abstr. Appl. Anal.*, **2012** (2012), 1–9.
12. H. F. Ma, P. S. Stanimirović, Characterizations, approximation and perturbations of the core-EP inverse, *Appl. Math. Comput.*, **359** (2019), 404–417. https://doi.org/10.1016/j.amc.2019.04.071
13. X. Mary, On generalized inverse and Green relations, *Linear Algebra Appl.*, **434** (2011), 1836–1844. https://doi.org/10.1016/j.laa.2010.11.045
14. D. Mosić, Core-EP inverse in rings with involution, *Publ. Math. Debrecen.*, **96** (2020), 427–443.
15. D. Mosić, P. S. Stanimirović, Expressions and properties of weak core inverse, *Appl. Math. Comput.*, **415** (2022). https://doi.org/10.1016/j.amc.2021.126704
16. R. Penrose, A generalized inverse for matrices, *Math. Proc. Camb. Philos. Soc.*, **51** (1955), 406–413.
17. K. M. Prasad, K. S. Mohana, Core-EP inverse, *Linear Multilinear Algebra*, **62** (2014), 792–802. https://doi.org/10.1080/03081087.2013.791690
18. J. K. Sahoo, R. Behera, P. S. Stanimirović, V. N. Katsikis, H. F. Ma, Core and core-EP inverses of tensors, *Comput. Appl. Math.*, **39** (2020). https://doi.org/10.1007/s40314-019-0983-5
19. P. S. Stanimirović, Limit representations of generalized inverses and related methods, *Appl. Math. Comput.*, **103** (1999), 51–68. https://doi.org/10.1016/S0096-3003(98)10048-6
20. P. S. Stanimirović, D. S. Cvetković-Ilíc, Successive matrix squaring algorithm for computing outer inverses, *Appl. Math. Comput.*, **203** (2008), 19–29. https://doi.org/10.1016/j.amc.2008.04.037
21. G. W. Stewart, On the continuity of the generalized inverse, *SIAM J. Appl. Math.*, **17** (1969), 33–45. https://doi.org/10.1137/0117004
22. K. S. Stojanović, D. Mosić, Generalization of the Moore-Penrose inverse, *RACSAM*, **114** (2020). https://doi.org/10.1007/s13398-020-00928-x
23. G. R. Wang, Y. M. Wei, S. Z. Qiao, *Generalized Inverses: Theory and Computations*, Beijing: Springer, 2018.

24. H. X. Wang, Core-EP decomposition and its applications, *Linear Algebra Appl.*, **508** (2016), 289–300. https://doi.org/10.1016/j.laa.2016.08.008
25. Y. M. Wei, A characterization and representation of the generalized inverse $A^{(2)}_{\mathcal{T},\mathcal{S}}$ and its applications, *Linear Algebra Appl.*, **280** (1998), 87–96. https://doi.org/10.1016/S0024-3795(98)00008-1
26. Y. M. Wei, D. S. Djordjević, On integral representation of the generalized inverse $A^{(2)}_{\mathcal{T},\mathcal{S}}$, *Appl. Math. Comput.*, **142** (2003), 189–194. https://doi.org/10.1016/S0096-3003(02)00296-5
27. Y. M. Wei, H. B. Wu, $(\mathcal{T},\mathcal{S})$ splitting methods for computing the generalized inverse $A^{(2)}_{\mathcal{T},\mathcal{S}}$ and rectangular systems, *Int. J. Comput. Math.*, **77** (2001), 401–424. https://doi.org/10.1080/00207160108805075
28. H. Yan, H. X. Wang, K. Z. Zuo, Y. Chen, Further characterizations of the weak group inverse of matrices and the weak group matrix, *AIMS Math.*, **6** (2021), 9322–9341. http://dx.doi.org/10.3934/math.2021542
29. Y. X. Yuan, K. Z. Zuo, Compute $\lim_{\lambda\to 0} X(\lambda I_p + YAX)^{-1}Y$ by the product singular value decomposition, *Linear Multilinear Algebra*, **64** (2016), 269–278. https://doi.org/10.1080/03081087.2015.1034641
30. M. M. Zhou, J. L. Chen, T. T. Li, D. G. Wang, Three limit representations of the core-EP inverse, *Filomat*, **32** (2018), 5887–5894.
31. K. Z. Zuo, Y. J. Cheng, The new revisitation of core EP inverse of matrices, *Filomat*, **33** (2019), 3061–3072. https://doi.org/10.2298/FIL1910061Z

AIMS Press