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Research article

Some stochastic orderings of multivariate skew-normal random vectors

Xueyan Li and Chuancun Yin*

School of Statistics and Data Science, Qufu Normal University, Qufu 273165, China

* **Correspondence:** Email: ccyin@qfnu.edu.cn; Tel: +865374453221.

Abstract: In this paper, we investigate some multivariate integral stochastic orderings of skew-normal random vectors. We derive the results of the sufficient and/or necessary conditions by applying an identity for $Ef(\mathbf{Y}) - Ef(\mathbf{X})$, where $\mathbf{X}$ and $\mathbf{Y}$ are multivariate skew-normal random vectors, f satisfies some weak regularity condition. The integral orders considered here are the componentwise convex, copositive, completely-positive orderings and their corresponding increasing ones as well as linear forms of stochastic orderings, which play a vital role in transforming the unmanageable multivariate components into an easy-to-handle univariate variable.

Keywords: componentwise convex ordering; completely-positive ordering; copositive ordering; integral stochastic ordering; linear ordering; multivariate skew-normal
Mathematics Subject Classification: 60E10, 60E15

1. Introduction

In recent years, more and more scholars have paid extensive attention to the research of the theory of stochastic orderings and have achieved a lot of research results. Stochastic orderings are the class of partial order relationships, which are defined on a family of random variables. It is a powerful tool to describe the size relationship between random variables and compare the degree of correlation of random variables. Nowadays, stochastic order relationships are widely used in many fields of probability and statistics, such as the statistical theory of economics and actuarial data, the comparison of stochastic processes in physics and other disciplines, etc. The comparison of two or more ordered experimental groups based on multivariate data is commonly used in research and in various applications in the medical field. For more detailed theoretical results, interested readers can refer to [1–3].

The problem of the stochastic order of the multivariate normal distribution has been fully researched and described in detail by Müller [4] and Arlotto and Scarsini [5] and so on. The stochastic orderings of the multivariate elliptical distribution were later introduced by [6–8] and later the results of the multivariate normal distribution were extended to the general multivariate elliptical distribution with the special cases of multivariate normal, multivariate logistic, multivariate-t and multivariate Laplace distributions by Yin [9]. Although these two distributions have good properties and characteristics, they are idealized symmetrical distributions. However, in practice, data often have skewness and heavy tails. In particular, it was fitted by using these two distributions but cannot achieve a more perfect effect. Therefore, researchers have promoted the elliptical distribution, either by mixing the two methods or by adding skewness to the distribution then obtained a series of asymmetric distributions, which can better fit the actual data.

Azzalini [10] proposed the skew-normal distribution and then Azzalini and Valle [11] extended it to the multivariate skew-normal families. There are some properties and applications of the multivariate skew-normal distribution were discussed in [12]. This distribution represents a mathematically tractable extension of the density of multivariate normal distribution, with adding parameters to adjust for skewness. The multivariate skew-normal distribution offers reasonable flexibility in fitting real data while maintaining some convenient formal properties of the normal density. A complete exposition of the theory of the skew-normal distribution can be found in [13].

Among the numerous stochastic orderings, Hessian, increasing Hessian orderings and linear orderings have been discussed quite extensively in the literature in recent years. For example, the results on skewness orderings on the multivariate skew-normal can be found in Arevalillo and Navarro [14]. Pu et al. [15] studied a class of multivariate generalized location-scale mixtures of elliptical distributions with respect to stochastic orderings. Amiri et al. [16], Amiri and Balakrishnan [17] considered the Hessian, increasing Hessian orderings and linear orderings, which is a practical tool for reducing dimension in multivariate stochastic comparisons. Amiri and Balakrishnan [17] established some stochastic comparison results for multivariate scale-shape mixture of skew-normal distributions by restricting the conditions of parameters. Thus, it was shown that there exists some equivalent correlation between stochastic orderings. Pertinent results of the comparisons of partial integral stochastic orderings of skew-normal distribution can be found in [18]. Similarly, related results can be found in [19], in which the orders for matrix variate skew-normal distribution were studied. Current research on integral stochastic orderings of skew-normal distributions is limited and some well-known stochastic orderings such as componentwise convex, copositive, completely-positive orderings and a variety of increasing orderings such as increasing supermodular, increasing directionally convex, increasing copositive, increasing completely-positive and increasing componentwise convex orderings as well as linear forms of stochastic orderings have not been well studied.

Whether the existence of sufficient and/or necessary conditions for those stochastic orderings is still an open question. This paper intends to solve these issues. The results can be also seen as supplements to the results in [18].

This paper is organized as follows: Section 2 contains an introduction to the skew-normal distribution and a review of knowledge about integral stochastic orderings. Section 3 derives necessary and/or sufficient conditions for stochastic orderings of the skew-normal distribution. In Section 4, necessary and sufficient conditions for the linear orderings of the skew-normal distribution are given. Section 5 gives some brief summaries.

2. Preliminaries

We firstly review the concept and property of the multivariate skew-normal distribution which is introduced by [12]. A n-dimensional random vector $\mathbf{Z}$ has the multivariate skew-normal distribution, denoted as $\mathbf{Z} \sim SN_n(\boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\alpha})$, if its probability density function has the following form:

$$f_{\mathbf{Z}}(\mathbf{z}) = 2\phi_n(\mathbf{z}; \boldsymbol{\xi}, \boldsymbol{\Omega})\Phi(\boldsymbol{\alpha}^\top \omega^{-1}(\mathbf{z} - \boldsymbol{\xi})),$$

where $\boldsymbol{\xi}$ is the mean vector, $\boldsymbol{\alpha}$ is the skewness parameter, $\boldsymbol{\xi}, \boldsymbol{\alpha} \in \mathbb{R}^n$, $\boldsymbol{\Omega} = [\omega_{ij}]$ is a $n \times n$ covariance matrix which has full rank, denote $\overline{\boldsymbol{\Omega}} = \omega^{-1}\boldsymbol{\Omega}\omega^{-1}$ is the corresponding correlation matrix, where $\omega = \mathrm{diag}(\omega_{11}, ..., \omega_{nn})^{\frac{1}{2}}$. The following notations will be used throughout the paper: The cumulative distribution function (CDF) of the univariate standard normal distribution is denoted by $\Phi(\cdot)$ and the probability density function (PDF) of the n-dimensional normal distribution is represented by $\phi_n(.; \boldsymbol{\xi}, \boldsymbol{\Omega})$.

The characteristic function of $\mathbf{Z}$ is (refer to [20]):

$$\begin{aligned} \boldsymbol{\Psi}_{\mathbf{Z}}(\mathbf{t}) &= 2\exp\left(\mathrm{i}\boldsymbol{\xi}\mathbf{t} - \frac{1}{2}\mathbf{t}^\top\boldsymbol{\Omega}\mathbf{t}\right)\Phi(\mathrm{i}\boldsymbol{\delta}^\top\mathbf{t}) \\ &= \exp\left(\mathrm{i}\boldsymbol{\xi}\mathbf{t} - \frac{1}{2}\mathbf{t}^\top\boldsymbol{\Omega}\mathbf{t}\right)\{1 + \mathrm{i}\tau(\boldsymbol{\delta}^\top\mathbf{t})\}, \mathbf{t} \in \mathbb{R}^{\mathrm{n}}, \end{aligned} \tag{2.1}$$

where

$$\boldsymbol{\delta} = (1 + \boldsymbol{\alpha}^\top\overline{\boldsymbol{\Omega}}\boldsymbol{\alpha})^{-\frac{1}{2}}\omega\overline{\boldsymbol{\Omega}}\boldsymbol{\alpha}, \tau(u) = \sqrt{\frac{2}{\pi}}\int_0^u \exp\left(\frac{\mathrm{z}^2}{2}\right)\mathrm{dz}.$$

Then, $\mathbf{Z}$ has the multivariate skew-normal distribution, also denoted as $\mathbf{Z} \sim SN_n(\boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\delta})$. Specially, the standard skew-normal distribution $\mathbf{Z}_0 \sim SN_n(0, \overline{\boldsymbol{\Omega}}, \boldsymbol{\delta})$, which can be abbreviated as $\mathbf{Z}_0 \sim SN_n(\overline{\boldsymbol{\Omega}}, \boldsymbol{\delta})$. Its mean vector and covariance matrix can be expressed as

$$E(\mathbf{Z}_0) = \sqrt{\frac{2}{\pi}}\boldsymbol{\delta}, \quad Cov(\mathbf{Z}_0) = \overline{\boldsymbol{\Omega}} - \frac{2}{\pi}\boldsymbol{\delta}\boldsymbol{\delta}^\top. \tag{2.2}$$

Consider the univariate skew-normal distribution $Z_1 \sim SN_1(\xi_1, \sigma_1^2, \delta_1^2)$. Then, Z_1 has a stochastic representation of the form (see [13])

$$Z_1 \stackrel{d}{=} \xi_1 + \sigma_1 X_1 \mid \{X_2 < \alpha X_1\},$$

where the random variables X_1 and X_2 are independent standard normal random variables and

$$\alpha = \frac{\delta_1}{\sigma_1\sqrt{1 - \left(\frac{\delta_1}{\sigma_1}\right)^2}}.$$

Lemma 2.1. *([18]) Suppose that* $\mathbf{X}$ *and* $\mathbf{Y}$ *are n-dimensional skew-normal random vectors*

$$\mathbf{X} \sim SN_n(\boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\delta}), \quad \mathbf{Y} \sim SN_n(\boldsymbol{\xi}^*, \boldsymbol{\Omega}^*, \boldsymbol{\delta}^*). \tag{2.3}$$

Define the density function $\varphi_\lambda(\cdot)$ *be distributed as*

$$SN_n(\lambda\boldsymbol{\xi}^* + (1-\lambda)\boldsymbol{\xi},\ \lambda\boldsymbol{\Omega}^* + (1-\lambda)\boldsymbol{\Omega},\ \lambda\boldsymbol{\delta}^* + (1-\lambda)\boldsymbol{\delta}, 0 \le \lambda \le 1,$$

where

$$\boldsymbol{\xi}_\lambda = \lambda\boldsymbol{\xi}^* + (1-\lambda)\boldsymbol{\xi},\ \boldsymbol{\Omega}_\lambda = \lambda\boldsymbol{\Omega}^* + (1-\lambda)\boldsymbol{\Omega},\ \boldsymbol{\delta}_\lambda = \lambda\boldsymbol{\delta}^* + (1-\lambda)\boldsymbol{\delta}.$$

Moreover, define $\varphi_\lambda^k(.)$, $k = 1, 2$ *by*

$$\varphi_\lambda^1(\mathbf{u}) = \varphi_\lambda(\mathbf{u}),\ \ \varphi_\lambda^2(\mathbf{u}) = \phi_n(\mathbf{u}; \boldsymbol{\xi}_\lambda, \boldsymbol{\Omega}_\lambda - \boldsymbol{\delta}_\lambda\boldsymbol{\delta}_\lambda^\top).$$

Assume that f*:* $\mathbb{R}^n \to \mathbb{R}$ *is a twice continuously differentiable function satisfying*

(1) $\lim_{x_j\to\pm\infty} f(\mathbf{x})\varphi_\lambda^k(\mathbf{x}) = 0$,

(2) $\lim_{x_j\to\pm\infty} f(\mathbf{x})\frac{\partial}{\partial x_i}\varphi_\lambda^k(\mathbf{x}) = 0$,

(3) $\lim_{x_j\to\pm\infty} \varphi_\lambda^k(\mathbf{x})\frac{\partial}{\partial x_i}f(\mathbf{x}) = 0$,

where $0 \le \lambda \le 1$, $\mathbf{x} \in \mathbb{R}^n$, $i, j = 1, \cdots, n$, $k = 1, 2$. *Then,*

$$E(f(\mathbf{Y}) - f(\mathbf{X})) = \int_0^1 \int_{\mathbb{R}^n} \left\{ \left((\boldsymbol{\xi}^* - \boldsymbol{\xi})^\top \nabla_f(\mathbf{x}) + \frac{1}{2}\mathrm{tr}\left((\boldsymbol{\Omega}^* - \boldsymbol{\Omega})\mathbf{H}_\mathrm{f}(\mathbf{x})\right) \right) \varphi_\lambda^1(\mathbf{x}) + \frac{2}{\sqrt{2\pi}}(\boldsymbol{\delta}^* - \boldsymbol{\delta})^\top \nabla_f(\mathbf{x})\varphi_\lambda^2(\mathbf{x}) \right\} d\mathbf{x}d\lambda. \tag{2.4}$$

Corollary 2.1. *Suppose that* f*,* $\mathbf{X}$ *and* $\mathbf{Y}$ *satisfy the conditions of Lemma 2.1, such that for* $\mathbf{x} \in \mathbb{R}^n$*,*

(1) $\Sigma_{i=1}^n(\xi_i^* - \xi_i)\frac{\partial}{\partial x_i}f(\mathbf{x}) \ge 0$.

(2) $\Sigma_{i=1}^n(\delta_i^* - \delta_i)\frac{\partial}{\partial x_i}f(\mathbf{x}) \ge 0$.

(3) $\Sigma_{i,j=1}^n(\omega_{ij}^* - \omega_{ij})\frac{\partial^2}{\partial x_i \partial x_j}f(\mathbf{x}) \ge 0$.

Then, $E(f(\mathbf{X})) \le E(f(\mathbf{Y}))$.

In the sequel, we introduce the concept of stochastic orderings. Integral stochastic ordering is the class of stochastic orderings that can be characterized by comparing the expectations of the random vectors $\mathbf{X}$ and $\mathbf{Y}$. Let $\mathcal{F}$ is the class of measurable function f: $\mathbb{R}^n \to \mathbb{R}$. If f satisfies $Ef(\mathbf{Y}) \ge Ef(\mathbf{X})$ for two random vectors $\mathbf{X}$ and $\mathbf{Y}$ whose expectations are assumed to exist and for $\forall f \in \mathcal{F}$ where $\mathcal{F}$ is a measurable mapping set, then it is called the integral stochastic ordering $\mathbf{X} \le_\mathcal{F} \mathbf{Y}$.

A function is supermodular if and only if its Hessian matrix has non-negative off-diagonal elements. f is increasing supermodular if and only if for all $\mathbf{x} \in \mathbb{R}^n$, there is $\nabla_f(\mathbf{x}) \ge 0$ and $\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \ge 0$, for $i \ne j$; f is increasing directionally convex if and only if for all $\mathbf{x} \in \mathbb{R}^n$, there is $\nabla_f(\mathbf{x}) \ge 0$ and $\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \ge 0$, for $1 \le i, j \le n$.

For the notions mentioned above, if f: $\mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable function then we write

$$\nabla_f(\mathbf{x}) = \left(\frac{\partial}{\partial x_1}f(\mathbf{x}), ..., \frac{\partial}{\partial x_n}f(\mathbf{x})\right)^\top, \quad \mathbf{H}_f(\mathbf{x}) = \left(\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}\right)_{n\times n}$$

as the gradient vector and the Hessian matrix of f, respectively. For the n-tuple vectors $\mathbf{a}$ and $\mathbf{b}$, we use the notations $\mathbf{a} \le \mathbf{b}$ when $a_i \le b_i$, and $\mathbf{a} \ge 0$ when $a_i \ge 0$, for $i = 1, 2, \cdots, n$.

Definition 2.1. If a $n \times n$ matrix A has quadratic form $\mathbf{x}^\top A\mathbf{x} \geq 0$ for all $\mathbf{x} \geq 0$ then A is said to be copositive. If there is a non-negative matrix $B_{m\times n}$ such that $A = B^\top B$ then A is said to be completely positive.

Let $\mathbb{S}$ be the space of $n \times n$-dimensional symmetric matrices and $\mathcal{H}$ be a closed convex cone in $\mathbb{S}$, which satisfies the inner product $\langle A, B\rangle = \mathrm{tr}(AB)$, for $A, B \in \mathbb{S}$. Then, we can define the function class as

$$\mathcal{F}_{\mathcal{H}} = \{f : \mathbb{R}^n \to \mathbb{R} : \mathbf{H}_f(\mathbf{x}) \in \mathcal{H}, \forall \mathbf{x} \in \mathbb{R}^n\}$$

and the class of increasing functions as

$$\mathbb{L} = \{f : \mathbb{R}^n \to \mathbb{R} : \nabla_f(\mathbf{x}) \geq 0, \forall \mathbf{x} \in \mathbb{R}^n\}.$$

Let $\mathbb{L}_{\mathcal{H}} = \mathcal{F}_{\mathcal{H}} \cap \mathbb{L}$.

Definition 2.2. If there is $\lambda x \in C$ for $x \in C, \forall \lambda \geq 0$ then a subset C of vector space V is called a cone. The cone C is convex if and only if $\alpha x + \beta y \in C$, for $\forall x, y \in C, \alpha, \beta \geq 0$. Besides, if C is closed under the inner product $\langle \cdot, \cdot \rangle$ then

$$C^* = \{y \in V : \langle x, y\rangle \geq 0, \forall x \in C\}$$

is called the dual of C. If $C = C^*$ is satisfied, then C is called self-dual.

We use C_{cop} to denote the cone of a copositive matrix and C_{cp} to denote the cone of a completely positive matrix. Let C^*_{cop} and C^*_{cp} are the duals of C_{cop} and C_{cp}, respectively. C_{psd} and C_+ are denoted as the cones of positive semi-definite matrix and non-negative matrix, respectively. C_{+off} and C_{+diag} are ordered as the cones of non-negative off-diagonal elements matrix and non-negative main diagonal elements matrix, respectively.

Lemma 2.2. *([5, 21, 22]) The cones C_{cop} and C_{cp} are closed and convex, and $C_{cp} = C^*_{cop}$, $C_{cop} = C^*_{cp}$. C_{psd} and C_+ are also closed and convex and self-dual $C_{psd} = C^*_{psd}$, $C_+ = C^*_+$. C_{+off} and C_{+diag} are closed and convex and their dual cones are*

$$C^*_{+off} = \{B \in \mathbb{S} : b_{ii} = 0, b_{ij} \geq 0, i \neq j \in \{1, ..., n\}\},$$

$$C^*_{+diag} = \{B \in \mathbb{S} : b_{ii} \geq 0, b_{ij} = 0, i \neq j \in \{1, ..., n\}\}.$$

Let f: $\mathbb{R}^n \to \mathbb{R}$ be a function with twice continuously derivative. Then, f is convex if and only if $\mathbf{H}_f \in C_{psd}$, f is directionally convex if and only if it satisfies $\mathbf{H}_f \in C_+$, f is supermodular if and only if it satisfies $\mathbf{H}_f \in C_{+off}$, f is componentwise convex if and only if it satisfies $\mathbf{H}_f \in C_{+diag}$.

We introduce some important stochastic orderings as following:

Definition 2.3. ([5]) If $f \in \mathcal{F} = \mathcal{F}_{\mathcal{H}}$ then this kind of stochastic orderings is called Hessian orderings. If $f \in \mathbb{L}_{\mathcal{H}}$ then this kind of stochastic orderings is called increasing Hessian orderings.

If for $\mathbf{a} \in \mathbb{R}^n$ and a scalar convex function ψ, the function f: $\mathbb{R}^n \to \mathbb{R}$ satisfies the condition $f(\mathbf{x}) = \psi(\mathbf{a}^\top\mathbf{X})$ then f is said to be linear-convex. If for $\mathbf{a} \in \mathbb{R}^n_+$ and a scalar convex function ψ, the function f: $\mathbb{R}^n \to \mathbb{R}$ satisfies the condition $f(\mathbf{x}) = \psi(\mathbf{a}^\top\mathbf{X})$ then f is said to be positive-linear-convex.

Definition 2.4. Suppose two random variables $\mathbf{X}$ and $\mathbf{Y}$.

(1) Usual random order: If $Ef(\mathbf{Y}) \geq Ef(\mathbf{X})$ holds true for any increasing function f then $\mathbf{X} \leq_{st} \mathbf{Y}$.

(2) Convex order: If $Ef(\mathbf{Y}) \geq Ef(\mathbf{X})$ holds true for any convex function f then $\mathbf{X} \leq_{cx} \mathbf{Y}$.

(3) Componentwise convex order: If $\mathcal{F}$ is a class of twice differentiable functions f: $\mathbb{R}^n \to \mathbb{R}$ satisfying $\frac{\partial^2}{\partial x_i^2} f(\mathbf{x}) \geq 0$, where $\mathbf{x} \in \mathbb{R}^n$ and $1 \leq i \leq n$ then $\mathbf{X} \leq_{ccx} \mathbf{Y}$.

(4) Completely positive order: If $\mathcal{F}$ is a class of functions f that satisfies the condition $\mathbf{H}_f(\mathbf{x}) \in C_{cp}$ then $\mathbf{X} \leq_{cp} \mathbf{Y}$.

(5) Copositive order: If $\mathcal{F}$ is a class of functions f satisfying the condition $\mathbf{H}_f(\mathbf{x}) \in C_{cop}$ then $\mathbf{X} \leq_{cop} \mathbf{Y}$.

(6) Increasing convex order: If $Ef(\mathbf{Y}) \geq Ef(\mathbf{X})$ holds true for any increasing convex function f then $\mathbf{X} \leq_{icx} \mathbf{Y}$.

(7) Increasing supermodular order: If $\mathcal{F}$ is a class of twice differentiable functions f: $\mathbb{R}^n \to \mathbb{R}$ satisfying $\nabla_f(\mathbf{x}) \geq 0$, for all $x \in \mathbb{R}^n$, and $\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \geq 0$, for $\mathbf{x} \in \mathbb{R}^n$, $1 \leq i < j \leq n$ then $\mathbf{X} \leq_{ism} \mathbf{Y}$.

(8) Increasing directionally convex order: If $\mathcal{F}$ is a class of twice differentiable functions f: $\mathbb{R}^n \to \mathbb{R}$, satisfying $\nabla_f(\mathbf{x}) \geq 0$, for all $\mathbf{x} \in \mathbb{R}^n$ and $\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \geq 0$, for $\mathbf{x} \in \mathbb{R}^n$, $1 \leq i, j \leq n$ then $\mathbf{X} \leq_{idcx} \mathbf{Y}$

(9) Increasing componentwise convex order: If $\mathcal{F}$ is a class of twice differentiable functions f: $\mathbb{R}^n \to \mathbb{R}$, satisfying $\nabla_f(\mathbf{x}) \geq 0$ and $\frac{\partial^2}{\partial x_i^2} f(\mathbf{x}) \geq 0$, for $\mathbf{x} \in \mathbb{R}^n$, $1 \leq i \leq n$ then $\mathbf{X} \leq_{iccx} Y$.

(10) Increasing copositive: If $\mathcal{F}$ is a class of increasing functions f satisfying condition $\mathbf{H}_f(\mathbf{x}) \in C_{cop}$ then $\mathbf{X} \leq_{icop} \mathbf{Y}$.

(11) Increasing completely-positive: If $\mathcal{F}$ is a class of increasing functions f satisfying condition $\mathbf{H}_f(\mathbf{x}) \in C_{cp}$ then $\mathbf{X} \leq_{icp} \mathbf{Y}$.

Then we introduce the definition of several linear stochastic orderings.

(a) If $\mathbf{a}^\top \mathbf{X} \leq_{st} \mathbf{a}^\top \mathbf{Y}$ holds true for any $\mathbf{a} \in \mathbb{R}^n$ then $\mathbf{X}$ is said to be less than $\mathbf{Y}$ in the sense of linear-usual stochastic order, which is denoted as $\mathbf{X} \leq_{lst} \mathbf{Y}$.

(b) If $\mathbf{a}^\top \mathbf{X} \leq_{st} \mathbf{a}^\top \mathbf{Y}$ holds true for any $\mathbf{a} \in \mathbb{R}^n_+$ then $\mathbf{X}$ is said to be less than $\mathbf{Y}$ in the sense of positive-linear-usual stochastic order, which is denoted as $\mathbf{X} \leq_{plst} \mathbf{Y}$.

(c) If $\mathbf{a}^\top \mathbf{X} \leq_{cx} \mathbf{a}^\top \mathbf{Y}$ holds true for any $\mathbf{a} \in \mathbb{R}^n$ then $\mathbf{X}$ is said to be less than $\mathbf{Y}$ in the sense of linear-convex order, which is denoted as $\mathbf{X} \leq_{lcx} \mathbf{Y}$.

(d) If $\mathbf{a}^\top \mathbf{X} \leq_{cx} \mathbf{a}^\top \mathbf{Y}$ holds true for any $\mathbf{a} \in \mathbb{R}^n_+$ then $\mathbf{X}$ is said to be less than $\mathbf{Y}$ in the sense of positive-linear-convex order, which is denoted as $\mathbf{X} \leq_{plcx} \mathbf{Y}$.

(e) If $\mathbf{a}^\top \mathbf{X} \leq_{icx} \mathbf{a}^\top \mathbf{Y}$ holds true for any $\mathbf{a} \in \mathbb{R}^n_+$ then $\mathbf{X}$ is said to be less than $\mathbf{Y}$ in the sense of increasing-positive-linear-convex order, which is denoted as $\mathbf{X} \leq_{plcx} \mathbf{Y}$.

3. Multivariate stochastic orderings results

This section establishes the sufficient and/or necessary conditions for the stochastic comparison between two random variables that obey the skew-normal distributions. The proofs of sufficient conditions of the stochastic comparison are fully used the Lemma 2.1 and the identity (2.4). When proving the necessary conditions, various methods are used to compare the parameters under the premise of considering the property of the stochastic orderings.

Lemma 3.1. *Suppose that a n-dimensional random vector* $\mathbf{X} \sim SN_n(\boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\delta})$*. Then,*

$$E(\mathbf{X}) = \boldsymbol{\xi} + \sqrt{\frac{2}{\pi}}\boldsymbol{\delta}.$$

If the second order moment exists,

$$Cov(\mathbf{X}) = \boldsymbol{\Omega} + \boldsymbol{\xi}\boldsymbol{\xi}^\top + \sqrt{\frac{2}{\pi}}\left(\boldsymbol{\xi}\boldsymbol{\delta}^\top + \boldsymbol{\delta}\boldsymbol{\xi}^\top\right).$$

Theorem 3.1. *Suppose that the random variables* $\mathbf{X}$ *and* $\mathbf{Y}$ *are defined as in (2.3).*

(1) If $\boldsymbol{\xi} = \boldsymbol{\xi}^*, \boldsymbol{\delta} = \boldsymbol{\delta}^*, \omega_{ii} \leq \omega_{ii}^*, 1 \leq i \leq n$ *and* $\omega_{ij} = \omega_{ij}^*, 1 \leq i < j \leq n$ *then* $\mathbf{X} \leq_{ccx} \mathbf{Y}$.

(2) If $\boldsymbol{\xi} = \boldsymbol{\xi}^*$ *then* $\mathbf{X} \leq_{ccx} \mathbf{Y}$ *if and only if* $\boldsymbol{\delta} = \boldsymbol{\delta}^*, \omega_{ii} \leq \omega_{ii}^*, 1 \leq i \leq n$ *and* $\omega_{ij} = \omega_{ij}^*, 1 \leq i < j \leq n$.

(3) If $\boldsymbol{\delta} = \boldsymbol{\delta}^*$ *then* $\mathbf{X} \leq_{ccx} \mathbf{Y}$ *if and only if* $\boldsymbol{\xi} = \boldsymbol{\xi}^*, \omega_{ii} \leq \omega_{ii}^*, 1 \leq i \leq n$ *and* $\omega_{ij} = \omega_{ij}^*, 1 \leq i < j \leq n$.

Proof. (1) If a convex function f is twice derivable, and $\mathbf{H}_f(\mathbf{x}) \in C_{+diag}$, then $\mathbf{X} \leq_{ccx} \mathbf{Y}$ by substituting the condition $\boldsymbol{\xi} = \boldsymbol{\xi}^*, \boldsymbol{\delta} = \boldsymbol{\delta}^*, \omega_{ii} \leq \omega_{ii}^*, 1 \leq i \leq n$, $\omega_{ij} = \omega_{ij}^*$ and $1 \leq i < j \leq n$ into Corollary 2.1.

(2)-(3) The sufficiency can be directly obtained through Corollary 2.1. The necessity is proved below. Let $\mathbf{X} \leq_{ccx} \mathbf{Y}$, considering a componentwise convex function which satisfies the Definition 2.4. Suppose that

$$f_1(\mathbf{x}) = x_i,\ f_2(\mathbf{x}) = -x_i,\ 1 \leq i \leq n.$$

Combining with $E(f(\mathbf{X})) \leq E(f(\mathbf{Y}))$, it quickly yields $E(f(\mathbf{X})) = E(f(\mathbf{Y}))$. Considering the condition $\boldsymbol{\xi} = \boldsymbol{\xi}^*$ and using Lemma 3.1, we have $\boldsymbol{\delta} = \boldsymbol{\delta}^*$ immediately. The same is true for the proof of (3). Consider the functions

$$f_3(\mathbf{x}) = x_i x_j,\ \ f_4(\mathbf{x}) = -x_i x_j,\ \ f_5(\mathbf{x}) = x_i^2,$$

where

$$1 \leq i \leq n,\ \ 1 \leq i < j \leq n.$$

Obviously, they all satisfy the definition of componentwise convex functions in Definition 2.4. Therefore, we have

$$E(\mathbf{X}_i\mathbf{X}_j) \leq E(\mathbf{Y}_i\mathbf{Y}_j),\ \ Cov(\mathbf{X}_i) \leq Cov(\mathbf{Y}_i).$$

Then, by combining with the expression about the second moment in Lemma 3.1, we can get the conclusion $\omega_{ii} \leq \omega_{ii}^*, 1 \leq i \leq n$, $\omega_{ij} = \omega_{ij}^*$ and $1 \leq i < j \leq n$. This ends the proof of Theorem 3.1. □

By considering the n-dimensional random variables $\mathbf{X}_0$ and $\mathbf{Y}_0$ which obey the standardized skew-normal distributions, we get the following theorem. The distributions of $\mathbf{X}_0$ and $\mathbf{Y}_0$ are respectively denoted as

$$\mathbf{X}_0 \sim SN_n(\overline{\boldsymbol{\Omega}}, \boldsymbol{\delta}),\ \ \mathbf{Y}_0 \sim SN_n(\overline{\boldsymbol{\Omega}^*}, \boldsymbol{\delta}^*). \tag{3.1}$$

Theorem 3.2. *Suppose that the random variables* $\mathbf{X}_0$ *and* $\mathbf{Y}_0$ *are defined as in (3.1). Then* $\mathbf{X}_0 \leq_{cp} \mathbf{Y}_0$ *if and only if* $\boldsymbol{\delta} = \boldsymbol{\delta}^*$ *and* $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ *is copositive.*

Proof. Sufficiency. Looking at the function f which satisfies the twice differentiable condition and $\mathbf{H}_f(\mathbf{x}) \in C_{cp}$, we take into account $\boldsymbol{\delta} = \boldsymbol{\delta}^*$ and $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ in Corollary 2.1, then we derive $E(f(\mathbf{X}_0)) \leq E(f(\mathbf{Y}_0))$, which means $\mathbf{X}_0 \leq_{cp} \mathbf{Y}_0$.

Necessity. Take the functions

$$f_1(\mathbf{x}) = x_i,\ f_2(\mathbf{x}) = -x_i,\ 1 \le i \le n,$$

f_1 and f_2 satisfy obviously $\mathbf{H}_f(\mathbf{x}) \in C_{cp}$ of the completely positive function in Definition 2.4. Considering the condition of $\mathbf{X}_0 \le_{cp} \mathbf{Y}_0$ and combining with condition $E(f(\mathbf{X}_0)) \le E(f(\mathbf{Y}_0))$, we deduce that their means are equal. Then, from the formula (2.2) we see that $\boldsymbol{\delta} = \boldsymbol{\delta}^*$. Let the function

$$f_3(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - E(\mathbf{X}))^\top A(\mathbf{x} - E(\mathbf{X})),$$

where A is any $n \times n$ dimensional symmetric matrix and $A \in C_{cp}$, it is clear that

$$H_{f_3}(\mathbf{x}) = A \in C_{cp},$$

for $\forall \mathbf{x} \in \mathbb{R}^n$. In order to prove that $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is copositive, that is

$$E((\mathbf{X}_0 - E(\mathbf{X}_0))^\top A(\mathbf{X}_0 - E(\mathbf{X}_0))) \le E((\mathbf{Y}_0 - E(\mathbf{Y}_0))^\top A(\mathbf{Y}_0 - E(\mathbf{Y}_0))).$$

Combining with the definition of standardized covariance in formula (2.2), it can be deduced that

$$\mathrm{tr}[(\overline{\boldsymbol{\Omega}} - \frac{2}{\pi}\boldsymbol{\delta}\boldsymbol{\delta}^\top)A] \le \mathrm{tr}[(\overline{\boldsymbol{\Omega}^*} - \frac{2}{\pi}\boldsymbol{\delta}^*\boldsymbol{\delta}^{*\top})A].$$

Also, we conclude

$$\mathrm{tr}[(\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}})A] \ge 0$$

by considering $\boldsymbol{\delta} = \boldsymbol{\delta}^*$. And because of $A \in C_{cp}$, $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}} \in C_{cp}^*$ where $C_{cp}^* = C_{cop}$. Consequently, we conclude that $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is copositive. This completes the proof of Theorem 3.2. □

The following theorem gives the condition of copositive order for the multivariate skew-normal distribution. The result (1) proves the sufficient condition for the general case, and the necessary condition for the standard skew-normal distribution is proved in (2).

Theorem 3.3. *Suppose that the random variables are defined as in (2.3) and (3.1).*

(1) If $\boldsymbol{\xi} = \boldsymbol{\xi}^*, \boldsymbol{\delta} = \boldsymbol{\delta}^*$ and $\boldsymbol{\Omega}^* - \boldsymbol{\Omega}$ is completely positive then $\mathbf{X} \le_{cop} \mathbf{Y}$.

(2) If $\mathbf{X}_0 \le_{cop} \mathbf{Y}_0$ then $\boldsymbol{\delta} = \boldsymbol{\delta}^*$ and $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is completely positive.

Proof. (1) Combining known conditions with Corollary 2.1, it can be obtained immediately.

(2) Suppose that $\mathbf{X}_0 \le_{cop} \mathbf{Y}_0$ holds. We take copositive functions

$$f_1(\mathbf{x}) = x_i,\ f_2(\mathbf{x}) = -x_i,\ 1 \le i \le n,$$

where f_1 and f_2 satisfy obviously $\mathbf{H}_f(\mathbf{x}) \in C_{cop}$ in Definition 2.4. Combining condition $E(f(\mathbf{X}_0)) \le E(f(\mathbf{Y}_0))$, it can be deduced that their means are equal. Then, from (2.2) we can see that $\boldsymbol{\delta} = \boldsymbol{\delta}^*$. Take the function

$$f_3(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - E(\mathbf{X}))^\top A(\mathbf{x} - E(\mathbf{X})),$$

where A is any $n \times n$ dimensional symmetric matrix and $A \in C_{cop}$, it is clear that

$$H_{f_3}(\mathbf{x}) = A \in C_{cop}$$

for $\forall \mathbf{x} \in \mathbb{R}^n$. In order to prove that $\overline{\mathbf{\Omega}^*} - \overline{\mathbf{\Omega}}$ is completely positive, that is

$$E((\mathbf{X}_0 - E(\mathbf{X}_0))^\top A(\mathbf{X}_0 - E(\mathbf{X}_0))) \leq E((\mathbf{Y}_0 - E(\mathbf{Y}_0))^\top A(\mathbf{Y}_0 - E(\mathbf{Y}_0))).$$

Combining with the definition of standardized covariance in formula (2.2), it can be deduced that

$$\mathrm{tr}[(\overline{\mathbf{\Omega}} - \frac{2}{\pi}\boldsymbol{\delta}\boldsymbol{\delta}^\top)A] \leq \mathrm{tr}[(\overline{\mathbf{\Omega}^*} - \frac{2}{\pi}\boldsymbol{\delta}^*\boldsymbol{\delta}^{*\top})A]$$

from which we conclude $\mathrm{tr}[(\overline{\mathbf{\Omega}^*} - \overline{\mathbf{\Omega}})A] \geq 0$ since $\boldsymbol{\delta} = \boldsymbol{\delta}^*$. And because of $A \in C_{cop}$, $\overline{\mathbf{\Omega}^*} - \overline{\mathbf{\Omega}} \in C^*_{cop}$ where $C^*_{cop} = C_{cp}$, Consequently, we conclude that $\overline{\mathbf{\Omega}^*} - \overline{\mathbf{\Omega}}$ is completely positive. This completes the proof of Theorem 3.3. □

The following theorem introduces several stochastic orderings of univariate skew-normal distribution, which will be used in the some theorem proof. Suppose that the univariate random variables X_1 and Y_1 have univariate skew-normal distributions, denoted as

$$X_1 \sim SN_1(\xi_1, \sigma_1^2, \delta_1), \quad Y_1 \sim SN_1(\xi_2, \sigma_2^2, \delta_2). \tag{3.2}$$

Lemma 3.2. *([18]) Suppose that the random variables are defined as in (3.2). Then,* $X_1 \leq_{st} Y_1$ *if and only if* $\xi_1 \leq \xi_2, \sigma_1 = \sigma_2^2, \delta_1 \leq \delta_2$.

Lemma 3.3. *Suppose that the random variables are defined as in (3.2).*

(1) If $\xi_1 \leq \xi_2, \sigma_1 \leq \sigma_2, \delta_1 \leq \delta_2$ *then* $X_1 \leq_{icx} Y_1$.

(2) If $X_1 \leq_{icx} Y_1$ *and* $\xi_1 = \xi_2$ *then* $\sigma_1 \leq \sigma_2, \delta_1 \leq \delta_2$.

Proof. (1) It is an immediate consequence of Corollary 2.1.

(2) Suppose that $X_1 \leq_{icx} Y_1$ and consider the increasing-convex function $f(\mathbf{x}) = x_i$, $i = 1, 2$. Then, we claim $E(X_1) \leq E(Y_1)$ because of $E(f(\mathbf{Y}) - f(\mathbf{X})) \geq 0$. By using Lemma 3.1 and the condition $\xi_1 = \xi_2$, we have $\delta_1 \leq \delta_2$. Also, we know

$$X_1 \sim SN_1(\xi_1, \sigma_1^2, \delta_1), \quad Y_1 \sim SN_1(\xi_2, \sigma_2^2, \delta_2).$$

We claim that $\sigma_1 \leq \sigma_2$. Suppose that $\sigma_1 > \sigma_2$. Then,

$$\begin{aligned}\lim_{t\to+\infty} \frac{E(Y_1 - t)_+}{E(X_1 - t)_+} &= \lim_{t\to+\infty} \frac{\int_t^{+\infty}(1 - F_{Y_1}(x))dx}{\int_t^{+\infty}(1 - F_{X_1}(x))dx} \\ &= \lim_{t\to+\infty} \frac{F_{Y_1}(t) - 1}{F_{X_1}(t) - 1} \\ &= \lim_{t\to+\infty} \frac{f_{Y_1}(t)}{f_{X_1}(t)} = 0,\end{aligned} \tag{3.3}$$

where,

$$f_{X_1}(t) = \frac{2}{\sigma_1}\phi((t-\xi_1)/\sigma_1)\Phi(\alpha_1(t-\xi_1)/\sigma_1),$$
$$f_{Y_1}(t) = \frac{2}{\sigma_2}\phi((t-\xi_2)/\sigma_2)\Phi(\alpha_2(t-\xi_2)/\sigma_2).$$

This contradicts $X_1 \le_{icx} Y_1$. Therefore, $\sigma_1 \le \sigma_2$. □

Lemma 3.4. *Suppose that the random variables are defined as in (3.2).*

(1) If $\xi_1 = \xi_2, \sigma_1 \le \sigma_2, \delta_1 = \delta_2$ then $X_1 \le_{cx} Y_1$.

(2) If $X_1 \le_{cx} Y_1$ and $\xi_1 = \xi_2$ then $\sigma_1 \le \sigma_2, \delta_1 = \delta_2$.

Proof. (1) Take the convex function f and substitute the known conditions into Corollary 2.1, which we can get the result immediately.

(2) It is well known that $X_1 \le_{cx} Y_1$ if and only if $X_1 \le_{icx} Y_1$ and $E(X_1) = E(Y_1)$, which combines with the mean formula in Lemma 3.1 and $\xi_1 = \xi_2$, we get $\delta_1 = \delta_2$. And we claim that $\sigma_1 \le \sigma_2$ by using Lemma 3.3. □

Theorem 3.4. *Suppose that the random variables are defined as in (2.3) and (3.1).*

(1) If $\boldsymbol{\xi} \le \boldsymbol{\xi}^*$, $\boldsymbol{\delta} \le \boldsymbol{\delta}^*$ and $\boldsymbol{\Omega}^* - \boldsymbol{\Omega}$ is copositive then $\mathbf{X} \le_{icp} \mathbf{Y}$.

(2) If $\mathbf{X}_0 \le_{icp} \mathbf{Y}_0$ then $\boldsymbol{\delta} \le \boldsymbol{\delta}^*$ and $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is copositive.

Proof. (1) For any $f \in \mathcal{F}$ where $\mathcal{F}$ is the class of increasing functions f which satisfy $\mathbf{H}_f(\mathbf{x}) \in C_{cp}$. Considering $\boldsymbol{\xi} \le \boldsymbol{\xi}^*$, $\boldsymbol{\delta} \le \boldsymbol{\delta}^*$ and $\boldsymbol{\Omega}^* - \boldsymbol{\Omega}$ is copositive with Corollary 2.1, it is quickly implies that $Ef(\mathbf{Y}) \ge Ef(\mathbf{X})$. Then, we draw the conclusion $\mathbf{X} \le_{icp} \mathbf{Y}$.

(2) Suppose that $\mathbf{X}_0 \le_{icp} \mathbf{Y}_0$. We take the twice differentiable increasingly function $f(\mathbf{x}) = x_i$, satisfying $\mathbf{H}_f(\mathbf{x}) \in C_{cp}$. Then, from the formula

$$E(f(\mathbf{Y}) - f(\mathbf{X})) \ge 0$$

we can know that

$$E(f(\mathbf{X}_0)) \le E(f(\mathbf{Y}_0)).$$

Therefore, we get $\boldsymbol{\delta} \le \boldsymbol{\delta}^*$ by combining with the (2.2). Let

$$f(\mathbf{x}) = g(\mathbf{a}^\top \mathbf{x}),\ \mathbf{a} \in \mathbb{R}^n_+,$$

where g is an increasing-convex function. Thus, $f \in \mathbb{L}_{C_{cp}}$. To sum up, we imply

$$E(g(\mathbf{a}^\top \mathbf{X}_0)) \le E(g(\mathbf{a}^\top \mathbf{Y}_0)),$$

i.e., $\mathbf{a}^\top \mathbf{X}_0 \le_{icx} \mathbf{a}^\top \mathbf{Y}_0$, where

$$\mathbf{a}^\top \mathbf{X}_0 \sim SN_1(\mathbf{a}^\top \overline{\boldsymbol{\Omega}} \mathbf{a},\ \mathbf{a}^\top \boldsymbol{\delta}),\ \mathbf{a}^\top \mathbf{Y}_0 \sim SN_1(\mathbf{a}^\top \overline{\boldsymbol{\Omega}^*} \mathbf{a},\ \mathbf{a}^\top \boldsymbol{\delta}^*).$$

Then, from the conclusion of Lemma 3.2, we get that

$$\mathbf{a}^\top(\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}})\mathbf{a} \ge 0,$$

i.e., $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is copositive. □

Theorem 3.5. *Suppose that the random variables are defined as in (2.3) and (3.1).*

(1) If $\boldsymbol{\xi} \leq \boldsymbol{\xi}^*$, $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*$ *and* $\boldsymbol{\Omega}^* - \boldsymbol{\Omega}$ *is completely positive then* $\mathbf{X} \leq_{icop} \mathbf{Y}$.

(2) If $\mathbf{X}_0 \leq_{icop} \mathbf{Y}_0$ *then* $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*$ *and* $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ *is completely positive.*

Proof. (1) Obviously, considering any increasing function $f \in \mathcal{F}$, $\mathbf{H}_f(\mathbf{x}) \in C_{cop}$ of f and Corollary 2.1 then $Ef(\mathbf{Y}) \geq Ef(\mathbf{X})$ is immediately available from $\boldsymbol{\xi} \leq \boldsymbol{\xi}^*$, $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*$ and $\boldsymbol{\Omega}^* - \boldsymbol{\Omega}$ is completely positive. Therefore, $\mathbf{X} \leq_{icop} \mathbf{Y}$.

(2) According to the Definition 2.4, we take the twice differentiable increasingly function $f(\mathbf{x}) = x_i$ satisfying $\mathbf{H}_f(\mathbf{x}) \in C_{cop}$. Then, the formula

$$E(f(\mathbf{Y}) - f(\mathbf{X})) \geq 0$$

shows

$$E(f(\mathbf{X}_0)) \leq E(f(\mathbf{Y}_0)).$$

Combining with formula (2.2), we can derive that $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*$. Consider

$$f(\mathbf{x}) = g(\mathbf{a}^\top \mathbf{x}),\ \mathbf{a} \in \mathbb{R}^n_+,$$

where g is an increasing-convex function, such that f is also an increasing-convex function, it yields that

$$\mathbf{H}_f(\mathbf{x}) = \mathbf{a}^\top \mathbf{a} g^{(2)}(\mathbf{a}^\top \mathbf{x}).$$

Note that $g^{(2)}(\mathbf{a}^\top \mathbf{x}) \geq 0$, because of the convexity of g, so $\mathbf{H}_f(\mathbf{x}) \geq 0$, i.e., $f \in \mathbb{L}_{C_{cop}}$. Assume that $\mathbf{X}_0 \leq_{icop} \mathbf{Y}_0$ then

$$E(g(\mathbf{a}^\top \mathbf{X}_0)) \leq E(g(\mathbf{a}^\top \mathbf{Y}_0)),$$

i.e., $\mathbf{a}^\top \mathbf{X}_0 \leq_{icx} \mathbf{a}^\top \mathbf{Y}_0$, where

$$\mathbf{a}^\top \mathbf{X}_0 \sim SN_1(\mathbf{a}^\top \overline{\boldsymbol{\Omega}} \mathbf{a},\ \mathbf{a}^\top \boldsymbol{\delta}),\ \mathbf{a}^\top \mathbf{Y}_0 \sim SN_1(\mathbf{a}^\top \overline{\boldsymbol{\Omega}^*} \mathbf{a},\ \mathbf{a}^\top \boldsymbol{\delta}^*).$$

The conclusion of Lemma 3.2 shows that $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ can be expressed by the product of any two non-negative matrices, that is to say $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is completely positive. □

Theorem 3.6. *Suppose that the random variables are defined as in (3.1), then* $\mathbf{X}_0 \leq_{ism} \mathbf{Y}_0$ *if and only if* $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*, \omega_{ii} = \omega^*_{ii}, 1 \leq i \leq n$, $\omega_{ij} \leq \omega^*_{ij}$ *and* $1 \leq i < j \leq n$.

Proof. Sufficiency. According to Lemma 1 in [17], we conclude that $\mathbf{X}_0 \leq_{\mathbb{L}_\mathcal{H}} \mathbf{Y}_0$ when $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*$, where random vectors $\mathbf{X}_0$ and $\mathbf{Y}_0$ are defined as (3.1). And $\mathcal{H} = C_{+off}$, its dual cone is

$$C^*_{+off} = \{B \in \mathbb{S} : b_{ii} = 0, b_{ij} \geq 0, i \neq j \in \{1, ..., n\}\}.$$

Then, by the supermodular ordering in Definition 2.4, $\mathbf{X}_0 \leq_{ism} \mathbf{Y}_0$ can be obtained.

Necessity. Suppose that $\mathbf{X}_0 \leq_{ism} \mathbf{Y}_0$, then $\mathbf{X}_{0i} \leq_{st} \mathbf{Y}_{0i}$, $1 \leq i \leq n$ and $\mathbf{X}_0 \leq_{iplcx} \mathbf{Y}_0$ can be known from Müller and Stoyan [1]. First, $\mathbf{X}_{0i} \leq_{st} \mathbf{Y}_{0i}$, $1 \leq i \leq n$ guarantees the results of Lemma 3.2. Then, it is shown that $\boldsymbol{\delta}_i \leq \boldsymbol{\delta}^*_i$, $\omega_{ii} = \omega^*_{ii}$ and $1 \leq i \leq n$. Moreover, $\mathbf{X}_0 \leq_{iplcx} \mathbf{Y}_0$, which means

$$\mathbf{a}^\top \mathbf{X}_0 \leq_{icx} \mathbf{a}^\top \mathbf{Y}_0,\ \mathbf{a} \in \mathbb{R}^n_+.$$

According to the conclusion given in Lemma 3.3, we can imply $\mathbf{a}^\top(\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}})\mathbf{a} \geq 0$, $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is copositive according to Definition 2.1. Combining the above derivation with the definition of the copositive matrix, it can be known that $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is a matrix whose diagonal elements are 0 and off-diagonal elements ≥ 0, which means that $\omega_{ij} \leq \omega^*_{ij}$ and $1 \leq i < j \leq n$. □

Theorem 3.7. *Suppose that the random variables are defined as in (3.1). Then,* $\mathbf{X}_0 - E(\mathbf{X}_0) \leq_{idcx} \mathbf{Y}_0 - E(\mathbf{Y}_0)$ *if and only if* $\omega_{ij} \leq \omega^*_{ij}$, $1 \leq i$ *and* $j \leq n$.

Proof. Sufficiency. Let $f \in \mathcal{F}$, where $\mathcal{F}$ be a class of twice differentiable and directionally convex functions. From Definition 2.4 we know that $\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \geq 0$. Then, according to the condition $\omega_{ij} \leq \omega^*_{ij}$, $1 \leq i, j \leq n$ and Corollary 2.1, we can get

$$\mathbf{X}_0 - E(\mathbf{X}_0) \leq_{idcx} \mathbf{Y}_0 - E(\mathbf{Y}_0).$$

Necessity. The condition

$$\mathbf{X}_0 - E(\mathbf{X}_0) \leq_{idcx} \mathbf{Y}_0 - E(\mathbf{Y}_0)$$

means

$$\mathbf{X}_0 - E(\mathbf{X}_0) \leq_{dcx} \mathbf{Y}_0 - E(\mathbf{Y}_0).$$

Then, according to the conclusion derived from Proposition 4.7 in [18], it can be known that $\omega_{ij} \leq \omega^*_{ij}$, $1 \leq i$ and $j \leq n$. □

4. Linear stochastic orderings results

Theorem 4.1. *Suppose that the random variables are defined as in (2.3). Then,* $\mathbf{X} \leq_{plst} \mathbf{Y}$ *if and only if* $\boldsymbol{\xi} \leq \boldsymbol{\xi}^*$, $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*$ *and* $\boldsymbol{\Omega}^* = \boldsymbol{\Omega}$.

Proof. Sufficiency. According to the conclusion derived from Proposition 4.3 in [18], we can see that $\mathbf{X} \leq_{st} \mathbf{Y}$ from the known conditions $\boldsymbol{\xi} \leq \boldsymbol{\xi}^*$, $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*$ and $\boldsymbol{\Omega}^* = \boldsymbol{\Omega}$. Then, we conclude $\mathbf{X} \leq_{plst} \mathbf{Y}$.

Necessity. $\mathbf{X} \leq_{plst} \mathbf{Y}$ implies $\mathbf{a}^\top \mathbf{X} \leq_{st} \mathbf{a}^\top \mathbf{Y}$ for $\mathbf{a} \in \mathbb{R}^n_+$. From the necessary and sufficient conditions of the usual stochastic ordering of the skew-normal distribution, it can be known that $\mathbf{a}^\top \boldsymbol{\xi}^* \geq \mathbf{a}^\top \boldsymbol{\xi}$, $\mathbf{a}^\top \boldsymbol{\delta}^* \geq \mathbf{a}^\top \boldsymbol{\delta}$ and $\mathbf{a}^\top (\boldsymbol{\Omega}^* - \boldsymbol{\Omega})\mathbf{a} = 0, \mathbf{a} \geq 0$. The conclusion is obviously available. □

Theorem 4.2. *Suppose that the random variables are defined as in (2.3) and (3.1).*

(1) If $\boldsymbol{\xi} = \boldsymbol{\xi}^*, \boldsymbol{\delta} = \boldsymbol{\delta}^*, \boldsymbol{\Omega}^* - \boldsymbol{\Omega}$ *is PSD then* $\mathbf{X} \leq_{lcx} \mathbf{Y}$.

(2) $\mathbf{X}_0 \leq_{lcx} \mathbf{Y}_0$ *if and only if* $\boldsymbol{\delta} = \boldsymbol{\delta}^*, \overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}} \geq 0$.

Proof. (1) From $\boldsymbol{\xi} = \boldsymbol{\xi}^*$ and $\boldsymbol{\delta} = \boldsymbol{\delta}^*$, $\boldsymbol{\Omega}^* - \boldsymbol{\Omega}$ is positive semi-definite, we can conclude $\mathbf{X} \leq_{cx} \mathbf{Y}$ and then $\mathbf{X} \leq_{lcx} \mathbf{Y}$.

(2) Because of $\mathbf{X}_0 \leq_{lcx} \mathbf{Y}_0$, we have $\mathbf{a}^\top \mathbf{X}_0 \leq_{cx} \mathbf{a}^\top \mathbf{Y}_0, \mathbf{a} \in \mathbb{R}^n_+$, where

$$\mathbf{a}^\top \mathbf{X}_0 \sim SN_1(\mathbf{a}^\top \overline{\boldsymbol{\Omega}} \mathbf{a}, \mathbf{a}^\top \boldsymbol{\delta}), \ \ \mathbf{a}^\top \mathbf{Y}_0 \sim SN_1(\mathbf{a}^\top \overline{\boldsymbol{\Omega}^*} \mathbf{a}, \mathbf{a}^\top \boldsymbol{\delta}^*).$$

Combining with the known necessary and sufficient conditions of convex order, we directly get $\mathbf{a}^\top \boldsymbol{\delta}^* = \mathbf{a}^\top \boldsymbol{\delta}$ and $\mathbf{a}^\top (\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}})\mathbf{a} \geq 0, \mathbf{a} \in \mathbb{R}^n$. That is, $\boldsymbol{\delta} = \boldsymbol{\delta}^*, \overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}} \geq 0$, which means $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is positive semi-definite. □

Theorem 4.3. *Suppose that the random variables are defined as in (2.3) and (3.1).*

(1) If $\boldsymbol{\xi} = \boldsymbol{\xi}^*, \boldsymbol{\Omega}^* - \boldsymbol{\Omega}$ *is copositive and* $\boldsymbol{\delta} = \boldsymbol{\delta}^*$ *then* $\mathbf{X} \leq_{plcx} \mathbf{Y}$.

(2) If $\mathbf{X}_0 \leq_{plcx} \mathbf{Y}_0$ *then* $\boldsymbol{\delta} = \boldsymbol{\delta}^*$ *and* $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ *is copositive.*

Proof. (1) By using the given conditions and Theorem 3.2, it is obvious that $\mathbf{X} \leq_{cp} \mathbf{Y}$ can be obtained. Consider a convex function g: $\mathbb{R}^n \to \mathbb{R}$ and let

$$f(\mathbf{x}) = g(\mathbf{a}^\top \mathbf{x}),\ \mathbf{a} \in \mathbb{R}_+^n,$$

then f: $\mathbb{R}^n \to \mathbb{R}$ is also a convex function. According to completely positive ordering in Definition 2.4, the condition $Ef(\mathbf{Y}) \geq Ef(\mathbf{X})$ is satisfied, that is,

$$E(g(\mathbf{a}^\top \mathbf{X})) \leq E(g(\mathbf{a}^\top \mathbf{Y})),$$

so $\mathbf{X} \leq_{plcx} \mathbf{Y}$ is launched.

(2) Suppose that $\mathbf{X}_0 \leq_{plcx} \mathbf{Y}_0$, then $\mathbf{a}^\top \mathbf{X}_0 \leq_{cx} \mathbf{a}^\top \mathbf{Y}_0, \mathbf{a} \in \mathbb{R}_+^n$. Combining with the known conclusion of univariate convex order in Lemma 3.4, it is shown that $\mathbf{a}^\top \mathbf{X}_0 \leq_{cx} \mathbf{a}^\top \mathbf{Y}_0$ is equivalent with $\mathbf{a}^\top \boldsymbol{\delta}^* = \mathbf{a}^\top \boldsymbol{\delta}$ and $\mathbf{a}^\top (\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}})\mathbf{a}$ is positive semi-definite. Then, we immediately conclude that $\boldsymbol{\delta} = \boldsymbol{\delta}^*$ and $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is copositive. □

Theorem 4.4. *Suppose that the random variables are defined as in (2.3) and (3.1).*

(1) If $\boldsymbol{\xi} \leq \boldsymbol{\xi}^*$, $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*$ and $\boldsymbol{\Omega}^* - \boldsymbol{\Omega}$ is copositive then $\mathbf{X} \leq_{iplcx} \mathbf{Y}$.

(2) If $\mathbf{X}_0 \leq_{iplcx} \mathbf{Y}_0$ then $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*$ and $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is copositive.

Proof. (1) Apply the conclusion derived in Theorem 3.4, if $\boldsymbol{\xi} \leq \boldsymbol{\xi}^*$, $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*$ and $\boldsymbol{\Omega}^* - \boldsymbol{\Omega}$ is copositive, then $\mathbf{X} \leq_{icp} \mathbf{Y}$ can be deduced. At this time, consider an increasing-convex function g: $\mathbb{R}^n \to \mathbb{R}$ and let $f(\mathbf{x}) = g(\mathbf{a}^\top \mathbf{x}), \mathbf{a} \in \mathbb{R}_+^n$. Then, f: $\mathbb{R}^n \to \mathbb{R}$ is also an increasing-convex function on $\mathbb{R}^n$ and f satisfies the expression of copositive ordering in Definition 2.4, such that $\frac{\partial}{\partial x_i} f(\mathbf{x}) \geq 0$ and

$$\mathbf{H}_f(\mathbf{x}) = \mathbf{a}^\top \mathbf{a} g^{(2)}(\mathbf{a}^\top \mathbf{x}) \in C_{cp}, \mathbf{a} \in \mathbb{R}_+^n.$$

Thus,

$$E(g(\mathbf{a}^\top \mathbf{X})) \leq E(g(\mathbf{a}^\top \mathbf{Y})),\ \mathbf{a} \in \mathbb{R}_+^n,$$

which means $\mathbf{X} \leq_{iplcx} \mathbf{Y}$.

(2) When $\mathbf{X}_0 \leq_{iplcx} \mathbf{Y}_0$, there is $\mathbf{a}^\top \mathbf{X}_0 \leq_{icx} \mathbf{a}^\top \mathbf{Y}_0, \mathbf{a} \in \mathbb{R}_+^n$ where

$$\mathbf{a}^\top \mathbf{X}_0 \sim SN_1(\mathbf{a}^\top \overline{\boldsymbol{\Omega}} \mathbf{a}, \mathbf{a}^\top \boldsymbol{\delta}),\ \ \mathbf{a}^\top \mathbf{Y}_0 \sim SN_1(\mathbf{a}^\top \boldsymbol{\delta}^*, \mathbf{a}^\top \overline{\boldsymbol{\Omega}^*} \mathbf{a}).$$

According to the conclusion of increasing-convex order in Lemma 3.3, we can get $\mathbf{a}^\top \boldsymbol{\delta} \leq \mathbf{a}^\top \boldsymbol{\delta}^*$ and $\mathbf{a}^\top (\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}})\mathbf{a} \geq 0, \mathbf{a} \geq 0$, i.e., $\boldsymbol{\delta} \leq \boldsymbol{\delta}^*$, $\overline{\boldsymbol{\Omega}^*} - \overline{\boldsymbol{\Omega}}$ is copositive. □

5. Conclusions

In this paper we consider the multivariate integral stochastic ordering of skew-normal distribution, including componentwise convex, copositive, completely positive orderings and increasing componentwise convex, increasing copositive, increasing completely positive, increasing directionally convex, increasing supermodular orderings, etc, as well as some important linear stochastic orderings. We obtain some necessary and/or sufficient conditions. As for future research directions, we will draw on the idea of a projection pursuit, which is a multivariate statistical method aimed at finding interesting data projections [23], in the studying of linear random orders. Furthermore, the results might be extended to other skew-symmetric distributions, such as the (generalized) skew-elliptical distributions, which are natural generalizations of skew-normal distributions [24].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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