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Research article

Pricing formulas of binary options in uncertain financial markets

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Abstract: Binary options have a payoff that is either a fixed value or nothing at all. In this paper, the generalized pricing formulas of binary options, including European binary call options, European binary put options, American binary call options and American binary put options, are investigated in uncertain financial markets. By applying the Liu's stock model to describe the stock price, the explicit pricing formulas of binary options are derived successfully. Besides, the corresponding numerical examples for the above four kinds of binary options are discussed in this paper.

Keywords: binary option; uncertain stock model; uncertain finance; uncertainty theory
Mathematics Subject Classification: 34H05, 91G30, 91G80

1. Introduction

Binary options, as a special kind of barrier options, are also called digital options, including binary call options and binary put options, and are the class of options where the return for holder is a fixed cash or nothing, in other words, the holder knows that it has a chance to obtain the amount of money in advance. Moreover, it has been the fastest growing trading product and is used to finish the hedging and speculation in the over-the-counter markets. In addition, it is an important tool for financial engineers to design the more complex derivative products.

Since binary options are the simplest and the most popular options for investors, the related research work under the framework of probability theory has been investigated by many scholars. For example, Buchen [1] investigated the binary options based on the stock price modelled by Itô stochastic differential equation, Reiner and Rubinstein [12] studied the pricing problem of binary options based on the Black-Scholes model and Ballestra [2] explored the pricing problem of binary options by means of the repeated spatial extrapolation. Specially, Hyong-Chol et al. [7] solved the pricing problem of i-th binary options.

Traditionally, the stock price was assumed to follow the stochastic differential equation. Is it really reasonable? In fact, this widely accepted presumption was challenged among others by Liu [11], and provided the uncertain differential equation driven by Liu process in uncertainty theory to describe the stock price. In other words, frequency is the empirical basis of probability theory, while belief degree is the empirical basis of the uncertainty theory, probability theory and uncertainty theory that complementary mathematical systems.

Based on the context of the uncertainty theory, the stock price was assumed to follow the uncertain differential equations, and many kinds of barrier options in uncertain financial markets have been investigated, such as the pricing problem of European barrier options was considered by Yao and Qin [18], the pricing problem of geometric Asian barrier options was explored by Gao et al. [5], the pricing problem of arithmetic Asian barrier options was initialled by Yang et al. [17] and the pricing problem of lookback barrier options was studied by Gao and Jia [6]. The above research work were based on the stock price that was described by Liu's stock model and the other different kinds of uncertain stock model were applied to obtain the price of barrier options. For example, the barrier options based on uncertain fractional first-hitting time model with Caputo type were presented by Jin et al. [9], the barrier options based on uncertain mean-reverting stock model were constructed by Tian et al. [14] and the knock-in barrier options based on uncertain stock model with floating interest rate were given by Jia and Chen [8].

Except for the barrier options explored in uncertain financial markets, European options in uncertain financial markets were studied by Liu [11], American options in uncertain financial markets were presented by Chen [3], Lookback options in uncertain financial markets were considered by Zhang et al. [21], Gao et al. [4] and Tian et al. [15], Asian options in uncertain markets were investigated by Sun and Chen [13] and Zhang and Liu [19], Power options in uncertain financial markets were developed by Zhang et al. [20].

This paper investigates the pricing problem of binary options in uncertain financial markets. The rest of this paper is structured as follows. Section 2 reviews some relevant basic contents. Section 3 investigates the pricing problem of binary options, including European binary call options, European binary put options, American binary call options and American binary put options, and their corresponding generalized pricing formulas and corresponding explicit pricing formulas are derived. Section 4 summarizes some results of this paper.

2. Preliminaries

Uncertainty theory was initialled by Liu [10], and investigated by many scholars. This section provides some significant contents for uncertain differential equations. Let Γ be a nonempty set (sometimes called universal set), and let $\mathcal{L}$ be a σ-algebra over Γ. The element Λ in $\mathcal{L}$ is called a measurable set. In order to rationally deal with belief degree, the uncertain measure $\mathcal{M}$ on the σ-algebra $\mathcal{L}$ is defined to follow the following four axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition 2.1. *An uncertain variable is a function from an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers, such that, for any Borel set B of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

Definition 2.2. *[10] As we consider the uncertain event B, and its complementary event* B^c*, then the duality axiom is*

$$\mathcal{M}\{B\} + \mathcal{M}\{B^c\} = 1.$$

Theorem 2.1. *[10] As we consider two uncertain events* B_1 *and* B_2*. If* $B_1 \subset B_2$*, then we have*

$$\mathcal{M}\{B_1\} \leq \mathcal{M}\{B_2\}.$$

Definition 2.3. *The uncertain variables* $\zeta_1, \zeta_2, \cdots, \zeta_m$ *are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{m} \{\zeta_i \in B_i\}\right\} = \bigwedge_{i=1}^{m} \mathcal{M}\{\zeta_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ *of real numbers.*

The uncertainty distribution of an uncertain variable ζ is defined as

$$\Theta(x) = \mathcal{M}\{\zeta \leq x\}$$

for any real number x. An uncertainty distribution $\Theta(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Theta(x) < 1$, and

$$\lim_{x \to -\infty} \Theta(x) = 0, \lim_{x \to +\infty} \Theta(x) = 1.$$

If ζ has a regular uncertainty distribution $\Theta(x)$, then the inverse function $\Theta^{-1}(\alpha)$ is called the inverse uncertainty distribution of ζ. By means of the inverse uncertain distribution (IUD), we can obtain the expected value

$$\mathrm{E}[\zeta] = \int_0^1 \Theta^{-1}(\theta)\mathrm{d}\theta. \tag{2.1}$$

Theorem 2.2. *[16] Let the uncertain differential equation*

$$\mathrm{d}S_t = a(t, S_t)\mathrm{d}t + b(t, S_t)\mathrm{d}C_t$$

has a solution S_t, and the related equation of the uncertain differential equation

$$\mathrm{d}S_t^\alpha = a(t, S_t^\alpha)\mathrm{d}t + |b(t, S_t^\alpha)|\Lambda^{-1}(\alpha)\mathrm{d}t$$

has a solution α-path S_t^α, where

$$\Lambda^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Then,

$$\mathcal{M}\{S_t \le S_t^\alpha, \forall t\} = \alpha,\ \mathcal{M}\{S_t > S_t^\alpha, \forall t\} = 1-\alpha,$$

and S_t has an IUD

$$\Theta_t^{-1}(\alpha) = S_t^\alpha.$$

Moreover, two important functions

$$\Upsilon_H(y) = \begin{cases} 1, & \text{if } y \ge H, \\ 0, & \text{if } y < H \end{cases}$$

and

$$L_{H_1H_2} = \begin{cases} H_1 - H_2, & \text{if } H_1 \ge H_2, \\ 0, & \text{if } H_1 < H_2 \end{cases}$$

are defined herein, where H, H_1 and H_2 represent the positive constant number.

3. Binary options

Binary options as a kind of barrier options have four cases, including European binary call options, European binary put options, American binary call options and American binary put options. Assume the binary options with a expiration time T and a exercise price H, and apply the Liu's stock model to describe the stock price, where the Liu's stock model has an IUD, that is

$$\Phi_t^{-1}(\alpha) = S_0\exp\left(h_1t + \frac{\sqrt{3}h_2t}{\pi}\ln\frac{\alpha}{1-\alpha}\right),\ S_0 \ge 0,\ h_2 > 0. \tag{3.1}$$

3.1. European binary call options

European binary call options are the options that pay zero if the terminal stock price is below the exercise price or pay a fixed cash. Apply the generalized uncertain stock model to describe the stock price, the payoff of European binary call options is given below:

$$\text{Payoff} = \begin{cases} C, & \textit{if } S(T) \ge H, \\ 0, & \textit{otherwise}. \end{cases}$$

At time T, the buyer of European binary call options pays

$$C \cdot \Upsilon_H(S(T)).$$

Assume that the price of the European binary call options is f_c^e, then, at time zero, the profit of buyer is

$$-f_c^e + C \cdot \exp(-h_3T) \cdot \Upsilon_H(S(T)).$$

Moreover, at time T, the seller of European binary call options receives

$$C \cdot \Upsilon_H(S(T)).$$

Then, at time zero, the profit of seller is

$$f_c^e - C \cdot \exp(-h_3 T) \cdot \Upsilon_H(S(T)).$$

According to the fairness principle, at time zero, the expected profit of the buyer and the seller should be the same, so we have

$$-f_c^e + C \cdot \exp(-h_3 T) \cdot \mathrm{E}[\Upsilon_H(S(T))] = f_c^e - C \cdot \exp(-h_3 T) \cdot \mathrm{E}[\Upsilon_H(S(T))].$$

Thus, the definition of European binary call options pricing formula is presented as below.

Definition 3.1. *Apply the generalized uncertain stock model to describe the stock price, then the pricing formula of European binary call options is*

$$f_c^e = C \cdot \exp(-h_3 T) \cdot \mathrm{E}[\Upsilon_H(S(T))].$$

Theorem 3.1. *Apply the Liu's stock model to model the stock price, then the price of European binary call options is*

$$f_c^e = C \cdot \exp(-h_3 T) \cdot (1 - \alpha_1),$$

where

$$\alpha_1 = \left(1 + \exp\left(\frac{\pi(h_1 T + \ln(S_0/H))}{\sqrt{3}h_2 T}\right)\right)^{-1}.$$

Proof. Firstly, we can easily obtain

$$\{\Upsilon_H(S(T)) \le \Upsilon_H(S(T^\alpha))\} \supseteq \{S(T) \le S(T^\alpha)\}$$

and

$$\{\Upsilon_H(S(T)) > \Upsilon_H(S(T^\alpha))\} \supseteq \{S(T) > S(T^\alpha)\}.$$

According to the Theorems 2.1 and 2.2, we have

$$\mathcal{M}\{\Upsilon_H(S(T)) \le \Upsilon_H(S(T^\alpha))\} \ge \mathcal{M}\{S(T) \le S(T^\alpha)\} = \alpha$$

and

$$\mathcal{M}\{\Upsilon_H(S(T)) > \Upsilon_H(S(T^\alpha))\} \ge \mathcal{M}\{S(T) > S(T^\alpha)\} = 1 - \alpha.$$

By using the Definition 2.2, we can obtain

$$\mathcal{M}\{\Upsilon_H(S(T)) \le \Upsilon_H(S(T^\alpha))\} + \mathcal{M}\{\Upsilon_H(S(T)) > \Upsilon_H(S(T^\alpha))\} = 1.$$

Thus,

$$\mathcal{M}\{\Upsilon_H(S(T)) \le \Upsilon_H(S(T^\alpha))\} = \alpha,$$

which indicates the uncertain variable

$$\Upsilon_H(S(T))$$

has an IUD

$$\Upsilon_H(S(T^\alpha)).$$

When $S(T^\alpha) \ge H$, we obtain

$$\alpha \ge \left(1 + \exp\left(\frac{\pi(h_1 T + \ln(S_0/H))}{\sqrt{3}h_2 T}\right)\right)^{-1}$$

and then set

$$\alpha_1 = \left(1 + \exp\left(\frac{\pi(h_1 T + \ln(S_0/H))}{\sqrt{3}h_2 T}\right)\right)^{-1}.$$

Therefore, the price is

$$\begin{aligned} f_c^e &= C \cdot \exp(-h_3 T) \cdot \mathrm{E}\left[\Upsilon_H(S(T))\right] \\ &= C \cdot \exp(-h_3 T) \cdot \int_0^1 \Upsilon_H(S(T^\alpha))\,\mathrm{d}\alpha \\ &= C \cdot \exp(-h_3 T) \cdot (1 - \alpha_1). \end{aligned}$$

Example 3.1. Set $h_1 = 0.04$, $h_2 = 0.02$ and $h_3 = 0.06$ in Liu's stock model, and let $S_0 = 4$, $T = 16$, $C = 10$ and $H = 8$ for European binary call options. Then, $f_c^e = 1.6283$.

As the exercise price is increasing and the other parameters keep unchanged in Example 3.1, then Figure 1 presents that the price of European binary call options f_c^e is decreasing.

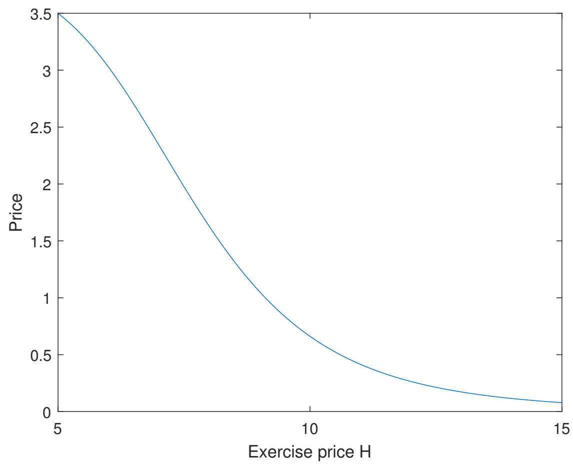

Figure 1. The relation of f_c^e and H in the Example 3.1.

3.2. European binary put options

European binary put options are the options that pay zero if the terminal stock price is over the exercise price or pay a fixed cash. Apply the generalized uncertain stock model to describe the stock price, the payoff of European binary put options is given below:

$$\text{Payoff} = \begin{cases} C, & \textit{if } S(T) \le H, \\ 0, & \textit{otherwise}. \end{cases}$$

At time T, the buyer of European binary put options pays

$$C \cdot (1 - \Upsilon_H(S(T))).$$

Assume that the price of the European binary put options is f_p^e, then, at time zero, the profit of buyer is

$$-f_p^e + C \cdot \exp(-h_3 T) \cdot (1 - \Upsilon_H(S(T))).$$

Moreover, at time T, the seller of European binary put options receives

$$C \cdot (1 - \Upsilon_H(S(T))).$$

Then, at time zero, the profit of seller is

$$f_p^e - C \cdot \exp(-h_3 T) \cdot (1 - \Upsilon_H(S(T))).$$

According to the fairness principle, at time zero, the expected profit of the buyer and the seller should be the same, so we have

$$-f_p^e + C \cdot \exp(-h_3 T) \cdot (1 - \mathrm{E}[\Upsilon_H(S(T))]) = f_p^e - C \cdot \exp(-h_3 T) \cdot (1 - \mathrm{E}[\Upsilon_H(S(T))]).$$

Thus, the definition of European binary put options pricing formula is presented as below.

Definition 3.2. *Apply the generalized uncertain stock model to describe the stock price, then the pricing formula of European binary put options is*

$$f_p^e = C \cdot \exp(-h_3 T) \cdot (1 - \mathrm{E}[\Upsilon_H(S(T))]).$$

Theorem 3.2. *Apply the Liu's stock model to describe the stock price, then the price of European binary put options is*

$$f_p^e = C \cdot \exp(-h_3 T) \cdot \alpha_2,$$

where

$$\alpha_2 = \left(1 + \exp\left(\frac{\pi(h_1 T + \ln(S_0/H))}{\sqrt{3}h_2 T}\right)\right)^{-1}.$$

Proof. Firstly, it exists an IUD

$$\Upsilon_H(S(T^\alpha))$$

for the uncertain variable

$$\Upsilon_H(S(T))$$

from the Theorem 3.1, where

$$S(T^\alpha) = S_0 \exp\left(h_1 T + \frac{\sqrt{3}h_2 T}{\pi} \ln \frac{\alpha}{1-\alpha}\right).$$

When $S(T^\alpha) \geq H$, we obtain

$$\alpha \geq \left(1 + \exp\left(\frac{\pi(h_1 T + \ln(S_0/H))}{\sqrt{3}h_2 T}\right)\right)^{-1}$$

and then set

$$\alpha_2 = \left(1 + \exp\left(\frac{\pi(h_1 T + \ln(S_0/H))}{\sqrt{3}h_2 T}\right)\right)^{-1}.$$

Therefore, the price is

$$\begin{aligned} f_p^e &= C \cdot \exp(-h_3 T) \cdot (1 - \mathrm{E}[\Upsilon_H(S(T))]) \\ &= C \cdot \exp(-h_3 T) \cdot \left(1 - \int_0^1 \Upsilon_H(S(T^\alpha))\,\mathrm{d}\alpha\right) \\ &= C \cdot \exp(-h_3 T) \cdot \alpha_2. \end{aligned}$$

Example 3.2. Set $h_1 = 0.02$, $h_2 = 0.05$ and $h_3 = 0.03$ in Liu's stock model, and let $S_0 = 8$, $T = 18$, $C = 12$ and $H = 6$ for the European binary put options. Then, $f_p^e = 1.4914$.

As the exercise price is increasing and the other parameters keep unchanged in Example 3.2, then Figure 2 presents that the price of European binary put options f_p^e is increasing.

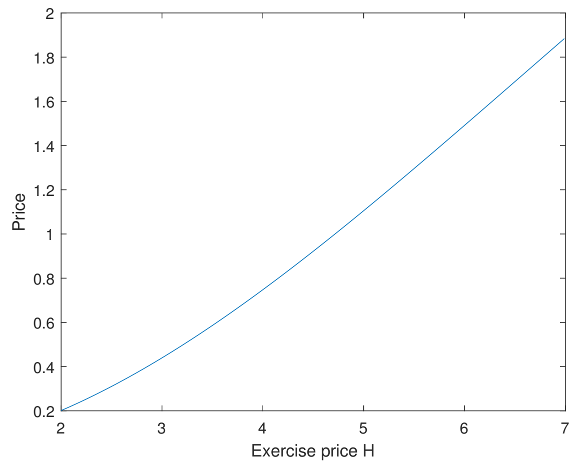

Figure 2. The relation of f_p^e and H in the Example 3.2.

3.3. American binary call options

American binary call options are the options that pay zero if the maximum value of stock price is below the exercise price or pay a fixed cash during the lifetime. Apply the generalized uncertain stock model to describe the stock price, the payoff of American binary call options is provided as below:

$$\text{Payoff} = \begin{cases} C, \ \textit{if} \max\limits_{0 \le t \le T} \{S(t)\} \ge H, \\ 0, \ \textit{otherwise}. \end{cases}$$

At time T, the buyer of American binary call options pays

$$C \cdot \Upsilon_H \left(\max_{0 \le t \le T} \{S(t)\} \right).$$

Assume that the price of the American binary call options is the character f_c^a, at time zero, then the profit of buyer is

$$-f_c^a + C \cdot \exp(-h_3 T) \cdot \Upsilon_H \left(\max_{0 \le t \le T} \{S(t)\} \right).$$

Moreover, at time T, the seller of American binary call options receives

$$C \cdot \Upsilon_H \left(\max_{0 \le t \le T} \{S(t)\} \right).$$

Then, at time zero, the profit of seller is

$$f_c^a - C \cdot \exp(-h_3 T) \cdot \Upsilon_H \left(\max_{0 \le t \le T} \{S(t)\} \right).$$

According to the fairness principle, at time zero, the expected profit of the buyer and the seller should be the same, so we have

$$-f_c^a + C \cdot \exp(-h_3 T) \cdot \mathrm{E}\left[\Upsilon_H \left(\max_{0 \le t \le T} \{S(t)\} \right) \right] = f_c^a - C \cdot \exp(-h_3 T) \cdot \mathrm{E}\left[\Upsilon_H \left(\max_{0 \le t \le T} \{S(t)\} \right) \right].$$

Thus, the definition of American binary call options pricing formula is presented below.

Definition 3.3. *Apply the generalized uncertain stock model to describe the stock price, then the pricing formula of American binary call options is*

$$f_c^a = C \cdot \exp(-h_3 T) \cdot \mathrm{E}\left[\Upsilon_H \left(\max_{0 \le t \le T} \{S(t)\} \right) \right].$$

Theorem 3.3. *Apply the Liu's stock model to describe the stock price, then set* $h = \max\{\alpha_0, \alpha_3\}$, *the price of American binary call options is*

$$f_c^a = C \cdot \exp(-\mu_3 T) \cdot (1 - h),$$

where

$$\alpha_0 = \left(\exp\left(\frac{\pi h_1}{\sqrt{3} h_2} + 1 \right) \right)^{-1}$$

and

$$\alpha_3 = \left(1 + \exp\left(\frac{\pi(\mu_1 T + \ln(S_0/H))}{\sqrt{3}\mu_2 T} \right) \right)^{-1}.$$

Proof. Firstly, we can easily obtain

$$\begin{aligned}&\left\{\Upsilon_H\left(\max_{0\le t\le T}\{S(t)\}\right)\le\Upsilon_H\left(\max_{0\le t\le T}\{S(t^\alpha)\}\right)\right\}\\ \supseteq&\left\{\max_{0\le t\le T}\{S(t)\}\le\max_{0\le t\le T}\{S(t^\alpha)\}\right\}\\ \supseteq&\{S(t)\le S(t^\alpha),\ \forall t\in[0,T]\}\end{aligned}$$

and

$$\begin{aligned}&\left\{\Upsilon_H\left(\max_{0\le t\le T}\{S(t)\}\right)>\Upsilon_H\left(\max_{0\le t\le T}\{S(t^\alpha)\}\right)\right\}\\ \supseteq&\left\{\max_{0\le t\le T}\{S(t)\}>\max_{0\le t\le T}\{S(t^\alpha)\}\right\}\\ \supseteq&\{S(t)>S(t^\alpha),\ \forall t\in[0,T]\}.\end{aligned}$$

According to the Theorems 2.1 and 2.2, we have

$$\mathcal{M}\left\{\Upsilon_H\left(\max_{0\le t\le T}\{S(t)\}\right)\le\Upsilon_H\left(\max_{0\le t\le T}\{S(t^\alpha)\}\right)\right\}\ge\mathcal{M}\{S(t)\le S(t^\alpha),\ \forall t\in[0,T]\}=\alpha$$

and

$$\mathcal{M}\left\{\Upsilon_H\left(\max_{0\le t\le T}\{S(t)\}\right)>\Upsilon_H\left(\max_{0\le t\le T}\{S(t^\alpha)\}\right)\right\}\ge\mathcal{M}\{S(t)>S(t^\alpha),\ \forall t\in[0,T]\}=1-\alpha.$$

By using the Definition 2.2, we can obtain

$$\mathcal{M}\left\{\Upsilon_H\left(\max_{0\le t\le T}\{S(t)\}\right)\le\Upsilon_H\left(\max_{0\le t\le T}\{S(t^\alpha)\}\right)\right\}+\mathcal{M}\left\{\Upsilon_H\left(\max_{0\le t\le T}\{S(t)\}\right)>\Upsilon_H\left(\max_{0\le t\le T}\{S(t^\alpha)\}\right)\right\}=1.$$

Thus,

$$\mathcal{M}\left\{\Upsilon_H\left(\max_{0\le t\le T}\{S(t)\}\right)\le\Upsilon_H\left(\max_{0\le t\le T}\{S(t^\alpha)\}\right)\right\}=\alpha,$$

which indicates the uncertain variable

$$\Upsilon_H\left(\max_{0\le t\le T}\{S(t)\}\right)$$

has an IUD

$$\Upsilon_H\left(\max_{0\le t\le T}\{S(t^\alpha)\}\right).$$

Meanwhile, we have

$$\mathrm{d}S_t^\alpha/\mathrm{d}t=S_t^\alpha\cdot\left(h_1+\frac{\sqrt{3}h_2}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Thus, we can set

$$\alpha_0=\left(\exp\left(\frac{\pi h_1}{\sqrt{3}h_2}+1\right)\right)^{-1}$$

and easily obtain that, if $\alpha > \alpha_0$, S_t^α is increasing with respect to t, then $\max_{0\le t\le T}\{S_t^\alpha\} \ge H$ means $S_T^\alpha \ge H$. When $S_T^\alpha \ge H$, we obtain

$$\alpha \ge \left(1 + \exp\left(\frac{\pi(h_1 T + \ln(S_0/H))}{\sqrt{3}h_2 T}\right)\right)^{-1}$$

and then set

$$\alpha_3 = \left(1 + \exp\left(\frac{\pi(h_1 T + \ln(S_0/H))}{\sqrt{3}h_2 T}\right)\right)^{-1}.$$

Conversely, if $\alpha \le \alpha_0$, S_t^α is decreasing with respect to t, then $\max_{0\le t\le T}\{S_t^\alpha\} \ge H$ means $S_0 \ge H$, this will be not established. Therefore, we set $h = \max\{\alpha_0, \alpha_3\}$, the price is

$$\begin{aligned} f_c^a &= C \cdot \exp(-h_3 T) \cdot \mathrm{E}\left[\Upsilon_H\left(\max_{0\le t\le T}\{S(t)\}\right)\right] \\ &= C \cdot \exp(-h_3 T) \cdot \int_0^1 \Upsilon_H\left(\max_{0\le t\le T}\{S(t^\alpha)\}\right) \mathrm{d}\alpha \\ &= C \cdot \exp(-h_3 T) \cdot (1 - h). \end{aligned}$$

Example 3.3. Set $h_1 = 0.05$, $h_2 = 0.02$ and $h_3 = 0.06$ in Liu's stock model, and let $S_0 = 4$, $T = 16$, $C = 10$ and $H = 6$ for the American binary call options. Then, $f_c^a = 3.4593$.

As the exercise price is increasing and the other parameters keep unchanged in Example 3.3, then Figure 3 presents that the price of American binary call option f_c^a is decreasing.

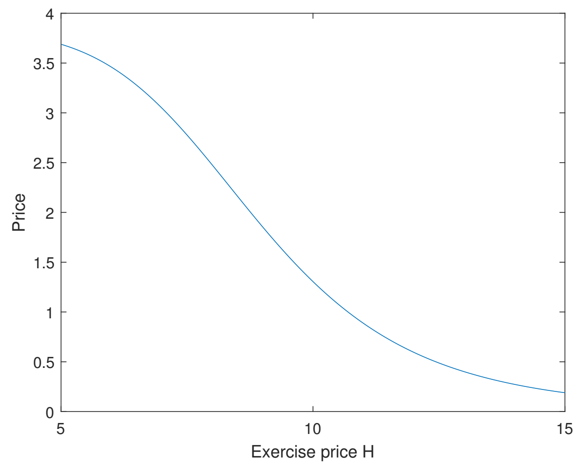

Figure 3. The relation of f_c^a and H in the Example 3.3.

3.4. American binary put options

American binary put options are the options that pay nothing if the minimum value of stock price is above the exercise price or pay a fixed cash during the life time. Apply the generalized uncertain stock model to describe the stock price, the payoff of American binary put options is provided as below:

$$\text{Payoff} = \begin{cases} C, & \textit{if } \min_{0\le t\le T}\{S(t)\} \le H, \\ 0, & \textit{otherwise}. \end{cases}$$

At time T, the buyer of American binary put options pays

$$C \cdot \left(1 - \Upsilon_H\left(\min_{0\le t\le T}\{S(t)\}\right)\right).$$

Assume that the price of the American binary put options is the character f_p^a, at time zero, then the profit of buyer is

$$-f_p^a + C \cdot \exp(-h_3 T) \cdot \left(1 - \Upsilon_H\left(\min_{0\le t\le T}\{S(t)\}\right)\right).$$

Moreover, at time T, the seller of American binary put options receives

$$C \cdot \left(1 - \Upsilon_H\left(\min_{0\le t\le T}\{S(t)\}\right)\right).$$

Then, at time zero, the profit of seller is

$$f_c^a - C \cdot \exp(-\mu_3 T) \cdot \left(1 - \Upsilon_H\left(\min_{0\le t\le T}\{S(t)\}\right)\right).$$

According to the fairness principle, at time zero, the expected of the buyer and the seller should be the same, so we have

$$-f_c^a + C \cdot \exp(-h_3 T) \cdot \left(1 - \mathrm{E}\left[\Upsilon_H\left(\min_{0\le t\le T}\{S(t)\}\right)\right]\right) = f_c^a - C \cdot \exp(-h_3 T) \cdot \left(1 - \mathrm{E}\left[\Upsilon_H\left(\min_{0\le t\le T}\{S(t)\}\right)\right]\right).$$

Thus, the definition of the American binary put option pricing formula is provided below.

Definition 3.4. *Apply the generalized uncertain stock model to describe the stock price, then the pricing formula of American binary put options is*

$$f_c^a = C \cdot \exp(-h_3 T) \cdot \left(1 - \mathrm{E}\left[\Upsilon_H\left(\min_{0\le t\le T}\{S(t)\}\right)\right]\right).$$

Theorem 3.4. *Apply Liu's stock model to describe the stock price, then the price of American binary put options is*

$$f_c^a = C \cdot \exp(-h_3 T) \cdot (\alpha_0 - (\alpha_0 - \alpha_4) \cdot L_{\alpha_0\alpha_4}),$$

where

$$\alpha_0 = \left(\exp\left(\frac{\pi h_1}{\sqrt{3}h_2} + 1\right)\right)^{-1}$$

and

$$\alpha_4 = \left(1 + \exp\left(\frac{\pi(h_1 T + \ln(S_0/H))}{\sqrt{3}h_2 T}\right)\right)^{-1}.$$

Proof. Firstly, we can easily have

$$\begin{aligned}
&\left\{\Upsilon_H\left(\min_{0\le t\le T}\{S(t)\}\right) \le \Upsilon_H\left(\min_{0\le t\le T}\{S(t^\alpha)\}\right)\right\}\\
\supseteq&\left\{\min_{0\le t\le T}\{S(t)\} \le \min_{0\le t\le T}\{S(t^\alpha)\}\right\}\\
\supseteq&\{S(t) \le S(t^\alpha),\ \forall t \in [0, T]\}
\end{aligned}$$

and

$$\left\{\Upsilon_H\left(\min_{0\le t\le T}\{S(t)\}\right) > \Upsilon_H\left(\min_{0\le t\le T}\{S(t^\alpha)\}\right)\right\}$$
$$\supseteq\left\{\min_{0\le t\le T}\{S(t)\} > \min_{0\le t\le T}\{S(t^\alpha)\}\right\}$$
$$\supseteq\{S(t) > S(t^\alpha),\ \forall t\in[0,T]\}.$$

According to the Theorems 2.1 and 2.2, we have

$$\mathcal{M}\left\{\Upsilon_H\left(\max_{0\le t\le T}\{S(t)\}\right) \le \Upsilon_H\left(\max_{0\le t\le T}\{S(t^\alpha)\}\right)\right\} \ge \mathcal{M}\{S(t)\le S(t^\alpha),\ \forall t\in[0,T]\} = \alpha$$

and

$$\mathcal{M}\left\{\Upsilon_H\left(\max_{0\le t\le T}\{S(t)\}\right) > \Upsilon_H\left(\max_{0\le t\le T}\{S(t^\alpha)\}\right)\right\} \ge \mathcal{M}\{S(t) > S(t^\alpha),\ \forall t\in[0,T]\} = 1-\alpha.$$

By using the Definition 2.2, we can obtain

$$\mathcal{M}\{\Upsilon_H(S(t)) \le \Upsilon_H(S(t^\alpha)), \forall t\in[0,T]\} + \mathcal{M}\{\Upsilon_H(S(t)) > \Upsilon_H(S(t^\alpha)),\ \forall t\in[0,T]\} = 1.$$

Thus,

$$\mathcal{M}\left\{\Upsilon_H\left(\min_{0\le t\le T}\{S(t)\}\right) \le \Upsilon_H\left(\min_{0\le t\le T}\{S(t^\alpha)\}\right)\right\} = \alpha,$$

which indicates the uncertain variable

$$\Upsilon_H\left(\min_{0\le t\le T}\{S(t)\}\right)$$

has an IUD

$$\Upsilon_H\left(\min_{0\le t\le T}\{S(t^\alpha)\}\right).$$

Meanwhile, we have

$$\mathrm{d}S_t^\alpha/\mathrm{d}t = S_t^\alpha\cdot\left(h_1+\frac{\sqrt{3}h_2}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Thus, we can set

$$\alpha_0 = \left(\exp\left(\frac{\pi h_1}{\sqrt{3}h_2}+1\right)\right)^{-1}$$

and easily obtain that, if $\alpha > \alpha_0$, S_t^α is increasing with respect to t, then $\min_{0\le t\le T}\{S_t^\alpha\} \ge H$ means $S_0 \ge H$, this will be always established. Conversely, if $\alpha \le \alpha_0$, S_t^α is decreasing with respect to t, then $\min_{0\le t\le T}\{S_t^\alpha\} \ge H$ means $S_T^\alpha \ge H$. When $S_T^\alpha \ge H$, we obtain

$$\alpha \ge \left(1+\exp\left(\frac{\pi(h_1T+\ln(S_0/H))}{\sqrt{3}h_2T}\right)\right)^{-1}$$

and then set

$$\alpha_4 = \left(1+\exp\left(\frac{\pi(h_1T+\ln(S_0/H))}{\sqrt{3}h_2T}\right)\right)^{-1},$$

if $\alpha_4 < \alpha_0$, then we obtain $\alpha_4 \leq \alpha \leq \alpha_0$, if $\alpha_4 \geq \alpha_0$, it becomes no solution. Therefore, the price is

$$\begin{aligned} f_p^a &= C \cdot \exp(-h_3 T) \cdot \left(1 - \mathrm{E}\left[\Upsilon_H\left(\min_{0 \leq t \leq T} \{S(t)\}\right)\right]\right) \\ &= C \cdot \exp(-h_3 T) \cdot \left(1 - \int_0^1 \Upsilon_H\left(\min_{0 \leq t \leq T} \{S(t^\alpha)\}\right) \mathrm{d}\alpha\right) \\ &= C \cdot \exp(-h_3 T) \cdot (\alpha_0 - (\alpha_0 - \alpha_4) \cdot L_{\alpha_0 \alpha_4}). \end{aligned}$$

Example 3.4. Set $h_1 = 0.03$, $h_2 = 0.02$ and $h_3 = 0.04$ in Liu's stock model, and let $S_0 = 10$, $T = 18$, $C = 20$ and $H = 8$ for American binary put options. Then, $f_c^a = 0.2038$.

As the exercise price is increasing and the other parameters keep unchanged in Example 3.4, then Figure 4 presents that the price of the binary put option f_p^a is increasing.

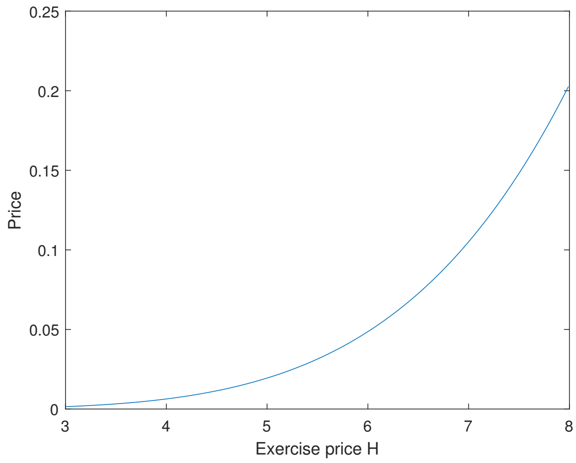

Figure 4. The relation of f_p^a and H in the Example 3.4.

4. Conclusions

In this paper, the generalized pricing formulas of binary options in uncertain financial markets, including European binary call options, European binary put options, American binary call options and American binary put options, were derived by means of the fairness principle. By applying the Liu's stock model to describe the stock price, the explicit pricing formulas for the above four kinds of binary options were obtained, respectively.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest in this article.

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