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Research article

New extrapolation projection contraction algorithms based on the golden ratio for pseudo-monotone variational inequalities

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Abstract: In real Hilbert spaces, for the purpose of trying to deal with the pseudo-monotone variational inequalities problem, we present a new extrapolation projection contraction algorithm based on the golden ratio in this study. Unlike ordinary inertial extrapolation, the algorithms are constructed based on a convex combined structure about the entire iterative trajectory. Extrapolation parameter ψ is selected in a more relaxed range instead of only taking the golden ratio $\phi = \frac{\sqrt{5}+1}{2}$ as the upper bound. Second, we propose an alternating extrapolation projection contraction algorithm to better increase the convergence effects of the extrapolation projection contraction algorithm based on the golden ratio. All our algorithms employ non-constantly decreasing adaptive step-sizes. The weak convergence results of the two algorithms are established for the pseudo-monotone variational inequalities. Additionally, the R-linear convergence results are investigated for strongly pseudo-monotone variational inequalities. Finally, we show the validity and superiority of the suggested methods with several numerical experiments. The numerical results show that alternating extrapolation does have obvious acceleration effect in practical application compared with no alternating extrapolation. Thus, the obvious effect of relaxing the selection range of parameter ψ on our two algorithms is clearly demonstrated.

Keywords: golden ratio; variational inequalities; linear rate; pseudo-monotone operator; projection algorithms; weak convergence
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1. Introduction

In the present investigation, H is a real Hilbert space and C is a nonempty, closed, and convex subset of H, $A : H \to H$ is a continus mapping. The variational inequality problem (abbreviated, VI(A, C)) is of the form: find $z^* \in C$ satisfied with

$$\langle Az^*, z - z^* \rangle \geq 0, \ \forall z \in C. \tag{1.1}$$

Numerous domains have important uses for variational inequalities. Many academics have studied and come up with a multitude of findings [1–4].

The problem VI(A, C) (1.1) is analogous to the problem of fixed points:

$$z^* = P_C(z^* - \lambda Az^*), \ \lambda > 0.$$

As a result, VI(A, C) (1.1) is possible solved by using the fixed point problem (see, e.g., [5, 6]). The following projection gradient algorithm is the simplest one:

$$z_{n+1} = P_C(z_n - \lambda Az_n). \tag{1.2}$$

However, this method's convergence necessitates a moderately strong supposition that A is a η-strongly monotone and L-Lipschitz continuous mapping, η is a positive constant and step-size $\lambda \in \left(0, \frac{2\eta}{L^2}\right)$. However, algorithm (1.2) does not work when A is monotone.

The extragradient algorithm of the following type was presented by Korpelevich in [7]: given $z_1 \in C$,

$$\begin{cases} y_n = P_C(z_n - \lambda_n Az_n), \\ z_{n+1} = P_C(z_n - \lambda_n Ay_n), \end{cases} \tag{1.3}$$

where $\lambda_n \in \left(0, \frac{1}{L}\right)$. A is relaxed to a monotone mapping based on algorithm (1.2). Moreover, it has been shown that the sequence $\{x_n\}$ will eventually arrive at a solution for the (1.1). However, P_C lacks a closed form formula and (1.3) requires calculating P_C twice in each iteration, which will result in an increase in the amount of computing that the procedure requires. There has been a lot of research done in this area by Censor et al ([8–10]). The issue that it can be tricky to calculate P_C was solved by using the projection onto the half space or intersection of half spaces rather than subset C. He first proposed projection and contraction method (PCM) in [11]. Cai et al. in [12] have studied the optimal step size η_n for PCM, and the method which takes the following form:

$$\begin{cases} y_n = P_C(z_n - \lambda Az_n), \\ d(z_n, y_n) = z_n - y_n - \lambda(Az_n - Ay_n), \\ z_{n+1} = z_n - \gamma\eta_n d(z_n, y_n), \end{cases} \tag{1.4}$$

where

$$\eta_n = \begin{cases} \frac{\langle z_n - y_n, d(z_n, y_n)\rangle}{\|d(z_n, y_n)\|^2}, & \|d(z_n, y_n)\| \neq 0, \\ 0, & \|d(z_n, y_n)\| = 0. \end{cases} \tag{1.5}$$

The benefit of this method is that A is as flexible as the algorithm (1.3) and only needs to calculate the projection once. The method's efficacy will be vastly enhanced by both theoretical and numerical experiments. After adding the optimal step size η_n, the speed of convergence is enhanced further. The focus of numerous professionals has been drawn to method (1.4) because of its great characteristics and results. Based on the method (1.4), numerous academics have achieved numerous significant advancements (see, e.g., [12–14] and others). Recently, Dong et al. in [13] added inertial to method (1.4) in order to obtain better convergence effect. In [15], Shehu and Iyiola incorporated alternating inertial and adaptive step-sizes:

$$\begin{cases} v_n = \begin{cases} u_n + \alpha_n (u_n - u_{n-1}), & n = \text{odd}, \\ u_n, & n = \text{even}, \end{cases} \\ \overline{u}_n = P_C (v_n - \lambda_n A v_n), \\ d(v_n, \overline{u}_n) = v_n - \overline{u}_n - \lambda_n (A v_n - A\overline{u}_n), \\ u_{n+1} = v_n - \gamma \eta_n d(v_n, \overline{u}_n), \end{cases} \tag{1.6}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu \|v_n - \overline{u}_n\|}{\|A v_n - A\overline{u}_n\|}, \lambda_n\right\}, & A v_n \neq A\overline{u}_n, \\ \lambda_n, & \text{otherwise}, \end{cases} \tag{1.7}$$

and

$$\eta_n = \begin{cases} \frac{\langle v_n - \overline{u}_n, d(v_n, \overline{u}_n)\rangle}{\|d(v_n, \overline{u}_n)\|^2}, & \|d(v_n, \overline{u}_n)\| \neq 0, \\ 0, & \|d(v_n, \overline{u}_n)\| = 0. \end{cases} \tag{1.8}$$

When the assumption of mapping A is relaxed to pseudo-monotone, convergence of the algorithm is proved. Additionally, they gave R-linear convergence analysis when A is a strongly pseudo-monotone mapping. In numerical experiments, the algorithm with alternating inertial in [15] performs better than the algorithm with general inertial in [13].

A fascinating concept has lately been created by Malitsky in [16] to solve mixed variational inequalities problem: find $z^* \in C$ satisfied with

$$\langle Az^*, z - z^* \rangle + g(z) - g(z^*) \geq 0, \ \forall z \in C, \tag{1.9}$$

where A is monotone mapping, g is a proper convex lower semicontinuous function. He proposed the following version of the golden ratio algorithm:

$$\begin{cases} \overline{z}_n = \frac{(\phi - 1) z_n + \overline{z}_{n-1}}{\phi}, \\ z_{n+1} = \text{prox}_{\lambda g} (\overline{z}_n - \lambda A z_n), \end{cases} \tag{1.10}$$

where ϕ is golden ratio, i.e. $\phi = \frac{\sqrt{5}+1}{2}$. In algorithm (1.10), $\overline{z}_n$ is actually a convex combination of all the previously generated iterates. It is straightforward to ascertain that when $g = \iota_C$, (1.9) is equivalent to (1.1). Then, the algorithm (1.10) may be written equivalently as:

$$\begin{cases} \overline{z}_n = \frac{(\phi - 1) z_n + \overline{z}_{n-1}}{\phi}, \\ z_{n+1} = P_C (\overline{z}_n - \lambda A z_n). \end{cases} \tag{1.11}$$

Numerous inertial algorithms have been published to address the issue of pseudo-monotone variational inequalities. Moreover, the golden ratio algorithms and their convergence have been researched for solving mixed variational inequalities problem when A is monotone. However, there are still few results about golden ratio for solving variational inequalities problem (1.1) when A is pseudo-monotone. The algorithm presented by Malitsky is very novel, and it provides us with some inspiration. Under more general circumstances, we hope to solve the variational inequalities problem (1.1) using the convex combination structure in this algorithm.

In this research, we combine the projection contraction method in [11] and golden ratio technique to present a new extrapolation projection contraction algorithm for the pseudo-monotone VI(A, C) (1.1). To speed up the convergence of the new extrapolation projection contraction algorithm, we also present an alternating extrapolation algorithm. We can greatly expand the selection range of step size in the combination structure, and expanding the range of step size has a significant effect on the results of numerical experiments. Although the golden ratio is not used in our algorithm in the end in [13], considering that this paper is inspired by Malitsky's golden ratio algorithm, the algorithms proposed in this paper is still recorded as projection contraction algorithms based on the golden ratio. In this paper, we primarily make the following improvements:

- We propose a projection contraction algorithm and an alternating extrapolation projection contraction algorithm based on the golden ratio. Weak convergence of two algorithms are established when A is pseudo-monotone, sequentially weakly continuous and L-Lipschitz continuous.
- We get R-linear convergence results of two algorithms when A is strongly pseudo-monotone.
- Our algorithms all use the new self-adaptive step-sizes which is not monotonically decreasing, like (1.7).
- In our algorithms, A is a pseudo-monotone mapping which is weaker than [13, 17, 18]. Additionally, it is not necessary to restrict the extrapolation parameter ψ in $\left(1, \frac{\sqrt{5}+1}{2}\right]$ as in [19,20], it can be to extend the value to $(1, +\infty)$.

The structure of the article is as follows:

Section 2: Related knowledge involved in the paper. Section 3: We give a projection contraction algorithm based on the golden ratio and the proofs of weak and R-linear convergence of the algorithm. Section 4: We also give an alternating extrapolation projection contraction algorithm based on the golden ratio, prove weak and R-linear convergence of the algorithm. Section 5: We give two numerical examples to verify the effectiveness of the algorithms.

2. Preliminaries

Let $\{z_n\}$ be a sequence in H. We denote $z_n \rightharpoonup z$ as $\{z_n\}$ weakly converges to z, while denote $z_n \to z$ as $\{z_n\}$ strongly converges to z.

Definition 2.1. *[21] $A : H \to H$ is known as:*
(a) η-strongly pseudo-monotone if

$$\langle Av, u - v\rangle \geq 0 \Rightarrow \langle Au, u - v\rangle \geq \eta \|u - v\|^2, \ \forall u, v \in H,$$

where $\eta > 0$;
(b) pseudo-monotone if

$$\langle Av, u - v\rangle \geq 0 \Rightarrow \langle Au, u - v\rangle \geq 0, \ \forall u, v \in H;$$

(c) L-Lipschitz continuous if there exists a constant $L > 0$ *such that*

$$\|Au - Av\| \leq L\|u - v\|, \ \forall u, v \in H;$$

(d) sequentially weakly continuous if for each sequence $\{u_n\}$ *:*

$$u_n \rightharpoonup u \Longrightarrow Au_n \rightharpoonup Au.$$

Definition 2.2. *[22]* P_C *is called the metric projection onto* C*, if for any point* $u \in H$*, there exists a unique point* $P_C u \in C$ *such that* $\|u - P_C u\| \leq \|u - v\|, \ \forall u \in C$.

Definition 2.3. *[23] Suppose a sequence* $\{z_n\}$ *in* H *converges in norm to* $z^* \in H$*. We say that* $\{z_n\}$ *converges to* z^* *R-linearly if* $\overline{\lim}_{n\to\infty} \|z_n - z^*\|^{\frac{1}{n}} < 1$.

Lemma 2.1. *[21], [22]* P_C *has the following properties:*
(i) $\langle u - v, P_C u - P_C v\rangle \geq \|P_C u - P_C v\|^2, \ \forall u, v \in H;$
(ii) $P_C u \in C$ and $\langle v - P_C u, P_C u - u\rangle \geq 0, \ \forall v \in C.$

Lemma 2.2. *[21] This following equation holds in* H *:*

$$\|\varrho u + (1-\varrho)v\|^2 = \varrho\|u\|^2 + (1-\varrho)\|v\|^2 - \varrho(1-\varrho)\|u - v\|^2, \ \forall \varrho \in \mathbb{R}, \ \forall u, v \in H. \tag{2.1}$$

Lemma 2.3. *[24] Suppose* A *is pseudo-monotone in VI*(A, C) *(1.1) and* S *is the solution set of VI*(A, C) *(1.1). Then* S *is closed, convex and*

$$S = \{z \in C : \langle Aw, w - z\rangle \geq 0, \forall w \in C\}.$$

Lemma 2.4. *[25] Let* $\{z_n\}$ *be a sequence in* H *such that the following two conditions hold:*
(i) for any $z \in C$, $\lim_{n\to\infty} \|z_n - z\|$ *exists;*
(ii) $\omega_w(z_n) \subset S$.
Then $\{z_n\}$ *converges weakly to a point in* C.

Lemma 2.5. *[19] Let* $\{a_n\}$ *and* $\{b_n\}$ *be non-negative real sequences which meet*

$$a_{n+1} \leq a_n - b_n, \ \forall n > N,$$

where N *is some non-negative integer. Then* $\lim_{n\to\infty} b_n = 0$ *and* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 2.6. *[26] Let* $\{\lambda_n\}$ *be non-negative number sequence such that*

$$\lambda_{n+1} \leq \xi_n \lambda_n + \tau_n, \quad \forall n \in \mathbb{N},$$

where $\{\xi_n\}$ *and* $\{\tau_n\}$ *meet*

$$\{\xi_n\} \subset [1, +\infty), \sum_{n=1}^{\infty} (\xi_n - 1) < +\infty, \ \tau_n > 0 \text{ and } \sum_{n=1}^{\infty} \tau_n < +\infty.$$

Then $\lim_{n\to\infty} \lambda_n$ *exists.*

3. Projection contraction algorithm based on the golden ratio

We provide a PCM algorithm with a new extrapolation step and the corresponding convergence analyses in this section.

Assumption 3.1. *In this paper, the following suppositions are true:*
(a) A: $H \to H$ is pseudo-monotone, sequentially weakly continuous and L-Lipschitz continuous.
(b) The solution set S is nonempty.
(c) $\{\xi_n\} \subset [1, +\infty)$, $\sum_{n=1}^{\infty} (\xi_n - 1) < +\infty$, $\tau_n > 0$ and $\sum_{n=1}^{\infty} \tau_n < +\infty$.

Algorithm 3.1. Projection contraction algorithm based on the golden ratio.

Step 0: Take the iterative parameters $\mu \in (0, 1)$, $\psi \in (1, +\infty)$, $\gamma \in (0, 2)$, and $\xi_1, \tau_1, \lambda_1 > 0$. Let $u_1 \in H$, $v_0 \in H$ be given starting points. Known sequences $\{\xi_n\}, \{\tau_n\}$. Set $n := 1$.
Step 1: Compute

$$v_n = \frac{\psi - 1}{\psi} u_n + \frac{1}{\psi} v_{n-1}. \tag{3.1}$$

Step 2: Compute

$$\overline{u}_n = P_C (v_n - \lambda_n A v_n), \tag{3.2}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|v_n - \overline{u}_n\|}{\|Av_n - A\overline{u}_n\|}, \xi_n\lambda_n + \tau_n\right\}, & Av_n \neq A\overline{u}_n, \\ \xi_n\lambda_n + \tau_n, & \text{otherwise.} \end{cases} \tag{3.3}$$

If $v_n = \overline{u}_n$, STOP. Otherwise, go to Step 3.
Step 3: Compute

$$d(v_n, \overline{u}_n) = (v_n - \overline{u}_n) - \lambda_n (Av_n - A\overline{u}_n), \tag{3.4}$$

$$\varphi_n = \langle v_n - \overline{u}_n, d(v_n, \overline{u}_n)\rangle,$$

$$u_{n+1} = v_n - \gamma\beta_n d(v_n, \overline{u}_n), \tag{3.5}$$

where

$$\beta_n = \begin{cases} \frac{\varphi_n}{\|d(v_n, \overline{u}_n)\|^2}, & \|d(v_n, \overline{u}_n)\| \neq 0, \\ 0, & \|d(v_n, \overline{u}_n)\| = 0. \end{cases} \tag{3.6}$$

Step 4: Set $n \leftarrow n + 1$, and go to Step 1.

Remark 3.1. *Observe in Algorithm 3.1 that if $Av_n \neq A\overline{u}_n$, then*

$$\frac{\mu \|v_n - \overline{u}_n\|}{\|Av_n - A\overline{u}_n\|} \geq \frac{\mu}{L}\frac{\|v_n - \overline{u}_n\|}{\|v_n - \overline{u}_n\|} = \frac{\mu}{L}.$$

Therefore, $\lambda_n \geq \min\left\{\frac{\mu}{L}, \lambda_1\right\} > 0$. By (3.3), we have $\lambda_{n+1} \leq \xi_n\lambda_n + \tau_n$. From Lemma 2.6 we obtain $\lim_{n\to\infty} \lambda_n = \lambda$ *when* $\{\xi_n\} \subset [1, +\infty)$, $\sum_{n=1}^{\infty} (\xi_n - 1) < +\infty$, and $\sum_{n=1}^{\infty} \tau_n < +\infty$.

Remark 3.2. *In our algorithms, it is not necessary to restrict the range of ψ to $\left(1, \frac{\sqrt{5}+1}{2}\right]$ or $(1, 2]$, ψ only needs to be greater than* 1*, which greatly relaxes the range of parameter to be chosen.*

Lemma 3.1. *Assume $\{u_n\}$ is the sequence generated by Algorithm 3.1 under the conditions of Assumption 3.1. Then $\{u_n\}$ is bounded and* $\lim_{n\to\infty} \|u_n - u^*\|$ *exists, where $u^* \in S$.*

Proof. It is available from the iterative formate

$$\begin{aligned} \|u_{n+1} - u^*\|^2 &= \|v_n - u^* - \gamma\beta_n d(v_n, \overline{u}_n)\|^2 \\ &= \|v_n - u^*\|^2 - 2\gamma\beta_n \langle v_n - u^*, d(v_n, \overline{u}_n)\rangle + \gamma^2\beta_n^2 \|d(v_n, \overline{u}_n)\|^2. \end{aligned} \tag{3.7}$$

According to (3.2) and Lemma 2.1(i),

$$\begin{aligned} &\langle \overline{u}_n - u^*, v_n - \overline{u}_n - \lambda_n Av_n\rangle \\ =&\langle P_C(v_n - \lambda_n Av_n) - P_C u^*, v_n - \lambda_n Av_n - u^* + u^* - \overline{u}_n\rangle \\ =&\langle P_C(v_n - \lambda_n Av_n) - P_C u^*, v_n - \lambda_n Av_n - u^*\rangle \\ &+ \langle P_C(v_n - \lambda_n Av_n) - P_C u^*, u^* - \overline{u}_n\rangle \\ \geq&\|P_C(v_n - \lambda_n Av_n) - P_C u^*\|^2 + \langle \overline{u}_n - u^*, u^* - \overline{u}_n\rangle \\ =&\|\overline{u}_n - u^*\|^2 - \|\overline{u}_n - u^*\|^2 \\ =&0. \end{aligned} \tag{3.8}$$

Since $\overline{u}_n \in C$ and $u^* \in S$, and Definition 2.1 (b), we have $\langle A\overline{u}_n, \overline{u}_n - u^*\rangle \geq 0$, thus,

$$\lambda_n \langle A\overline{u}_n, \overline{u}_n - u^*\rangle \geq 0. \tag{3.9}$$

Making use of (3.8) and (3.9), we gain

$$\langle \overline{u}_n - u^*, d(v_n, \overline{u}_n)\rangle = \langle \overline{u}_n - u^*, v_n - \overline{u}_n - \lambda_n Av_n + \lambda_n A\overline{u}_n\rangle \geq 0,$$

so,

$$\langle v_n - u^*, d(v_n, \overline{u}_n)\rangle \geq \varphi_n. \tag{3.10}$$

Putting (3.10) in (3.7), we get

$$\|u_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - \gamma(2 - \gamma)\beta_n\varphi_n. \tag{3.11}$$

By (3.5) and (3.6), we gain

$$\beta_n\varphi_n = \|\beta_n d(v_n, \overline{u}_n)\|^2 = \frac{1}{\gamma^2}\|v_n - u_{n+1}\|^2. \tag{3.12}$$

Putting (3.12) in (3.11),

$$\|u_{n+1}-u^*\|^2 \le \|v_n-u^*\|^2 - \frac{2-\gamma}{\gamma}\|v_n-u_{n+1}\|^2 . \tag{3.13}$$

From (3.1) and Lemma 2.2,

$$\begin{aligned}\|u_n-u^*\|^2 &= \frac{\psi}{\psi-1}\|v_n-u^*\|^2 - \frac{1}{\psi-1}\|v_{n-1}-u^*\|^2 + \frac{\psi}{(\psi-1)^2}\|v_n-v_{n-1}\|^2\\ &= \frac{\psi}{\psi-1}\|v_n-u^*\|^2 - \frac{1}{\psi-1}\|v_{n-1}-u^*\|^2 + \frac{1}{\psi}\|u_n-v_{n-1}\|^2 .\end{aligned} \tag{3.14}$$

Combing (3.13) and (3.14),

$$\begin{aligned}\|u_{n+1}-u^*\|^2 - \|u_n-u^*\|^2 \le &-\frac{1}{\psi-1}\|v_n-u^*\|^2 + \frac{1}{\psi-1}\|v_{n-1}-u^*\|^2\\ &-\frac{1}{\psi}\|u_n-v_{n-1}\|^2 - \frac{2-\gamma}{\gamma}\|v_n-u_{n+1}\|^2 ,\end{aligned} \tag{3.15}$$

so we can obtain

$$a_{n+1} \le a_n - b_n,$$

where

$$\begin{aligned}a_n &= \|u_n-u^*\|^2 + \frac{1}{\psi-1}\|v_{n-1}-u^*\|^2 ,\\ b_n &= \frac{2-\gamma}{\gamma}\|v_n-u_{n+1}\|^2 + \frac{1}{\psi}\|u_n-v_{n-1}\|^2 .\end{aligned}$$

From the above proof, we have obtained $a_n \ge 0$ and $b_n \ge 0$. According to Lemma 2.5, we can get $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} a_n$ exists. Thus, we can get further $\lim_{n\to\infty}\|v_n-u_{n+1}\|^2 = 0$.

Inferring from the definition of v_n, we get

$$\begin{aligned}a_{n+1} &= \|u_{n+1}-u^*\|^2 + \frac{1}{\psi-1}\|v_n-u^*\|^2\\ &= \frac{\psi}{\psi-1}\|v_{n+1}-u^*\|^2 + \frac{\psi}{(\psi-1)^2}\|v_{n+1}-v_n\|^2\\ &\quad -\frac{1}{\psi-1}\|v_n-u^*\|^2 + \frac{1}{\psi-1}\|v_n-u^*\|^2\\ &= \frac{\psi}{\psi-1}\|v_{n+1}-u^*\|^2 + \frac{1}{\psi}\|u_{n+1}-v_n\|^2 .\end{aligned} \tag{3.16}$$

We already know that

$$\lim_{n\to\infty}\|v_n-u_{n+1}\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} a_n \text{ exists,} \tag{3.17}$$

we can easily get $\lim_{n\to\infty}\|v_{n+1}-u^*\|^2$ exists. From this it can be concluded that $\lim_{n\to\infty}\|u_{n+1}-u^*\|^2$ exists and $\{u_n\}$, $\{v_n\}$ are bounded. □

Lemma 3.2. *Suppose $\{\overline{u}_n\}$ and $\{v_n\}$ are generated by Algorithm 3.1. Then under Assumption 3.1,* $\lim_{n\to\infty} \|v_n - \overline{u}_n\| = 0$.

Proof. Noting

$$\begin{aligned} \varphi_n &= \|v_n - \overline{u}_n\|^2 - \lambda_n \langle v_n - \overline{u}_n, Av_n - A\overline{u}_n\rangle \\ &\geq \|v_n - \overline{u}_n\|^2 - \lambda_n \|v_n - \overline{u}_n\| \|Av_n - A\overline{u}_n\| \\ &\geq \left(1 - \frac{\lambda_n \mu}{\lambda_{n+1}}\right) \|v_n - \overline{u}_n\|^2 . \end{aligned} \tag{3.18}$$

Available from (3.4),

$$\begin{aligned} \|d(v_n, \overline{u}_n)\| &\leq \|v_n - \overline{u}_n\| + \lambda_n \|Av_n - A\overline{u}_n\| \\ &\leq \left(1 + \frac{\lambda_n}{\lambda_{n+1}}\mu\right) \|v_n - \overline{u}_n\| . \end{aligned} \tag{3.19}$$

Choosing a fixed number ρ in $(\mu, 1)$. Since $\lim_{n\to\infty} \lambda_n = \lambda$, we have $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}}\mu = \mu < \rho$. Then $\exists\, n_0$ such that $\frac{\lambda_n}{\lambda_{n+1}}\mu < \rho$, $\forall n \geq n_0$. Therefore, $\forall n \geq n_0$, we have

$$\|d(v_n, \overline{u}_n)\| < (1 + \rho) \|v_n - \overline{u}_n\| ,$$

and

$$\varphi_n > (1 - \rho) \|v_n - \overline{u}_n\|^2 .$$

Thus,

$$\beta_n = \frac{\varphi_n}{\|d(v_n, \overline{u}_n)\|^2} > \frac{(1 - \rho) \|v_n - \overline{u}_n\|^2}{(1 + \rho)^2 \|v_n - \overline{u}_n\|^2} = \frac{1 - \rho}{(1 + \rho)^2}, \tag{3.20}$$

and so, $\forall n \geq n_0$, we can get

$$\|v_n - \overline{u}_n\|^2 < \frac{1}{1 - \rho}\varphi_n = \frac{1}{(1 - \rho)\beta_n \gamma^2} \|v_n - u_{n+1}\|^2 < \frac{(1 + \rho)^2}{(1 - \rho)^2 \gamma^2} \|v_n - u_{n+1}\|^2 . \tag{3.21}$$

From (3.21), we get $\lim_{n\to\infty} \|v_n - \overline{u}_n\| = 0$. □

Lemma 3.3. *Assume that $\{u_n\}$ is generated by Algorithm 3.1, then $\omega_w(u_n) \subset S$.*

Proof. Since $\{u_n\}$ is bounded, $\omega_w(u_n) \neq \emptyset$. Arbitrarily choose $q \in \omega_w(u_n)$, then there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $u_{n_k} \rightharpoonup q$. Then $\overline{u}_{n_k-1} \rightharpoonup q$, $v_{n_k-1} \rightharpoonup q$. From Lemma 2.1(ii) and (3.2) we have

$$\langle v_n - \lambda_n A v_n - \overline{u}_n, \overline{u}_n - u\rangle \geq 0,\ \forall u \in C,$$

thus,

$$\langle Av_n, u - \overline{u}_n\rangle \geq \frac{1}{\lambda_n} \langle v_n - \overline{u}_n, u - \overline{u}_n\rangle ,\ \forall u \in C. \tag{3.22}$$

From (3.22) we can obtain

$$\langle Av_n, u - v_n\rangle \geq \langle Av_n, \overline{u}_n - v_n\rangle + \frac{1}{\lambda_n} \langle v_n - \overline{u}_n, u - \overline{u}_n\rangle ,\ \forall u \in C.$$

So

$$\left\langle Av_{n_k-1}, u - v_{n_k-1}\right\rangle \geq \left\langle Av_{n_k-1}, \overline{u}_{n_k-1} - v_{n_k-1}\right\rangle + \frac{1}{\lambda_{n_k-1}}\left\langle v_{n_k-1} - \overline{u}_{n_k-1}, u - \overline{u}_{n_k-1}\right\rangle, \ \forall u \in C. \tag{3.23}$$

Fixing $u \in C$ and passing $k \to \infty$ in (3.23), noting $\left\|v_{n_k} - \overline{u}_{n_k}\right\| \to 0$, $\{\overline{u}_{n_k}\}$ and $\{Av_{n_k}\}$ are bounded, we obtain

$$\underline{\lim}_{k\to\infty} \left\langle Av_{n_k-1}, u - v_{n_k-1}\right\rangle \geq 0. \tag{3.24}$$

Choosing a decreasing sequence $\{\epsilon_k\}$ such that $\epsilon_k > 0$ and $\lim_{k\to\infty} \epsilon_k = 0$. For each ϵ_k,

$$Av_{N_k} \neq 0 \text{ and } \left\langle Av_{n_j-1}, u - v_{n_j-1}\right\rangle + \epsilon_k \geq 0, \ \forall j \geq N_k, \tag{3.25}$$

where N_k is smallest non-negative integer that satisfies (3.25). As $\{\epsilon_k\}$ is decreasing, $\{N_k\}$ is increasing. For simplicity, it is useful to write N_k as n_{N_k}. Setting

$$\vartheta_{N_k-1} = \frac{Av_{N_k-1}}{\left\|Av_{N_k-1}\right\|^2},$$

one gets $\left\langle Av_{N_k-1}, \vartheta_{N_k-1}\right\rangle = \left\langle Av_{N_k-1}, \frac{Av_{N_k-1}}{\left\|Av_{N_k-1}\right\|^2}\right\rangle = 1$. Then, by (3.25) for each k,

$$\begin{aligned} &\left\langle Av_{N_k-1}, u + \epsilon_k \vartheta_{N_k-1} - v_{N_k-1}\right\rangle \\ =&\left\langle Av_{N_k-1}, u - v_{N_k-1}\right\rangle + \epsilon_k \left\langle Av_{N_k-1}, \vartheta_{N_k-1}\right\rangle \\ \geq& 0. \end{aligned} \tag{3.26}$$

From Definition 2.1(b), we have

$$\left\langle A\left(u + \epsilon_k \vartheta_{N_k-1}\right), u + \epsilon_k \vartheta_{N_k-1} - v_{N_k-1}\right\rangle \geq 0. \tag{3.27}$$

Since $v_{n_k-1} \rightharpoonup q$ as $k \to \infty$ and Definition 2.1(d), we obtain that $Av_{n_k-1} \rightharpoonup Aq$. Suppose $Aq \neq 0$ (if $Aq = 0, \ q \in S$). Following that, employing the norm's sequentially weakly lower semicontinuity, we gain

$$0 < \|Aq\| \leq \underline{\lim}_{k\to\infty} \left\|Av_{n_k-1}\right\|.$$

Because $\{N_k\} \subset \{n_k\}$, and $\lim_{k\to\infty} \epsilon_k = 0$,

$$0 \leq \overline{\lim}_{k\to\infty} \left\|\epsilon_k \vartheta_{N_k-1}\right\| = \overline{\lim}_{k\to\infty} \left(\epsilon_k \frac{1}{\left\|Av_{N_k-1}\right\|}\right) \leq \frac{\overline{\lim}_{k\to\infty} \epsilon_k}{\underline{\lim}_{k\to\infty} \left\|Av_{N_k-1}\right\|} \leq \frac{0}{\|Aq\|} = 0,$$

and this means $\lim_{k\to\infty} \left\|\epsilon_k \vartheta_{N_k-1}\right\| = 0$. Inputting $k \to \infty$ into (3.27), we get

$$\langle Au, u - q\rangle \geq 0, \ \forall u \in C.$$

From Lemma 2.3, $q \in S$, then $\omega_w(u_n) \subset S$. □

Theorem 3.1. *Assume $\{u_n\}$ is the sequence generated by Algorithm 3.1 under the conditions of Assumption 3.1. There exists $u^\star \in S$ such that $u_n \rightharpoonup u^\star$.*

Proof. From Lemmas 3.1 and 3.3, we get $\lim_{n\to\infty} \|u_n - u^*\|$ exists and $\omega_w(u_n) \subset S$. From Lemma 2.4, $u_n \rightharpoonup u^\star \in S$. □

Theorem 3.2. *Suppose $\{u_n\}$ is generated by Algorithm 3.1 under the condition of A is η-strongly pseudo-monotone with $\eta > 0$. Then $\{u_n\}$ converges R-linearly to the unique solution u^* of VI(A, C) (1.1).*

Proof. Since $\overline{u}_n \in C$, from Definition 2.1(a), we have

$$\langle A\overline{u}_n, \overline{u}_n - u^* \rangle \geq \eta \|\overline{u}_n - u^*\|^2.$$

Multiply λ_n on both sides of above inequality, we get

$$\lambda_n \langle A\overline{u}_n, \overline{u}_n - u^* \rangle \geq \lambda_n \eta \|\overline{u}_n - u^*\|^2. \tag{3.28}$$

(3.8) plus (3.28), we obtain

$$\begin{aligned} \langle \overline{u}_n - u^*, d(v_n, \overline{u}_n) \rangle &= \langle \overline{u}_n - u^*, v_n - \overline{u}_n - \lambda_n A v_n + \lambda_n A\overline{u}_n \rangle \\ &\geq \lambda_n \eta \|\overline{u}_n - u^*\|^2, \end{aligned} \tag{3.29}$$

so

$$\langle v_n - u^*, d(v_n, \overline{u}_n) \rangle \geq \varphi_n + \lambda_n \eta \|\overline{u}_n - u^*\|^2. \tag{3.30}$$

Putting (3.30) into (3.7), we obtain

$$\|u_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - \gamma(2-\gamma)\beta_n \varphi_n - 2\gamma\beta_n\lambda_n\eta \|\overline{u}_n - u^*\|^2. \tag{3.31}$$

Using (3.18) in (3.31), we have

$$\|u_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - \gamma(2-\gamma)\beta_n \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) \|v_n - \overline{u}_n\|^2 - 2\gamma\beta_n\lambda_n\eta \|\overline{u}_n - u^*\|^2, \tag{3.32}$$

where

$$\begin{aligned} &- \gamma(2-\gamma)\beta_n \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) \|v_n - \overline{u}_n\|^2 - 2\gamma\beta_n\lambda_n\eta \|\overline{u}_n - u^*\|^2 \\ \leq &- \gamma\beta_n \min\left\{(2-\gamma)\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right), 2\lambda_n\eta\right\} \left(\|v_n - \overline{u}_n\|^2 + \|\overline{u}_n - u^*\|^2\right) \\ \leq &- \gamma\beta_n \min\left\{\frac{1}{2}(2-\gamma)\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right), \lambda_n\eta\right\} \|v_n - u^*\|^2 \\ < &- \gamma\frac{1-\rho}{(1+\rho)^2} \min\left\{\frac{1}{2}(2-\gamma)(1-\rho), \frac{1}{2}\lambda\eta\right\} \|v_n - u^*\|^2, \ \forall n \geq n_0'. \end{aligned} \tag{3.33}$$

The last inequality is true because there exists n_0' such that $\beta_n > \frac{1-\rho}{(1+\rho)^2}$, $\lambda_n > \frac{\lambda}{2}$ and $\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) > 1 - \rho$, $\forall n \geq n_0'$. Putting (3.33) in (3.32), we get

$$\|u_{n+1} - u^*\|^2 < \left(1 - \gamma\frac{1-\rho}{(1+\rho)^2} \min\left\{\frac{1}{2}(2-\gamma)(1-\rho), \frac{1}{2}\lambda\eta\right\}\right) \|v_n - u^*\|^2. \tag{3.34}$$

Since $\beta_n > \frac{1-\rho}{(1+\rho)^2}$ and $\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) > 1 - \rho$, we obtain

$$0 < 1 - \gamma\frac{1-\rho}{(1+\rho)^2}\min\left\{\frac{1}{2}(2-\gamma)(1-\rho), \frac{1}{2}\lambda\eta\right\} < 1.$$

Let $\delta^2 = 1 - \gamma\frac{1-\rho}{(1+\rho)^2}\min\left\{\frac{1}{2}(2-\gamma)(1-\rho), \frac{1}{2}\lambda\eta\right\}$, we have $0 < \delta^2 < 1$ and

$$\|u_{n+1} - u^*\|^2 < \delta^2 \|v_n - u^*\|^2, \ \forall n \geq n_0'. \tag{3.35}$$

Putting (3.14) into (3.35), after collation, we get

$$\frac{\psi}{\psi - 1}\|v_{n+1} - u^*\|^2 < \left(\delta^2 + \frac{1}{\psi - 1}\right)\|v_n - u^*\|^2, \ \forall n \geq n_0'. \tag{3.36}$$

Since $0 < \delta^2 < 1$, $\delta^2 + \frac{1}{\psi-1} < 1 + \frac{1}{\psi-1} = \frac{\psi}{\psi-1}$. And we can get $0 < \frac{\delta^2+\frac{1}{\psi-1}}{\frac{\psi}{\psi-1}} < 1$. Therefore,

$$\|v_{n+1} - u^*\|^2 < r^2\|v_n - u^*\|^2, \ \forall n \geq n_0',$$

where $r = \sqrt{\frac{\delta^2+\frac{1}{\psi-1}}{\frac{\psi}{\psi-1}}}$. By induction, we get

$$\|v_{n+1} - u^*\|^2 < r^{2\left(n-n_0'+1\right)}\left\|v_{n_0'} - u^*\right\|^2, \ \forall n \geq n_0'.$$

By (3.35),

$$\|u_{n+1} - u^*\|^2 < \delta^2 r^{2\left(n-n_0'\right)}\left\|v_{n_0'} - u^*\right\|^2, \ \forall n \geq n_0'.$$

And we have

$$\|u_{n+1} - u^*\|^{\frac{1}{n}} < r^{\frac{n-n_0'}{n}}\left(\delta\left\|v_{n_0'} - u^*\right\|\right)^{\frac{1}{n}}, \ \forall n \geq n_0'.$$

So

$$\overline{\lim_{n\to\infty}}\|u_{n+1} - u^*\|^{\frac{1}{n}} \leq r < 1.$$

Therefore, $\{u_n\}$ converges R-linearly to the unique solution u^*. □

4. Alternating extrapolation projection contraction algorithm based on the golden ratio

In this part, we offer an algorithm for settling the problem of variational inequalities based on the golden ratio and provide the proofs of weak and R-linear convergence.

Algorithm 4.1. Alternating extrapolation projection contraction algorithm based on the golden ratio.

Step 0: Take the iterative parameters $\mu \in (0,1)$, $\psi \in (1,+\infty)$, $\gamma \in (0,2)$ and $\xi_1, \tau_1, \lambda_1 > 0$. Let $u_1 \in H$,

$v_0 \in H$ be given starting points. Known sequences $\{\xi_n\}, \{\tau_n\}$. Set $n := 1$.
Step 1: Compute

$$v_n = \begin{cases} \frac{\psi-1}{\psi} u_n + \frac{1}{\psi} v_{n-1}, & n = \text{odd}, \\ u_n, & n = \text{even}. \end{cases} \tag{4.1}$$

Step 2: Compute

$$\overline{u}_n = P_C (v_n - \lambda_n A v_n), \tag{4.2}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\mu \|v_n - \overline{u}_n\|}{\|Av_n - A\overline{u}_n\|}, \xi_n \lambda_n + \tau_n \right\}, & Av_n \neq A\overline{u}_n, \\ \xi_n \lambda_n + \tau_n, & \text{otherwise}. \end{cases} \tag{4.3}$$

If $v_n = \overline{u}_n$, STOP. Otherwise, go to Step 3.
Step 3: Compute

$$d(v_n, \overline{u}_n) = (v_n - \overline{u}_n) - \lambda_n (Av_n - A\overline{u}_n), \tag{4.4}$$

$$u_{n+1} = v_n - \gamma \beta_n d(v_n, \overline{u}_n), \tag{4.5}$$

$$\varphi_n = \langle v_n - \overline{u}_n, d(v_n, \overline{u}_n) \rangle$$

where

$$\beta_n = \begin{cases} \frac{\varphi_n}{\|d(v_n, \overline{u}_n)\|^2}, & \|d(v_n, \overline{u}_n)\| \neq 0, \\ 0, & \|d(v_n, \overline{u}_n)\| = 0. \end{cases} \tag{4.6}$$

Step 4: Set $n \leftarrow n + 1$, and go to Step 1.

Lemma 4.1. *Assume $\{u_n\}$ is the sequence generated by Algorithm 4.1 under the conditions of Assumption 3.1. Then $\{u_{2n}\}$ is bounded and* $\lim_{n\to\infty} \|u_{2n} - u^*\|$ *exists, where $u^* \in S$.*

Proof. Following the proof line (3.7)–(3.13) of Lemma 3.1 and $\|v_{2n} - u^*\|^2 = \|u_{2n} - u^*\|^2$, we obtain

$$\|u_{2n+1} - u^*\|^2 \leq \|u_{2n} - u^*\|^2 - \frac{2-\gamma}{\gamma} \|v_{2n} - u_{2n+1}\|^2. \tag{4.7}$$

From (3.13) we have

$$\|u_{2n+2} - u^*\|^2 \leq \|v_{2n+1} - u^*\|^2 - \frac{2-\gamma}{\gamma} \|v_{2n+1} - u_{2n+2}\|^2. \tag{4.8}$$

By the definition of v_n,

$$\|v_{2n+1} - u^*\|^2 = \frac{\psi-1}{\psi} \|u_{2n+1} - u^*\|^2 + \frac{1}{\psi} \|v_{2n} - u^*\|^2 - \frac{\psi-1}{\psi^2} \|v_{2n} - u_{2n+1}\|^2. \tag{4.9}$$

Combing (4.9) and (4.8), we obtain

$$\begin{aligned}\|u_{2n+2}-u^*\|^2-\|u_{2n}-u^*\|^2 \leq & \frac{\psi-1}{\psi}\left(\|u_{2n+1}-u^*\|^2-\|u_{2n}-u^*\|^2\right) \\ & -\frac{2-\gamma}{\gamma}\|v_{2n+1}-u_{2n+2}\|^2-\frac{\psi-1}{\psi^2}\|v_{2n}-u_{2n+1}\|^2 .\end{aligned} \tag{4.10}$$

From (4.7) we have

$$\|u_{2n+1}-u^*\|^2-\|u_{2n}-u^*\|^2 \leq -\frac{2-\gamma}{\gamma}\|v_{2n}-u_{2n+1}\|^2 . \tag{4.11}$$

Combining (4.10) and (4.11), we get

$$\begin{aligned}&\|u_{2n+2}-u^*\|^2-\|u_{2n}-u^*\|^2 \\ \leq & -\frac{\psi-1}{\psi}\cdot\left(\frac{2-\gamma}{\gamma}+\frac{1}{\psi}\right)\|v_{2n}-u_{2n+1}\|^2-\frac{2-\gamma}{\gamma}\|v_{2n+1}-u_{2n+2}\|^2 \\ \leq & 0.\end{aligned} \tag{4.12}$$

Therefore $\|u_{2n+2}-u^*\| \leq \|u_{2n}-u^*\|$. This proves that $\{u_{2n}\}$ is bounded and $\lim_{n\to\infty}\|u_{2n}-u^*\|$ exists. □

Lemma 4.2. *Under Assumption 3.1, suppose* $\{u_{2n}\}$ *and* $\{\overline{u}_{2n}\}$ *are generated by Algorithm 4.1. Then* $\lim_{n\to\infty}\|u_{2n}-\overline{u}_{2n}\|=0$.

Proof. From (4.12) and $u_{2n}=v_{2n}$, we get that $\{\|u_{2n}-u^*\|\}$ is bounded and

$$\lim_{n\to\infty}\|u_{2n}-u_{2n+1}\|=0.$$

From (3.18) and (3.19), we have

$$\varphi_{2n} \geq \left(1-\frac{\lambda_{2n}}{\lambda_{2n+1}}\mu\right)\|v_{2n}-\overline{u}_{2n}\|^2 , \tag{4.13}$$

and

$$\|d\left(v_{2n},\overline{u}_{2n}\right)\| \leq \left(1+\frac{\lambda_{2n}}{\lambda_{2n+1}}\mu\right)\|v_{2n}-\overline{u}_{2n}\| . \tag{4.14}$$

Combining (4.13) and (4.14), we can obtain

$$\begin{aligned}\|u_{2n+1}-v_{2n}\| = \gamma\beta_{2n}\|d\left(v_{2n},\overline{u}_{2n}\right)\| &= \gamma\frac{\varphi_{2n}}{\|d\left(v_{2n},\overline{u}_{2n}\right)\|} \\ &\geq \gamma\left(\frac{1-\frac{\lambda_{2n}\mu}{\lambda_{2n+1}}}{1+\frac{\lambda_{2n}\mu}{\lambda_{2n+1}}}\right)\|v_{2n}-\overline{u}_{2n}\| \\ &> \gamma\frac{1-\rho}{1+\rho}\|v_{2n}-\overline{u}_{2n}\| ,\ \forall n \geq n_0.\end{aligned} \tag{4.15}$$

Using $u_{2n}=v_{2n}$ and $\lim_{n\to\infty}\|u_{2n}-u_{2n+1}\|=0$ in (4.15), we get

$$\lim_{n\to\infty}\|u_{2n}-\overline{u}_{2n}\|=0.$$

□

Lemma 4.3. *Assume that $\{u_{2n}\}$ is generated by Algorithm 4.1, then $\omega_w(u_{2n}) \subset S$.*

Proof. $\forall p \in \omega_w(u_{2n})$, then exists a subsequence $\{u_{2n_k}\} \subset \{u_{2n}\}$, such that $u_{2n_k} \rightharpoonup p$. By Lemma 2.1(ii) and (4.2) we have

$$\langle u_{2n_k} - \lambda_{2n_k} A u_{2n_k} - \bar{u}_{2n_k}, \bar{u}_{2n_k} - u \rangle \geq 0, \ \forall u \in C,$$

thus,

$$\langle A u_{2n_k}, u - \bar{u}_{2n_k} \rangle \geq \frac{1}{\lambda_{n_k}} \langle u_{2n_k} - \bar{u}_{2n_k}, u - \bar{u}_{2n_k} \rangle, \forall u \in C,$$

and

$$\frac{1}{\lambda_{2n_k}} \langle u_{2n_k} - \bar{u}_{2n_k}, u - \bar{u}_{2n_k} \rangle + \langle A u_{2n_k}, \bar{u}_{2n_k} - u_{2n_k} \rangle \leq \langle A u_{2n_k}, u - u_{2n_k} \rangle, \forall u \in C. \tag{4.16}$$

Similar to Lemma 3.3, the following proof steps are omitted as they are redundant. Thus, we come to the conclusion,

$$\langle Au, u - p \rangle \geq 0, \ \forall u \in C.$$

Using Lemma 2.3, we get $p \in S$. □

Theorem 4.1. *Assume $\{u_n\}$ is the sequence generated by Algorithm 4.1 under the conditions of Assumption 3.1. There exists $q \in S$ such that $u_n \rightharpoonup q$.*

Proof. $\{u_{2n}\}$ is bounded implies that $\{u_{2n}\}$ has weakly convergent subsequences. Then, we can choose a subsequence of $\{u_{2n}\}$, denoted by $\{u_{2n_k}\}$ such that $u_{2n_k} \rightharpoonup q \in H$. We obtain $\lim_{n\to\infty} \|u_{2n} - q\|$ exists and $q \in S$ from Lemma 4.1 and 4.3. The proof of the whole sequence $u_{2n} \rightharpoonup q \in S$ which is the same as Lemma 4.4 in [15]. Hence, $u_n \rightharpoonup q \in S$. □

Theorem 4.2. *Suppose $\{u_n\}$ is generated by Algorithm 4.1 under the condition of A is η-strongly pseudo-monotone with $\eta > 0$. Then $\{u_n\}$ converges R-linearly to the unique solution u^* of VI(A, C) (1.1).*

Proof. From (3.35), $\forall n \geq n_0'$, we have

$$\|u_{2n+1} - u^*\|^2 < \delta^2 \|v_{2n} - u^*\|^2 = \delta^2 \|u_{2n} - u^*\|^2, \tag{4.17}$$

and

$$\|u_{2n+2} - u^*\|^2 < \delta^2 \|v_{2n+1} - u^*\|^2, \tag{4.18}$$

where $\delta^2 = 1 - \gamma \frac{1-\rho}{(1+\rho)^2} \min\left\{\frac{1}{2}(2-\gamma)(1-\rho), \frac{1}{2}\lambda\eta\right\}$ and $0 < \delta^2 < 1$. Combining (4.9) and (4.18),

$$\|u_{2n+2} - u^*\|^2 < \delta^2 \left(\frac{\psi - 1}{\psi} \|u_{2n+1} - u^*\|^2 + \frac{1}{\psi} \|u_{2n} - u^*\|^2 - \frac{\psi - 1}{\psi^2} \|v_{2n} - u_{2n+1}\|^2 \right). \tag{4.19}$$

Putting (4.17) in (4.19), we have

$$\begin{aligned}
&\|u_{2n+2} - u^*\|^2 \\
&< \delta^2 \left(\frac{\psi - 1}{\psi} \delta^2 \|u_{2n} - u^*\|^2 + \frac{1}{\psi} \|u_{2n} - u^*\|^2 - \frac{\psi - 1}{\psi^2} \|v_{2n} - u_{2n+1}\|^2 \right) \\
&\leq \delta^2 \left(\frac{\psi - 1}{\psi} \delta^2 + \frac{1}{\psi} \right) \|u_{2n} - u^*\|^2 \\
&< \delta^2 \|u_{2n} - u^*\|^2, \ \forall n \geq n_0'.
\end{aligned} \tag{4.20}$$

So

$$\|u_{2n+2}-u^*\|^2 < \delta^2 \|u_{2n}-u^*\|^2, \ \forall n \geq n_0^{'}. \tag{4.21}$$

By induction, we have

$$\|u_{2n+2}-u^*\|^2 < \delta^{2\left(n-n_0^{'}+1\right)} \left\|u_{2n_0^{'}}-u^*\right\|^2, \ \forall n \geq n_0^{'}.$$

Thus,

$$\begin{aligned}\|u_{2n+3}-u^*\|^2 &< \delta^2 \|u_{2n+2}-u^*\|^2 \\ &< \|u_{2n+2}-u^*\|^2 \\ &< \delta^{2\left(n-n_0^{'}+1\right)} \left\|u_{2n_0^{'}}-u^*\right\|^2, \ \forall n \geq n_0^{'}.\end{aligned} \tag{4.22}$$

Therefore, $\{u_n\}$ converges R-linearly to the unique solution u^* . □

5. Numerical examples

The following sections provide some computational experiments and comparisons between our algorithms considered in Sections 3 and 4 and other algorithms. All codes were written in MATLAB R2016b and performed on a PC Desktop AMD Ryzen R7-5600U CPU @ 3.00 GHz, RAM 16.00 GB.

We make a comparison of our Algorithm 3.1, Algorithm 4.1, Algorithm 2 in [15] and Algorithm 1 in [27], Time in the table indicates CPU Time. In this section, we set maximum number of iterations $n_{max} = 6 \times 10^5$, $\xi_n = 1 + \frac{1}{n^2}$ and $\tau_n = \frac{1}{n^2}$.

Example 5.1. *Define* $A : \mathbb{R}^m \to \mathbb{R}^m$ *by*

$$Au = (M+\beta)(Nu+q),$$

where $M = e^{-u^T Qu}$*, N is a positive semi-definite matrix, Q is a positive definite matrix,* $q \in \mathbb{R}^m$ *and* $\beta > 0$*. In addition to being easy to obtain, A is pseudo-monotone, differentiable and Lipschitz continuous. Take* $C = \{u \in \mathbb{R}^m \mid Bu \leq b\}$*, where B is a* $k^* \times m$ *matrix and* $b \in \mathbb{R}_+^{k^*}$ *with* $k^* = 10$*. Select the initial point* $u_1 = (1, 1, \ldots, 1)^T$ *for all algorithms. Initial points of Algorithm 3.1 and Algorithm 4.1,* v_0 *is generated randomly in* $\mathbb{R}^m$*. We take* $\psi = \frac{\sqrt{5}+1}{2}$*,* $\mu = 0.6$ *in Algorithm 3.1 and Algorithm 4.1. We take* $\theta_n = \frac{2-\gamma}{1.01\gamma}$ *in Algorithm 2 in [15] and* $\theta = 0.45(1-\mu)$ *in Algorithm 1 in [27]. Thus, we take different values for* λ_1 *and* γ *respectively to compare with the algorithms in the other two papers. In this example, we take* $\|\overline{u}_n - v_n\| < 10^{-3}$ *as the stopping criterion.*

In Table 1, we give a comparison of our Algorithms 3.1 and 4.1 with Algorithm 1 in [27] and Algorithm 2 in [15] in different dimensions for $\gamma = 1.5, \lambda_1 = 0.05$ *and a comparison Figure 1 for* $m = 100$*. It is illustrated that our two algorithms have some superiority.*

Table 1. Example 5.1 with $\gamma = 1.5, \lambda_1 = 0.05$ and various values of m.

Problem size		Alg 3.1		Alg 4.1		Alg 2 in [15]		Alg 1 in [27]	
k^*	m	Iter	Time	Iter	Time	Iter	Time	Iter	Time
10	100	1821	0.3262	1162	0.2071	1979	0.3926	7644	1.4647
	150	1568	0.3271	1050	0.2163	4716	0.9535	29390	6.3288
	200	1645	0.4181	1164	0.3074	6754	1.6716	——	——
	300	2034	0.7110	1192	0.4525	18566	6.0136	——	——
	500	2641	2.2280	1584	1.4692	55310	27.8013	——	——
	1000	3885	9.3913	2332	5.8732	——	——	——	——

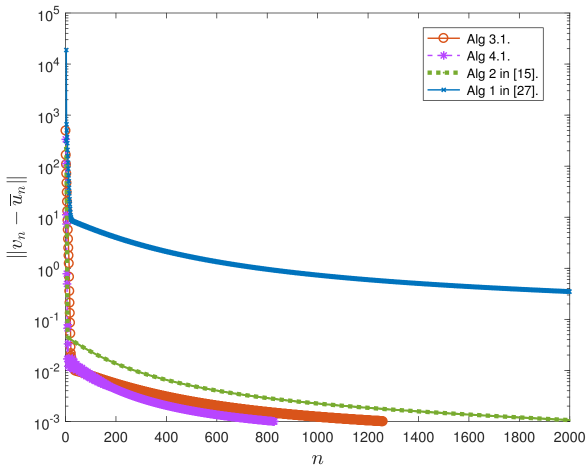

Figure 1. Relationship between error value and iteration times in Example 5.1 with $k^* = 10, m = 100$.

In Tables 2 and 3, we give a comparison of Algorithm 3.1 and Algorithm 4.1 for the same number of dimensions with different γ, respectively. We find that the larger γ is for both algorithms in the same dimension, the fewer the iterations and the shorter the CPU Time, where $\gamma \in (0, 2)$.

Table 2. Algorithm 3.1 with different γ.

γ	$m = 200$		$m = 400$		$m = 800$		$m = 1000$	
	Iter	Time	Iter	Time	Iter	Time	Iter	Time
0.25	9996	2.4136	14529	1.5986	20320	27.2981	23441	42.2450
0.5	4746	1.1729	7244	3.3542	10136	14.8189	11069	21.5681
1	2438	0.6266	3402	1.7116	5049	8.4571	5736	12.3817
1.25	1939	0.6079	2866	1.4242	4126	7.2440	4515	10.2743
1.5	1546	0.4253	2315	1.1200	3386	6.0285	3878	9.1699

Table 3. Algorithm 4.1 with different γ.

γ	$m = 200$		$m = 400$		$m = 800$		$m = 1000$	
	Iter	Time	Iter	Time	Iter	Time	Iter	Time
0.25	7212	1.7554	10193	4.6670	14085	19.7867	17188	32.4055
0.5	3063	6.7968	4836	2.3662	7362	11.7332	8108	16.3521
1	1626	0.4460	2292	1.1557	3170	5.8066	3324	8.0294
1.25	1429	0.7354	1468	0.8067	2602	5.0285	3212	7.8888
1.5	964	0.2697	1300	0.7061	2104	4.1989	2586	6.4205

Example 5.2. *[28] Define a mapping A by*

$$Au = \left(M^T M + N + P\right) u.$$

The matrices N and P are randomly generated skew-symmetric matrix and positive diagonal matrix, respectively. Assume $C := \{u \in \mathbb{R}^m \mid Mu \leq p\}$, *where matrix* $M \in \mathbb{R}^{k \times m}$ *and vector* $p \in \mathbb{R}^k$ *are randomly generated. Thus, all entries in p are non-negative. Here VI(A, C) (1.1) has a unique solution* $u^* = 0$. *Set* $\psi = \frac{\sqrt{5}+1}{2}$, $\mu = \frac{1}{\sqrt{2}}$ *in Algorithm 3.1, 4.1. We choose* $\theta_n = \frac{2-\gamma}{1.01\gamma}$ *in Algorithm 2 in [15] and* $\theta = 0.45\,(1-\mu)$ *in Algorithm 1 in [27]. Additionally, we take different values for* λ_1 *and* γ *, respectively, to compare with the algorithms in the other two papers. We use the stopping criterion* $\|\overline{u}_n - y_n\| \leq 10^{-3}$.

In Table 4, we give a comparison of our Algorithm 3.1 and Algorithm 4.1 with Algorithm 1 in [27] and Algorithm 2 in [15] in different dimensions for $\gamma = 1.5, \lambda_1 = 0.05$ *and a comparison Figure 2 for* $k = 30, m = 60$.

In Figures 3 and 4 we compared the impact of Algorithm 3.1 and Algorithm 4.1 with varying ψ.

Table 4. Example 5.2 with $\gamma = 1.5, \lambda_1 = 0.05$ and various values of k, m.

Problem size		Alg 3.1		Alg 4.1		Alg 2 in [15]		Alg 1 in [27]	
k	m	Iter	Time	Iter	Time	Iter	Time	Iter	Time
30	60	1146	0.2566	437	0.0990	2227	0.5476	7656	12.8395
	100	1429	0.3478	510	0.1237	6731	2.2071	23695	277.4316
	120	1680	0.4009	586	0.1407	7585	2.4785	27968	245.5148
50	50	1359	0.4956	501	0.1841	1166	0.5706	4987	15.0018
	100	1434	0.5570	476	0.1862	5817	3.1011	22307	300.3848
	150	1439	0.6184	420	0.1832	14681	8.5531	52346	1.8247e+03
100	100	1399	1.0800	498	0.3947	4115	4.7235	16011	514.9273
	200	1445	1.3125	405	0.3740	19535	28.4272	——	——
500	500	1561	0.7445	448	0.2162	25091	17.1813	——	——
	1000	887	8.8188	255	2.5293	——	——	——	——
1000	1000	957	18.0159	270	5.0675	——	——	——	——
	2000	647	22.4660	206	7.0048	——	——	——	——

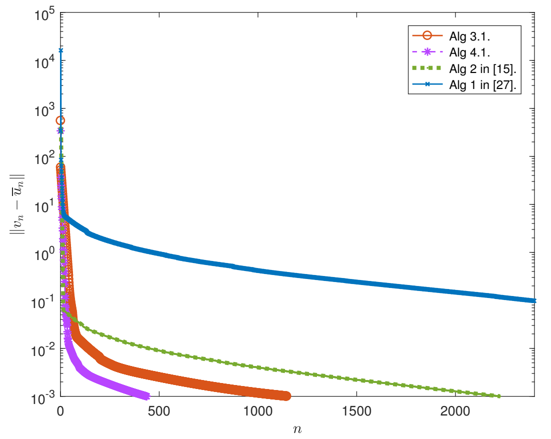

Figure 2. Relationship between error value and iteration times in Example 5.2 with $\gamma = 1.5, \lambda_1 = 0.05, k = 30, m = 60$.

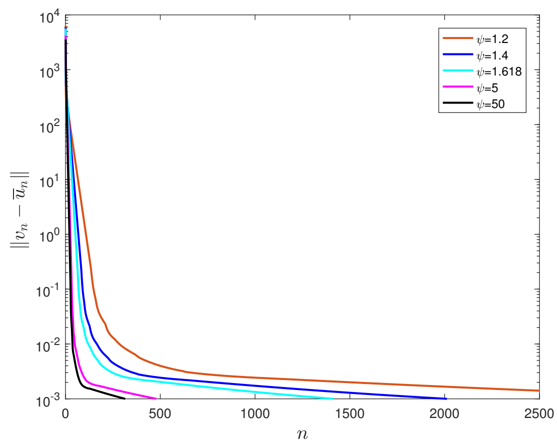

Figure 3. Algorithm 3.1 with varying ψ.

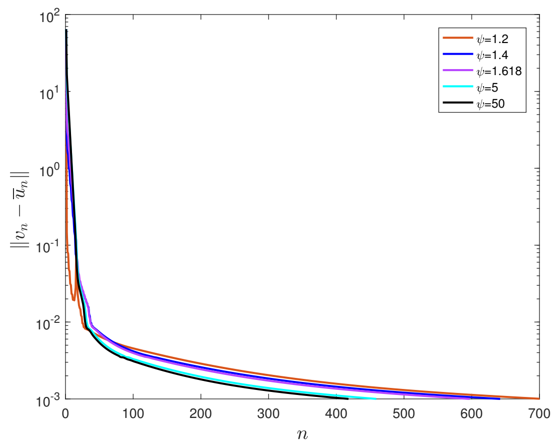

Figure 4. Algorithm 4.1 with varying ψ.

Remark 5.1. *From Figures 1 and 2, we can see that the projection contraction algorithms based on golden ratio have numerical advantages over inertial extrapolation. Alternating extrapolation projection contraction algorithm is better than projection contraction algorithm based on golden ratio. Thus, it can be seen from Figures 3 and 4 that our algorithms with larger ψ converges faster.*

6. Conclusions

We present a projection contraction algorithm and an alternating extrapolation projection contraction algorithm based on the golden ratio for solving pseudo-monotone variational inequalities problem in real Hilbert spaces. We give proofs of weak convergence of the two algorithms when the operator is pseudo-monotone. Thus, we obtain R-linear convergence when A is strongly pseudo-monotone mapping. We have extended the range of the ψ from $\left(1, \frac{\sqrt{5}+1}{2}\right]$ to $(1, +\infty)$, and the proofs of both algorithms are given in the absence of Lipschitz constant. We give numerical examples and show the superiority of our algorithms. Then, we discover that our algorithms suffer less impact under the same unfavorable conditions and has a relatively stable rate of convergence.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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