Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 23247–23271.
DOI:10.3934/math.20231182
Received: 29 March 2023
Revised: 26 June 2023
Accepted: 28 June 2023
Published: 21 July 2023

Research article

Classification of spacelike conformal Einstein hypersurfaces in Lorentzian space $\mathbb{R}^{n+1}_1$

Yayun Chen[1] **and Tongzhu Li**[2,*]

[1] Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China
[2] Beijing Key Laboratory on MCAACI, Beijing 100081, China

* **Correspondence:** Email: litz@bit.edu.cn.

Abstract: Let $f : M^n \to \mathbb{R}^{n+1}_1$ be an n-dimensional umbilic-free spacelike hypersurface in the $(n+1)$-dimensional Lorentzian space $\mathbb{R}^{n+1}_1$ with an induced metric I. Let II be the second fundamental form and H the mean curvature of f. One can define the conformal metric $g = \frac{n}{n-1}(\|II\|^2 - nH^2)I$ on $f(M^n)$, which is invariant under the conformal transformation group of $\mathbb{R}^{n+1}_1$. If the Ricci curvature of g is constant, then the spacelike hypersurface f is called a conformal Einstein hypersurface. In this paper, we completely classify the n-dimensional spacelike conformal Einstein hypersurfaces up to a conformal transformation of $\mathbb{R}^{n+1}_1$.

Keywords: conformal metric; conformal sectional curvature; conformal Einstein hypersurface; conformal transformation group
Mathematics Subject Classification: 53A30, 53B30

1. Introduction

Let $\mathbb{R}^{n+2}_s$ be the real vector space R^{n+2} with the Lorentzian product $\langle , \rangle_s$ given by the following:

$$\langle X, Y\rangle_s = -\sum_{i=1}^{s} x_i y_i + \sum_{j=s+1}^{n+2} x_j y_j.$$

Let $\mathbb{R}^{n+2}$ denote the $(n+2)$-dimensional Euclidean space and a dot $\cdot$ represent its inner product. For any $a > 0$, the standard sphere $\mathbb{S}^{n+1}(a)$, the hyperbolic space $\mathbb{H}^{n+1}(-a)$, the de sitter space $\mathbb{S}^{n+1}_1(a)$ and the anti-de sitter space $\mathbb{H}^{n+1}_1(-a)$ are defined by the following:

$$\mathbb{S}^{n+1}(a) = \{x \in \mathbb{R}^{n+2} | x\cdot x = a^2\},\ \ \mathbb{H}^{n+1}(-a) = \{x \in \mathbb{R}^{n+2}_1 | \langle x, x\rangle_1 = -a^2\},$$
$$\mathbb{S}^{n+1}_1(a) = \{x \in \mathbb{R}^{n+2}_1 | \langle x, x\rangle_1 = a^2\},\ \ \mathbb{H}^{n+1}_1(-a) = \{x \in \mathbb{R}^{n+2}_2 | \langle x, x\rangle_2 = -a^2\}.$$

Let $M_1^{n+1}(c)$ be the Lorentz space form with constant sectional curvature c with respect to its standard Lorentzian metric. When $c = 0$, $M_1^{n+1}(c) = \mathbb{R}_1^{n+1}$. When $c = 1$, $M_1^{n+1}(c) = \mathbb{S}_1^{n+1}(1)$. When $c = -1$, $M_1^{n+1}(c) = \mathbb{H}_1^{n+1}(-1)$.

A diffeomorphism $\Phi : M_1^{n+1}(c) \to M_1^{n+1}(c)$ is called a conformal transformation, if $\Phi^* h = e^{2\tau} h$ for some smooth function τ on $M_1^{n+1}(c)$, where h denotes the standard Lorentzian metric of $M_1^{n+1}(c)$. All conformal transformations form a transformation group, which is called the conformal group of $M_1^{n+1}(c)$. In [2], X. Ji et al. studied the conformal geometry of spacelike hypersurfaces in the Lorentz space form $M_1^{n+1}(c)$. They defined the conformal metric g and the conformal second fundamental form B on a spacelike hypersurface, which determined the spacelike hypersurface up to a conformal transformation of $M_1^{n+1}(c)$. Since the conformal geometry of spacelike hypersurfaces in Lorentzian space forms $M_1^{n+1}(c)$ is uniform by the conformal map (2.1), in this paper, we only consider the conformal geometry of spacelike hypersurfaces in the Lorentzian space $\mathbb{R}_1^{n+1}$.

Let $f : M^n \to \mathbb{R}_1^{n+1}$ be an n-dimensional umbilic-free spacelike hypersurface in $\mathbb{R}_1^{n+1}$, and let $I = \langle df, df\rangle_1$ be the induced metric, II be the second fundamental form and H be the mean curvature. The conformal metric g and the conformal second fundamental form B of the hypersurface are defined by, respectively,

$$g = \rho^2 \langle df, df\rangle_1 = \frac{n}{n-1}(\|II\|^2 - nH^2)I, \quad B = \rho \sum_{ij}(II - HI), \tag{1.1}$$

which form a complete conformal invariant of the spacelike hypersurface when the dimension of the spacelike hypersurface $n \geq 3$ (see Section 2). In the conformal geometry of spacelike hypersufaces, a notable class of spacelike hypersurfaces are those with constant conformal sectional curvature (i.e., constant sectional curvature with respect to the conformal metric g). In [2], the authors have classified the spacelike hypersurfaces with constant conformal sectional curvature up to a conformal transformation of $\mathbb{R}_1^{n+1}$.

Theorem 1.1. *Let $f : M^n \to \mathbb{R}_1^{n+1}$, $(n \geq 3)$, be an umbilic-free spacelike hypersurface with constant conformal sectional curvature δ in $\mathbb{R}_1^{n+1}$. Then, f is locally conformally equivalent to one of the following hypersurfaces:*

1) a cylinder over a curvature-spiral in a Lorentzian 2-plane $\mathbb{R}_1^2$ (where $\delta \leq 0$);
2) a cone over a curvature-spiral in a de sitter 2-sphere $\mathbb{S}_1^2 \subset \mathbb{R}_1^3$ (where $\delta < 0$);
3) a rotational hypersurface over a curvature-spiral in a Lorentzian hyperbolic 2-plane $\mathbb{R}_{1+}^2 \subset \mathbb{R}_1^2$ (the constant curvature δ could be positive, negative or zero); and
4) a cone over the hyperbolic torus $\mathbb{H}^1(-\sqrt{a^2-1}) \times \mathbb{S}^1(a)$, $a > 1$ (where $\delta = 0$).

The curvature-spiral $\gamma(s)$ in a 2-dimensional Lorentzian space form $M_1^2(c)$ is determined by the following intrinsic equation:

$$\left[\frac{d}{ds}\frac{1}{\kappa}\right]^2 + c\left[\frac{1}{\kappa}\right]^2 = -\delta, \tag{1.2}$$

where s is the arc-length parameter, and κ denotes the geodesic curvature of the spacelike curve γ, and δ is a real constant. The definition of the Lorentzian hyperbolic n-plane $\mathbb{R}_{1+}^n \subset \mathbb{R}_1^n$ is given in Section 3.

Another notable class of spacelike hypersurfaces are those with a constant conformal Ricci curvature (i.e., constant Ricci curvature with respect to the conformal metric g), which is called a conformal Einstein hypersurface. Clearly, the spacelike hypersurface with a constant conformal sectional curvature is a conformal Einstein hypersurface, but the converse may not be true when the dimension of the spacelike hypersurface $n \geq 4$. In this paper, our goal is to classify these conformal Einstein hypersurfaces of dimension $n \geq 4$. We note that some of such examples come from cones, cylinders, or rotational hypersurfaces over the spacelike (λ, n)-surfaces in the 3-dimensional Lorentzian space forms $\mathbb{S}^3_1(1), \mathbb{R}^3_1, \mathbb{R}^3_{1+}$, respectively, so we first give the definition of the spacelike (λ, n)-surface as follows.

Definition 1.1. *Let $u : M^2 \to M^3_1(c)$ be an umbilic-free spacelike surface in $M^3_1(c)$, and let I_u, H_u, K_u be the induced metric, the mean curvature, the Gauss curvature of u, respectively. Let $Hess$ be the Hessian operator with respect to I_u and ∇ the gradient with respect to I_u. For a positive integer $n \geq 4$, let*

$$\Lambda = \frac{1}{\sqrt{4H_u^2 - \frac{2n}{n-1}(K_u + c)}}.$$

The surface u is called an (λ, n)-surface for some λ=constant, if the Hessian matrix and the gradient of the function Λ satisfy the following equations:

$$Hess(\Lambda) = \frac{(n-3)c\Lambda - K_u\Lambda}{n-2} I_u, \quad |\nabla\Lambda|^2 = \frac{\Lambda^2[n(n-3)c - 2K_u]}{(n-1)(n-2)} - \frac{\lambda}{n-1}.$$

Our main result is given as follows.

Theorem 1.2. *Let $f : M^n \to \mathbb{R}^{n+1}_1 (n \geq 3)$ be a spacelike conformal Einstein hypersurface without umbilical points in $\mathbb{R}^{n+1}_1$. Then, f is locally conformally equivalent to one of the following spacelike hypersurfaces:*

1) spacelike hypersurfaces with constant conformal sectional curvature;
2) the spacelike hypersurface

$$f : \mathbb{H}^k\Big(-\sqrt{\frac{k-1}{n-2}}\Big) \times \mathbb{H}^{n-k}\Big(-\sqrt{\frac{n-k-1}{n-2}}\Big) \to \mathbb{H}^{n+1}_1(-1), \quad 1 < k < n-1;$$

3) a cylinder over a spacelike (λ, n)-surface in $\mathbb{R}^3_1$, $(n \geq 4)$;
4) a cone over a spacelike (λ, n)-surface in $\mathbb{S}^3_1(1)$, $(n \geq 4)$; and
5) a rotational hypersurface over a spacelike (λ, n)-surface in $\mathbb{R}^3_+$, $(n \geq 4)$.

The rest of this paper is organized as follows. In Section 2, we study the conformal geometry of spacelike hypersurfaces in $\mathbb{R}^{n+1}_1$. In Section 3, we construct some examples of the spacelike conformal Einstein hypersurfaces. In Section 4, we give the proof of the classification Theorem 1.2.

2. Conformal geometry of spacelike hypersurfaces

In this section, we recall some conformal invariants of a spacelike hypersurface and give a congruent theorem of the spacelike hypersurfaces under the conformal transformation group of $\mathbb{R}^{n+1}_1$. For details readers refer to [2–4].

Let C^{n+2} be the cone in $\mathbb{R}^{n+3}_2$ and $\mathbb{Q}^{n+1}_1$ the conformal compactification space in $\mathbb{R}P^{n+3}$ defined by the following:

$$C^{n+2} = \{X \in \mathbb{R}^{n+3}_2 | \langle X, X\rangle_2 = 0, X \neq 0\}, \quad \mathbb{Q}^{n+1}_1 = \{[X] \in \mathbb{R}P^{n+2} | \langle X, X\rangle_2 = 0\}.$$

Let $O(n+3,2)$ be the Lorentzian group of the $\mathbb{R}^{n+3}_2$ keeping $\langle,\rangle_2$ invariant. $O(n+3,2)$ is also a transformation group of $\mathbb{Q}^{n+1}_1$ and the action is defined by the following:

$$T([X]) = [XT], \quad X \in C^{n+2}, \quad T \in O(n+3,2).$$

Topologically, $\mathbb{Q}^{n+1}_1$ is identified with the compact space $\mathbb{S}^n \times \mathbb{S}^1/\mathbb{S}^0$, which is endowed by a standard Lorentzian metric $h = g_{\mathbb{S}^n} \oplus (-g_{\mathbb{S}^1})$, where $g_{\mathbb{S}^k}$ denotes the standard metric of the k-dimensional sphere $\mathbb{S}^k$. Therefore, $\mathbb{Q}^{n+1}_1$ has the conformal metric class $[h]$ and $[O(n+3,2)]$ is the conformal transformation group of $\mathbb{Q}^{n+1}_1$(see [1,5]).

Let $X = (x_1, \cdots, x_{n+3}) \in \mathbb{R}^{n+3}_2$, $P = \{[X] \in \mathbb{Q}^{n+1}_1 | x_1 = x_{n+3}\}$, $P_- = \{[X] \in \mathbb{Q}^{n+1}_1 | x_{n+3} = 0\}$, $P_+ = \{[X] \in \mathbb{Q}^{n+1}_1 | x_1 = 0\}$, we can define the following conformal diffeomorphisms:

$$\begin{array}{ll} \sigma_0 : \mathbb{R}^{n+1}_1 \to \mathbb{Q}^{n+1}_1 \backslash P, & u \mapsto [(\frac{1+\langle u,u\rangle_1}{2}, u, \frac{\langle u,u\rangle_1 - 1}{2})], \\ \sigma_1 : \mathbb{S}^{n+1}_1(1) \to \mathbb{Q}^{n+1}_1 \backslash P_+, & u \mapsto [(1,u)], \\ \sigma_{-1} : \mathbb{H}^{n+1}_1(-1) \to \mathbb{Q}^{n+1}_1 \backslash P_-, & u \mapsto [(u,1)]. \end{array} \tag{2.1}$$

We may regard $\mathbb{Q}^{n+1}_1$ as the common compactification of $\mathbb{R}^{n+1}_1, \mathbb{S}^{n+1}_1(1), \mathbb{H}^{n+1}_1(-1)$.

Let $f : M^n \to M^{n+1}_1(c)$ be a spacelike hypersurface. Using σ_c, we obtain the hypersurface $\sigma_c \circ f : M^n \to \mathbb{Q}^{n+1}_1$ in $\mathbb{Q}^{n+1}_1$. From [1,2], we have the following theorem.

Theorem 2.1. *[2] Two hypersurfaces $f, \bar{f} : M^n \to M^{n+1}_1(c)$ are conformally equivalent if and only if there exists $T \in O(n+3,2)$ such that $\sigma_c \circ f = T(\sigma_c \circ \bar{f}) : M^n \to \mathbb{Q}^{n+1}_1$.*

Let $f : M^n \to \mathbb{R}^{n+1}_1$ be an umbilic-free spacelike hypersurface, II be the second fundamental form, and H be the mean curvature; then, the conformal position vector $Y : M^n \to \mathbb{R}^{n+3}_2$ of the spacelike hypersurface f is defined by the following:

$$Y = \rho^2\Big(\frac{\langle f,f\rangle_1 + 1}{2}, f, \frac{\langle f,f\rangle_1 - 1}{2}\Big), \quad \rho^2 = \frac{n}{n-1}(|II|^2 - n|H|^2).$$

Theorem 2.2. *[2] Two spacelike hypersurfaces $f, \bar{f} : M^n \to \mathbb{R}^{n+1}_1$ are conformally equivalent if and only if there exists $T \in O(n+3,2)$ such that $\bar{Y} = YT$, where $Y, \bar{Y}$ are the conformal position vector of $f, \bar{f}$, respectively.*

From Theorem 2.2, it immediately follows that

$$g = \langle dY, dY\rangle_2 = \rho^2 \langle df, df\rangle_1$$

is a conformal invariant, which is called the conformal metric of f.

Let $\{E_1, \cdots, E_n\}$ be a local orthonormal basis of M^n with respect to g, with dual basis $\{\omega_1, \cdots, \omega_n\}$. Denote $Y_i = E_i(Y)$ and define the following:

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y\rangle_2 Y,$$

where Δ is the Laplace operator of g; then we have

$$\langle N, Y\rangle_2 = 1,\ \langle N, N\rangle_2 = 0,\ \langle N, Y_k\rangle_2 = 0,\ \langle Y_i, Y_j\rangle_2 = \delta_{ij}, \quad 1 \le i, j, k \le n.$$

We may decompose $\mathbb{R}^{n+3}_2$ such that

$$\mathbb{R}^{n+3}_2 = \mathrm{span}\{Y, N\} \oplus \mathrm{span}\{Y_1, \cdots, Y_n\} \oplus \mathbb{V},$$

where $\mathbb{V} \perp \mathrm{span}\{Y, N, Y_1, \cdots, Y_n\}$. We call $\mathbb{V}$ the conformal normal bundle of f, which is a linear bundle. Let ξ be a local section of $\mathbb{V}$ and $\langle \xi, \xi\rangle_2 = -1$. ξ is called the conformal normal vector field of the spacelike hypersurface. Therefore, $\{Y, N, Y_1, \cdots, Y_n, \xi\}$ forms a moving frame in $\mathbb{R}^{n+3}_2$ along M^n. We write the structure equations as follows:

$$\begin{aligned}
\mathrm{d}Y &= \sum_i \omega_i Y_i,\\
\mathrm{d}N &= \sum_{ij} A_{ij}\omega_j Y_i + \sum_i C_i \omega_i \xi,\\
\mathrm{d}Y_i &= -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_j B_{ij}\omega_j \xi,\\
\mathrm{d}\xi &= \sum_i C_i \omega_i Y + \sum_{ij} B_{ij}\omega_j Y_i,
\end{aligned} \tag{2.2}$$

where $\omega_{ij}(= -\omega_{ji})$ are the connection 1-forms on M^n with respect to $\{\omega_1, \cdots, \omega_n\}$. It is clear that $A = \sum_{ij} A_{ij}\omega_j \otimes \omega_i$, $B = \sum_{ij} B_{ij}\omega_j \otimes \omega_i$, and $C = \sum_i C_i \omega_i$ are globally defined conformal invariants. We call A, B and C the conformal 2-tensor, the conformal second fundamental form and the conformal 1-form, respectively. The covariant derivatives of these tensors are defined by the following:

$$\begin{aligned}
\sum_j C_{i,j}\omega_j &= dC_i + \sum_k C_k \omega_{kj},\\
\sum_k A_{ij,k}\omega_k &= dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki},\\
\sum_k B_{ij,k}\omega_k &= dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}.
\end{aligned}$$

By exterior differentiation of the structure Eq (2.2), we can get the integrable conditions of the structure equations

$$A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}, \tag{2.3}$$
$$A_{ij,k} - A_{ik,j} = B_{ij}C_k - B_{ik}C_j, \tag{2.4}$$
$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2.5}$$
$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{jk}A_{ki}), \tag{2.6}$$
$$R_{ijkl} = B_{il}B_{jk} - B_{ik}B_{jl} + A_{ik}\delta_{jl} + A_{jl}\delta_{ik} - A_{il}\delta_{jk} - A_{jk}\delta_{il}. \tag{2.7}$$

Furthermore, we have

$$\begin{aligned}&\mathrm{tr}(A)=\frac{1}{2n}(n^2\kappa-1),\quad R_{ij}=\mathrm{tr}(A)\delta_{ij}+(n-2)A_{ij}+\sum_k B_{ik}B_{kj},\\&(1-n)C_i=\sum_j B_{ij,j},\quad \sum_{ij}B_{ij}^2=\frac{n-1}{n},\quad \sum_i B_{ii}=0,\end{aligned}\tag{2.8}$$

where κ is the normalized scalar curvature of g. From (2.8), we see that when $n\geq 3$, all coefficients in the structure equations are determined by the conformal metric g and the conformal second fundamental form B, thus we get the congruent theorem of spacelike hypersurfaces.

Theorem 2.3. *[2] Two spacelike hypersurfaces $f,\bar{f}:M^n\to\mathbb{R}_1^{n+1}(n\geq 3)$ are conformally equivalent if and only if there exists a diffeomorphism $\varphi:M^n\to M^n$ which preserves the conformal metric g and the conformal second fundamental form B.*

The second covariant derivative of the conformal second fundamental form B_{ij} is defined by the following:

$$\sum_m B_{ij,km}\omega_m=dB_{ij,k}+\sum_m B_{mj,k}\omega_{mi}+\sum_m B_{im,k}\omega_{mj}+\sum_m B_{ij,m}\omega_{mk}.\tag{2.9}$$

Thus, we have the following Ricci identities

$$B_{ij,kl}-B_{ij,lk}=\sum_m B_{mj}R_{mikl}+\sum_m B_{im}R_{mjkl}.\tag{2.10}$$

Next, we give the relations between the conformal invariants and the isometric invariants of a spacelike hypersurface in $\mathbb{R}_1^{n+1}$.

Assume that $f:M^n\to\mathbb{R}_1^{n+1}$ is an umbilic-free spacelike hypersurface. Let $\{e_1,\cdots,e_n\}$ be an orthonormal local basis with respect to the induced metric $I=\langle df,df\rangle_1$ with dual basis $\{\theta_1,\cdots,\theta_n\}$. Let e_{n+1} be a normal vector field of f, $\langle e_{n+1},e_{n+1}\rangle_1=-1$. Let $II=\sum_{ij}h_{ij}\theta_i\otimes\theta_j$ denote the second fundamental form and $H=\frac{1}{n}\sum_i h_{ii}$ denote the mean curvature. Therefore, the conformal metric g and conformal normal vector field ξ have the following expressions:

$$\begin{aligned}&g=e^{2\tau}I,\quad e^{2\tau}=\frac{n}{n-1}(|II|^2-n|H|^2),\\&\xi=-Hy+(\langle f,e_{n+1}\rangle_1,e_{n+1},\langle f,e_{n+1}\rangle_1).\end{aligned}\tag{2.11}$$

By a direct calculation, we get the following expressions of the conformal invariants:

$$\begin{aligned}&A_{ij}=e^{-2\tau}[\tau_i\tau_j-h_{ij}H-\tau_{i,j}+\frac{1}{2}(-|\nabla\tau|^2+|H|^2)\delta_{ij}],\\&B_{ij}=e^{-\tau}(h_{ij}-H\delta_{ij}),\\&C_i=e^{-2\tau}(H\tau_i-H_i-\sum_j h_{ij}\tau_j),\end{aligned}\tag{2.12}$$

where $\tau_i=e_i(\tau)$ and $|\nabla\tau|^2=\sum_i\tau_i^2$, and $\tau_{i,j}$ is the Hessian of τ for I and $H_i=e_i(H)$.

Thus, $\{E_1=e^{-\tau}e_1,\cdots,E_n=e^{-\tau}e_n\}$ is an orthonormal local basis with respect to the conformal metric g and $\{\omega=e^{\tau}\theta_1,\cdots,\omega_n=e^{-\tau}\theta_n\}$ is the dual basis. Let $\{\theta_{ij}|1\leq i,j\leq n\}$ denote the connection of the induced metric $I = \langle df, df\rangle_1$ with respect to the basis $\{\theta_1, \cdots, \theta_n\}$ and $\{\omega_{ij} | 1 \leq i, j \leq n\}$ the connection of the conformal metric g with respect to the basis $\{\omega_1, \cdots, \omega_n\}$, then we have the following:

$$\omega_{ij} = \theta_{ij} + e_i(\tau)\theta_j - e_j(\tau)\theta_i. \tag{2.13}$$

Let $\{b_1, \cdots, b_n\}$ be the eigenvalues of the conformal second fundamental form B, which are called the conformal principal curvatures of f. Let $\{\lambda_1, \cdots, \lambda_n\}$ be the principal curvatures of f. From (2.12), we have

$$b_i = e^{-\tau}(\lambda_i - H), \quad i = 1, \cdots, n. \tag{2.14}$$

Clearly, the number of distinct conformal principal curvatures is the same as that of the principal curvatures of f.

3. Examples of spacelike conformal Einstein hypersurfaces

In this section, we construct some examples of spacelike conformal Einstein hypersurfaces in a Lorentzian space form $M_1^{n+1}(c)$. Using σ_c, we obtain the hypersurface $\sigma_{\bar{c}}^{-1} \circ \sigma_c \circ f : M^n \to \mathbb{R}_1^{n+1}$ in $\mathbb{R}_1^{n+1}$ for the spacelike hypersurface f in another Lorentzian space form $M_1^{n+1}(c)$, furthermore, the conformal invariants of the spacelike hypersurfaces in $M_1^{n+1}(c)$ are invariant under the diffeomorphisms σ_c (see Section 2 in [4]). Thus we can regard these spacelike hypersurfaces in $M_1^{n+1}(c)$ as in $\mathbb{R}_1^{n+1}$.

Example 3.1. *For a constant $a > 0$, let $x_1 : \mathbb{H}^k(-a) \to \mathbb{R}_1^{k+1}$ be the standard embedding and $y : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be the identity. We define the spacelike hypersurface as follows:*

$$f = (x_1, y) : \mathbb{H}^k(-a) \times \mathbb{R}^{n-k} \to \mathbb{R}_1^{n+1}, \quad 1 \leq k \leq n-1.$$

Let $\xi = (\frac{1}{a}x_1, \overrightarrow{0})$ be a normal vector field of f. Thus,

$$I = \langle dx, dx\rangle_1 = g_{\mathbb{H}^k(-a)} + g_{\mathbb{R}^{n-k}}, \quad II = -\langle dx, d\xi\rangle_1 = \frac{-1}{a} g_{\mathbb{H}^k(-a)},$$

where $g_{\mathbb{H}^k(-a)}$ denotes the standard metric on $\mathbb{H}^k(-a)$ and $g_{\mathbb{R}^{n-k}}$ the standard metric on $\mathbb{R}^{n-k}$. Let $\{e_1, \cdots, e_k\}$ be a local orthonormal basis on $T\mathbb{H}^k(-a)$ and $\{e_{k+1}, \cdots, e_n\}$ be a local orthonormal basis on $T\mathbb{R}^{n-k}$; then under the local orthonormal basis $\{e_1, \cdots, e_n\}$ on $T(\mathbb{H}^k(-a) \times \mathbb{R}^{n-k})$, $(h_{ij}) = diag(\frac{-1}{a}, \cdots, \frac{-1}{a}, 0, \cdots, 0)$. From (2.12), we have that the conformal 1-form $C = 0$ and under the local orthonormal basis,

$$(B_{ij}) = diag(\underbrace{b_1, \cdots, b_1}_{k}, \underbrace{b_2, \cdots, b_2}_{n-k}), \quad (A_{ij}) = diag(\underbrace{a_1, \cdots, a_1}_{k}, \underbrace{a_2, \cdots, a_2}_{n-k}),$$

where

$$b_1 = \sqrt{\frac{(n-1)(n-k)}{n^2 k}}, \; b_2 = -\sqrt{\frac{(n-1)k}{n^2(n-k)}}, \; a_1 = \frac{(n-1)(k-2n)}{2n^2(n-k)}, \; a_2 = \frac{(n-1)k}{2n^2(n-k)}.$$

From (2.8) and above data, the Ricci curvature R_{ij} with respect to the conformal metric g are given by the following:

$$R_{11} = \cdots = R_{kk} = \frac{(n-1)(1-k)}{(n-k)k}, \quad R_{k+1k+1} = \cdots = R_{nn} = 0.$$

Thus, the spacelike hypersurface $f:\mathbb{H}^k(-a)\times\mathbb{R}^{n-k}\to\mathbb{R}^{n+1}_1$ is a conformal Einstein hypersurface if and only if $k=1$, which is of constant conformal sectional curvature $\delta=0$. In fact, the conformal Einstein hypersurface is a cylinder over the curvature-spiral with a constant geodesic curvature.

Example 3.2. *Let $x_1:\mathbb{S}^k(1)\to\mathbb{R}^{k+1}$ and $x_2:\mathbb{H}^{n-k}(-1)\to\mathbb{R}^{n-k+1}_1$ be two standard embeddings. For a constant $a>0$, we define the spacelike hypersurface as follows:*

$$f=(\sqrt{1+a^2}x_1,ax_2):\mathbb{S}^k(\sqrt{1+a^2})\times\mathbb{H}^{n-k}(-a)\to\mathbb{S}^{n+1}_1(1),\ \ 1\le k\le n-1.$$

Let $\xi=(ax_1,\ \sqrt{1+a^2}x_2)$ be a normal vector field of f. Thus,

$$I=(1+a^2)g_{\mathbb{S}^k(1)}+a^2g_{\mathbb{H}^{n-k}(-1)},\ \ II=-a\sqrt{1+a^2}(g_{\mathbb{S}^k(1)}+g_{\mathbb{H}^{n-k}(-1)}).$$

Let $\{e_1,\cdots,e_k\}$ be a local orthonormal basis on $T\mathbb{S}^k(\sqrt{1+a^2})$ and $\{e_{k+1},\cdots,e_n\}$ be a local orthonormal basis on $T\mathbb{H}^{n-k}(-a)$; then under the local orthonormal basis $\{e_1,\cdots,e_n\}$, $(h_{ij})=diag(\frac{-a}{\sqrt{1+a^2}},\cdots,\frac{-a}{\sqrt{1+a^2}},\frac{-\sqrt{1+a^2}}{a},\cdots,\frac{-\sqrt{1+a^2}}{a})$. From (2.12), we have that $C=0$ and under the local orthonormal basis,

$$(B_{ij})=diag(\underbrace{b_1,\cdots,b_1}_{k},\underbrace{b_2,\cdots,b_2}_{n-k}),\ \ (A_{ij})=diag(\underbrace{a_1,\cdots,a_1}_{k},\underbrace{a_2,\cdots,a_2}_{n-k}),$$

where

$$b_1=\frac{1}{n}\sqrt{\frac{(n-1)(n-k)}{k}},\ \ b_2=\frac{-1}{n}\sqrt{\frac{(n-1)k}{n-k}},$$
$$a_1=\frac{n-1}{k(n-k)}\frac{(n-k)^2+n^2a^2}{2n^2},\ \ a_2=\frac{n-1}{k(n-k)}\frac{k^2-n^2a^2-n^2}{2n^2}.$$

By direct calculation using the Eq (2.8), the spacelike hypersurface f is not a conformal Einstein hypersurface.

Example 3.3. *Let $x_1:\mathbb{H}^k(-1)\to\mathbb{R}^{k+1}_1$ and $x_2:\mathbb{H}^{n-k}(-1)\to\mathbb{R}^{n-k+1}_1$ be two standard embeddings. For $0<a<1$, we define the spacelike hypersurface as follows:*

$$f=(\sqrt{1-a^2}x_1,ax_2):\mathbb{H}^k(-\sqrt{1-a^2})\times\mathbb{H}^{n-k}(-a)\to\mathbb{H}^{n+1}_1(-1),\ \ 1\le k\le n-1.$$

Let $\xi=(-ax_1,\ \sqrt{1-a^2}x_2)$ be a normal vector field of f. Thus,

$$I=(1-a^2)g_{\mathbb{H}^k(-1)}+a^2g_{\mathbb{H}^{n-k}(-1)},\ \ II=a\sqrt{1-a^2}(g_{\mathbb{H}^k(-1)}-g_{\mathbb{H}^{n-k}(-1)}).$$

Let $\{e_1,\cdots,e_k\}$ be a local orthonormal basis on $T\mathbb{H}^k(-a)$ and $\{e_{k+1},\cdots,e_n\}$ be a local orthonormal basis on $T\mathbb{H}^{n-k}(-\sqrt{1-a^2})$; then under the local orthonormal basis $\{e_1,\cdots,e_n\}$, $(h_{ij})=diag(\frac{a}{\sqrt{1-a^2}},\cdots,\frac{a}{\sqrt{1-a^2}},\frac{-\sqrt{1-a^2}}{a},\cdots,\frac{-\sqrt{1-a^2}}{a})$. From (2.12), we have that $C=0$ and under the local orthonormal basis

$$(B_{ij})=diag(\underbrace{b_1,\cdots,b_1}_{k},\underbrace{b_2,\cdots,b_2}_{n-k}),\ \ (A_{ij})=diag(\underbrace{a_1,\cdots,a_1}_{k},\underbrace{a_2,\cdots,a_2}_{n-k}),$$

where

$$b_1 = \frac{1}{n}\sqrt{\frac{(n-1)(n-k)}{k}},\quad b_2 = \frac{-1}{n}\sqrt{\frac{(n-1)k}{n-k}},$$
$$a_1 = \frac{n-1}{k(n-k)}\frac{(n-k)^2 - n^2a^2}{2n^2},\quad a_2 = \frac{n-1}{k(n-k)}\frac{n^2a^2 - n^2 + k^2}{2n^2}.$$

By direct calculation using the Eq (2.8), the Ricci curvature with respect to the conformal metric g is given by the following:

$$R_{11} = \cdots = R_{kk} = \frac{(n-1)(n-k-1)}{2nk} + \frac{(n-1)(1-k)a^2}{(n-k)k},$$

$$R_{k+1k+1} = \cdots = R_{nn} = \frac{(n-1)(k^2+k-n(n-1))}{2nk(n-k)} + \frac{(n-1)(n-k-1)a^2}{(n-k)k}.$$

Thus, the spacelike hypersurface $f : \mathbb{H}^k(-\sqrt{1-a^2}) \times \mathbb{H}^{n-k}(-a) \to \mathbb{H}_1^{n+1}(-1)$ is a conformal Einstein hypersurface if and only if $a = \sqrt{\frac{n-k-1}{n-2}}$, i.e.,

$$f : \mathbb{H}^k\Big(-\sqrt{\frac{k-1}{n-2}}\Big) \times \mathbb{H}^{n-k}\Big(-\sqrt{\frac{n-k-1}{n-2}}\Big) \to \mathbb{H}_1^{n+1}(-1),\quad 1 < k < n-1.$$

Example 3.4. *For $1 \leq p, q \leq n$ with $p + q < n$ and a constant $a > 1$, we define the spacelike hypersurface*

$$f : \mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \to \mathbb{R}_1^{n+1},$$

defined by

$$f(u', u'', t, u''') = (tu', tu'', u'''),$$

where $u' \in \mathbb{H}^q(-\sqrt{a^2-1}), u'' \in \mathbb{S}^p(a), u''' \in \mathbb{R}^{n-p-q-1}$.

Let $b = \sqrt{a^2-1}$. One of the normal vector to f can be taken as $e_{n+1} = (\frac{a}{b}u', \frac{b}{a}u'', 0)$. The first and second fundamental form of f are given by the following:

$$I = t^2(\langle du', du'\rangle_1 + \langle du'', du''\rangle) + dt\cdot dt + \langle du''', du'''\rangle,$$

$$II = -\langle dx, de_{n+1}\rangle_1 = -t(\frac{a}{b}\langle du', du'\rangle_1 + \frac{b}{a}\langle du'', du''\rangle).$$

Thus, the mean curvature of f satisfies $H = \frac{-pb^2 - qa^2}{nabt}$ and $e^{2\tau} = \frac{n}{n-1}[\sum_{ij} h_{ij}^2 - nH^2] = \frac{p(n-p)b^4 - 2pqa^2b^2 + q(n-q)a^4}{(n-1)a^2b^2t^2} := \frac{\alpha^2}{t^2}$. The conformal metric is as follows:

$$g = \alpha^2\langle du', du'\rangle_1 + \alpha^2\langle du'', du''\rangle + \frac{\alpha^2}{t^2}(dt\cdot dt + \langle du''', du'''\rangle).$$

From (2.12), we have $C = 0$ and

$$(B_{ij}) = diag(\underbrace{b_1, \cdots, b_1}_{q}, \underbrace{b_2, \cdots, b_2}_{p}, \underbrace{b_3, \cdots, b_3}_{n-p-q}),$$

$$(A_{ij}) = diag(\underbrace{a_1, \cdots, a_1}_{q}, \underbrace{a_2, \cdots, a_2}_{p}, \underbrace{a_3, \cdots, a_3}_{n-p-q}),$$

where $b_1 = \frac{pb^2-(n-q)a^2}{nab\alpha}$, $b_2 = \frac{qa^2-(n-p)b^2}{nab\alpha}$, $b_3 = \frac{pb^2+qa^2}{nab\alpha}$, and

$$a_1 = \frac{(pb^2+qa^2)^2 - (pb^2+qa^2)2na^2 + n^2a^2b^2}{2n^2a^2b^2\alpha^2},$$

$$a_2 = \frac{(pb^2+qa^2)^2 - (pb^2+qa^2)2nb^2 + n^2a^2b^2}{2n^2a^2b^2\alpha^2}, \quad a_3 = \frac{(pb^2+qa^2)^2 - n^2a^2b^2}{2n^2a^2b^2\alpha^2}.$$

By direct calculation, using the Eq (2.8), we can see that the spacelike hypersurface f is a conformal Einstein hypersurface if and only if $p = q = 1$ and $n = 3$, which is of a constant conformal sectional curvature $\delta = 0$.

A spacelike hypersurface with constant conformal principal curvatures and vanishing conformal 1-form is called a conformal isoparametric hypersurface. By the main theorem in [4], Examples 3.1–3.4 are all spacelike conformal isoparametric hypersurfaces. Thus, we have following results.

Proposition 3.1. *Let $f : M^n \to \mathbb{R}^{n+1}_1$ be a spacelike conformal isoparametric hypersurface. If f is conformal Einstein, then f is locally conformally equivalent to one of the following examples:*

1) the cylinder $f : \mathbb{H}^1(-a) \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}_1$;
2) the spacelike hypersurface

$$f : \mathbb{H}^k\Big(-\sqrt{\frac{k-1}{n-2}}\Big) \times \mathbb{H}^{n-k}\Big(-\sqrt{\frac{n-k-1}{n-2}}\Big) \to \mathbb{H}^{n+1}_1(-1), \quad 1 < k < n-1;$$

3) the spacelike hypersurface

$$f : \mathbb{H}^1(-\sqrt{a^2-1}) \times \mathbb{S}^1(a) \times \mathbb{R}^+ \to \mathbb{R}^4_1.$$

Particularly, the spacelike hypersurfaces in (1) and (2) have only two distinct principal curvatures.

Example 3.5. *Let $u : M^2 \to \mathbb{R}^3_1$ be a spacelike surface in $\mathbb{R}^3_1$. We define the cylinder f over the spacelike surface u in $\mathbb{R}^{n+1}_1$ by*

$$f = (u, id) : M^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^3_1 \times \mathbb{R}^{n-2} = \mathbb{R}^{n+1}_1,$$

where $id : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ denotes the identity map.

Let η be the unit normal vector of u. Then, $e_{n+1} = (\eta, \vec{0})$ is the unit normal vector of f. The induced metric I and the second fundamental form II of f are given by

$$I = I_u + g_{\mathbb{R}^{n-2}}, \quad II = II_u, \tag{3.1}$$

where I_u, II_u are the induced metric and the second fundamental forms of u, respectively. Let λ_1, λ_2 be the principal curvatures of the spacelike surface u. The principal curvatures of the cylinder f are obviously $\lambda_1, \lambda_2, 0, \ldots, 0$. The conformal metric g of the cylinder f is

$$g = \frac{n}{n-1}(|II|^2 - nH^2)I = \left(4H_u^2 - \frac{2n}{n-1}K_u\right)(I_u + g_{\mathbb{R}^{n-2}}), \tag{3.2}$$

where H_u, K_u are the mean curvature and the Gauss curvature of u, respectively.

Example 3.6. *Let $u : M^2 \longrightarrow \mathbb{S}^3_1 \subset \mathbb{R}^4$ be a spacelike surface in $\mathbb{S}^3_1$. We define the cone over the spacelike surface u in $\mathbb{R}^{n+1}_1$ by the following:*

$$f : M^2 \times R^+ \times \mathbb{R}^{n-3} \longrightarrow \mathbb{R}^{n+1}_1, \quad f(x,t,y) = (tu(x), y).$$

By direct calculation, the induced metric and the second fundamental forms of the cone f are, respectively,

$$I = t^2 I_u + g_{\mathbb{R}^{n-2}}, \; II = t\, II_u,$$

where $I_u, II_u, I_{\mathbb{R}^{n-2}}$ are understood as before. Let λ_1, λ_2 be the principal curvatures of the spacelike surface u. The principal curvatures of the hypersurface f are $\frac{1}{t}\lambda_1, \frac{1}{t}\lambda_2, 0, \ldots, 0$. Thus, the conformal metric g of the cone f is as follows:

$$\begin{aligned} g = \rho^2 I &= \frac{1}{t^2}\left[4H_u^2 - \frac{2n}{n-1}(K_u - 1)\right](t^2 I_u + g_{\mathbb{R}^{n-2}}) \\ &= \left[4H_u^2 - \frac{2n}{n-1}(K_u - 1)\right](I_u + g_{\mathbb{H}^{n-2}}), \end{aligned} \tag{3.3}$$

where H_u, K_u are the mean curvature and Gauss curvature of u, respectively. From (2.12), we know that the conformal position vector of the cone f is as follows:

$$Y = \left[4H_u^2 - \frac{2n}{n-1}(K_u - 1)\right]\Big(\frac{t^2 + \langle y, y\rangle + 1}{2t}, u, \frac{y}{t}, \frac{t^2 + \langle y, y\rangle - 1}{2t}\Big).$$

Note that

$$i(t,y) = (\frac{t^2 + \langle y, y\rangle + 1}{2t}, \frac{y}{t}, \frac{t^2 + \langle y, y\rangle - 1}{2t}) : \mathbb{R}^+ \times \mathbb{R}^{n-2} \to \mathbb{H}^{n-1} \subset \mathbb{R}^n_1 \tag{3.4}$$

is nothing but the identity map of $\mathbb{H}^{n-1}$, since $\mathbb{R}^+ \times \mathbb{R}^{n-2} = \mathbb{H}^{n-1}$ is the upper half-space endowed with the standard hyperbolic metric.

The n-dimensional Lorentzian hyperbolic plane $\mathbb{R}^n_{1+} \subset \mathbb{R}^n_1$ is defined by

$$\mathbb{R}^n_{1+} = \{(x_1, x_1, \cdots, x_n) \in \mathbb{R}^n_1 | x_n > 0\},$$

and endowed by the Lorentzian metric $ds^2 = \frac{1}{x_n^2}(-dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + \cdots + dx_n \otimes dx_n)$. The sectional curvature of $\mathbb{R}^n_{1+}$ is -1 with respect to the Lorentzian metric ds^2. For $p = (x_1, x_1, \cdots, x_n) \in \mathbb{R}^n_{1+}$, let $\bar{x} = (x_1, x_1, \cdots, x_{n-1})$, then $p = (\bar{x}, x_n)$. There exists a standard isometric mapping $\phi : \mathbb{R}^n_{1+} \to \mathbb{H}^n_1(-1)$ defined by

$$\phi(x_1, \cdots, x_n) = \phi(\bar{x}, x_n) = \Big(\frac{x_n^2 + \langle \bar{x}, \bar{x}\rangle_1 + 1}{2x_n}, \frac{\bar{x}}{x_n}, \frac{x_n^2 + \langle \bar{x}, \bar{x}\rangle_1 - 1}{2x_n}\Big). \tag{3.5}$$

Example 3.7. *Let $u = (x_1, x_2, x_3) : M^2 \longrightarrow \mathbb{R}^3_{1+}$ be a spacelike surface in the 3-dimensional Lorentzian hyperbolic plane $\mathbb{R}^3_{1+}$. We define the rotational hypersurface over the spacelike u in $\mathbb{R}^{n+1}_1$ as follows:*

$$f : M^2 \times \mathbb{S}^{n-2} \longrightarrow \mathbb{R}^{n+1}_1, \quad f(x_1, x_2, x_3, \theta) = (x_1, x_2, x_3\theta),$$

where $\theta : \mathbb{S}^{n-2} \longrightarrow \mathbb{R}^{n-1}$ is the standard sphere.

Let η be the unit normal vector of the spacelike surface u given by $\eta = (\eta_1, \eta_2, \eta_3)$. Then, the unit normal vector of the rotational hypersurface f in $\mathbb{R}^{n+1}_1$ is as follows:

$$\xi = \frac{1}{x_3}(\eta_1, \eta_2, \eta_3\theta).$$

By direct calculation, the induced metric and the second fundamental form of u are, respectively,

$$I_u = \frac{1}{x_3^2}(-dx_1 \cdot dx_1 + dx_2 \cdot dx_2 + dx_3 \cdot dx_3),$$
$$II_u = -\langle \tau_*(du), \tau_*(d\eta)\rangle = \frac{1}{x_3^2}(-dx_1 \cdot d\eta_1 + dx_2 \cdot d\eta_2 + dx_3 \cdot d\eta_3) - \frac{\eta_3}{x_3} I_u.$$

Thus, we can write out the induced metric and the second fundamental form of f,

$$I = x_3^2(I_u + g_{\mathbb{S}^{n-2}}), \quad II = x_3 II_u - \eta_3 I_u - \eta_3 g_{\mathbb{S}^{n-2}}.$$

Let λ_1, λ_2 be the principal curvatures of u. Then, the principal curvatures of the rotational hypersurface f are

$$\frac{\lambda_1}{x_3} - \frac{\eta_3}{x_3^2}, \frac{\lambda_2}{x_3} - \frac{\eta_3}{x_3^2}, \frac{-\eta_3}{x_3^2}, \ldots, \frac{-\eta_3}{x_3^2}.$$

Thus,

$$\rho^2 = \frac{n}{n-1}(|II|^2 - nH^2) = \frac{1}{x_3^2}\left[4H_u^2 - \frac{2n}{n-1}(K_u + 1)\right],$$

where H_u, K_u are the mean curvature and Gauss curvature of u, respectively. Therefore, the conformal metric of the rotational hypersurface f is as follows:

$$g = \rho^2 I = \left[4H_u^2 - \frac{2n}{n-1}(K_u + 1)\right](I_u + g_{\mathbb{S}^{n-2}}). \tag{3.6}$$

From Examples 3.5–3.7, the cylinder, the cone and the rotational hypersurface can be written by

$$f : M^2 \times N^{n-2}(c) \longrightarrow \mathbb{R}^{n+1}_1,$$

when f is a cylinder over a spacelike surface $u(M^2) \subset \mathbb{R}^3_1$, $c = 0$ and $N^{n-2}(c) = \mathbb{R}^{n-2}$; a cone over a spacelike surface $u(M^2) \subset \mathbb{S}^3_1$, $c = -1$ and $N^{n-2}(c) = R^+ \times \mathbb{R}^{n-3} = \mathbb{H}^{n-2}$; and a rotational hypersurface over a spacelike surface $u(M^2) \subset \mathbb{R}^3_{1+}$, $c = 1$ and $N^{n-2}(c) = \mathbb{S}^{n-2}$. Let the induced metric, the Gauss curvature, and the mean curvature of the spacelike surface u, be denoted by I_u, K_u, and H_u, respectively. From (3.2), (3.3) and (3.6), the conformal metric of the cylinder, the cone and the rotational hypersurface f can be unified in a single formula:

$$g = \left[4H_u^2 - \frac{2n}{n-1}(K_u + c)\right](I_u + g_{N^{n-2}(c)}) := \phi^2(I_u + g_{N^{n-2}(c)}), \tag{3.7}$$

where $g_{N^{n-2}(c)}$ is the Riemannian metric of an $(n-2)$-dimensional space form of constant curvature c.

Proposition 3.2. *Let $f : M^n \to \mathbb{R}^{n+1}_1 (n \geq 4)$ be the cylinder, or the cone, or the rotational hypersurface over an umbilic-free spacelike surface $u : M^2 \to M^3_1(c)$, which was constructed as Examples 3.5–3.7. Then, the spacelike hypersurface f is a spacelike conformal Einstein hypersurface with the Ricci curvature λ if and only if u is a spacelike (λ, n)-surface in $M^3_1(c)$.*

Proof. Now we take the local orthonormal basis $\{e_1, e_2\}$ on TM^2 with respect to I_u, consisting of principal vectors. Let $\{e_3, \ldots, e_n\}$ be a local orthonormal basis on $TN^{n-2}(c)$, then, $\{e_1, e_2, e_3, \ldots, e_n\}$ is a orthonormal basis on $T(M^2 \times N^{n-2}(c))$ with respect to the product metric $I_u + I_{N^{n-2}(c)}$.

Let $\tilde{R}_{ijkl}$ denote the curvature tensor with respect to $I_u + I_{N^{n-2}(c)}$, and R_{ijkl} denote the curvature tensor with respect to the conformal metric g. Let $\mu = \frac{1}{\phi} = \frac{1}{\sqrt{4H_u^2 - \frac{2n}{n-1}(K_u + c)}}$, then by direct computation (also see [6]), we have the following:

$$\begin{aligned} R_{ijij} &= \mu^2 \tilde{R}_{ijij} + \mu\mu_{ii} + \mu\mu_{jj} - |\nabla\mu|^2, i \neq j, \\ R_{ijik} &= \mu^2 \tilde{R}_{ijik} + \mu\mu_{jk},\ i \neq j,\ j \neq k,\ k \neq i, \end{aligned} \tag{3.8}$$

where μ_{ij} and $\nabla\mu$ are the Hessian matrix and the gradient of μ with respect to the metric $I_u + I_{N^{n-2}(c)}$. Since the metric $I_u + I_{N^{n-2}(c)}$ is a Riemannian product metric, thus

$$\mu_i = 0,\ \mu_{ij} = 0,\ R_{1i1i} = R_{2i2i} = 0,\ i, j \geq 3.$$

Thus, we have

$$|\nabla\mu|^2 = \mu_1^2 + \mu_2^2 = |\nabla_u\mu|^2,\ \Delta\mu = \mu_{11} + \mu_{22} = \Delta_u\mu,$$

where $\Delta_u\mu$ and $\nabla_u\mu$ are the Hessian matrix and the gradient of μ with respect to the metric I_u.

Now, we assume that the Ricci curvature with respect to the conformal metric g is λ, then from the first equation of (3.8),

$$\begin{aligned} \lambda &= \sum_{k\neq 1} R_{1k1k} = \mu^2 \sum_{k \neq 1} \tilde{R}_{1k1k} + (n-1)\mu\mu_{11} + \sum_{k\neq 1} \mu\mu_{kk} - (n-1)|\nabla\mu|^2 \\ &= \mu^2 K_u + \mu\Delta\mu + (n-2)\mu\mu_{11} - (n-1)|\nabla\mu|^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \lambda &= \mu^2(n-3)c + \mu\Delta\mu - (n-1)|\nabla\mu|^2, \\ \lambda &= \mu^2 K_u + \mu\Delta\mu + (n-2)\mu\mu_{11} - (n-1)|\nabla\mu|^2, \\ \lambda &= \mu^2 K_u + \mu\Delta\mu + (n-2)\mu\mu_{22} - (n-1)|\nabla\mu|^2. \end{aligned} \tag{3.9}$$

From above equations, we have

$$\begin{aligned} &\mu_{12} = 0, \\ &\Delta\mu = \frac{2\mu}{n-2}[(n-3)c - K_u] = 2\mu_{11} = 2\mu_{22}, \\ &|\nabla\mu|^2 = \mu_1^2 + \mu_2^2 = \frac{\mu^2[n(n-3)c - 2K_u]}{(n-1)(n-2)} - \frac{\lambda}{n-1}. \end{aligned} \tag{3.10}$$

Thus,

$$Hess_u(\mu)(e_i, e_j) = \frac{(n-3)c\mu - \mu K_\mu}{n-2} I_u(e_i, e_j),\quad e_i, e_j \in TM^2,$$

where $Hess_u$ is the Hessian matrix with respect to the metric I_u. By the Definition 1.1, we know that u is a spacelike (λ, n)-surface in $M_1^3(c)$.

Let u be a spacelike (λ, n)-surface in $M_1^3(c)$. We note that the conformal metric of f is given by (3.7), by direct calculation we know that f is a spacelike conformal Einstein hypersurface with Ricci curvature λ. Thus, we finish the proof. □

4. The proof of the main Theorem 1.2

Let $f : M^n \to \mathbb{R}_1^{n+1} (n \geq 4)$ be a spacelike conformal Einstein hypersurface without umbilical points. Since three dimensional Einstein manifolds are of constant sectional curvature, in this section, we assume $n \geq 4$. Because of the local nature of our results, we can assume that the multiplicities of all principal curvatures are locally constant. In fact, there always exists an open dense subset U of M^n on which the multiplicities of the principal curvatures are locally constant.

We assume that the spacelike Einstein hypersurface has $(s + t)$ distinct principal curvatures. Since the multiplicities of all principal curvatures are locally constant, we can choose a local orthonormal basis $\{E_1, \ldots, E_n\}$ with respect to the conformal metric g such that

$$(B_{ij}) = diag\{b_1, b_2, \cdots, b_n\} = diag\{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_s, \bar{b}_{s+1}, \ldots, \bar{b}_{s+1}, \ldots, \bar{b}_{s+t}, \ldots, \bar{b}_{s+t}\}.$$

Here, the conformal principal curvatures $\bar{b}_1, \ldots, \bar{b}_s$ are simple, the multiplicities of the conformal principal curvatures $\bar{b}_{s+1}, \ldots, \bar{b}_{s+t}$ are greater than one. Under this local orthonormal basis, let the index set

$$[i] = \{m | b_m = b_i\}.$$

As the spacelike hypersurface is a conformal Einstein, from (2.8), we have the following:

$$R_{ij} = \lambda\delta_{ij} = \sum_k B_{ik}B_{kj} + \mathrm{tr(A)}\delta_{\mathrm{ij}} + \mathrm{(n-2)A_{ij}}. \tag{4.1}$$

Thus, under the basis $\{E_1, \ldots, E_n\}$, we have

$$(A_{ij}) = diag\{a_1, \ldots, a_n\},\ a_i = \frac{1}{n-2}(\lambda - b_i^2 - \mathrm{tr(A)}), 1 \leq \mathrm{i} \leq \mathrm{n}. \tag{4.2}$$

Since f is a spacelike Einstein hypersurface, λ and tr(A) are constant.

By the covariant derivative for the Eq (4.1), we get that

$$A_{ij,k} = \frac{-1}{n-2}\Big(\sum_m B_{im,k}B_{mj} + \sum_m B_{im}B_{mj,k}\Big).$$

Thus, under the basis $\{E_1, \ldots, E_n\}$, we have

$$-(b_i + b_j)B_{ij,k} = (n-2)A_{ij,k}. \tag{4.3}$$

Lemma 4.1. *Under the basis $\{E_1, \ldots, E_n\}$, the conformal invariants of f have the following relations:*

$$\begin{aligned}
&(1)\ C_i = 0;\ i > s,\\
&(2)\ B_{ij,k} = 0,\ i \neq j, j \neq k, k \neq i,\ B_{ii,j} = 0, i \neq j, i, j \in [i],\\
&(3)\ B_{jj,i} = \frac{b_i + (n-1)b_j}{b_i - b_j}C_i,\ B_{ij,j} = \frac{nb_j}{b_i - b_j}C_i,\ [i] \neq [j],\\
&(4)\ \omega_{ij} = \frac{B_{ij,i}}{b_i - b_j}\omega_i + \frac{B_{ij,j}}{b_i - b_j}\omega_j = \frac{nb_jC_i}{(b_i - b_j)^2}\omega_j - \frac{nb_iC_j}{(b_i - b_j)^2}\omega_i, [i] \neq [j].
\end{aligned} \tag{4.4}$$

Proof. Using $dB_{ij}+\sum_k B_{kj}\omega_{ki}+\sum_k B_{ik}\omega_{kj}=\sum_k B_{ij,k}\omega_k$, let $[i]=[j], i\neq j$, so $b_i=b_j$, we get

$$B_{ij,k}=0,\ [i]=[j], i\neq j,\ 1\leq k\leq n. \tag{4.5}$$

Particularly, $B_{ij,j}=0$. Using (2.4) and (2.5),

$$A_{ij,j}-A_{jj,i}=-b_jC_i,\ B_{ij,j}-B_{jj,i}=-C_i,$$

from (4.3), we obtain

$$\frac{n}{n-2}b_jC_i=0,$$

If $b_j\neq 0$, then $C_i=0$. If $b_j=0$, then $E_i(b_j)=B_{jj,i}=0$. Combining $B_{ij,j}=0$, thus $C_i=0$. Therefore,

$$C_i=0,\ i>s,$$

which proves the Eq (1) in Lemma 4.1.

If $i\neq j, j\neq k, i\neq k$, then $B_{ij,k}=B_{ik,j}$, $A_{ij,k}=A_{ik,j}$. Moreover, if $b_i\neq b_j$ *or* $b_i\neq b_k$, using (4.3) we obtain the following:

$$B_{ij,k}=A_{ij,k}=0, i\neq j, j\neq k, i\neq k. \tag{4.6}$$

If $b_i=b_j$, combining (4.5) and (4.6), we obtain the following:

$$B_{ij,k}=0,\ i,j>s,\ i\neq j; 1\leq k\leq n.$$

Thus, we obtain the Eq (2) in Lemma 4.1.

If $[i]\neq[j]$, using (2.4), (2.5) and (4.3), we obtain that

$$-b_jC_i=A_{ij,j}-A_{jj,i}=-\frac{b_i+b_j}{n-2}B_{ij,j}+\frac{2b_j}{n-2}B_{jj,i}=-\frac{b_i+b_j}{n-2}B_{ij,j}+\frac{2b_j}{n-2}(B_{ij,j}+C_i),$$

and

$$B_{jj,i}=\frac{b_i+(n-1)b_j}{b_i-b_j}C_i,\ B_{ij,j}=\frac{nb_j}{b_i-b_j}C_i,\ b_i\neq b_j.$$

Thus, we obtain the Eq (3) in Lemma 4.1.

Using $dB_{ij}+\sum_k B_{kj}\omega_{ki}+\sum_k B_{ik}\omega_{kj}=\sum_k B_{ij,k}\omega_k$, we have

$$(b_i-b_j)\omega_{ij}=\sum_k B_{ij,k}\omega_k.$$

Since $b_i\neq b_j$, we have

$$\omega_{ij}=\frac{B_{ij,i}}{b_i-b_j}\omega_i+\frac{B_{ij,j}}{b_i-b_j}\omega_j=\frac{nb_jC_i}{(b_i-b_j)^2}\omega_j-\frac{nb_iC_j}{(b_i-b_j)^2}\omega_i,$$

that completes proof of the Lemma 4.1. □

Proposition 4.1. *Let $f: M^n\to\mathbb{R}^{n+1}_1 (n\geq 4)$ be a spacelike conformal Einstein hypersurface without an umbilical point. If the conformal 1-form $C=0$, then f is locally conformally equivalent to one of the following examples:*

1) the cylinder $f : \mathbb{H}^1(-a) \times \mathbb{R}^{n-1} \to \mathbb{R}_1^{n+1}$; and
2) the spacelike hypersurface

$$f : \mathbb{H}^k\Big(-\sqrt{\frac{k-1}{n-2}}\Big) \times \mathbb{H}^{n-k}\Big(-\sqrt{\frac{n-k-1}{n-2}}\Big) \to \mathbb{H}_1^{n+1}(-1),\ 1 < k < n-1.$$

Particularly, f has only two distinct principal curvatures.

Proof. Since $C = 0$, from Lemma 4.1, we know that $B_{jj,i} = 0$, $i \neq j$. Since $\mathrm{tr(B)} = 0$, we have $\sum_m B_{mm,i} = 0$ and $B_{ii,i} = 0$. Thus $B_{ij,k} = 0$. Therefore, the conformal second fundamental form of f is parallel. Particularly, the conformal principal curvatures are constant; thus, the spacelike conformal Einstein hypersurfaces are conformal isoparametric hypersurfaces. By Proposition 3.1, we finish the proof. □

Theorem 4.1. *Let $f : M^n \to \mathbb{R}_1^{n+1}$ $(n \geq 4)$ be a spacelike conformal Einstein hypersurface without umbilical points; then, f has three distinct principal curvatures at most.*

Proof. We assume that $s + t \geq 4$. Next, we prove that there exists a contradiction.

Now we fix the indices i, j, k such that $[i] \neq [j], [j] \neq [k], [k] \neq [i]$, then

$$B_{ij,k} = 0, i \in [i], j \in [j], k \in [k].$$

Noting $E_k(b_i) = B_{ii,k}$, and using definition of $C_{i,j}$ and Lemma 4.1, we can obtain the following:

$$\begin{aligned} B_{ij,jk} &= E_k(B_{ij,j}) + B_{kj,j}\omega_{ki}(E_k) \\ &= n\frac{b_k + (n-1)b_j}{(b_i - b_j)(b_k - b_j)}C_iC_k + \frac{nb_j}{b_i - b_j}C_{i,k}. \end{aligned}$$

Similarly, we have

$$B_{ij,kj} = \frac{n^2 b_j}{(b_i - b_j)(b_k - b_j)}C_iC_k.$$

From Ricci identity $B_{ij,jk} - B_{ij,kj} = (b_i - b_j)R_{jijk} = 0$, thus we obtain

$$C_iC_k + b_jC_{i,k} = 0. \tag{4.7}$$

Since $s + t \geq 4$, there is $[l]$ such that $[l] \neq [i], [j], [k]$. Similarly, we have

$$C_iC_k + b_lC_{i,k} = 0. \tag{4.8}$$

From (4.8) and (4.7), we can get

$$(b_j - b_l)C_{i,k} = 0,\ \ C_iC_k = 0.$$

This implies that there are at least $n-1$ zero elements in $\{C_1, \ldots, C_n\}$, and we assume that

$$C_2 = \ldots = C_n = 0.$$

If the multiplicity of b_1 is greater than one, then from Lemma 4.1, we have $C_1 = 0$ and

$$B_{ij,k} = 0, 1 \leq i, j, k \leq n,$$

thus B is parallel. From Proposition 4.1, we know that M^n has two distinct principal curvatures. This is a contradiction.

Now, we assume that the multiplicity of b_1 is one. Since $s+t \geq 4$, we take $i, j, k > 1$. Noting $[i] \neq [j], [j] \neq [k], [k] \neq [i]$, so we have the following:

$$C_i = C_j = C_k = 0, \omega_{ij} = 0,\ \omega_{ik} = 0,\ \omega_{jk} = 0,$$
$$\omega_{1i} = \frac{nb_iC_1}{(b_1-b_i)^2}\omega_i,\ \omega_{1j} = \frac{nb_jC_1}{(b_1-b_j)^2}\omega_j,\ \omega_{1k} = \frac{nb_kC_1}{(b_1-b_k)^2}\omega_k.$$

Using $d\omega_{ij} - \sum_l \omega_{il} \wedge \omega_{lj} = -\frac{1}{2}\sum_{kl} R_{ijkl}\omega_k \wedge \omega_l$, we obtain the following:

$$\begin{aligned} R_{ijij} &= -b_ib_j + a_i + a_j = \frac{-n^2 b_ib_j}{(b_1-b_i)^2(b_1-b_j)^2}C_1^2, \\ R_{ikik} &= -b_ib_k + a_i + a_k = \frac{-n^2 b_ib_k}{(b_1-b_i)^2(b_1-b_k)^2}C_1^2, \end{aligned} \tag{4.9}$$

where $i \in [i], j \in [j], k \in [k]$.

Subtracting the second formula of (4.9) from the first one, we obtained

$$b_i(b_k - b_j) + (a_j - a_k) = \frac{n^2 b_i C_1^2 (b_j - b_k)(b_jb_k - b_1^2)}{(b_1-b_i)^2(b_1-b_k)^2(b_1-b_j)^2}. \tag{4.10}$$

From (4.1), we have $a_k - a_j = \frac{b_j^2 - b_k^2}{n-2}$. Combining it with (4.10), we obtain

$$\frac{(n-2)b_i + b_j + b_k}{n-2} = \frac{n^2 b_i C_1^2 (b_1^2 - b_jb_k)}{(b_1-b_i)^2(b_1-b_k)^2(b_1-b_j)^2}. \tag{4.11}$$

Similarly,

$$\frac{(n-2)b_j + b_i + b_k}{n-2} = \frac{n^2 b_j C_1^2 (b_1^2 - b_ib_k)}{(b_1-b_j)^2(b_1-b_k)^2(b_1-b_i)^2}. \tag{4.12}$$

Using (4.11) and (4.12), we have

$$\frac{n^2C_1^2b_1^2}{(b_1-b_j)^2(b_1-b_k)^2(b_1-b_i)^2} = \frac{n-3}{n-2}. \tag{4.13}$$

If $s+t \geq 5$, then there exists another conformal principal curvature b_l and

$$\frac{n^2C_1^2b_1^2}{(b_1-b_j)^2(b_1-b_l)^2(b_1-b_i)^2} = \frac{n-3}{n-2}. \tag{4.14}$$

Combining the Eqs (4.13) and (4.14), we can get that $b_l = b_k$, which is a contradiction. Thus,

$$s+t = 4,\ n^2b_1^2C_1^2 = \frac{n-3}{n-2}(b_1-b_j)^2(b_1-b_k)^2(b_1-b_i)^2.$$

This and (4.9) yield the following equations:

$$\begin{aligned}
\frac{-b_ib_j+a_i+a_j}{(b_1-b_k)^2}&=\frac{-(n-3)b_ib_j}{(n-2)b_1^2},\\
\frac{-b_ib_k+a_i+a_k}{(b_1-b_j)^2}&=\frac{-(n-3)b_ib_k}{(n-2)b_1^2},\\
\frac{-b_jb_k+a_j+a_k}{(b_1-b_i)^2}&=\frac{-(n-3)b_jb_k}{(n-2)b_1^2}.
\end{aligned}\tag{4.15}$$

Combining (2.8) and (4.1), it is easy to prove that the conformal principal curvatures $\{b_1, b_i, b_j, b_k\}$ are constant. Thus $B_{ii,1} = B_{jj,1} = B_{kk,1} = 0$ and $C_1 = 0$. Therefore, the conformal 1-form $C = 0$ and from Proposition 4.1 we know that $s + t = 2$, which is a contradiction. Thus, we complete proof of the Theorem 4.1. □

Since $s + t \leq 3$, we consider two cases:
Case 1. $s + t = 2$;
Case 2. $s + t = 3$.

First, we consider Case 1, $s + t = 2$, we have the following results.

Theorem 4.2. *Let $f : M^n \to \mathbb{R}_1^{n+1}$ $(n \geq 4)$ be a spacelike conformal Einstein hypersurface with two distinct principal curvatures, then f is locally conformally equivalent to one of the following hypersurfaces:*

1) the spacelike hypersurfaces with constant conformal sectional curvature;
2) the spacelike hypersurface

$$f : \mathbb{H}^k\Big(-\sqrt{\frac{k-1}{n-2}}\Big)\times\mathbb{H}^{n-k}\Big(-\sqrt{\frac{n-k-1}{n-2}}\Big)\to\mathbb{H}_1^{n+1}(-1),\ 1<k<n-1.$$

Proof. We assume that the spacelike conformal Einstein hypersurface has two distinct conformal principal curvatures b_1, b_2. If the multiplicities of the conformal principal curvatures b_1, b_2 are greater than 1, then the conformal 1-form $C = 0$. By Proposition 4.1, we finish the proof.

If one of conformal principal curvatures b_1, b_2 is simple, then the spacelike hypersurface is conformally flat. Since the spacelike hypersurface is conformal Einstein, then the spacelike conformal Einstein hypersurface is of constant conformal sectional curvature. Thus we finish the proof. □

Next, we consider Case 2, $s + t = 3$, that is, the spacelike conformal Einstein hypersurface has three distinct conformal principal curvatures b_1, b_2, b_3. If the multiplicities of the conformal principal curvatures b_1, b_2, b_3 are greater than 1, then the conformal 1-form $C = 0$ by Lemma 4.1. By Proposition 4.1, we know that such a hypersurface does not exist. Thus, we need to consider the following two subcases, (1) $\{b_1, \ldots, b_n\} = \{b_1, \mu, \ldots, \mu, \nu, \ldots, \nu\}$, (2) $\{b_1, \ldots, b_n\} = \{b_1, b_2, \mu, \ldots, \mu\}$. The following proposition means that the subcase (1) cannot occur.

Proposition 4.2. *Let $f : M^n \to \mathbb{R}_1^{n+1}(n \geq 4)$ be a spacelike hypersurface. If f has three distinct principal curvatures and one of the principal curvatures is simple, i.e.,*

$$\{b_1,\ldots,b_n\}=\{b_1,\underbrace{\mu,\ldots,\mu}_{s},\underbrace{\nu,\ldots,\nu}_{t}\},1+s+t=n,s,t\geq 2.$$

Then, the conformal Ricci curvature of f can not be constant.

Proof. Let $i\in\{m|b_m=\mu\}$, $j\in\{m|b_m=\nu\}$, from Lemma 4.1, we have

$$\begin{aligned}&C_2=\ldots=C_n=0,\\&B_{1i,i}=\frac{n\mu}{b_1-\mu}C_1,B_{1j,j}=\frac{n\nu}{b_1-\nu}C_1,\\&\omega_{1i}=\frac{B_{1i,i}}{b_1-\mu}\omega_i,\ \omega_{1j}=\frac{B_{1j,j}}{b_1-\nu}\omega_j,\end{aligned}\tag{4.16}$$

Since $B_{jj,1}=B_{1j,j}+C_1$, from (4.16), we obtain

$$B_{ii,1}=\frac{b_1+(n-1)\mu}{b_1-\mu}C_1,\ B_{jj,1}=\frac{b_1+(n-1)\nu}{b_1-\nu}C_1.\tag{4.17}$$

Since $\mathrm{tr(B)}=0$, $\nabla_{E_1}\mathrm{tr(B)}=\mathrm{tr}(\nabla_{\mathrm{E}_1}\mathrm{B})=0$ (i.e., $\sum_m B_{mm,1}=0$). Combining it with $b_1+s\mu+t\nu=0$ and $b_1^2+s\mu^2+t\nu^2=\frac{n-1}{n}$, yields the following:

$$B_{11,1}=-sB_{ii,1}-tB_{jj,1}=\frac{nb_1^2-\frac{n-1}{n}}{(b_1-\mu)(b_1-\nu)}C_1.\tag{4.18}$$

Using $d\omega_{ij}-\sum_l\omega_{il}\wedge\omega_{lj}=-\frac{1}{2}\sum_{kl}R_{ijkl}\omega_k\wedge\omega_l$, we obtain

$$R_{ijij}=\frac{-n^2\mu\nu C_1^2}{(b_1-\mu)^2(b_1-\nu)^2}.\tag{4.19}$$

Using the definition of $B_{ij,kl}$ and Lemma 4.1, we have

$$\begin{aligned}B_{1i,i1}&=\frac{b_1B_{ii,1}-\mu B_{11,1}}{(b_1-\mu)^2}nC_1+\frac{n\mu}{b_1-\mu}C_{1,1},B_{1i,1i}=(B_{11,1}-B_{ii,1}-B_{1i,i})\frac{n\mu C_1}{(b_1-\mu)^2},\\B_{1j,j1}&=\frac{b_1B_{jj,1}-\nu B_{11,1}}{(b_1-\nu)^2}nC_1+\frac{n\nu}{b_1-\nu}C_{1,1},B_{1j,1j}=(B_{11,1}-B_{jj,1}-B_{1j,j})\frac{n\nu C_1}{(b_1-\nu)^2}.\end{aligned}$$

Using Ricci identity $B_{1i,i1}-B_{1i,1i}=(\mu-b_1)R_{1i1i}$ and Lemma 4.1, we obtain the following:

$$\begin{aligned}(b_1-\mu)^2R_{1i1i}&=\frac{nC_1}{b_1-\mu}[2\mu B_{11,1}-(b_1+\mu)B_{ii,1}-\mu B_{1i,i}]-n\mu C_{1,1},\\(b_1-\nu)^2R_{1j1j}&=\frac{nC_1}{b_1-\nu}[2\nu B_{11,1}-(b_1+\nu)B_{jj,1}-\nu B_{1j,j}]-n\nu C_{1,1},\end{aligned}\tag{4.20}$$

From (4.20), (4.16) and (4.18), we can obtain the following:

$$\begin{aligned}&(b_1-\mu)^2\nu R_{1i1i}-(b_1-\nu)^2\mu R_{1j1j}=\frac{n(\mu-\nu)C_1^2}{(b_1-\mu)^2(b_1-\nu)^2}\chi,\\&\chi:=b_1^2[\mu^2+\nu^2+b_1^2-2b_1(\mu+\nu)-4(n-1)\mu\nu]+(3n-2)b_1\mu\nu(\mu+\nu)\\&\ +(2n^2-2n+1)\mu^2\nu^2.\end{aligned}$$

Combining it with (4.19), we have

$$(b_1-\mu)^2\nu R_{1i1i}-(b_1-\nu)^2\mu R_{1j1j}+\frac{\mu-\nu}{n\mu\nu}\chi R_{ijij}=0. \tag{4.21}$$

Using (4.2), (4.21) and $b_1+s\mu+t\nu=0, b_1^2+s\mu^2+t\nu^2=\frac{n-1}{n}$, we know that b_1,μ, and ν are constant. From Lemma 4.1, we get $C_1=0$. Therefore, $C=0$. Using Proposition 4.1, we know that f has only two distinct principal curvatures, which is a contradiction, finishing the proof. □

Next let $f: M^n\to\mathbb{R}_1^{n+1}(n\geq 4)$ be a spacelike conformal Einstein hypersurface with three distinct principal curvatures, two of then being simple, i.e.,

$$\{b_1,\ldots,b_n\}=\{b_1,b_2,\mu,\ldots,\mu\}.$$

Using (4.2), we have

$$(A_{ij})=diag\{a_1,\ldots,a_n\}=\{a_1,a_2,a,\ldots,a\}.$$

In the following section, we assume the index $3\leq\alpha,\beta,\gamma\leq n$. From Lemma 4.1, we have

$$\begin{aligned}
&C_3=\ldots=C_n=0,\\
&B_{1\alpha,\alpha}=\frac{n\mu}{b_1-\mu}C_1,\ B_{2\alpha,\alpha}=\frac{n\nu}{b_2-\nu}C_1,\\
&B_{12,1}=\frac{nb_1}{b_2-b_1}C_2,\ B_{12,2}=\frac{nb_2}{b_1-b_2}C_1,\\
&\omega_{1\alpha}=\frac{B_{1i,i}}{b_1-\mu}\omega_\alpha,\ \omega_{2\alpha}=\frac{B_{2i,i}}{b_2-\nu}\omega_\alpha,\ \omega_{12}=\frac{B_{12,1}}{b_1-b_2}\omega_1+\frac{B_{12,2}}{b_1-b_2}\omega_2.
\end{aligned} \tag{4.22}$$

Thus, we can deduce the following results,

$$E_\alpha(b_1)=E_\alpha(b_2)=E_\alpha(\mu)=0,\ E_\alpha(C_1)=E_\alpha(C_2)=0. \tag{4.23}$$

Using $d\omega_{1\alpha}-\sum_m\omega_{1m}\wedge\omega_{m\alpha}=-\frac{1}{2}\sum_{kl}R_{1\alpha kl}\omega_k\wedge\omega_l$ and (4.22), we get

$$\begin{aligned}
&E_1(\frac{B_{1\alpha,\alpha}}{b_1-\mu})+(\frac{B_{1\alpha,\alpha}}{b_1-\mu})^2-\frac{B_{12,1}}{b_1-b_2}\frac{B_{2\alpha,\alpha}}{b_2-\mu}=-R_{1\alpha1\alpha}=b_1\mu-a_1-a_\alpha,\\
&E_2(\frac{B_{1\alpha,\alpha}}{b_1-\mu})=\frac{B_{2\alpha,\alpha}}{b_2-\mu}\frac{B_{12,2}}{b_1-b_2}-\frac{B_{1\alpha,\alpha}}{b_1-\mu}\frac{B_{2\alpha,\alpha}}{b_2-\mu}.
\end{aligned} \tag{4.24}$$

Similarly, from $d\omega_{2\alpha}-\sum_m\omega_{2m}\wedge\omega_{m\alpha}=-\frac{1}{2}\sum_{kl}R_{2\alpha kl}\omega_k\wedge\omega_l$

$$\begin{aligned}
&E_1(\frac{B_{2\alpha,\alpha}}{b_2-\mu})=-\frac{B_{1\alpha,\alpha}}{b_1-\mu}\frac{B_{12,1}}{b_1-b_2}-\frac{B_{1\alpha,\alpha}}{b_1-\mu}\frac{B_{2\alpha,\alpha}}{b_2-\mu},\\
&E_2(\frac{B_{2\alpha,\alpha}}{b_2-\mu})+(\frac{B_{2\alpha,\alpha}}{b_2-\mu})^2-\frac{B_{12,2}}{b_1-b_2}\frac{B_{1\alpha,\alpha}}{b_1-\mu}=-R_{2\alpha2\alpha}=b_2\mu-a_2-a_\alpha.
\end{aligned} \tag{4.25}$$

Under the orthonormal basis $\{E_1,\cdots,E_n\}$. $\{Y,N,Y_1,\cdots,Y_n,\xi\}$ forms a moving frame in $\mathbb{R}_2^{n+3}$ along M^n. We define

$$\begin{aligned}
&F=\xi-\mu,\ \ X_1=-\frac{B_{1\alpha,\alpha}}{b_1-\mu}Y+Y_1,\ X_2=-\frac{B_{2\alpha,\alpha}}{b_2-\mu}Y+Y_2,\\
&P=aY-N+\mu F-\frac{B_{1\alpha,\alpha}}{b_1-\mu}X_1-\frac{B_{2\alpha,\alpha}}{b_2-\mu}X_2,\\
&K=2a-\mu^2+(\frac{B_{1\alpha,\alpha}}{b_1-\mu})^2+(\frac{B_{2\alpha,\alpha}}{b_2-\mu})^2.
\end{aligned} \tag{4.26}$$

Since the conformal principal curvatures b_1 and b_1 are simple; thus, the principal vector fields E_1 and E_2 are well defined. Since the vectors Y, N, Y_1, Y_2, ξ are well defined alone the hypersurface, thus the vectors F, X_1, X_2, P, K are also well defined. It is easy to get that

$$\langle F, F\rangle_2 = -1,\ \langle X_1, X_1\rangle_2 = \langle X_2, X_2\rangle_2 = 1,\ \langle P, P\rangle_2 = -K.$$

$$\langle F, X_1\rangle_2 = \langle F, X_2\rangle_2 = \langle F, P\rangle_2 = \langle X_1, X_2\rangle_2 = \langle X_1, P\rangle_2 = \langle X_2, P\rangle_2 = 0.$$

By direct calculation, from (2.2), (4.23), (4.24) and (4.25), we have the following equations:

$$\begin{aligned}
&E_1(F) = (b_1-\mu)X_1,\ E_2(F) = (b_2-\mu)X_2,\ E_\alpha(F) = 0,\\
&E_1(X_1) = P + (\frac{B_{2\alpha,\alpha}}{b_2-\mu} + \frac{B_{12,1}}{b_1-b_2})X_2 + (b_1-\mu)F,\\
&E_2(X_1) = (\frac{B_{12,2}}{b_1-b_2} + \frac{B_{1\alpha,\alpha}}{b_1-\mu})X_2,\ E_\alpha(X_1) = 0.
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
&E_1(X_2) = -(\frac{B_{12,1}}{b_1-b_2} + \frac{B_{2\alpha,\alpha}}{b_2-\mu})X_1,\\
&E_2(X_2) = P + (\frac{B_{1\alpha,\alpha}}{b_1-\mu} - \frac{B_{12,2}}{b_1-b_2})X_1 + (b_2-\mu)F,\ E_\alpha(X_2) = 0,\\
&E_1(P) = -\frac{B_{1\alpha,\alpha}}{b_1-\mu}P + KX_1,\ E_2(P) = -\frac{B_{2\alpha,\alpha}}{b_2-\mu}P + KX_2,\ E_\alpha(P) = 0.
\end{aligned} \tag{4.28}$$

We define

$$T = aY + N - \mu\xi + \frac{B_{1\alpha,\alpha}}{b_1-\mu}Y_1 + \frac{B_{2\alpha,\alpha}}{b_2-\mu}Y_2.$$

Then,

$$T + P = KY,\ \langle P, P\rangle_2 = -K,\ \langle T, T\rangle_2 = K. \tag{4.29}$$

By direct calculation, from (2.2), (4.23)–(4.25), we have the following equations:

$$\begin{aligned}
&E_1(T) = -\frac{B_{1\alpha,\alpha}}{b_1-\mu}T,\ E_2(T) = -\frac{B_{2\alpha,\alpha}}{b_2-\mu}T,\ E_\alpha(T) = KY_\alpha,\\
&E_1(Y_\alpha) = \sum_\gamma \omega_{\alpha\gamma}(E_1)Y_\gamma,\ E_2(Y_\alpha) = \sum_\gamma \omega_{\alpha\gamma}(E_2)Y_\gamma,\\
&E_\alpha(Y_\alpha) = -T + \sum_\gamma \omega_{\alpha\gamma}(E_\alpha)Y_\gamma,\ E_\beta(Y_\alpha) = \sum_\gamma \omega_{\alpha\gamma}(E_\beta)Y_\gamma,\ \alpha \neq \beta.
\end{aligned} \tag{4.30}$$

From (4.27) and (4.28), we know that the subspace $V_1 = span\{F, X_1, X_2, P\}$ is fixed along M^n. From (4.30), we know that the subspace $V_2 = span\{T, Y_3, \cdots, Y_n\}$ is fixed along M^n. Since $T \perp V_1$, thus

$$V_1 \perp V_2.$$

From the fourth equation in (4.22), we know that the distributions

$$\mathbb{D}_1 = Span\{E_1, E_2\},\ \mathbb{D}_2 = Span\{E_3, \cdots, E_n\}$$

are integrable. Let $\tilde{M}^2$ be an integral submanifold of $\mathbb{D}_1$, by (4.27) and (4.28) the vector F induces a 2-dimensional submanifold in $\mathbb{H}_1^{n+2}$

$$F : \tilde{M}^2 \to \mathbb{H}_1^{n+2}.$$

By direct calculation, from (2.2), (4.23)–(4.25), we have

$$E_1(K) = -2\frac{B_{1\alpha,\alpha}}{b_1 - \mu}K,\ E_2(K) = -2\frac{B_{2\alpha,\alpha}}{b_2 - \mu}K,\ E_\alpha(K) = 0. \tag{4.31}$$

Regarding (4.31) as a linear first order ODE for K, we know that $K \equiv 0$ or $K \neq 0$ on the connected hypersurface M^n. Thus, we need to consider the following subcases: (1) $K = 0$ on M^n; (2) $K < 0$ on M^n; (3) $K > 0$ on M^n. Next, we treat them case by case.

Proposition 4.3. *Let $f : M^n \to \mathbb{R}_1^{n+1} (n \geq 4)$ be a spacelike conformal Einstein hypersurface with three distinct principal curvatures. If $K = 0$, then f is locally conformally equivalent to a cylinder over a spacelike (λ, n)-surface in $\mathbb{R}_1^3$, $(n \geq 4)$.*

Proof. Since $K = 0$, then $\langle P, P\rangle_2 = 0$, from (4.28) we have

$$E_1(P) = \frac{B_{1\alpha,\alpha}}{\mu - b_1}P,\ E_2(P) = \frac{B_{2\alpha,\alpha}}{\mu - b_2}P.$$

Therefore, P has a fixed direction, and we can write, up to a conformal transformation

$$\begin{aligned} &P = \psi(1, -1, 0, \cdots, 0) = \psi e,\ \psi \in C^\infty(M^n),\\ &Span\{F, X_1, X_2, P\}\\ &\ = Span\{e, (0, 0, 1, 0, \cdots, 0), (0, 0, 0, 1, 0, \cdots, 0), (0, 0, 0, 0, 1, 0, \cdots, 0)\}. \end{aligned}$$

Let the spacelike hypersurface $f : M^n \to \mathbb{R}_1^{n+1}$ have the principal curvatures

$$\lambda_1, \lambda_2, \lambda, \cdots, \lambda.$$

From $\langle P, F\rangle_2 = \langle e, F\rangle_2 = 0$, we get

$$\lambda = 0.$$

Similarly, from $\langle e, X_1\rangle_2 = \langle e, X_2\rangle_2 = \langle e, Y_\alpha\rangle_2 = 0$, we get that

$$\frac{B_{1\alpha,\alpha}}{\mu - b_1}\rho + E_1(\rho) = 0,\ \frac{B_{2\alpha,\alpha}}{\mu - b_2}\rho + E_2(\rho) = 0,\ E_\alpha(\rho) = 0.$$

Thus, we have

$$E_1(\log\rho) = \frac{B_{1\alpha,\alpha}}{b_1 - \mu},\ E_2(\log\rho) = \frac{B_{2\alpha,\alpha}}{b_2 - \mu},\ E_\alpha(\rho) = 0. \tag{4.32}$$

Let $\{e_i = \rho E_i, 1 \leq i \leq n\}$, then $\{e_1, \cdots, e_n\}$ is a orthonormal basis of TM^n with respect to the induced metric of f, $\{\theta_1, \cdots, \theta_n\}$ its dual basis and $\{\theta_{ij}\}$ connection form with respect to basis $\{\theta_1, \cdots, \theta_n\}$. Then, from (2.13), we obtain

$$\theta_{1\alpha} = 0,\ \theta_{2\alpha} = 0. \tag{4.33}$$

Therefore, the spacelike hypersurface $f : M^n \to \mathbb{R}_1^{n+1}$ is conformally equivalent to the cylinder hypersurface given by Example (3.5). By Proposition 3.2, we finish the proof. □

Proposition 4.4. *Let $f : M^n \to \mathbb{R}^{n+1}_1 (n \geq 4)$ be a spacelike conformal Einstein hypersurface with three distinct principal curvatures. If $K < 0$, then f is locally conformally equivalent to a cone over a spacelike (λ, n)-surface in the Lorentzian space form $\mathbb{S}^3_1(1)$, $(n \geq 4)$.*

Proof. Since $K < 0$, by (4.29) the vector field P is a spacelike vector field in $\mathbb{R}^{n+3}_2$. Thus, up to a conformal transformation we can write the following:

$$\begin{aligned} V_1 &= span\{F, X_1, X_2, P\} \\ &= span\{(0,1,0,...,0),\ (0,0,1,0,...,0),\ (0,0,0,1,...,0),\ (0,0,0,0,1,0,...,0)\}. \end{aligned}$$

Since the spacelike hypersurface f has principal curvatures

$$\{\lambda_1,\ \lambda_2,\ \lambda, \cdots, \lambda\},$$

and $e = (1,0,...,0,1) \perp V_1$, we have $\langle F, e\rangle_2 = 0$ which implies that

$$\lambda = 0.$$

Let

$$\bar{P} = \frac{P}{\sqrt{-K}}, \theta = \frac{T}{\sqrt{-K}},$$

then $\langle \bar{P}, \bar{P}\rangle_2 = 1, \langle \theta, \theta\rangle_2 = -1$. Eqs (4.27) and (4.28) mean that

$$\bar{P} : \tilde{M}^2 \to \mathbb{S}^3_1 \subset \mathbb{R}^4_1 = V_1$$

is a spacelike surface, and the Eq (4.30) mean that

$$\theta : L \to \mathbb{H}^{n-2} \subset \mathbb{R}^{n-1}_1$$

is a standard embedding and the sectional curvature of $\theta(L)$ is -1. Since $dimL = dim\mathbb{H}^{n-2} = n-2$, we know that $\theta : L \to \mathbb{H}^{n-2}$ is a standard isometric isomorphism. By (3.4), we have the standard isometric isomorphism

$$\theta : L \to \mathbb{H}^{n-2} = \mathbb{R}^+ \times \mathbb{R}^{n-3}.$$

Since $P + T = KY$,

$$Y = \frac{1}{\sqrt{-K}}(\bar{P}, \theta) : M^n = \tilde{M}^2 \times L \to \mathbb{S}^3_1 \times \mathbb{H}^{n-2} = \mathbb{S}^3_1 \times \mathbb{R}^+ \times \mathbb{R}^{n-3} \subset \mathbb{R}^{n+3}_1.$$

Therefore, $g = \langle dY, dY\rangle_2 = \frac{-1}{K}(I + I_{\mathbb{H}^{n-1}})$. Thus, the spacelike hypersurface f is conformally equivalent to the cone hypersurface given by Example 3.6. By Proposition 3.2, we finish the proof. □

Proposition 4.5. *Let $f : M^n \to \mathbb{R}^{n+1}_1 (n \geq 4)$ be a spacelike conformal Einstein hypersurface with three distinct principal curvatures. If $K > 0$, then f is locally conformally equivalent to a rotational hypersurface over a spacelike (λ, n)-surface in the Lorentzian space form $\mathbb{R}^3_{1+}$.*

Proof. Since $K > 0$, then $\langle P, P\rangle_2 < 0$. Thus, up to a conformal transformation, we can write the following:

$$\begin{aligned} V_1 &= span\{F, X_1, X_2, P\} \\ &= span\{(1,0,\cdots,0),\ (0,1,0,\cdots,0),\ (0,\cdots,0,1,0),\ (0,\cdots,0,1)\}. \end{aligned}$$

Thus, $e = (1, 0, ..., 0, 1) \in V_1$ and $\langle Y_\alpha, e\rangle_2 = 0,\ 2 \le \alpha \le n$, which imply that $E_\alpha(\tau) = 0,\ 2 \le \alpha \le n$. Setting

$$\bar{P} = \frac{P}{\sqrt{K}},\quad \theta = \frac{T}{\sqrt{K}},$$

then $\langle \bar{P}, \bar{P}\rangle_2 = -1, \langle\theta, \theta\rangle_2 = 1$. Eqs (4.27) and (4.28) mean that

$$\bar{P} : \tilde{M}^2 \to \mathbb{H}^3_1 \subset \mathbb{R}^4_1 = V_1$$

is a spacelike surface. Eq (4.30) means that

$$\theta : L \to \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$$

is a standard embedding and the sectional curvature of $\theta(L)$ is 1. Since $dimL = n-2$, $\theta : L \to \mathbb{S}^{n-2}$ is a standard isometric isomorphism. Since $P + T = KY$,

$$Y = \frac{1}{\sqrt{K}}(\bar{P}, \theta) : \tilde{M}^2 \times L \to \mathbb{H}^3_1 \times \mathbb{S}^{n-2}.$$

Denote $\bar{P} = (u_1, u_2, u_3, u_4) \in \mathbb{H}^3_1$, then

$$Y = \frac{u_1 - u_4}{\sqrt{K}}\left(\frac{u_1}{u_1 - u_4}, \frac{u_2}{u_1 - u_4}, \frac{u_3}{u_1 - u_4}, \frac{u_4}{u_1 - u_4}, \frac{\theta}{u_1 - u_4}\right).$$

Thus the spacelike hypersurface $f : \tilde{M}^2 \times \mathbb{S}^{n-2} \to \mathbb{R}^{n+1}_1$ is now given by

$$f = \left(\frac{u_2}{u_1 - u_4}, \frac{u_3}{u_1 - u_4}, \frac{\theta}{u_1 - u_4}\right).$$

Note that

$$\varphi(u_1, u_2, u_3, u_4) = \left(\frac{u_2}{u_1 - u_4}, \frac{u_3}{u_1 - u_4}, \frac{1}{u_1 - u_4}\right)$$

is the inverse mapping of the local isometric correspondence $\phi : \mathbb{R}^3_{1+} \to \mathbb{H}^3_1$ by (3.5). Thus, the spacelike hypersurface f is conformally equivalent to the rotational hypersurface given by Example 3.7. By Proposition 3.2, we finish the proof. □

Combining Propositions 4.3–4.5, we have the following theorem:

Theorem 4.3. *Let $f : M^n \to \mathbb{R}^{n+1}_1 (n \ge 4)$ be a spacelike conformal Einstein hypersurface with three distinct principal curvatures. Then, f is locally conformally equivalent to one of the following examples:*

1) a cylinder over a (λ, n)-surface in $\mathbb{R}^3_1$;
2) a cone over a (λ, n)-surface in $\mathbb{S}^3_1$;
3) a rotation hypersurface over a (λ, n)-surface in $\mathbb{R}^3_{1+}$.

Combining Theorems 4.1 and 4.3, we finish the proof of the main Theorem 1.2.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Authors are supported by the Grant No. 12071028 of NSFC.

Conflict of interest

The authors declare that they have no competing interests.

References

1. M. Cahen, Y. Kerbrat, Domaines symmétriques des quadriques projectives, *J. Math. Pure Appl.*, **62** (1983), 327–348.
2. X. Ji, T. Z. Li, H. F. Sun, Spacelike hypersurfaces with constant conformal sectional curvature in R_1^{n+1}, *Pac. J. Math.*, **300** (2019), 17–37. http://doi.org/10.2140/pjm.2019.300.17
3. T. Z. Li, C. X. Nie, Conformal geometry of hypersurfaces in Lorentz space forms, *Geometry*, **2013** (2013), 549602. http://doi.org/10.1155/2013/549602
4. T. Z. Li, C. X. Nie, Spacelike Dupin hypersurfaces in Lorentzian space forms, *J. Math. Soc. Japan*, **70** (2018), 463–480. http://doi.org/10.2969/jmsj/07027573
5. B. O'Neill, *Semi-Riemannian geometry*, New York: Academic Press, 1983.
6. T. S. Yau, Remarks on conformal transformations, *J. Differential Geom.*, **8** (1973), 369–381. http://doi.org/10.4310/jdg/1214431798