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Research article

Extended suprametric spaces and Stone-type theorem

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Abstract: Extended suprametric spaces are defined, and the contraction principle is established using elementary properties of the greatest lower bound instead of the usual iteration procedure. Thereafter, some topological results and the Stone-type theorem are derived in terms of suprametric spaces. Also, we have shown that every suprametric space is metrizable. Further, we prove the existence of a solution of Ito-Doob type stochastic integral equations using our main fixed point theorem in extended suprametric spaces.

Keywords: an extended suprametric space; metrization; fixed point and Ito-Doob type stochastic integral equations
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1. Introduction

A topological space can be presumed as an axiomatization of the notion of a point's closeness to a set. When a point is a member of the closure of a set, it is said to be close to the set. According to the theory of metric spaces, which is an axiomatization of the idea of a pair of points being close to one another in a metric space, the distance between any two points is assessed by a real number, and its basic properties are outlined by a set of axioms. Nevertheless, the class of metric spaces is inextricably linked with the fascinating class of metrizable spaces, which is a class of topological spaces and plays a significant role in applications of modern and general topology, as well as in the development of proper topological structures and relations. We place a high priority on metrizable spaces because they are utilized in numerous interesting topological spaces in multiple mathematical disciplines. Numerous researchers have been working on its extension, generalization or improvement because of its wide range of applications in numerous fields of mathematics. Topological spaces and metric spaces are both extensively used topics. As a special case of topological spaces, metric spaces are actually of interest, and the suggested axioms of certain spaces are geometrically meaningful. This makes metrizability a fascinating topic for topological spaces. Unsurprisingly, some spaces are not metrizable. As a result, researchers try to build more general and metrizable functions. Metric spaces are a unique type of topological space. In metric spaces, sequences are used to characterize topological properties. Sequences are completely inadequate for such convenience in topological spaces. Seeking classes that are largely independent of topological spaces and metric spaces is simple, and with members, sequences play an important role in assessing their topological properties.

A variety of spaces have recently been built, as well as some new types of modified metric spaces. Weakening the axioms of certain modified metric spaces or of metric spaces, in general, is the crucial step in creating these spaces. It is frequently not stated what the topological characteristics of new modified metric spaces are, and it is frequently not taken into account how these modified metric spaces relate to previously modified metric spaces in terms of fixed point theorems. In this article, extended suprametric spaces are introduced, and the contraction principle is established using elementary properties of the greatest lower bound instead of the usual iteration procedure. Thereafter, some topological results and the Stone-type theorem are derived in terms of suprametric spaces. Also, we have shown that every suprametric space is metrizable.

The metric space has been generalized in numerous research works to more abstract spaces, including the b-metric spaces of Bakhtin [1] and Bourbaki [2], the partial metric spaces of Matthews [3] and the rectangle metric spaces of Branciari [4]. The b-metric was developed due to the theories of Bakhtin [1] and Bourbaki [2]. Czerwik [5] established an axiom that was weaker than the triangular inequality and specifically defined a b-metric space in order to develop the Banach contraction result. Numerous authors have generalized the b-metric space as a result of being inspired by its idea and have produced a variety of fixed-point results (see, for example, [6–9]).

2. Extended suprametric spaces and contraction principle

Definition 2.1. Let $\mathcal{X}$ be a non-empty set.

(a) A suprametric on the set $\mathcal{X}$ is a function $\mathscr{D} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ which satisfies the following conditions:

(i) $(\forall\, \mathbb{x}, \mathbb{y} \in \mathcal{X})\ \mathscr{D}(\mathbb{x}, \mathbb{y}) = 0 \Leftrightarrow \mathbb{x} = \mathbb{y}$;

(ii) $(\forall\, \mathbb{x}, \mathbb{y} \in \mathcal{X})\ \mathscr{D}(\mathbb{x}, \mathbb{y}) = \mathscr{D}(\mathbb{y}, \mathbb{x})$;

(iii) $(\exists \zeta \in \mathbb{R}^+)(\forall\, \mathbb{x}, \mathbb{y}, \mathbb{z} \in \mathcal{X})\ \mathscr{D}(\mathbb{x}, \mathbb{z}) \le \mathscr{D}(\mathbb{x}, \mathbb{y}) + \mathscr{D}(\mathbb{y}, \mathbb{z}) + \zeta \mathscr{D}(\mathbb{x}, \mathbb{y})\mathscr{D}(\mathbb{y}, \mathbb{z})$.

(b) An extended suprametric on the set $\mathcal{X}$ is a function $\mathscr{D} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ which satisfies conditions (i) and (ii) of item **(a)** and the following condition:

(iv) there exists a function $\gamma : \mathcal{X} \times \mathcal{X} \to [1, +\infty)$ such that :
$(\forall\, \mathbb{x}, \mathbb{y}, \mathbb{z} \in \mathcal{X})\ \mathscr{D}(\mathbb{x}, \mathbb{z}) \le \mathscr{D}(\mathbb{x}, \mathbb{y}) + \mathscr{D}(\mathbb{y}, \mathbb{z}) + \gamma(\mathbb{x}, \mathbb{z})\mathscr{D}(\mathbb{x}, \mathbb{y})\mathscr{D}(\mathbb{y}, \mathbb{z})$.

If $\mathscr{D}$ is a suprametric (respectively, an extended suprametric) on $\mathcal{X}$, then the ordered pair $(\mathcal{X}, \mathscr{D})$ is called a suprametric space [10] (respectively, an extended suprametric space).

Example 2.2. Assume that $\mathcal{X}$ is a set of natural numbers. Define $\mathscr{D} : \mathcal{X}\times\mathcal{X} \to \mathbb{R}^+$ by $\mathscr{D}(\mathtt{x},\mathtt{y}) = (\mathtt{x}-\mathtt{y})^2$ and $\gamma : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ by $\gamma(\mathtt{x},\mathtt{y}) = e^{\mathtt{x}+\mathtt{y}}$. Then $(\mathcal{X},\mathscr{D})$ is an extended suprametric space.

Proof. Clearly, conditions (i) and (ii) of item **(a)** holds for all $\mathtt{x},\mathtt{y} \in \mathcal{X}$.
For any $\mathtt{x},\mathtt{y},\mathtt{z} \in \mathcal{X}$, consider

$$\begin{aligned}\mathscr{D}(\mathtt{x},\mathtt{z}) &= (\mathtt{x}-\mathtt{z})^2\\ &= (\mathtt{x}-\mathtt{y}+\mathtt{y}\mathtt{z})^2\\ &= (\mathtt{x}-\mathtt{y})^2 + (\mathtt{y}-\mathtt{z})^2 + 2(\mathtt{x}-\mathtt{y})(\mathtt{y}-\mathtt{z})\\ &\le (\mathtt{x}-\mathtt{y})^2 + (\mathtt{y}-\mathtt{z})^2 + 2(\mathtt{x}-\mathtt{y})^2(\mathtt{y}-\mathtt{z})^2\\ &< (\mathtt{x}-\mathtt{y})^2 + (\mathtt{y}-\mathtt{z})^2 + e^{\mathtt{x}+\mathtt{z}}(\mathtt{x}-\mathtt{y})^2(\mathtt{y}-\mathtt{z})^2.\end{aligned}$$

Therefore, $\mathscr{D}(\mathtt{x},\mathtt{z}) \le \mathscr{D}(\mathtt{x},\mathtt{y}) + \mathscr{D}(\mathtt{y},\mathtt{z}) + \gamma(\mathtt{x},\mathtt{z})\mathscr{D}(\mathtt{x},\mathtt{y})\mathscr{D}(\mathtt{y},\mathtt{z})$ for all $\mathtt{x},\mathtt{y},\mathtt{z} \in \mathcal{X}$. Hence $(\mathcal{X},\mathscr{D})$ is an extended suprametric space.

Remark 2.3. If $\gamma(\mathtt{x},\mathtt{y}) = \zeta$ for $\zeta \ge 1$ then we obtain the definition of a suprametric space.

Definition 2.4. A sequence $\{\mathtt{x}_n\}$ in $\mathcal{X}$ is said to be a convergent sequence in an extended suprametric space $(\mathcal{X},\mathscr{D})$ if for every $\varepsilon > 0$ there is $M = M(\varepsilon) \in \mathbb{N}$ such that $\mathscr{D}(\mathtt{x}_n,\mathtt{x}) < \varepsilon$, for all $n \ge M$ and $\mathtt{x} \in \mathcal{X}$. In this case, we write it as $\lim_{n\to\infty} \mathtt{x}_n = \mathtt{x}$.

Definition 2.5. A sequence $\{\mathtt{x}_n\}$ in $\mathcal{X}$ is said to be a Cauchy sequence in an extended suprametric space $(\mathcal{X},\mathscr{D})$ for every $\varepsilon > 0$ there is $M = M(\varepsilon) \in \mathbb{N}$ such that $\mathscr{D}(\mathtt{x}_m,\mathtt{x}_n) < \varepsilon$, for all $m,n \ge M$.

Definition 2.6. An extended suprametric space $(\mathcal{X},\mathscr{D})$ is complete if and only if every Cauchy sequence in $\mathcal{X}$ is convergent.

Remark 2.7. Assume that $(\mathcal{X},\mathscr{D})$ is an extended suprametric space. If $\mathscr{D}$ is continuous, then every convergent sequence has a unique limit.

Remark 2.8. Whenever a sequence $\{\mathtt{x}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete extended suprametric space then there exists $\mathtt{x}^\star \in \mathcal{X}$ such that $\lim_{n\to\infty} \mathscr{D}(\mathtt{x}_n,\mathtt{x}^\star) = 0$ and that every subsequence $\{\mathtt{x}_{n(\hbar)}\}_{\hbar\in\mathbb{N}}$ converges to $\mathtt{x}^\star$.

Let $\mathscr{K}$ represent a non-empty collection of positive real numbers that is bounded below. Then c is an infimum of $\mathscr{K}$ and any number in $\mathscr{K}$ that exceeds c can't be the lower bound of $\mathscr{K}$ according to the infimum property of $\mathbb{R}$ (see [11]).
A straightforward result of the infimum's properties is as follows:

Lemma 2.9. Let $\mathscr{K} = \{a/a$ is a nonnegative real number$\}$ be a nonempty set with zero as its greatest lower bound (shortly, *glb*). Then $\lim_{n\to\infty} g_n = 0$, where the sequence $\{g_n\}_{n=1}^{\infty}$ exists in $\mathscr{K}$.

Theorem 2.10. Assume that $(\mathcal{X},\mathscr{D})$ is a complete extended suprametric space such that $\mathscr{D}$ is continuous and $\mathcal{Z} : \mathcal{X} \to \mathcal{X}$ be a mapping. Consider that there exists $\theta \in [0,1)$ such that

$$\mathscr{D}(\mathcal{Z}\mathtt{x},\mathcal{Z}\mathtt{y}) \le \theta\mathscr{D}(\mathtt{x},\mathtt{y}), \quad \text{for all } \mathtt{x},\mathtt{y} \in \mathcal{X}. \tag{2.1}$$

Then, $\mathcal{Z}$ has a unique fixed point, and for every $\mathtt{x}_0 \in \mathcal{X}$ the iterative sequence defined by $\mathtt{x}_n = \mathcal{Z}\mathtt{x}_{n-1}$, $n \in \mathbb{N}$ converges to this fixed point.

Proof. Define the sequence $\mathbb{x}_n$ by $\mathbb{x}_n = \mathcal{Z}\mathbb{x}_{n-1}$ for all $n \in \mathbb{N}$ for some arbitrary $\mathbb{x}_0 \in \mathcal{X}$. Now, we demonstrate that the fixed point's existence can be effectively established by utilizing only the basic properties of an extended suprametric space and an infimum, omitting the usual iteration process.

Case-1: For $\theta = 0$.

This case is trivial since for $\theta = 0$, $\mathcal{Z}$ is a constant map, so $\mathcal{Z}$ has a fixed point.

Case-2: For $0 < \theta < 1$.

We define $\mathcal{D} = \{\mathscr{D}(\mathbb{x}, \mathcal{Z}\mathbb{x}) / \mathbb{x} \in \mathcal{X}\}$ and put $p = \inf \mathcal{D}$. Suppose that $p > 0$. Since $0 < \theta < 1$, we have $\frac{p}{\theta} > p$, so there exists $\mathbb{x}_p \in \mathcal{D}$ such that $\mathscr{D}(\mathbb{x}_p, \mathcal{Z}\mathbb{x}_p) < \frac{p}{\theta}$. Then

$$\mathscr{D}(\mathcal{Z}\mathbb{x}_p, \mathcal{Z}^2\mathbb{x}_p) \leq \theta\mathscr{D}(\mathbb{x}_p, \mathcal{Z}\mathbb{x}_p) < p, \text{ a contradiction.}$$

The contradiction obtained shows that $p = 0$. Now, since $p = 0$, there exists a sequence $\{\mathbb{x}_n\}$ of members of $\mathcal{X}$ such that $\lim_{n\to\infty} \mathscr{D}(\mathbb{x}_n, \mathcal{Z}\mathbb{x}_n) = 0$. Thus, for all $\varepsilon > 0$, there exists $\hbar \in \mathbb{N}$ such that for all $n \geq \hbar$, as a result

$$\mathscr{D}(\mathbb{x}_n, \mathbb{x}_{n+1}) < \varepsilon. \tag{2.2}$$

Now we will demonstrate that the sequence $\{\mathbb{x}_n\}$ is Cauchy.

Using the existing assumptions and (2.2), and for large enough integers a, b such that $b > a > \hbar$, consider

$$\begin{aligned}
\mathscr{D}(\mathbb{x}_a, \mathbb{x}_b) &\leq \mathscr{D}(\mathbb{x}_a, \mathbb{x}_{a+1}) + \mathscr{D}(\mathbb{x}_{a+1}, \mathbb{x}_b) + \gamma(\mathbb{x}_a, \mathbb{x}_b)\mathscr{D}(\mathbb{x}_a, \mathbb{x}_{a+1})\mathscr{D}(\mathbb{x}_{a+1}, \mathbb{x}_b) \\
&\leq \theta^{a-\hbar}\mathscr{D}(\mathbb{x}_\hbar, \mathbb{x}_{\hbar+1}) + \mathscr{D}(\mathbb{x}_{a+1}, \mathbb{x}_b) + \gamma(\mathbb{x}_a, \mathbb{x}_b)\theta^{a-\hbar}\mathscr{D}(\mathbb{x}_\hbar, \mathbb{x}_{\hbar+1})\mathscr{D}(\mathbb{x}_{a+1}, \mathbb{x}_b) \\
&\leq \theta^{a-\hbar}\varepsilon + \mathscr{D}(\mathbb{x}_{a+1}, \mathbb{x}_b) + \gamma(\mathbb{x}_a, \mathbb{x}_b)\theta^{a-\hbar}\varepsilon\mathscr{D}(\mathbb{x}_{a+1}, \mathbb{x}_b) \\
&\leq \theta^{a-\hbar}\varepsilon + [1 + \gamma(\mathbb{x}_a, \mathbb{x}_b)\varepsilon\theta^{a-\hbar}]\mathscr{D}(\mathbb{x}_{a+1}, \mathbb{x}_b).
\end{aligned} \tag{2.3}$$

Similarly,

$$\begin{aligned}
\mathscr{D}(\mathbb{x}_{a+1}, \mathbb{x}_b) &\leq \mathscr{D}(\mathbb{x}_{a+1}, \mathbb{x}_{a+2}) + \mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b) + \gamma(\mathbb{x}_{a+1}, \mathbb{x}_b)\mathscr{D}(\mathbb{x}_{a+1}, \mathbb{x}_{a+2})\mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b) \\
&\leq \theta^{a-\hbar+1}\mathscr{D}(\mathbb{x}_{\hbar+1}, \mathbb{x}_{\hbar+2}) + \mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b) + \gamma(\mathbb{x}_{a+1}, \mathbb{x}_b)\theta^{a-\hbar+1}\mathscr{D}(\mathbb{x}_{\hbar+1}, \mathbb{x}_{\hbar+2})\mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b) \\
&\leq \theta^{a-\hbar+1}\varepsilon + (1 + \gamma(\mathbb{x}_{a+1}, \mathbb{x}_b)\varepsilon\theta^{a-\hbar+1})\mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b).
\end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we get

$$\begin{aligned}
\mathscr{D}(\mathbb{x}_a, \mathbb{x}_b) &\leq \varepsilon\theta^{a-\hbar} + [1 + \gamma(\mathbb{x}_a, \mathbb{x}_b)\varepsilon\theta^{a-\hbar}]\mathscr{D}(\mathbb{x}_{a+1}, \mathbb{x}_b) \\
&\leq \varepsilon\theta^{a-\hbar} + [1 + \gamma(\mathbb{x}_a, \mathbb{x}_b)\varepsilon\theta^{a-\hbar}]\{\varepsilon\theta^{a-\hbar+1} + [1 + \gamma(\mathbb{x}_{a+1}, \mathbb{x}_b)\theta^{a-\hbar+1}]\mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b)\} \\
&\leq \varepsilon\theta^{a-\hbar} + [1 + \gamma(\mathbb{x}_a, \mathbb{x}_b)\varepsilon\theta^{a-\hbar}]\{\varepsilon\theta^{a-\hbar+1} + \mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b) + \varepsilon\theta^{a-\hbar+1}\gamma(\mathbb{x}_{a+1}, \mathbb{x}_b)\mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b)\} \\
&\leq \varepsilon\theta^{a-\hbar} + \varepsilon\theta^{a-\hbar+1} + \mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b) + \varepsilon\theta^{a-\hbar+1}\gamma(\mathbb{x}_{a+1}, \mathbb{x}_b)\mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b) \\
&\quad + \varepsilon^2\theta^{a-\hbar}\theta^{a-\hbar+1}\gamma(\mathbb{x}_a, \mathbb{x}_b) + \varepsilon\theta^{a-\hbar}\gamma(\mathbb{x}_a, \mathbb{x}_b)\mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b) + \varepsilon^2\theta^{a-\hbar}\theta^{a-\hbar+1}\gamma(\mathbb{x}_a, \mathbb{x}_b)\gamma(\mathbb{x}_{a+1}, \mathbb{x}_b)\mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b) \\
&\leq \varepsilon\theta^{a-\hbar} + \varepsilon\theta^{a-\hbar+1}[1 + \gamma(\mathbb{x}_a, \mathbb{x}_b)\varepsilon\theta^{a-\hbar}] \\
&\quad + [1 + \gamma(\mathbb{x}_{a+1}, \mathbb{x}_b)\varepsilon\theta^{a-\hbar+1} + \gamma(\mathbb{x}_a, \mathbb{x}_b)\varepsilon\theta^{a-\hbar} + \gamma(\mathbb{x}_a, \mathbb{x}_b)\gamma(\mathbb{x}_{a+1}, \mathbb{x}_b)\varepsilon^2\theta^{a-\hbar}\theta^{a-\hbar+1}]\mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b) \\
&\leq \varepsilon\theta^{a-\hbar} + \varepsilon\theta^{a-\hbar+1}[1 + \gamma(\mathbb{x}_a, \mathbb{x}_b)\varepsilon\theta^{a-\hbar}][(1 + \varepsilon\theta^{a-\hbar}\gamma(\mathbb{x}_a, \mathbb{x}_b))(1 + \varepsilon\theta^{a-\hbar+1}\gamma(\mathbb{x}_{a+1}, \mathbb{x}_b))]\mathscr{D}(\mathbb{x}_{a+2}, \mathbb{x}_b).
\end{aligned}$$

Performing this process repeatedly and using (2.2) in each term of the sum until we reach continuing this process and using (2.2) in every term of the sum, until we obtain

$$\mathscr{D}(\mathbb{x}_a, \mathbb{x}_b) \leq \varepsilon\theta^{a-\hbar}\Sigma_{i=0}^{b-a-1}\theta^i\Pi_{\eta=0}^{i-1}[1 + \varepsilon\gamma(\mathbb{x}_{a+\eta}, \mathbb{x}_b)\theta^{a-\hbar+\eta}]$$

since $\theta \in [0, 1)$, it follows that

$$\mathscr{D}(\mathbb{x}_a, \mathbb{x}_b) \leq \varepsilon\theta^{a-\hbar}\Sigma_{i=0}^{b-a-1}\theta^i\Pi_{\eta=0}^{i-1}[1 + \varepsilon\gamma(\mathbb{x}_{a+\eta}, \mathbb{x}_b)\theta^{\eta}]. \tag{2.5}$$

Let $\mathfrak{U}_i = \theta^i\Pi_{\eta=0}^{i-1}[1 + \varepsilon\gamma(\mathbb{x}_{a+\eta}, \mathbb{x}_b)\theta^{\eta}]$.
By Ratio test $\Sigma_{i=0}^{\infty}\mathfrak{U}_i$ is converges, since $\lim_{i\to\infty}\left|\frac{\mathfrak{U}_{i+1}}{\mathfrak{U}_i}\right| < 1$ as $\theta \in [0, 1)$.

Hence from (2.5), we can deduce that, $\mathscr{D}(\mathbb{x}_a, \mathbb{x}_b)$ tends to zero as a, b tend to infinity, that suggests the sequence $\{\mathbb{x}_n\}$ is Cauchy. Therefore by completeness of $\mathcal{X}$, as a result of this $\{\mathbb{x}_n\}$ converges to some $\mathbb{x}^\star \in \mathcal{X}$(say).
We will now prove that $\mathbb{x}^\star$ is a fixed point of $\mathcal{Z}$.
By utilizing conditions (2) and (3) of Remark 2.7 and (2.1), we get

$$\mathscr{D}(\mathcal{Z}\mathbb{x}_{n(\hbar)}, \mathcal{Z}\mathbb{x}^\star) \leq \theta\mathscr{D}(\mathbb{x}_{n(\hbar)}, \mathbb{x}^\star).$$

Therefore as $\hbar \to \infty$, thus, we conclude $\mathbb{x}^\star = \mathcal{Z}\mathbb{x}^\star$. Therefore, our assertion is true.
To prove uniqueness, let us assume $\mathbb{x}_a$ and $\mathbb{x}_b$ are two fixed points of $\mathcal{Z}$, additionally from (2.5), we get,

$$\begin{aligned}\mathscr{D}(\mathbb{x}_a, \mathbb{x}_b) &= \mathscr{D}(\mathcal{Z}\mathbb{x}_a, \mathcal{Z}\mathbb{x}_b)\\ &\leq \theta\mathscr{D}(\mathbb{x}_a, \mathbb{x}_b)\\ &< \mathscr{D}(\mathbb{x}_a, \mathbb{x}_b), \quad \text{a contradiction.}\end{aligned}$$

Hence $\mathbb{x}_a = \mathbb{x}_b$. This completes the proof of the theorem. □

3. Stone-type theorem in suprametric spaces

From the outset of General Topology, metrization has been and continues to be one of its most crucial fields. In the literature, there are numerous metrization theorems (see [12–18]). The metrizability hypotheses vary greatly from one metrization theorem to another, even though the thesis is always the same. Furthermore, not only are the proofs very dissimilar, but it is also difficult to draw any conclusions about one metrization theorem from another. The Stone-type theorem is derived in terms of suprametric spaces in this section. In addition, we have shown that every suprametric space is metrizable.

Assume that $(\mathcal{X}, \mathscr{D})$ is a suprametric space, $\mathbb{x}_0 \in \mathcal{X}$ and ϱ a positive number, the set $\mathscr{B}(\mathbb{x}_0, \varrho) = \{\mathbb{x} \in \mathcal{X} / \mathscr{D}(\mathbb{x}_0, \mathbb{x}) < \varrho\}$ is called the *open ball* with centre $\mathbb{x}_0 \in \mathcal{X}$ and radius $\varrho > 0$ or, briefly the ϱ-ball about $\mathbb{x}_0$ moreover, $\mathscr{B}[\mathbb{x}_0, \varrho] = \{\mathbb{x} \in \mathcal{X} / \mathscr{D}(\mathbb{x}_0, \mathbb{x}) \leq \varrho\}$ is the *closed ball*. For a set $\mathscr{A} \subset \mathcal{X}$ and a positive number ϱ, by the ϱ-ball about $\mathscr{A}$ we mean the set $\mathscr{B}(\mathscr{A}, \varrho) = \bigcup_{\mathbb{x}\in\mathscr{A}} \mathscr{B}(\mathbb{x}, \varrho)$; let us note that $\mathbb{x} \in \mathscr{B}(\mathbb{x}, \varrho)$ so that $\mathscr{A} \subset \mathscr{B}(\mathscr{A}, \varrho)$. Let us also observe that if $\mathbb{x}_1 \in \mathscr{B}(\mathbb{x}_0, \varrho)$, then $\mathscr{B}(\mathbb{x}_0, \varrho_1) \subset \mathscr{B}(\mathbb{x}_0, \varrho)$ for $\varrho_1 = \frac{\varrho - \mathscr{D}(\mathbb{x}_0, \mathbb{x}_1)}{1+\zeta\mathscr{D}(\mathbb{x}_0, \mathbb{x}_1)} > 0$.

A subset Y of $\mathcal{X}$ is said to be *open* if for any point $\mathbb{y} \in Y$, there is $\varrho > 0$ such that $\mathscr{B}(\mathbb{y}, \varrho) \subset Y$. Let $\mathfrak{I}_s = \{Y \subseteq \mathcal{X} / \forall\, \mathbb{x} \in Y,$ there exists $\varrho > 0$ such that $\mathscr{B}(\mathbb{x}, \varrho) \subset Y\}$. One can easily see that $\mathfrak{I}_s$ is a topology on $\mathcal{X}$.

For every $\mathbb{x} \in \mathcal{X}$ we define a collection of families of subsets $\mathscr{M}(\mathbb{x})$ of $\mathcal{X}$ that have the following characteristics:

$(\mathscr{B}_{p1})$ For every $\mathbb{x} \in \mathcal{X}$, $\mathscr{M}(\mathbb{x}) \neq \emptyset$ and for every $\mathcal{Q} \in \mathscr{M}(\mathbb{x})$, $\mathbb{x} \in \mathcal{Q}$.

$(\mathscr{B}_{p2})$ If $\mathbb{x} \in \mathcal{Q} \in \mathscr{M}(\mathbb{y})$ then there is a $\mathscr{R} \in \mathscr{M}(\mathbb{x})$ such that $\mathscr{R} \subset \mathcal{Q}$.

$(\mathscr{B}_{p3})$ For any $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathscr{M}(\mathbb{x})$ there exists a $\mathcal{Q} \in \mathscr{M}(\mathbb{x})$ such that $\mathcal{Q} \subset \mathcal{Q}_1 \cap \mathcal{Q}_2$;
where $\mathscr{M}(\mathbb{x}) = \{\mathscr{B}(\mathbb{x}, \varrho)/\varrho > 0\}$.

Let $\mathfrak{I}$ be the family of all subsets of $\mathcal{X}$ that are unions of subfamilies of $\mathscr{M}(\mathbb{x})$. That is, let $\mathcal{Q} \in \mathfrak{I}$ iff $\mathcal{Q} = \bigcup \mathscr{B}_0$ for a subfamily $\mathscr{B}_0$ of $\mathscr{M}(\mathbb{x})$. Clearly $\mathscr{M}(\mathbb{x})$ is a base for the space $(\mathcal{X}, \mathfrak{I})$ and $\mathfrak{I}$ is the topology generated by the base $\mathscr{M}(\mathbb{x})$. Since $\emptyset = \bigcup \mathscr{B}_0$ for $\mathscr{B}_0 = \emptyset$, $\mathcal{X} = \bigcup \mathscr{B}_0$ for $\mathscr{B}_0 = \mathscr{M}(\mathbb{x})$.

Take $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathfrak{I}$, which yield $\mathcal{Q}_1 = \bigcup_{s \in \mathcal{S}} \mathcal{Q}_s$ and $\mathcal{Q}_2 = \bigcup_{t \in \mathcal{T}} \mathcal{Q}_t$, where $\mathcal{Q}_s, \mathcal{Q}_t \in \mathscr{M}(\mathbb{x})$ for $s \in \mathcal{S}$ and $t \in \mathcal{T}$. $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \bigcup_{s \in \mathcal{S}, t \in \mathcal{T}} \mathcal{Q}_s \cap \mathcal{Q}_t$, since for every $\mathbb{x} \in \mathcal{Q}_s \cap \mathcal{Q}_t$ there exists a $\mathcal{Q}(\mathbb{x}) \in \mathscr{M}(\mathbb{x})$ such that $\mathbb{x} \in \mathcal{Q}(\mathbb{x}) \subset \mathcal{Q}_s \cap \mathcal{Q}_t$, that suggests $\mathcal{Q}_s \cap \mathcal{Q}_t = \bigcup \mathscr{B}_0$ for $\mathscr{B}_0 = \{\mathcal{Q}(\mathbb{x})/\mathbb{x} \in \mathcal{Q}_s \cap \mathcal{Q}_t\}$. As a result, we construct a topology $\mathfrak{I}$ on the set $\mathcal{X}$ is called the topology induced by the metric $\mathscr{D}$. One can easily check that $\mathfrak{I}$ and $\mathfrak{I}_s$ coincides. Clearly the family of all open balls is a base for $(\mathcal{X}, \mathfrak{I})$. The family of all $\frac{1}{i}$ balls about $\mathbb{x}_0$, where $i = 1, 2, 3...$ is a base for $(\mathcal{X}, \mathfrak{I})$ at the point $\mathbb{x}_0$, that would suggest the space $(\mathcal{X}, \mathfrak{I})$ is first countable.

The topology $\mathfrak{I}$ induced by the suprametric $\mathscr{D}$ on a set $\mathcal{X}$ was introduced by Maher Berzig [10] in Proposition 1.2.

The topology $\mathfrak{I}$ is Hausdorff space according to [10], since for every pair $\mathbb{x}, \mathbb{y}$ of distinct points of $\mathcal{X}$, it follows from the (iii) of Definition 2.1 that $\mathscr{B}(\mathbb{x}, \frac{\varrho}{2})$ and $\mathscr{B}(\mathbb{y}, \frac{\varrho}{2+\zeta\varrho})$ are disjoint neighbourhoods of $\mathbb{x}$ and $\mathbb{y}$, where $\varrho = \mathscr{D}(\mathbb{x}, \mathbb{y}) > 0$. Also note that every suprametric space is continuous [10].

Example 3.1. We define a suprametric on set of all infinite sequence $\{p_i\}$ of real numbers satisfying the condition $\Sigma_{i=0}^{\infty} p_i^2 < \infty$ (say $\mathbb{H}$).
That is, $\mathbb{H} = \{$set of all infinite sequence $\{p_i\}$of real numbers$/\Sigma_{i=1}^{\infty} p_i^2 < \infty\}$ which is called Hilbert space.

Let us define,

$$\mathscr{D}(p, q) = \sqrt{\Sigma_{i=0}^{\infty}(p_i - q_i)^2} \text{ for } p = p_i, q = q_i.$$

We will prove that $\mathscr{D}$ is a suprametric. First of all to prove that $\mathscr{D}$ is well-defined. That is, the series in the definition of $\mathscr{D}$ is convergent.

Let us note that for every pair of points $p = p_i, q = q_i$ in $\mathbb{H}$ and any positive integer ℓ, we have

$$\Sigma_{i=1}^{\ell}(p_i - q_i)^2 = \left(\sqrt{\Sigma_{i=1}^{\infty} p_i^2} + \sqrt{\Sigma_{i=1}^{\infty} q_i^2}\right)^2.$$

Thus the series in the notion of $\mathscr{D}$ is convergent and $\mathscr{D}(p, q)$ is well-defined. Now we will prove that $\mathscr{D}$ is a suprametric. Clearly, $\mathscr{D}$ satisfies $(\mathscr{D}_1), (\mathscr{D}_2)$ of suprametric definition. We shall show that condition (iii) of Definition 2.1 is also satisfied.
Let $p = p_i, q = q_i$ and $\varrho = \varrho_i$ be any points of $\mathbb{H}$ and let

$$p^{\ell} = \{p_1, p_2, ..., p_{\ell}, 0, 0, ...\};$$

$$q^{\ell} = \{q_1, q_2, ..., q_{\ell}, 0, 0, ...\};$$

$$\varrho^{\ell} = \{\varrho_1, \varrho_2, ..., \varrho_{\ell}, 0, 0, ...\};$$

and $u_i = p_i - q_i,\ v_i = q_i - \varrho_i,\ w_i = p_i - \varrho_i$.
By Cauchy inequality we have

$$\begin{aligned}
[\mathscr{D}(p^{\ell}, \varrho^{\ell})]^2 &= \Sigma_{i=1}^{\ell} w_i^2 \\
&= \Sigma_{i=1}^{\ell} (u_i + v_i)^2 \\
&= \Sigma_{i=1}^{\ell} u_i^2 + 2\Sigma_{i=1}^{\ell} u_i v_i + \Sigma_{i=1}^{\ell} v_i^2 \\
&\leq \Sigma_{i=1}^{\ell} u_i^2 + 2\sqrt{\Sigma_{i=1}^{\ell} u_i^2}\sqrt{\Sigma_{i=1}^{\ell} v_i^2} + \Sigma_{i=1}^{\ell} v_i^2 \\
&\leq \Sigma_{i=1}^{\ell} u_i^2 + 2\sqrt{\Sigma_{i=1}^{\ell} u_i^2}\sqrt{\Sigma_{i=1}^{\ell} v_i^2} + \Sigma_{i=1}^{\ell} v_i^2 + 2\zeta\sqrt{\Sigma_{i=1}^{\ell} u_i^2}\Sigma_{i=1}^{\ell} v_i^2 \\
&\quad + 2\zeta\Sigma_{i=1}^{\ell} u_i^2\sqrt{\Sigma_{i=1}^{\ell} v_i^2} + \zeta^2\Sigma_{i=1}^{\ell} u_i^2\Sigma_{i=1}^{\ell} v_i^2 \\
&= \left(\sqrt{\Sigma_{i=1}^{\ell} u_i^2} + \sqrt{\Sigma_{i=1}^{\ell} v_i^2} + \zeta\sqrt{\Sigma_{i=1}^{\ell} u_i^2}\sqrt{\Sigma_{i=1}^{\ell} v_i^2}\right)^2 \\
&= \left(\sqrt{\Sigma_{i=1}^{\infty} u_i^2} + \sqrt{\Sigma_{i=1}^{\infty} v_i^2} + \zeta\sqrt{\Sigma_{i=1}^{\infty} u_i^2}\sqrt{\Sigma_{i=1}^{\infty} v_i^2}\right)^2 \\
&= \left(\sqrt{\Sigma_{i=1}^{\infty}(p_i - q_i)^2} + \sqrt{\Sigma_{i=1}^{\infty}(q_i - \varrho_i)^2} + \zeta\sqrt{\Sigma_{i=1}^{\infty}(p_i - q_i)^2}\sqrt{\Sigma_{i=1}^{\infty}(q_i - \varrho_i)^2}\right)^2.
\end{aligned}$$

For $\ell = 1, 2, ...,$ we have

$$\mathscr{D}(p^{\ell}, \varrho^{\ell}) \leq \mathscr{D}(p^{\ell}, q^{\ell}) + \mathscr{D}(q^{\ell}, \varrho^{\ell}) + \zeta\mathscr{D}(p^{\ell}, q^{\ell})\mathscr{D}(q^{\ell}, \varrho^{\ell})$$

which implies,

$$\mathscr{D}(p, \varrho) \leq \mathscr{D}(p, q) + \mathscr{D}(q, \varrho) + \zeta\mathscr{D}(p, q)\mathscr{D}(q, \varrho).$$

Hence $\mathscr{D}$ is a suprametric on $\mathbb{H}$.

Definition 3.2. [19] Suppose $\mathscr{H} = \{\mathscr{H}_q : q \in \mathscr{S}\}$ be a family of subsets of topological space $\mathcal{X}$.

- If for all $\mathbb{x} \in \mathcal{X}$ there will be a neighbourhood $\mathcal{Q}_{\mathbb{x}}$ of $\mathbb{x}$, so that the family $\{q \in \mathscr{S} : \mathcal{Q}_{\mathbb{x}} \cap \mathscr{H}_q \neq \emptyset\}$ is finite then $\mathscr{H}$ is known as *locally finite*.
- If for all $\mathbb{x} \in \mathcal{X}$, there will be a neighbourhood $\mathcal{Q}_{\mathbb{x}}$ of $\mathbb{x}$, so that the family $\{p \in \mathscr{S} : \mathscr{H}_p \cap \mathcal{Q}_{\mathbb{x}} \neq \emptyset\}$ will have at most one element then $\mathscr{H}$ is known as *discrete*. It is obvious that any finite family is locally finite.
- For every locally finite $\mathscr{H}_i$, if $\mathscr{H} = \bigcup_{i=\mathbb{N}} \mathscr{H}_i$, then the family $\mathscr{H}$ is called *σ-locally finite*.
- For every $\mathscr{H}_i$ is discrete, if $\mathscr{H} = \bigcup_{i=\mathbb{N}} \mathscr{H}_i$ then the family $\mathscr{H}$ is called *σ-discrete*
- If $\bigcup_{p\in\mathscr{S}} \mathscr{H}_p = \mathcal{X}$, then the family $\mathscr{H}$ is called a *cover* of $\mathcal{X}$.
- If for all $i \in I$ there will be $p \in \mathscr{S}$, so that $\mathfrak{B}_i \subset \mathscr{A}_p$, then a cover $\mathfrak{B}$ of subsets of $\mathcal{X}$ is known as a *refinement* of the cover $\mathscr{H}$, where $\mathfrak{B} = \{\mathfrak{B}_i / i \in I\}$.

Definition 3.3. Let $\mathscr{O}$ and $\mathscr{L}$ are two disjoint non-empty closed subsets of $\mathcal{X}$. If $\mathscr{O} \subset \mathcal{Q}$ and $\mathscr{L} \subset \mathscr{R}$ for two non-empty disjoint open sets $\mathcal{Q}$ and $\mathscr{R}$ in the topological space $(\mathcal{X}, \mathfrak{I})$. Then $\mathcal{X}$ is said to be a regular space.

Definition 3.4. Assume that $(\mathcal{X}, \mathscr{D})$ is a suprametric space and $\mathcal{X}$ is a super set of $\mathscr{W}$. Accordingly,

(1) $\mathscr{W}$ is said to be an *open set* whenever $\mathscr{W} \in \mathfrak{I}$.

(2) $\mathscr{W}$ is said to be a *closed set* whenever $\mathcal{X} \backslash \mathscr{W} \in \mathfrak{I}$.

(3) $\mathtt{x} \in \mathcal{X}$ is called to be a *limit point* of $\mathscr{W}$ whenever there is $\varrho > 0$ such that $(\mathscr{W}(\mathtt{x}, \varrho) \backslash \{\mathtt{x}\}) \cap \mathscr{W}$ having an infinite number of points of $\mathscr{W}$.

(4) Denoted by $\mathscr{W}'$, the collection of all limit points of $\mathscr{W}$ is known as the derived set of $\mathscr{W}$.

Proposition 3.5. *Closed balls* are *closed set* in a suprametric space $(\mathcal{X}, \mathscr{D})$.

Proof. Take $\mathtt{x} \in \mathcal{X}$, $\varrho > 0$, and the closed ball $\mathscr{B}[\mathtt{x}, \varrho]$. In order to show that $\mathscr{B}[\mathtt{x}, \varrho]$ is closed, then it suffices to depict $\mathcal{X} \setminus \mathscr{B}[\mathtt{x}, \varrho] = \mathcal{F}$(fix) is open. Let $\mathtt{y} \in \mathcal{F}$. Thus $\mathscr{D}(\mathtt{x}, \mathtt{y}) = \varrho'$(say) $> \varrho$. We must now find some $s > 0$ such that $\mathscr{B}(\mathtt{y}, s) \subset \mathcal{F}$.

Choose $s > 0$ such that $s < \frac{\varrho' - \varrho}{1 + \zeta\varrho}$. Let $a \in \mathscr{B}(\mathtt{y}, s)$, thus $\mathscr{D}(a, \mathtt{y}) < s$. Besides $\mathtt{x} \in \mathcal{X}, \mathtt{y} \in \mathcal{F}$ and $a \in \mathscr{B}(\mathtt{y}, s)$, we now have $\mathscr{D}(\mathtt{x}, \mathtt{y}) \leq \mathscr{D}(\mathtt{x}, a) + \mathscr{D}(a, \mathtt{y}) + \zeta\mathscr{D}(\mathtt{x}, a)\mathscr{D}(a, \mathtt{y})$.

$$
\begin{aligned}
\Rightarrow \mathscr{D}(\mathtt{x}, \mathtt{y}) &\geq \frac{\mathscr{D}(\mathtt{x}, \mathtt{y}) - \mathscr{D}(a, \mathtt{y})}{1 + \zeta\mathscr{D}(a, \mathtt{y})} \\
&> \frac{\varrho' - s}{1 + \zeta s} \\
&> \frac{\varrho' - \left(\frac{\varrho' - \varrho}{1 + \zeta\varrho}\right)}{1 + \zeta\left(\frac{\varrho' - \varrho}{1 + \zeta\varrho}\right)} \\
&= \frac{\varrho(\varrho'\zeta + 1)}{1 + \zeta\varrho'} \\
&= \varrho.
\end{aligned}
$$

Thus $\mathscr{D}(\mathtt{x}, a) > \varrho$ if $a \in \mathscr{B}(\mathtt{y}, s)$, having $0 < s < \frac{\varrho' - \varrho}{1 + \zeta\varrho}$. Which yields $\mathcal{F}$ is an open set. Consequently $\mathscr{B}[\mathtt{x}, \varrho]$ is a closed set. □

Theorem 3.6. Assume that $(\mathcal{X}, \mathscr{D})$ is a suprametric space. Whenever $\mathscr{E}$ be a closed subset of $\mathcal{X}$ and $\mathtt{x} \in \mathcal{X} \setminus \mathscr{E}$ then there are two disjoint open sets $\mathcal{Q}$ and $\mathscr{R}$ containing $\mathscr{E}$ and $\mathtt{x}$.

Proof. As $\mathtt{x} \in \mathcal{X} \setminus \mathscr{E}$, $\mathscr{E}$ is closed. Which yields, $\mathscr{D}(\mathtt{x}, a) > 0$ for all $a \in \mathscr{E}$.
Let $2\varrho = \inf\{\mathscr{D}(\mathtt{x}, a)/a \in \mathscr{E}\}$, where $\varrho > 0$. Let us assume the open ball $\mathscr{B}(\mathtt{x}, \frac{\varrho}{2}) = \mathscr{R}$ (fix) and the open set $\mathcal{Q} = \bigcup_{a \in \mathscr{E}} \mathscr{B}(a, \frac{3\varrho}{2 + \zeta\varrho})$. Therefore $\mathscr{E} \subset \mathcal{Q}$.
We will prove that $\mathcal{Q} \cap \mathscr{R} = \emptyset$. Assume that there is $\xi \in \mathcal{Q} \cap \mathscr{R}$. Thus for any $a \in \mathscr{E}$,

$$\mathscr{D}(\mathtt{x}, a) \leq \mathscr{D}(\mathtt{x}, \xi) + \mathscr{D}(\xi, a) + \zeta\mathscr{D}(\mathtt{x}, \xi)\mathscr{D}(\xi, a)$$

$$< \frac{\varrho}{2} + \frac{3\varrho}{2+\zeta\varrho} + \zeta\frac{\varrho}{2}\frac{3\varrho}{2+\zeta\varrho}$$
$$= 2\varrho.$$

Our assumption is contradicted by this, so $\mathcal{Q}$ and $\mathcal{R}$ are two disjoint non-empty open sets in $\mathcal{X}$, each containing $\mathcal{E}$ and $\mathbb{x}$. □

Theorem 3.7. Let $\mathcal{O}$ and $\mathcal{L}$ be two disjoint non-empty closed subsets of a suprametric space $(\mathcal{X}, \mathcal{D})$. Then $\mathcal{O} \subset \mathcal{Q}$ and $\mathcal{L} \subset \mathcal{R}$ for two disjoint open sets $\mathcal{R}$ and $\mathcal{Q}$ in $\mathcal{X}$.

Proof. Let $a \in \mathcal{O}$, $b \in \mathcal{L}$ and $a \neq b$ which implies $\mathcal{D}(a,b) > 0$.
Let $2\varrho = glb\{\mathcal{D}(a,b)/a \in \mathcal{O},\ b \in \mathcal{L}\}$. Let $\mathcal{R} = \bigcup_{b\in\mathcal{L}} \mathcal{B}(b, \frac{\varrho}{2})$ which consists $\mathcal{L}$, where $\mathcal{R}$ is an open set. Now for $a \in \mathcal{O},\ b \in \mathcal{L}$ and $\varrho > 0$.
Consider $\mathcal{Q} = \bigcup_{a\in\mathcal{O}} \mathcal{B}(a, \frac{3\varrho}{2+\zeta\varrho})$. Hence $\mathcal{Q}$ is open and $\mathcal{O} \subset \mathcal{Q}$.
Now we will prove that $\mathcal{Q}$ and $\mathcal{R}$ are disjoint. If this is not the case, there exist $\xi \in \mathcal{Q} \cap \mathcal{R}$. Whereupon for every $a \in \mathcal{O}$ and $b \in \mathcal{L}$, $\mathcal{D}(a,\xi) < \frac{3\varrho}{2+\zeta\varrho}, \mathcal{D}(b,\xi) < \frac{\varrho}{2}$.
So for $a \in \mathcal{O}, b \in \mathcal{L}$ and $\xi \in \mathcal{Q} \cap \mathcal{R}$,

$$\mathcal{D}(a,b) \leq \mathcal{D}(a,\xi) + \mathcal{D}(\xi,b) + \zeta\mathcal{D}(a,\xi)\mathcal{D}(\xi,b)$$
$$< \frac{3\varrho}{2+\zeta\varrho} + \frac{\varrho}{2} + \zeta\frac{3\varrho}{2+\zeta\varrho}\frac{\varrho}{2}$$
$$= \frac{3\varrho}{2+\zeta\varrho}(1 + \frac{\zeta\varrho}{2}) + \frac{\varrho}{2}$$
$$= 2\varrho, \text{ a contradiction.}$$

Our assumption is contradicted by this, so $\mathcal{Q} \cap \mathcal{R} = \emptyset$. This completes the proof. □

Definition 3.8. A topological space $\mathcal{X}$ is called *suprametrizable* if there exists a suprametric $\mathcal{D}$ on $\mathcal{X}$ which induces the topology of $\mathcal{X}$.

Theorem 3.9. [13] (The Stone Theorem) A metrizable space has an open refinement for each open cover that is both σ-discrete and locally finite.

Theorem 3.10. [14] (The Bing Metrization Theorem) A topological space is metrizable if and only if it is regular and has a σ-discrete base.

Theorem 3.11. [15] (Collins-Roscoe Metrization Theorem) Let $\mathcal{X}$ be a T_1-space such that, for every $\mathbb{x} \in \mathcal{X}$, there exists a countable neighborhood base $\mathcal{B}(\mathbb{x}) = \{W(n,\mathbb{x}) : n \in \mathbb{N}\}$ at $\mathbb{x}$ satisfying the following conditions:

(1) $(\forall n \in \mathbb{N})\ W(n+1,\mathbb{x}) \subseteq W(n,\mathbb{x})$;

(2) $(\forall \mathbb{x} \in \mathcal{X})(\forall n \in \mathbb{N})(\exists r \in \mathbb{N})(n \leq r \wedge (\forall\ \mathbb{x} \in W(r,\mathbb{x})))(\mathbb{x} \in W(n,\mathbb{y}) \wedge W(n,\mathbb{y}) \subseteq W(n,\mathbb{y}))$.

Then $\mathcal{X}$ is metrizable.

Theorem 3.12. (Stone-type Theorem) Assume $(\mathcal{X}, \mathcal{D})$ is a suprametric space. Then, for every open cover of $\mathcal{X}$, there is an open refinement that is both locally finite and σ-discrete.

Proof. Let $\mathcal{W} = \{W_\flat\}_{\flat\in\mathcal{S}}$ be an open cover of $\mathcal{X}$. Take a suprametric $\mathscr{D}$ on the space $\mathcal{X}$ and $<$ is a well-order relation on $\mathcal{S}$. Let $\mathscr{X}_i = \{\mathscr{R}_{\flat,i}\}_{\flat\in\mathcal{S}} \subseteq \mathcal{X}$

$$\mathscr{R}_{\flat,i} = \bigcup_{\delta\in\mathscr{C}} \mathscr{B}(\delta, \frac{1}{2^i}), \tag{3.1}$$

where the union is taken over all points $\delta \in \mathcal{X}$ gratifying the below mentioned assertions.

(c_1) In order for $\delta \in W_\flat$ to exist, $\flat$ must be the smallest element in S.

(c_2) $\delta \notin \mathscr{R}_{p,\eta}$ for $\eta < i$ and $p \in \mathcal{S}$.

(c_3) $\mathscr{B}\left(\delta, \frac{3}{2^i}(1+\frac{\zeta}{2^i}) + \frac{\zeta^2}{2^{3i}}\right) \subset W_\flat$.

The set $\mathscr{R}_{\flat,i}$ is open, as it is given in Proposition 1.1 of [10]. Note that $\zeta \geq 0$ by Definition 1.1 of [10]. Since $\mathscr{B}(\delta, \frac{1}{2^i}) \subset \mathscr{B}\left(\delta, \frac{3}{2^i}(1+\frac{\zeta}{2^i}) + \frac{\zeta^2}{2^{3i}}\right)$, from condition (c_3), we get $\mathscr{R}_{\flat,i} \subset W_\flat$. Let $\mathbb{x}$ be a point of $\mathcal{X}$, take the smallest element $\flat \in \mathcal{S}$ such that $\mathbb{x} \in W_\flat$ and a natural number i such that

$$\mathscr{B}\left(\mathbb{x}, \frac{3}{2^i}(1+\frac{\zeta}{2^i}) + \frac{\zeta^2}{2^{3i}}\right) \subset W_\flat.$$

It implies that $\mathbb{x} \in \mathscr{C}$ if and only if $\mathbb{x} \notin \mathscr{R}_{p,\eta}$ for all $\eta < i$ and all $p \in \mathcal{S}$. In this case $\mathbb{x} \in \mathscr{R}_{\flat,i}$. Then we either have $\mathbb{x} \in \mathscr{R}_{p,\eta}$ for some $\eta < i$ and some $p \in \mathcal{S}$ or $\mathbb{x} \in \mathscr{R}_{\flat,i}$. Hence the union $\mathscr{X} = \bigcup_{i=1}^{\infty} \mathscr{R}_i$ is an open refinement of the cover $\{W_\flat\}_{\flat\in\mathcal{S}}$.

We shall prove that for every i. If $\mathbb{x}_1 \in \mathscr{R}_{\flat_1,i}, \mathbb{x}_2 \in \mathscr{R}_{\flat_2,i}$ and $\flat_1 \neq \flat_2$ then

$$\mathscr{D}(\mathbb{x}_1, \mathbb{x}_2) > \frac{1}{2^i}, \tag{3.2}$$

and this will show that the families $\mathscr{R}_i$ are discrete, because every $\frac{1}{2^{i+1}}$ ball meets at most one member of $\mathscr{R}_i$.

Let $\mathbb{x}_1 \in \mathscr{R}_{\flat_1,i}$ and $\mathbb{x}_2 \in \mathscr{R}_{\flat_2,i}$ $\forall i \in \mathbb{N}$ having $\flat_1 \neq \flat_2$. Let us assume that $\flat_1 < \flat_2$. By the definition of $\mathscr{R}_{\flat_1,i}$ and $\mathscr{R}_{\flat_2,i}$, there exist points δ_1, δ_2 satisfying conditions (c_1)–(c_3) such that $\mathbb{x}_\hbar \in \mathscr{B}(\delta_\hbar, \frac{1}{2^i}) \subset \mathscr{R}_{\flat_\hbar,i}$ for $\hbar = 1, 2$.

From condition (c_3), as a result of this, $\mathscr{B}\left(\delta_1, \frac{3}{2^i}(1+\frac{\zeta}{2^i}) + \frac{\zeta^2}{2^{3i}}\right) \subset W_{\flat_1}$ and from (c_1) we see that $\delta_2 \notin W_{\flat_1}$, so that $\mathscr{D}(\delta_1, \delta_2) \geq \frac{3}{2^i}(1+\frac{\zeta}{2^i}) + \frac{\zeta^2}{2^{3i}}$.

Hence

$$\begin{aligned}
\mathscr{D}(\mathbb{x}_1, \mathbb{x}_2) &\geq \frac{\mathscr{D}(\delta_1,\delta_2) - \mathscr{D}(\delta_1,\mathbb{x}_1) - \mathscr{D}(\mathbb{x}_2,\delta_2) - \zeta\mathscr{D}(\delta_1,\mathbb{x}_1)\mathscr{D}(\mathbb{x}_2,\delta_2)}{1 + \zeta\mathscr{D}(\mathbb{x}_2,\delta_2) + \zeta\mathscr{D}(\delta_1,\mathbb{x}_1) + \zeta^2\mathscr{D}(\delta_1,\mathbb{x}_1)\mathscr{D}(\delta_2,\mathbb{x}_2)} \\
&\geq \frac{\frac{3}{2^i}(1+\frac{\zeta}{2^i}) + \frac{\zeta^2}{2^{3i}} - \frac{1}{2^i} - \frac{1}{2^i} - \frac{\zeta}{2^{2i}}}{1 + \frac{\zeta}{2^i} + \frac{\zeta}{2^i} + \frac{\zeta^2}{2^{2i}}} \\
&\geq \frac{\frac{3}{2^i} + \frac{3\zeta}{2^{2i}} + \frac{\zeta^2}{2^{3i}} - \frac{2}{2^i} - \frac{\zeta}{2^{2i}}}{1 + \frac{2\zeta}{2^i} + \frac{\zeta^2}{2^{2i}}}
\end{aligned}$$

$$= \frac{\frac{1}{2^i} + \frac{2\zeta}{2^{2i}} + \frac{\zeta^2}{2^{3i}}}{\left(1 + \frac{\zeta}{2^i}\right)^2}$$

$$= \frac{\frac{1}{2^i}\left(1 + \frac{\zeta}{2^i}\right)^2}{\left(1 + \frac{\zeta}{2^i}\right)^2}$$

$$= \frac{1}{2^i}.$$

Hence $\mathscr{D}(\mathbb{x}_1, \mathbb{x}_2) \geq \frac{1}{2^i}$.

If there exists $\mathbb{x} \in \mathcal{X}$ such that $\mathbb{x}_1, \mathbb{x}_2 \in \mathscr{B}\left(\mathbb{x}, \frac{-1+\sqrt{1+\frac{\zeta}{2^i}}}{\zeta}\right)$ then we have

$$\begin{aligned}
\frac{1}{2^i} &\leq \mathscr{D}(\mathbb{x}_1, \mathbb{x}_2) \\
&\leq \mathscr{D}(\mathbb{x}_1, \mathbb{x}) + \mathscr{D}(\mathbb{x}, \mathbb{x}_2) + \zeta \mathscr{D}(\mathbb{x}_1, \mathbb{x})\mathscr{D}(\mathbb{x}, \mathbb{x}_2) \\
&\leq \frac{-1 + \sqrt{1 + \frac{\zeta}{2^i}}}{\zeta} + \frac{-1 + \sqrt{1 + \frac{\zeta}{2^i}}}{\zeta} + \zeta\left(\frac{-1 \pm \sqrt{1 + \frac{\zeta}{2^i}}}{\zeta}\right)^2 \\
&= 2\left(\frac{-1 + \sqrt{1 + \frac{\zeta}{2^i}}}{\zeta}\right) + \frac{\left(-1 + \sqrt{1 + \frac{\zeta}{2^i}}\right)^2}{\zeta} \\
&= \frac{1}{2^i}.
\end{aligned}$$

Since this goes against our assumption, so radius $\frac{-1+\sqrt{1+\frac{\zeta}{2^i}}}{\zeta}$ of every ball coincides at only one element of $\mathscr{R}_i$. This can be written as $\mathscr{X} = \bigcup_{i \in \mathbb{N}} \mathscr{R}_i$ is σ-discrete.

Assume that $i \in \mathbb{N}$, additionally for each $p \in \mathcal{S}$, $i \geq \eta + \hbar$ and $\delta \in \mathcal{C}$ which gives $\delta \notin \mathscr{R}_{p,\eta}$. Here whenever $\mathscr{B}\left(\mathbb{x}, \frac{1}{2^\hbar}\right) \subset \mathscr{R}_{p,\eta}$ then $\delta \notin \mathscr{B}\left(\mathbb{x}, \frac{1}{2^\hbar}\right)$, which yields that $\mathscr{D}(\mathbb{x}, \delta) \geq \frac{1}{2^\hbar}$. Note that $\eta + \hbar \geq \hbar + 1$ and $i \geq \hbar + 1$ then $\frac{1}{2^{\eta+\hbar}} \leq \frac{1}{2^{\hbar+1}}$ and $\frac{1}{2^i} \leq \frac{1}{2^{\hbar+1}}$.

Let $\psi = \max\left\{\frac{1}{2^{\eta+\hbar}}, \frac{1}{2^i}\right\}$ and $\psi < \frac{-1+\sqrt{1+\frac{\zeta}{2^\hbar}}}{\zeta} \leq \frac{1}{2^{\hbar+1}}$.

Suppose to the contrary that there exists $\mathbb{y} \in \mathscr{B}\left(\mathbb{x}, \frac{1}{2^{\eta+\hbar}}\right) \cap \mathscr{B}\left(\delta, \frac{1}{2^i}\right)$ then

$$\begin{aligned}
\mathscr{D}(\mathbb{x}, \delta) &\leq \mathscr{D}(\mathbb{x}, \mathbb{y}) + \mathscr{D}(\mathbb{y}, \delta) + \zeta \mathscr{D}(\mathbb{x}, \delta)\mathscr{D}(\mathbb{y}, \delta) \\
&\leq \frac{1}{2^{\eta+\hbar}} + \frac{1}{2^i} + \zeta\left(\frac{1}{2^{\eta+\hbar}}\right)\left(\frac{1}{2^i}\right)
\end{aligned}$$

$$\leq \frac{-1+\sqrt{1+\frac{\zeta}{2^{\hbar}}}}{\zeta} + \frac{-1+\sqrt{1+\frac{\zeta}{2^{\hbar}}}}{\zeta} + \zeta\left(\frac{-1+\sqrt{1+\frac{\zeta}{2^{\hbar}}}}{\zeta}\right)^2$$
$$\leq \frac{1}{2^{\hbar}}$$

which concludes that $\mathscr{B}\left(\mathbb{x}, \frac{1}{2^{\eta+\hbar}}\right) \cap \mathscr{B}\left(\delta, \frac{1}{2^i}\right) = \emptyset$.

This implies $\mathscr{B}\left(\mathbb{x}, \frac{1}{2^{\eta+\hbar}}\right) \cap \mathscr{R}_{\mathfrak{b},i} \neq \emptyset$ for $i \geq \eta + \hbar$ and $\mathfrak{b} \in \mathcal{S}$ with $\mathscr{B}\left(\mathbb{x}, \frac{1}{2^{\hbar}}\right) \subset \mathscr{R}_{p,\eta}$. Let $\mathbb{x} \in \mathcal{X}$, as such $\mathscr{R}$ is refinement of $\mathcal{W}$, there exist l, η and p so $\mathscr{D}(\mathbb{x}, \frac{1}{2^l}) \subset \mathscr{R}_{p,\eta}$ and thus there is $\hbar, \eta$ and p so $\mathscr{D}(\mathbb{x}, \frac{1}{2^{\hbar}}) \subset \mathscr{R}_{p,\eta}$. Thus the ball $\mathscr{B}\left(\mathbb{x}, \frac{1}{2^{\eta+\hbar}}\right)$ fulfils at most $\eta + \hbar - 1$ members of $\mathscr{R}$. Which concludes, $\mathscr{R}$ is σ-locally finite as $\mathscr{X}_i$ is locally finite. □

Corollary 3.13. Assume that $(\mathcal{X}, \mathscr{D})$ is a suprametric space. Thus, there exists σ-discrete base for $\mathcal{X}$.

Proof. Let $\mathscr{F}_i = \{\mathscr{B}(\mathbb{x}, \frac{1}{i}) : \mathbb{x} \in \mathcal{X}\}$ for all $i \in \mathbb{N}$. Which yields $\mathcal{X}$ has an open cover $\mathscr{F}_i$. Making use of Theorem 3.12, $\mathscr{F}_i$ has an open σ-discrete refinement $\mathfrak{B}_i$. We get our claim that $\mathcal{X}$ has a σ-discrete base $\bigcup_{i\in\mathbb{N}} \mathfrak{B}_i$, as one can easily prove that $\bigcup_{i\in\mathbb{N}} \mathfrak{B}_i$ is a base of $\mathcal{X}$. □

Theorem 3.14. Every suprametric space $(\mathcal{X}, \mathscr{D})$ is metrizable.

Proof. Now, we present two proofs of the metrizability of suprametric spaces using two different approaches. In the first approach, Stone's theorem is used, while in the second, the Collins-Roscoe metrization theorem is used.

Approach-I.

One can easily conclude that $\mathcal{X}$ is regular space with σ-discrete base from Corollary 3.13 and Theorem 3.6. Thus $\mathcal{X}$ is metrizable by making use of Theorem 3.10.

Approach-II.

Now, assume that $\mathscr{D}$ is a suprametric on a set $\mathcal{X}$, and $\zeta \in \mathbb{R}^+$ is a fixed constant such that:

$$(\forall \mathbb{x}, \mathbb{y}, \mathbb{z} \in \mathcal{X})\ \mathscr{D}(\mathbb{x}, \mathbb{z}) \leq \mathscr{D}(\mathbb{x}, \mathbb{y}) + \mathscr{D}(\mathbb{y}, \mathbb{z}) + \zeta \mathscr{D}(\mathbb{x}, \mathbb{y})\mathscr{D}(\mathbb{y}, \mathbb{z}).$$

Let $\mathfrak{I}_{\mathscr{D}}$ be the topology induced by $\mathscr{D}$. It is trivial that $(\mathcal{X}, \mathfrak{I}_{\mathscr{D}})$ is a T_1-space because if $\mathbb{x}, \mathbb{y} \in \mathcal{X}$ and $\mathbb{x} \neq \mathbb{y}$, then $\mathscr{D}(\mathbb{x}, \mathbb{y}) > 0$ (the author of [10] noticed that $(\mathcal{X}, \mathfrak{I}_{\mathscr{D}})$ is even a Hausdorff space). For every $\mathbb{x} \in \mathcal{X}$ and $n \in \mathbb{N}$, we put $W(n, \mathbb{x}) = \{\mathbb{y} \in \mathcal{X} : \mathscr{D}(\mathbb{x}, \mathbb{y}) < \frac{1}{2^n}\}$. Clearly, for every $\mathbb{x} \in \mathcal{X}$, the family $\{W(n, \mathbb{x}) : n \in \mathbb{N}\}$ is a local neighborhood base at $\mathbb{x}$ in $(\mathcal{X}, \mathfrak{I}_{\mathscr{D}})$. Of course, for every $\mathbb{x} \in \mathcal{X}$ and $n \in \mathbb{N}$, $W(n+1, \mathbb{x}) \subseteq W(n, \mathbb{x})$. Let us fix $\mathbb{x} \in \mathcal{X}$ and $n \in \mathbb{N}$. We can fix $r \in \mathbb{N}$ such that $n \leq r$ and $\frac{1}{2^{r-1}} + \frac{\zeta}{2^{2r}} < \frac{1}{2^n}$. Consider any $\mathbb{y} \in W(r, n)$. Then $\mathbb{x} \in W(r, \mathbb{x}) \subseteq W(n, \mathbb{y})$. To show that $W(r, \mathbb{y}) \subseteq W(n, \mathbb{x})$, we consider any $\mathbb{z} \in W(r, \mathbb{y})$. Then $\mathscr{D}(\mathbb{x}, \mathbb{z}) \leq \mathscr{D}(\mathbb{x}, \mathbb{y}) + \mathscr{D}(\mathbb{y}, \mathbb{z}) + \zeta \mathscr{D}(\mathbb{x}, \mathbb{y})\mathscr{D}(\mathbb{y}, \mathbb{z}) < \frac{1}{2^r} + \frac{1}{2^r} + \zeta \frac{1}{2^{2r}} < \frac{1}{2^r}$. This implies that $\mathbb{z} \in W(n, \mathbb{x})$, so $W(r, \mathbb{y}) \subseteq W(n, \mathbb{x})$. It follows from the Collins-Roscoe Metrization Theorem 3.11 that $(\mathcal{X}, \mathfrak{I}_{\mathscr{D}})$ is metrizable. □

Remark 3.15. It is well known that Stone's theorem is unprovable in ZF (see Theorem 2 of [20]). Additionally, in ZF, it is unprovable to demonstrate that any metric space has a σ-locally finite basis. (see From 232 in the book: Howard, Paul; Rubin, Jean E. (1998). Consequences of the axiom of choice.

Mathematical Surveys and Monographs. Vol. 59. Providence, Rhode Island: American Mathematical Society. ISBN 9780821809778. [21]). Therefore Theorem 3.12 and Corollary 3.13 of the manuscript are both unprovable in ZF. As a result, this manuscript's Theorem 3.12, which involves the axiom of choice, leads to the conclusion that any suprametrizable space is metrizable. Furthermore, the Collins-Roscoe Metrization Theorem, which is an alternate and more straightforward argument, is used to demonstrate that suprametrizable space is metrizable.

Remark 3.16. Theorem 3.12 and Corollary 3.13 are straightforward from the perspective of Approach-II. Furthermore, any suprametrizable space is normal, which follows directly from Approach-II. Therefore, according to Approach-II, Theorems 3.6 and 3.7 are not necessary in order to establish that X is metrizable.

4. Application

We will examine the existence and uniqueness of a random solution to a stochastic integral equation of the following type in this section:

$$\mathfrak{z}(\mathrm{t},\lambda) = \int_0^{\mathrm{t}} \varphi(\tau,\mathfrak{z}(\tau,\lambda))d\tau + \int_0^{\mathrm{t}} \psi(\tau,\mathfrak{z}(\tau,\lambda))d\delta(\tau), \tag{4.1}$$

where $\mathrm{t} \in [0,1]$. The subsequent integral, defined in regard to scalar Brownian motion procedures $\{\delta(\mathrm{t})\}$, where $\mathrm{t} \in [0,1]$, is an Ito-type stochastic integral. The primary integral is a Lebesgue integral. Keep in mind that $C^*([0,1],\mathscr{L}_2(\phi,\mathscr{S},\mathscr{K})) \subset C_c(\mathbb{R}^+,\mathscr{L}_2(\phi,\mathscr{S},\mathscr{K}))$. The operators will be defined as $\mathscr{U}^*$ and $\mathscr{U}^\star$ from $C^*([0,1],\mathscr{L}_2(\phi,\mathscr{S},\mathscr{K}))$ into $C^*([0,1],\mathscr{L}_2(\phi,\mathscr{S},\mathscr{K}))$ by

$$\mathscr{U}^*\mathfrak{z}(\mathrm{t},\lambda) = \int_0^{\mathrm{t}} \mathfrak{z}(\mathrm{t},\lambda)d\tau \tag{4.2}$$

and

$$\mathscr{U}^\star\mathfrak{z}(\mathrm{t},\lambda) = \int_0^{\mathrm{t}} \mathfrak{z}(\mathrm{t},\lambda)d\delta(\tau). \tag{4.3}$$

Here $\mathfrak{z}(\mathrm{t},\lambda) \in C^*([0,1],\mathscr{L}_2(\phi,\mathscr{S},\mathscr{K}))$.

Lemma 4.1. [22] $\mathscr{U}^*$ and $\mathscr{U}^\star$ are continuous operators from $C^*([0,1],\mathscr{L}_2(\phi,\mathscr{S},\mathscr{K}))$ into $C^*([0,1],\mathscr{L}_2(\phi,\mathscr{S},\mathscr{K}))$. These operators are characterized by (4.2) and (4.3), accordingly.

Let T be a linear operator, and let $\mathscr{A}$ and $\mathscr{B}$ be a pair of Banach spaces. The preceding lemma, which is relevant to the examination of this section, is given. It is employed in the main theorem.

Lemma 4.2. [22] Let T be a continuous operator from $C_c(\mathbb{R}^+,\mathscr{L}_2(\phi,\mathscr{S},\mathscr{K}))$ into itself. If $\mathscr{A}$ and $\mathscr{B}$ are Banach spaces stronger than C_c and the pair $(\mathscr{A},\mathscr{B})$ is admissible with respect to T, then T is a continuous operator from $\mathscr{A}$ to $\mathscr{B}$.

Definition 4.3. [22] The pair of spaces $(\mathscr{A},\mathscr{B})$ will be called admissible with respect to the operator $T : C_c(\mathbb{R}^+,\mathscr{L}_2(\phi,\mathscr{S},\mathscr{K})) \to C_c(\mathbb{R}^+,\mathscr{L}_2(\phi,\mathscr{S},\mathscr{K}))$ if and only if $T(\mathscr{B}) \subset \mathscr{A}$.

Definition 4.4. [22] By stating that the Banach space $\mathscr{B}$ is stronger than the space $C_c(\mathbb{R}^+,\mathscr{L}_2(\phi,\mathscr{S},\mathscr{K}))$we mean that every convergent sequence in $\mathscr{B}$, with respect to its norm, will also converge in C_c.

The aforementioned theorem identifies the necessary conditions for a second-order stochastic process, a unique random solution to Eq (4.1), to exist.

Theorem 4.5. Let $\chi = C([0,1], \mathscr{L}_2(\phi, \mathscr{S}, \mathscr{K}))$ be the space of all continuous and bounded functions on $[0,1]$ with values in $\mathscr{L}_2(\phi, \mathscr{S}, \mathscr{K})$. Note that χ is extended suprametric space by considering $\mathscr{D}(\mathfrak{z}(t,\lambda), \hat{\mathfrak{z}}(t,\lambda)) = \sup|\mathfrak{z}(t,\lambda) - \hat{\mathfrak{z}}(t,\lambda)|^2$ with $\gamma(\mathfrak{z}(t,\lambda), \hat{\mathfrak{z}}(t,\lambda)) = e^{\mathfrak{z}(t,\lambda)+\hat{\mathfrak{z}}(t,\lambda)}$, where $\gamma : \chi \times \chi \to [1, \infty)$.

Considering the aforementioned assumptions, take into account the stochastic integral equation (4.1).

(1) $\mathscr{A}$ and $\mathscr{B}$ are subsets in $C^*([0,1], \mathscr{L}_2(\phi, \mathscr{S}, \mathscr{K}))$ which are stronger than $C^*([0,1], \mathscr{L}_2(\phi, \mathscr{S}, \mathscr{K}))$ such that $(\mathscr{A}, \mathscr{B})$ is admissible with respect to the operators $\mathscr{U}^*$ and $\mathscr{U}^\star$;

(2) $\mathfrak{z}(t,\lambda) \to \varphi(\tau, \mathfrak{z}(\tau,\lambda))$ is an operator on $\mathcal{S} = \{\mathfrak{z}(t,\lambda)/\mathfrak{z}(t,\lambda) \in \mathscr{B}$ and $|\mathfrak{z}(t,\lambda)| \leq \kappa\}$ with values in $\mathscr{A}$ satisfying
$$|\varphi(\tau, \mathfrak{z}(\tau,\lambda)) - \varphi(\tau, \hat{\mathfrak{z}}(\tau,\lambda))|_{\mathscr{A}} \leq \eta_1 |\mathfrak{z}(\tau,\lambda) - \hat{\mathfrak{z}}(\tau,\lambda)|_{\mathscr{B}};$$

(3) $\mathfrak{z}(t,\lambda) \to \psi(\tau, \mathfrak{z}(\tau,\lambda))$ is an operator on $\mathcal{S}$ into $\mathscr{A}$ satisfying
$$|\psi(\tau, \mathfrak{z}(\tau,\lambda)) - \psi(\tau, \hat{\mathfrak{z}}(\tau,\lambda))|_{\mathscr{A}} \leq \eta_2 |\mathfrak{z}(\tau,\lambda) - \hat{\mathfrak{z}}(\tau,\lambda)|_{\mathscr{B}};$$

(4) $(\ell_1\eta_1 + \ell_2\eta_2)^2 < 1$ and $|\varphi(t,0)|_{\mathscr{A}} + |\psi(t,0)|_{\mathscr{A}} \leq e(1 - \ell_1\eta_1 - \ell_2\eta_2)$;

(5) Lemma 4.2 holds in view of $\mathscr{U}^*$ and $\mathscr{U}^\star$, i.e.,
$$|(\mathscr{U}^*\mathfrak{z})(t,\lambda)|_{\mathscr{B}} \leq \ell_1 |\mathfrak{z}(t,\lambda)|_{\mathscr{A}} \text{ and}$$
$$|(\mathscr{U}^\star\mathfrak{z})(t,\lambda)|_{\mathscr{B}} \leq \ell_2 |\mathfrak{z}(t,\lambda)|_{\mathscr{A}},$$
where $\ell_1, \ell_2 < 1$.

Then Eq (4.1) has a unique random solution.

Proof. We split the proof into four steps.
Step-1. Constructing an operator $\mathscr{M} : \mathcal{S} \to \mathscr{B}$ to which we can apply our Theorem 2.10, for $\mathfrak{z}(t,\lambda) \in \mathcal{S}$,
$$\mathscr{M}\mathfrak{z}(t,\lambda) = \int_0^t \varphi(\tau, \mathfrak{z}(\tau,\lambda))d\tau + \int_0^t \psi(\tau, \mathfrak{z}(\tau,\lambda))d\delta(\tau). \tag{4.4}$$

The set $\mathcal{S}$ is a closed subset of $\chi = C([0,1], \mathscr{L}_2(\phi, \mathscr{S}, \mathscr{K}))$ endowed with suprametric $\mathscr{D}$. So $(\chi, \mathscr{D})$ is complete.
Step-2. We will show that $\mathscr{M}$ is a contraction operator on χ.
Let $\mathfrak{z}(t,\lambda), \hat{\mathfrak{z}}(t,\lambda) \in \chi$.
Consider,
$$|\mathscr{M}\mathfrak{z}(t,\lambda) - \mathscr{M}\hat{\mathfrak{z}}(t,\lambda)|^2_{\mathscr{B}} = \Bigg[\Bigg|\int_0^t \Big[\varphi(\tau, \mathfrak{z}(\tau,\lambda)) - \varphi(\tau, \hat{\mathfrak{z}}(\tau,\lambda))\Big]_{\mathscr{B}} d\tau$$

$$
\begin{aligned}
&+ \int_0^t \Big[\psi(\tau, \mathfrak{z}(\tau,\lambda)) - \psi(\tau, \hat{\mathfrak{z}}(\tau,\lambda))\Big] d\delta(\tau)\Big|\Bigg]^2_{\mathscr{B}} \\
&\leq \Bigg[\int_0^t |\varphi(\tau, \mathfrak{z}(\tau,\lambda)) - \varphi(\tau, \hat{\mathfrak{z}}(\tau,\lambda))|_{\mathscr{A}} d\tau \\
&+ \int_0^t |\psi(\tau, \mathfrak{z}(\tau,\lambda)) - \psi(\tau, \hat{\mathfrak{z}}(\tau,\lambda))| d\delta(\tau)\Bigg]^2_{\mathscr{A}} \\
&\leq \Big[\ell_1|\varphi(t, \mathfrak{z}(t,\lambda)) - \varphi(t, \hat{\mathfrak{z}}(t,\lambda))| + \ell_2|\psi(t, \mathfrak{z}(t,\lambda)) - \psi(t, \hat{\mathfrak{z}}(t,\lambda))|\Big]^2 \\
&\leq [\ell_1\eta_1|\mathfrak{z}(t,\lambda) - \hat{\mathfrak{z}}(t,\lambda)|_{\mathscr{B}} + \ell_2\eta_2|\mathfrak{z}(t,\lambda) - \hat{\mathfrak{z}}(t,\lambda)|_{\mathscr{B}}]^2 \\
&\leq [(\ell_1\eta_1 + \ell_2\eta_2)|\mathfrak{z}(t,\lambda) - \hat{\mathfrak{z}}(t,\lambda)|_{\mathscr{B}}]^2 \\
&\leq (\ell_1\eta_1 + \ell_2\eta_2)^2|\mathfrak{z}(t,\lambda) - \hat{\mathfrak{z}}(t,\lambda)|^2_{\mathscr{B}}. \qquad (4.5)
\end{aligned}
$$

Step-3. Observe that $\mathscr{M} : \chi \to \chi$, for $\mathfrak{z}(t,\lambda) \in \chi$, we need to show that $(\mathscr{M}\mathfrak{z}) \in \chi$.
For any element $\mathfrak{z}(t,\lambda) \in \chi$, we have

$$
\begin{aligned}
|(\mathscr{M}\mathfrak{z})(t,\lambda)|_{\mathscr{B}} &= \left|\int_0^t \varphi(\tau, \mathfrak{z}(\tau,\lambda)) d\tau + \int_0^t \psi(\tau, \mathfrak{z}(\tau,\lambda)) d\delta(\tau)\right| \\
&\leq \left|\int_0^t \varphi(\tau, \mathfrak{z}(\tau,\lambda)) d\tau\right|_{\mathscr{B}} + \left|\int_0^t \psi(\tau, \mathfrak{z}(\tau,\lambda)) d\delta(\tau)\right|_{\mathscr{B}} \\
&\leq \ell_1|\varphi(t, \mathfrak{z}(t,\lambda))|_{\mathscr{A}} + \ell_2|\psi(t, \mathfrak{z}(t,\lambda))|_{\mathscr{A}} \\
&\leq \ell_1\eta_1|\mathfrak{z}(t,\lambda)|_{\mathscr{B}} + \ell_2\eta_2|\mathfrak{z}(t,\lambda)|_{\mathscr{B}} + \ell_1|\varphi(t,0)|_{\mathscr{A}} + \ell_2|\psi(t,0)|_{\mathscr{A}}.
\end{aligned}
$$

Since $\mathfrak{z}(t,\lambda) \in \mathscr{S}$, it follows that

$$|(\mathscr{M}\mathfrak{z})(t,\lambda)|_{\mathscr{B}}| \leq \kappa(\ell_1\eta_1 + \ell_2\eta_2) + |\varphi(t,0)|_{\mathscr{A}} + |\psi(t,0)| \leq \kappa.$$

Thus $(\mathscr{M}\mathfrak{z}) \in \mathscr{S}$.
Step-4. From Eq (4.5), $|\mathscr{M}\mathfrak{z}(t,\lambda) - \mathscr{M}\hat{\mathfrak{z}}(t,\lambda)|^2 \leq (\ell_1\eta_1 + \ell_2\eta_2)^2|\mathfrak{z}(t,\lambda) - \hat{\mathfrak{z}}(t,\lambda)|^2$.
Taking supremum on both sides, we get $\mathscr{D}(\mathscr{M}\mathfrak{z}(t,\lambda), \mathscr{M}\hat{\mathfrak{z}}(t,\lambda)) \leq \theta\mathscr{D}(\mathfrak{z}(t,\lambda), \hat{\mathfrak{z}}(t,\lambda))$ where $\theta = (\ell_1\eta_1 + \ell_2\eta_2)^2 < 1$.
Thus, all the conditions of Theorem 2.10 satisfied. Thus, the existence and uniqueness of a random solution of Eq (4.1) follow from the Theorem 2.10. □

Open Question: Let $(\mathcal{X}, \mathscr{D})$ be a extended suprametric space. If $\mathscr{D}$ is continuous in one variable, then $\mathcal{X}$ is metrizable?

5. Conclusions

In this article, we focus on the extended suprametric space which opens new rooms for researchers. We consider that this new structure shall lead to the help of the solutions of certain differential equations and hence, produce new applications. In addition, we foresee that it shall allow us to achieve more refined results in existing applications.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they do not have any conflict interests.

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