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Research article

On a class of analytic functions closely related to starlike functions with respect to a boundary point

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Abstract: In this article, we introduce a new class of analytic functions in the open unit disc that are closely related to functions that are starlike with respect to a boundary point. For this new class of functions, we obtain representation theorem, interesting coefficient estimates and also certain differential subordination implications involving this new class.

Keywords: analytic; univalent; starlike function with respect to a boundary point; coefficient estimates; differential subordination
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1. Introduction

Consider $\mathbb{D} = \{z : |z| < 1\}$, an open unit disc in the complex plane $\mathbb{C}$, $\mathcal{H}$ as the collection of all analytic functions in $\mathbb{D}$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}$$

and $\mathcal{S}$ be a subclass of $\mathcal{A}$ containing all univalent functions. A function $f \in \mathcal{A}$ is starlike if $f(\mathbb{D})$ is starlike with respect to origin and the class of all starlike functions $f \in \mathcal{A}$ is denoted by $\mathcal{ST}$. Similarly, a function $f \in \mathcal{A}$ is convex if $f(\mathbb{D})$ is convex with respect to all points of $f(\mathbb{D})$ and the class of all convex functions $f \in \mathcal{A}$ is denoted by $\mathcal{CV}$. An analytic function $p : \mathbb{D} \to \mathbb{C}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad |p_n| \leq 2 \tag{1.1}$$

satisfying $\Re\left(p(z)\right) > 0$ for all $z \in \mathbb{D}$ is known as a function with positive real part. The class of such functions, denoted by $\mathcal{P}$, is known as the class of Carathéodory functions. Note that $\Re\left(p(z)\right) > 0$ can be written as $|\arg\left(p(z)\right)| < \dfrac{\pi}{2}$. Connections between $\mathcal{ST}$ and $\mathcal{P}$ and $\mathcal{CV}$ and $\mathcal{P}$ are as follows: a function $f \in \mathcal{ST}$ if and only if $\dfrac{zf'(z)}{f(z)} \in \mathcal{P},\ z \in \mathbb{D}$ and a function $f \in \mathcal{CV}$ if and only if $1 + \dfrac{zf''(z)}{f'(z)} \in \mathcal{P},\ z \in \mathbb{D}$. Thus, the properties of $\mathcal{ST}$ and $\mathcal{CV}$ functions can be obtained from the properties of functions in the class $\mathcal{P}$. Note that the Möbius function

$$L_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + ... = 1 + 2\sum_{n=1}^{\infty} z^n, \quad z \in \mathbb{D}, \tag{1.2}$$

is analytic and univalent in the open unit disc $\mathbb{D}$ and it maps $\mathbb{D}$ onto the right half-plane and is in the class $\mathcal{P}$.

Even though many authors extensively explored the concept of starlikeness of a given order for a long time, Robertson [16] was the pioneer in introducing the concept of an analytic univalent functions mapping an open unit disc onto a starlike domain with respect to the boundary point. He constructed the subclass $\mathcal{G}^*$ of $\mathcal{H}$ of functions $g, g(0) = 1$ mapping $\mathbb{D}$ onto a starlike domain with respect to $g(1) = \lim_{r \to 1^-} g(r) = 0$ and $\Re\left(e^{i\rho} g(z)\right) > 0$ for some real ρ and all $z \in \mathbb{D}$. Assume also that the constant function 1 belongs to the class $\mathcal{G}^*$. He conjectured that the class $\mathcal{G}^*$ coincides with the class $\mathcal{G}$,

$$\mathcal{G} = \left\{ g \in \mathcal{H} : g(z) \neq 0, g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \quad z \in \mathbb{D} \right\}, \tag{1.3}$$

such that

$$\Re\left(2z\frac{g'(z)}{g(z)} + \frac{1+z}{1-z}\right) > 0, \quad z \in \mathbb{D} \tag{1.4}$$

and proved that $\mathcal{G} \subset \mathcal{G}^*$. Later, this conjecture was confirmed by Lyzzaik [11] who proved $\mathcal{G}^* \subset \mathcal{G}$. Furthermore, if $g \in \mathcal{G}, g \neq 1$ then g is univalent close-to-convex in $\mathbb{D}$, as proved by Robertson [16]. It is worth mentioning that the analytic characterization (1.4) was known earlier to Styer [19].

In [3], a class $\mathcal{G}(M)$, $M > 1$, consisting of all analytic and non-vanishing functions of the form (1.3), such that

$$\Re\left(2z\frac{g'(z)}{g(z)} + z\frac{P'(z;M)}{P(z;M)}\right) > 0, \quad z \in \mathbb{D},$$

which is a closely related function to the class $\mathcal{G}$ was introduced by Jakubowski [3]. Here,

$$P(z;M) = \frac{4z}{\left(\sqrt{(1-z)^2 + \frac{4z}{M}} + 1 - z\right)^2}, \quad z \in \mathbb{D}$$

is the Pick function. The class $\mathcal{G}(1)$ was also considered in [3], where

$$\mathcal{G}(1) = \left(g \in \mathcal{H} : g(z) \neq 0, \Re\left(1 + 2z\frac{g'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}\right).$$

Todorov [20] linked the class $\mathcal{G}$ with a functional $f(z)/(1-z)$ for $z \in \mathbb{D}$ and obtained a structural formula and coefficient estimates. Silverman and Silvia [17] introduced an interesting class $\mathcal{G}(\beta) \subset \mathcal{G}^*, 0 < \beta \leq 1$, consisting of all analytic function g of the form (1.3) and satisfying

$$\Re\left(z\frac{g'(z)}{g(z)} + (1-\beta)\frac{1+z}{1-z}\right) > 0, \quad z \in \mathbb{D}.$$

Clearly, $\mathcal{G}\left(\frac{1}{2}\right) = \mathcal{G}$. For $-1 < A \leq 1$ and $-A < B \leq 1$, Jakubowski and Włodarczyk [4] defined the class $\mathcal{G}(A, B)$, of all g of the form (1.3), satisfying $\Re\left(2z\frac{g'(z)}{g(z)} + \frac{1+Az}{1-Bz}\right) > 0, \quad z \in \mathbb{D}$ (see also the work of Sivasubramanian [18]). Related works on the class $\mathcal{G}$ were considered earlier by [1,6–10,14]. We remark at this point, that the function

$$\frac{1}{2}\ln\frac{1+z}{1-z} = z + \frac{z^3}{3} + \cdots + \frac{z^{2n+1}}{2n+1} + \cdots$$

is univalent in $\mathbb{D}$. In this article, we are interested in introducing and investigating a new class as follows.

Definition 1.1. *Let $\mathcal{G}_c$ be the class consisting of all functions of the form (1.3) satisfying*

$$\Re\left\{2z\,\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c\right\} > 0, \; z \in \mathbb{D}, \tag{1.5}$$

where $0 < c \leq 2$.

If $c = 1$, the class $\mathcal{G}_1 = \mathcal{G}$ was introduced and investigated by Robertson [16]. For this new class of functions, we obtain representation theorem, interesting coefficient estimates and also certain differential subordination implications involving this new class.

Example 1.1. *The function*

$$g_1(z) = \frac{1}{1-z}\exp\left\{-\frac{1}{2}\int_0^z\left(\frac{\left(\frac{1+t}{1-t}\right)^c - 1}{t}\right)dt\right\}$$

is in the class $\mathcal{G}_c$, where $0 < c \leq 2$.

Proof. Taking logarithm on both sides and by a simple differentiation, one can easily get

$$2\frac{g_1'(z)}{g_1(z)} + \frac{1}{z}\left(\frac{1+z}{1-z}\right)^c = \frac{1+z}{z(1-z)}.$$

Hence,

$$\Re\left\{2z\frac{g_1'(z)}{g_1(z)} + \left(\frac{1+z}{1-z}\right)^c\right\} = \Re\left\{\frac{1+z}{1-z}\right\} > 0$$

implies $g_1 \in \mathcal{G}_c$.

Example 1.2. *The function*

$$g_2(z) = \exp\left\{-\frac{1}{2}\int_0^z \left(\frac{\left(\frac{1+t}{1-t}\right)^c - 1}{t}\right) dt\right\}$$

is in the class $\mathcal{G}_c$, where $0 < c \le 2$.

Proof. Taking logarithm on both sides and by a simple differentiation, one can easily get

$$2\,\frac{g_2'(z)}{g_2(z)} + \frac{1}{z}\left(\frac{1+z}{1-z}\right)^c = \frac{1}{z}.$$

Therefore,

$$\Re\left\{2z\,\frac{g_2'(z)}{g_2(z)} + \left(\frac{1+z}{1-z}\right)^c\right\} = 1 > 0$$

implies that $g_2 \in \mathcal{G}_c$.

Similarly, we can show that

Example 1.3. *The function*

$$g_3(z) = \frac{1}{\sqrt{1-z}} \exp\left\{-\frac{1}{2}\int_0^z \left(\frac{\left(\frac{1+t}{1-t}\right)^c - 1}{t}\right) dt\right\}$$

is in the class $\mathcal{G}_c$, where $0 < c \le 2$.

Example 1.4. *The function*

$$g_4(z) = \frac{1}{\sqrt{1+z}} \exp\left\{-\frac{1}{2}\int_0^z \left(\frac{\left(\frac{1+t}{1-t}\right)^c - 1}{t}\right) dt\right\}$$

is in the class $\mathcal{G}_c$, where $0 < c \le 2$.

The above examples show that there are many functions present in the class $\mathcal{G}_c$ proving that the class $\mathcal{G}_c$ is non empty.

2. Representation theorems

Theorem 2.1. *Let $0 < c \le 2$. Furthermore, let*

$$\beta_c(z) = -\frac{1}{2c}\int_0^z \left(\frac{\left(\frac{1+t}{1-t}\right)^c - 1}{t}\right) dt. \tag{2.1}$$

A function g is in $\mathcal{G}_c$ if and only if there exists a starlike function $s \in \mathcal{ST}$ such that

$$g(z) = \left(\frac{s(z)}{z}\right)^{\frac{1}{2}} \exp\{c\,\beta_c(z)\}. \tag{2.2}$$

Proof. Suppose that $s \in \mathcal{ST}$ and g is given by (2.2). Then, g is analytic and $g(0) = 1$. Therefore, from (1.5), we have for some function g satisfying (2.2), there exists a starlike function s such that

$$(g(z))^2 \left(z \, \exp\{-2c\beta_c(z)\}\right) = s(z).$$

Hence,

$$2 \log g(z) + \log z - 2c\beta_c(z) = \log s(z).$$

By a simple differentiation followed by simplification we get,

$$2\frac{zg'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c = \frac{zs'(z)}{s(z)},$$

where

$$\left(\frac{1+z}{1-z}\right)^c = 1 + 2cz + 2c^2z^2 + 2\,\frac{2c^3+c}{3}z^3 + 2\,\frac{c^4+2c^2}{3}z^4 + \cdots .$$

Therefore,

$$\Re\left\{2z\,\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c\right\} = \Re\left(z\,\frac{s'(z)}{s(z)}\right) > 0.$$

Hence, $g \in \mathcal{G}_c$.

On the other hand, suppose $g \in \mathcal{G}_c$ and

$$s(z) = z\, g^2(z) \exp\{-2c\beta_c(z)\}.$$

Then, $s(0) = 0$, $s'(0) = 1$ and

$$\Re\left(\frac{zs'(z)}{s(z)}\right) = \Re\left(2z\,\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c\right).$$

The above expression is positive as $g \in \mathcal{G}_c$ which implies $s \in \mathcal{ST}$.

For the choices of $c = 1$ and $c = 2$ we get the following corollaries as listed below.

Corollary 2.1. *[16] A function g is in $\mathcal{G}_1$ if and only if there exists a function $s \in \mathcal{ST}$ such that* $(g(z))^2 = \left(\frac{s(z)}{z}\right)(1-z)^2$.

Corollary 2.2. *A function g is in $\mathcal{G}_2$ if and only if there exists a function $s \in \mathcal{ST}$ such that* $g(z) = \left(\frac{s(z)}{z}\right)^{\frac{1}{2}} \exp\left(\frac{-2}{1-z}\right)$.

Theorem 2.2. *(Herglotz representation theorem) Let $0 < c \leq 2$ and let g be an analytic function in $\mathbb{D}$ such that $g(0) = 1$. Then, $g \in \mathcal{G}_c$ if and only if*

$$g(z) = \exp\left[-\int_{-\pi}^{\pi} \log(1 - ze^{-it})d\mu(t) - \frac{1}{2}\int_0^z \left(\frac{\left(\frac{1+t}{1-t}\right)^c - 1}{t}\right)dt\right], \tag{2.3}$$

where $\mu(t)$ is a probability measure on $[-\pi, \pi]$.

Proof. Let $0 < c \leq 2$. If $g \in \mathcal{G}_c$, we can write

$$2z\,\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c = \int_{-\pi}^{\pi} \frac{1+z\,e^{-it}}{1-z\,e^{-it}}d\mu(t).$$

By a simple integration and simplification we get,

$$2\ \log g(z) = -\int_0^z \left(\frac{\left(\frac{1+t}{1-t}\right)^c - 1}{t}\right)dt - 2\int_{-\pi}^{\pi} \log\left(1 - ze^{-it}\right)d\mu(t).$$

Upon simplification of the above equation, one can obtain (2.3). The converse part can be proved by similar lines as in the necessary part and hence the details are omitted.

For $c = 1$ and $c = 2$ we get the following corollaries.

Corollary 2.3. *[16] Let g be an analytic function in $\mathbb{D}$ such that $g(0) = 1$. Then, $g \in \mathcal{G}_1$ if and only if*

$$g(z) = (1-z)\ \exp\left(-\int_{-\pi}^{\pi} \log(1 - ze^{-it})d\mu(t)\right).$$

Corollary 2.4. *Let g be an analytic function in $\mathbb{D}$ such that $g(0) = 1$. Then, $g \in \mathcal{G}_2$ if and only if*

$$g(z) = \exp\ -\left(\frac{2}{1-z} + \int_{-\pi}^{\pi} \log(1 - ze^{-it})d\mu(t)\right).$$

Theorem 2.3. *Let $0 < c \leq 2$. A function $g \in \mathcal{G}_c$ if and only if there exists a function $p \in \mathcal{P}$ such that*

$$g(z) = \frac{1}{\sqrt{z}}\ \exp\left(\frac{1}{2}\left(\int_0^z \frac{p(\zeta)}{\zeta}d\zeta - \int_0^z \left(\frac{\left(\frac{1+t}{1-t}\right)^c - 1}{t}\right)dt\right)\right). \tag{2.4}$$

Proof. Let $g \in \mathcal{G}_c$. Then, by the definition of $\mathcal{G}_c$, for some function $p \in \mathcal{P}$,

$$2\frac{g'(z)}{g(z)} + \frac{1}{z}\left(\frac{1+z}{1-z}\right)^c = \frac{p(z)}{z}.$$

Upon integration and simplification, we get

$$\log z\,(g(z))^2 = \int_0^z \frac{p(\zeta)}{\zeta}d\zeta - \int_0^z \left(\frac{\left(\frac{1+t}{1-t}\right)^c - 1}{t}\right)dt$$

which proves the necessary part of the theorem. Conversely, assume $p \in \mathcal{P}$ and $p(0) = 1$ and let g be as in (2.4). Then, g is analytic in $\mathbb{D}$ and by applying simple calculations we can easily prove that $g \in \mathcal{G}_c$.

For $c = 1$ and $c = 2$, we get the following corollaries.

Corollary 2.5. *Let $g \in \mathcal{G}_1$ if and only if there exists a function $p \in \mathcal{P}$ such that*

$$g(z) = \frac{1-z}{\sqrt{z}} \, \exp\left(\frac{1}{2} \int_0^z \frac{p(\zeta)}{\zeta} d\zeta \right).$$

Corollary 2.6. *Let $g \in \mathcal{G}_2$ if and only if there exists a function $p \in \mathcal{P}$ such that*

$$g(z) = \frac{1}{\sqrt{z}} \, \exp\left(\frac{1}{2} \int_0^z \frac{p(\zeta)}{\zeta} d\zeta - \frac{2z}{1-z} \right).$$

Theorem 2.4. *Let $0 < c \leq 2$. A function $g \in \mathcal{G}_c$ if and only if there exists a function $p \in \mathcal{H}$ such that $p \prec \dfrac{1+z}{1-z}$ and for $z \in \mathbb{D}$,*

$$g(z) = \frac{1}{\sqrt{z}} \, \exp\left(\frac{1}{2} \left(\int_0^z \frac{p(\zeta)}{\zeta} d\zeta - \int_0^z \left(\frac{\left(\frac{1+t}{1-t}\right)^c - 1}{t} \right) dt \right) \right).$$

Proof. The proof follows from Theorem 2.3.

3. Coefficient estimates for the class $\mathcal{G}_c$

Theorem 3.1. *Let $0 < c \leq 2$ and $z \in \mathbb{D}$. If $g \in \mathcal{G}_c$, we have the following sharp inequalities.*

$$|c + d_1| \leq 1, \tag{3.1}$$

$$\left| c^2 + 2d_2 - d_1^2 \right| \leq 1, \tag{3.2}$$

and

$$\left| 2c^3 + c + 9d_3 - 9d_1 d_2 + 3d_1^3 \right| \leq 3. \tag{3.3}$$

Further, for $\alpha \in \mathbb{R}$, let

$$\mathcal{H}(\alpha, c) = 4d_2 - 8\alpha c d_1 - 2d_1^2(1 + 2\,\alpha) + 2c^2(1 - 2\alpha). \tag{3.4}$$

Then,

$$|\mathcal{H}(\alpha, c)| \leq \begin{cases} 2\left(1 - 2\alpha|c + d_1|^2\right), & \textit{if } \alpha \leq \frac{1}{2}, \\ 2\left(1 - 2(1-\alpha)|c + d_1|^2\right), & \textit{if } \alpha \geq \frac{1}{2}. \end{cases} \tag{3.5}$$

Proof. Let

$$p(z) = 2z\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z} \right)^c, \; z \in \mathbb{D}.$$

On expanding the right hand side of the above function p, we get

$$p(z) = 1 + 2\,(c + d_1)\, z + 2\left(c^2 + 2d_2 - d_1^2\right) z^2 + \frac{2}{3}\left(2c^3 + c + 9d_3 - 9d_1 d_2 + 3d_1^3\right) z^3 + \cdots. \tag{3.6}$$

By making use of the known inequality $|p_i| \leq 2$ for all $p \in \mathcal{P}$, we can get the sharp inequalities given in (3.1)–(3.3). From (1.1) and (3.6) and from the known fact that

$$|p_2 - \alpha p_1^2| \leq \begin{cases} 2 - \alpha\, |p_1|^2\,, & \text{if } \alpha \leq \frac{1}{2}, \\ 2 - (1-\alpha)\, |p_1|^2\,, & \text{if } \alpha \geq \frac{1}{2}, \end{cases}$$

we can obtain (3.5)

For $c = 1$ and $c = 2$, we have the following corollaries as stated below.

Corollary 3.1. *[2] Let $z \in \mathbb{D}$. If $g \in \mathcal{G}_1$, we have the following inequalities.*

$$|1 + d_1| \leq 1,\ \left|1 + 2d_2 - d_1^2\right| \leq 1,\ \left|1 + 3d_3 - 3d_1 d_2 + d_1^3\right| \leq 1.$$

Further,

$$|\mathcal{H}(\alpha, 1)| \leq \begin{cases} 2\left(1 - 2\alpha|1 + d_1|^2\right), & \textit{if } \alpha \leq \frac{1}{2}, \\ 2\left(1 - 2(1-\alpha)|1 + d_1|^2\right), & \textit{if } \alpha \geq \frac{1}{2}. \end{cases}$$

All of these inequalities are sharp.

Corollary 3.2. *Let $z \in \mathbb{D}$. If $g \in \mathcal{G}_2$, the following inequalities hold.*

$$\left|1 + \frac{d_1}{2}\right| \leq \frac{1}{2},\ \left|4 + 2d_2 - d_1^2\right| \leq 1,\ \left|6 + 3d_3 - 3d_1 d_2 + d_1^3\right| \leq 1.$$

Also,

$$|\mathcal{H}(\alpha, 2)| \leq \begin{cases} 2\left(1 - 2\alpha|2 + d_1|^2\right), & \textit{if } \alpha \leq \frac{1}{2}, \\ 2\left(1 - 2(1-\alpha)|2 + d_1|^2\right), & \textit{if } \alpha \geq \frac{1}{2}. \end{cases}$$

All of these inequalities are sharp.

Theorem 3.2. *Let $0 < c \leq 2$ and let the function $g(z)$ be of the form (1.3) belong to the class $\mathcal{G}_c$. Then, for $n = 2, 3, \cdots$, the following estimates*

$$\begin{aligned} &\left| nd_n - c(n-2)d_{n-1} + \cdots + \left[1 + (-1)^{n-1}\right] \frac{c(c-1)\cdots(c-n+2)}{2(n-1)!} d_1 \right. \\ &\qquad\qquad \left. + \left[1 - (-1)^n\right] \frac{c(c-1)(c-2)\cdots(c-n+1)}{2n!} \right|^2 \\ &\leq 1 + \sum_{k=1}^{n-1} \left| (k+1)d_k - c(k-1)d_{k-1} + \cdots + \left[1 + 3(-1)^{k-1}\right] \frac{c(c-1)(c-2)\cdots(c-k+2)}{2(k-1)!} d_1 \right. \\ &\qquad\qquad \left. + \left[1 + (-1)^k\right] \frac{c(c-1)\cdots(c-k+1)}{2k!} \right|^2 \end{aligned}$$

hold.

Proof. Let the function g of the form (1.3) belong to the class $\mathcal{G}_c$. Then, there exists a function $p \in \mathcal{P}$ such that,

$$p(z) = 2z\,\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c,\ z \in \mathbb{D}. \tag{3.7}$$

Since $p \in \mathcal{P}$, there exists a function ω of the form

$$\omega(z) = \frac{p(z)-1}{p(z)+1},\ z \in \mathbb{D},$$

where ω is analytic in $\mathbb{D}, \omega(0) = 0, |\omega(z)| \leq 1$ for $z \in \mathbb{D}$. Furthermore,

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)}. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\left((1-z)^c g(z) + 2z(1-z)^c g'(z) + (1+z)^c g(z)\right)\omega(z) = (1-z)^c\left(2zg'(z) - g(z)\right) + (1+z)^c g(z).$$

Let

$$\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n.$$

Considering the expansion of the function of g as power series, we get

$$\begin{aligned}
&(2 + \textstyle\sum_{n=1}^{\infty}\left(d_n - cd_{n-1} + \frac{c(c-1)}{2!}d_{n-2} + \cdots + (-1)^{n-1}\frac{c(c-1)\cdots(c-n+2)}{(n-1)!}d_1 + (-1)^n\frac{c(c-1)\cdots(c-n+1)}{n!}\right)z^n \\
&+ 2\textstyle\sum_{n=1}^{\infty}\left(nd_n - c(n-1)d_{n-1} + \frac{c(c-1)}{2!}(n-2)d_{n-2} + \cdots + (-1)^{n-1}\frac{c(c-1)\cdots(c-n+2)}{(n-1)!}d_1\right)z^n \\
&+ \textstyle\sum_{n=1}^{\infty}\left(d_n + cd_{n-1} + \frac{c(c-1)}{2!}d_{n-2} + \cdots + \frac{c(c-1)\cdots(c-n+2)}{(n-1)!}d_1 + \frac{c(c-1)\cdots(c-n+1)}{n!}\right)z^n)\left(\textstyle\sum_{n=1}^{\infty}\omega_n z^n\right) \\
&= 2\textstyle\sum_{n=1}^{\infty}\left(nd_n - c(n-1)d_{n-1} + \frac{c(c-1)}{2!}(n-2)d_{n-2} + \cdots + (-1)^{n-1}\frac{c(c-1)\cdots(c-n+2)}{(n-1)!}d_1\right)z^n \\
&+ \textstyle\sum_{n=1}^{\infty}\left(d_n + cd_{n-1} + \frac{c(c-1)}{2!}d_{n-2} + \cdots + \frac{c(c-1)\cdots(c-n+2)}{(n-1)!}d_1 + \frac{c(c-1)\cdots(c-n+1)}{n!}\right)z^n \\
&- \textstyle\sum_{n=1}^{\infty}\left(d_n - cd_{n-1} + \frac{c(c-1)}{2!}d_{n-2} + \cdots + (-1)^{n-1}\frac{c(c-1)\cdots(c-n+2)}{(n-1)!}d_1 + (-1)^n\frac{c(c-1)\cdots(c-n+1)}{n!}\right)z^n.
\end{aligned}$$

For $z \in \mathbb{D}$, this can be again simplified to bring into the form,

$$\begin{aligned}
&\Bigg(1 + \Bigg(\sum_{n=1}^{\infty}\Bigg((n+1)d_n - c(n-1)d_{n-1} + \cdots + \left[1 + 3(-1)^{n-1}\right]\frac{c(c-1)(c-2)\cdots(c-n+2)}{2(n-1)!}d_1 \\
&\qquad + \ [1 + (-1)^n]\,\frac{c(c-1)\cdots(c-n+1)}{2n!}\Bigg)z^n\Bigg)\left(\sum_{n=1}^{\infty}\omega_n z^n\right)\Bigg) \\
&= \sum_{n=2}^{\infty}\Bigg(nd_n - c(n-2)d_{n-1} + \cdots + \left[1 + (-1)^{n-1}\right]\frac{c(c-1)\cdots(c-n+2)}{2(n-1)!}d_1 \\
&\qquad + \ [1 - (-1)^n]\,\frac{c(c-1)(c-2)\cdots(c-n+1)}{2(n)!}\Bigg)z^n.
\end{aligned} \tag{3.9}$$

For $n = 2, 3, \cdots$, let

$$\begin{aligned}
p_n(c) = {} & (n+1)d_n - c(n-1)d_{n-1} + \cdots + \left[1 + 3(-1)^{n-1}\right]\frac{c(c-1)(c-2)\cdots(c-n+2)}{2(n-1)!}d_1 \\
& + [1 + (-1)^n]\,\frac{c(c-1)\cdots(c-n+1)}{2n!}
\end{aligned} \tag{3.10}$$

and

$$s_n(c) = nd_n - c(n-2)d_{n-1} + \cdots + \left[1 + (-1)^{n-1}\right] \frac{c(c-1)\cdots(c-n+2)}{2(n-1)!} d_1$$
$$+ \left[1 - (-1)^n\right] \frac{c(c-1)(c-2)\cdots(c-n+1)}{2(n)!}. \tag{3.11}$$

From (3.9)–(3.11), we have

$$\sum_{n=1}^{\infty} \omega_n z^n + \sum_{n=2}^{\infty} \left(p_1(c)\omega_{n-1} + \cdots p_{n-1}(c)\omega_1\right) z^n = \sum_{n=1}^{\infty} s_n(c) z^n, \quad z \in \mathbb{D}. \tag{3.12}$$

Equating the coefficient of z, we have

$$2\omega_1 = d_1 + 2c.$$

Since $|\omega_1| \leq 1$, we obtain

$$\left|\frac{d_1}{2} + c\right| \leq 1$$

and for $n = 2, 3, \cdots,$

$$\omega_n + p_1(c)\omega_{n-1} + \cdots + p_{n-1}\omega_1 = s_n(c).$$

From the Eqs (3.9)–(3.12), we have

$$\left(1 + \sum_{k=1}^{n-1} p_k(c)\right)\left(\sum_{k=1}^{\infty} \omega_k z^k\right) = \sum_{k=1}^{n} s_k(c) z^k + \sum_{k=n+1}^{\infty} E_k z^k,$$

where E_k are the appropriate coefficients. Since $|\omega(z)| < 1$ for $z \in \mathbb{D}$,

$$\left|\sum_{k=1}^{n} s_k(c) z^k + \sum_{k=n+1}^{\infty} E_k z^k\right|^2 < \left|1 + \sum_{k=1}^{n-1} p_k(c) z^k\right|^2.$$

By simplifying this, we have

$$\sum_{k=1}^{n} |s_k(c)|^2 \leq 1 + \sum_{k=1}^{n-1} |p_k(c)|^2.$$

Since $|s_k(c)|^2 \geq 0$ for $k = 1, 2, \cdots, n-1$, we obtain

$$|s_n(c)|^2 \leq 1 + \sum_{k=1}^{n-1} |p_k(c)|^2, n = 2, 3, \cdots.$$

This essentially completes the proof of Theorem 3.2.

Let us consider the class $\mathcal{B}$ defined by $\mathcal{B} = \{\omega \in \mathcal{H} : |\omega(z)| \leq 1, z \in \mathbb{D}\}$ and $\mathcal{B}_0$ be the subclass of $\mathcal{B}$ consisting of functions ω such that $\omega(0) = 0$. The elements of $\mathcal{B}_0$ are also known as Schwarz functions.

Lemma 3.1. *[5] If $\omega \in \mathcal{B}_0$ is of the form $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n,\ z \in \mathbb{D}$, then for $\nu \in \mathbb{C}$,*

$$\left|\omega_2 - \nu\omega_1^2\right| \leq \max\{1, |\nu|\}.$$

Lemma 3.2. *If $\omega \in \mathcal{B}_0$ is of the form $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n,\ z \in \mathbb{D}$, then for any real numbers q_1 and q_2, the following sharp estimates holds:*

$$\left|\omega_3 + q_1\ \omega_1\ \omega_2 + q_2\ \omega_1^3\right| \leq H(q_1, q_2), \tag{3.13}$$

where

$$H(q_1,q_2) := \begin{cases} 1 \ \mathit{if}\ (q_1,q_2) \in D_1 \cup D_2 \\ |q_2| \ \mathit{if}\ (q_1,q_2) \in \cup_{k=3}^{7} D_k \\ \frac{2}{3}\left(|q_1|+1\right)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} \ \mathit{if}\ (q_1,q_2) \in D_8 \cup D_9 \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} \ \mathit{if}\ (q_1,q_2) \in D_{10} \cup D_{11} / \{\pm 2, 1\} \\ \frac{2}{3}\left(|q_1|-1\right)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} \ \mathit{if}\ (q_1,q_2) \in D_{12} \end{cases}$$

and the sets $D_k, k = 1, 2, \cdots$ are defined in [15].

Now we obtain a few upper bounds for early coefficients and for the Fekete-Szegö functional in the class $\mathcal{G}_c$.

Theorem 3.3. *Let $g \in \mathcal{G}_c, 0 < c \leq 2$. Then,*

$$|c + d_1| \leq 1, \tag{3.14}$$

$$|d_1| \leq 1 + c, \tag{3.15}$$

$$|c^2 + 2d_2 - d_1^2| \leq 1, \tag{3.16}$$

$$|d_2| \leq 1 + c, \tag{3.17}$$

$$|3d_3 - 3d_1 d_2 + d_1^3| \leq \frac{4c^3 + 2c + 3}{6} \tag{3.18}$$

and

$$|d_3| \leq \frac{3 + 2c + 18c^2 + 10c^3}{18}. \tag{3.19}$$

Furthermore, for $\delta \in \mathbb{R}$

$$|d_2 - \delta d_1^2| \leq \frac{1}{2}\, max\{1, 2|1-\delta|\} + c|2\delta - 1| + c^2|-\delta|. \tag{3.20}$$

Proof. From the class $\mathcal{G}_c$, there exists $\omega \in \mathcal{B}_0$ of the form $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n,\ z \in \mathbb{D}$, such that

$$2z\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c = \frac{1+\omega(z)}{1-\omega(z)}. \tag{3.21}$$

Since

$$2z\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c$$

$$= 1 + 2\,(c + d_1)z)\,z + 2\left(c^2 + 2d_2 - d_1^2\right)z^2 + \frac{2}{3}\left(2c^3 + c + 9d_3 - 9d_1d_2 + 3d_1^3\right) + \cdots \tag{3.22}$$

and

$$\frac{1+\omega(z)}{1-\omega(z)} = 1 + 2\omega_1 z + \left(2\omega_1^2 + 2\omega_2\right)z^2 + \left(2\omega_1^3 + 4\omega_1\omega_2 + \omega_3\right)z^3 + \cdots . \tag{3.23}$$

By comparing the corresponding coefficients from (3.21)–(3.23), we have the following:

$$2\,(c + d_1) = 2\omega_1, \tag{3.24}$$

$$2\left(c^2 + 2d_2 - d_1^2\right) = 2\left(\omega_1^2 + \omega_2\right) \tag{3.25}$$

and

$$\frac{2}{3}\left(2c^3 + c + 9d_3 - 9d_1d_2 + 3d_1^3\right) = 2\omega_1^2 + 4\omega_1\omega_2 + \omega_3. \tag{3.26}$$

Since $|\omega_1| \leq 1$, from (3.24),

$$|c + d_1| \leq 1$$

and hence

$$|d_1| \leq 1 + c.$$

From (3.25) and by Lemma 3.1, we have

$$\left|c^2 + 2d_2 - d_1^2\right| = \left|\omega_2 + \omega_1^2\right| \leq 1$$

and

$$2d_2 = \omega_2 + 2\omega_1^2 - 2c\omega_1,$$

which in turn gives

$$2\,|d_2| \leq \max\{1, 2\} + 2c\,|\omega_1|$$

and hence

$$|d_2| \leq 1 + c.$$

From (3.26) and by Lemma 3.2, we get

$$\left|3d_3 - 3d_1d_2 + d_1^3\right| \leq \frac{1}{2} + \frac{2}{3}c^3 + \frac{1}{3}c.$$

Hence, we obtain

$$\left|3d_3 - 3d_1d_2 + d_1^3\right| \leq \frac{4c^3 + 2c + 3}{6}.$$

Substituting the formulas for d_1 and d_2, we obtain

$$|3d_3| \;=\; \left|\frac{1}{2}\left(\omega_3 + 7\omega_1\omega_2 + 6\omega_1^3\right) - \frac{3}{2}c\left(\omega_2 + 2\omega_1^2\right) + 3c^2\omega_1 - 3\alpha\omega_1 + \frac{1}{3}c^3 + \frac{1}{3}c\right|$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left|\omega_3+7\omega_1\omega_2+6\omega_1^3\right|+\frac{3}{2}c\left|\omega_2+2\omega_1^2\right|+3c^2\left|\omega_1\right|+3c\left|\omega_1\right|+\frac{1}{3}c^3+\frac{1}{3}c \\
&\leq \frac{1}{2}H(7,6)+\frac{3}{2}c\max\{1,|-2|\}+3c^2+3c+\frac{1}{3}c^3+\frac{1}{3}c.
\end{aligned}
$$

Therefore,

$$
|d_3| \leq \frac{3+38c+18c^2+2c^3}{18}.
$$

Furthermore, we get

$$
\left|d_2-\delta d_1^2\right| \leq \frac{1}{2}\left|\omega_2+2(1-\delta)\omega_1^2\right|+c(2\delta-1)\left|\omega_1\right|+c^2\left|-\delta\right|.
$$

The proof of the theorem is completed by virtue of Lemma 3.1.

It can be remarked here that if ν is a real number, Lemma 3.1 can be improved in the following way and can be found in [2] .

Lemma 3.3. *If $\omega \in \mathcal{B}_0$ is of the form $\omega(z)=\sum_{n=1}^{\infty}\omega_n z^n,\ z \in \mathbb{D}$, then*

$$
\left|\omega_2-\nu\omega_1^2\right| \leq \begin{cases} -\nu, \ \nu \leq -1, \\ 1, \ -1 \leq \nu \leq 1, \\ \nu, \ \nu \geq 1. \end{cases} \tag{3.27}
$$

For $\nu<-1$ or $\nu>1$, equality holds if and only if $\omega(z)=z$ or one of its rotations. For $-1<\nu<1$, equality holds if and only if $\omega(z)=z^2,\ z\in\mathbb{D}$ or one of its rotations. For $\nu=-1$ equality holds if and only if $\omega(z)=\frac{z(\lambda+z)}{(1+\lambda z)}, z \in \mathbb{D}$ or one of its rotations, while for $\nu=1$ equality holds if and only if $w(z)=\frac{-z(\lambda+z)}{(1+\lambda z,)}, 0 \leq \lambda \leq 1, z \in \mathbb{D}$ or one of its rotations.

We can improve the results obtained in (3.20), in view of Lemma 3.3 as follows: For $\delta \in \mathbb{R}$, we get,

$$
\left|d_2-\delta d_1^2\right| \leq \begin{cases} (1+c)-2(1+c)^2\delta, \ \delta \leq 0, \\ \dfrac{1-2c+2(c^2-2c)\delta}{2}, \ 0 \leq \delta \leq \dfrac{1}{2}, \\ \dfrac{1-2c+2(c^2+2c)\delta}{2}, \ \dfrac{1}{2} \leq \delta \leq \dfrac{3}{2}, \\ (1+c)^2\delta-2(1+c), \ \delta \geq \frac{3}{2}. \end{cases} \tag{3.28}
$$

Theorem 3.4. *Let $0<r<1$. If $g \in \mathcal{G}_c$ then for $|z|=r<1$,*

$$
\sqrt{\frac{-f_\alpha(-r)}{r}} \leq |g(z)|\, e^{-2c\beta_c'(r)} \leq \sqrt{\frac{f_\alpha(-r)}{r}}. \tag{3.29}
$$

Proof. Let us define

$$
f(z)=z\,(g(z))^2\exp\{-2z\beta_c'(z)\}. \tag{3.30}
$$

Note that the function g is non-vanishing in $\mathbb{D}$. Therefore, f is analytic in $\mathbb{D}$ and a simple computation shows that

$$
z\frac{f'(z)}{f(z)}=2z\frac{g'(z)}{g(z)}+\left(\frac{1+z}{1-z}\right)^c, \ z \in \mathbb{D}. \tag{3.31}
$$

From (3.31), we can see that $g \in \mathcal{G}_c$ if and only if $f \prec \frac{1+z}{1-z}, z \in \mathbb{D}$. By applying the result of Ma and Minda [12], we infer that

$$-f_\alpha(-r) \leq |f(z)| \leq f_\alpha(-r),\ |z| = r. \tag{3.32}$$

Hence,

$$-f_\alpha(-r) \leq \left| z\,(g(z))^2\ \exp\{-2z\beta_c'(z)\} \right| \leq f_\alpha(-r),\ |z| = r, \tag{3.33}$$

which gives (3.29).

For $c = 1$ and $c = 2$ we have the following corollaries.

Corollary 3.3. *For* $0 < r < 1$*, if* $g \in \mathcal{G}_1$ *then we have for* $|z| = r < 1$

$$\sqrt{\frac{-f_\alpha(-r)}{r}}(1-r) \leq |g(z)| \leq \sqrt{\frac{f_\alpha(-r)}{r}}(1+r). \tag{3.34}$$

Corollary 3.4. *For* $0 < r < 1$*, if* $g \in \mathcal{G}_2$ *then we have for* $|z| = r < 1$*,*

$$\sqrt{\frac{-f_\alpha(-r)}{r}} \exp \frac{2}{1-r} \leq |g(z)| \leq \sqrt{\frac{f_\alpha(-r)}{r}} \exp \frac{2}{1+r}. \tag{3.35}$$

Theorem 3.5. *Let* $g \in \mathcal{G}_c$*. Then,*

$$|d_1| < 1,\ |d_2| < 1,\ |d_3| < 1,\ |d_4| < 1 \tag{3.36}$$

and

$$|d_2^2 - d_3| \leq 1. \tag{3.37}$$

Proof. Since $g \in \mathcal{G}_c$, there is a Schwarz function ω satisfying that

$$2z\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \cdots . \tag{3.38}$$

If $\omega(z) = z$, we have

$$2z\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c = 1 + 2z + 2z^2 + 2z^3 + \cdots . \tag{3.39}$$

Equating the coefficients using (3.22),

$$d_1 = 1 - c,\ d_2 = 1 - c,\ d_3 = \frac{6 - 11c + 6c^2 - 2c^3}{9},\ d_4 = 1 - \frac{32}{36}c - \frac{1}{9}c^2 + \frac{5}{9}c^3 - \frac{23}{36}c^4.$$

Since $0 < c \leq 2$, (3.36) holds.

Also, since $0 < c \leq 2$,

$$\begin{aligned} \left|d_2^2 - d_3\right| &= \left|(1-c)^2 - \left(1 - \frac{8}{9}c + \frac{5}{9}c^3\right)\right| \\ &= \left|c^2 - \frac{10}{9}c - \frac{5}{9}c^3\right| \end{aligned}$$

implies that (3.37) holds.

4. Differential subordination involving the class $\mathcal{G}_c$

In this section we are connecting the class $\mathcal{G}_c$, $0 < c \leq 2$ by taking the equivalent condition of $\mathcal{G}_c$, $0 < c \leq 2$ associated with $L_0(z) = \dfrac{1+z}{1-z}$. Note that the definition $\mathcal{G}_c, 0 < c \leq 2$ can be rewritten in the equivalent form as $g \in \mathcal{G}_c$ if

$$2z\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c \prec L_0(z),\ z \in \mathbb{D}. \tag{4.1}$$

To prove the differential subordination results, we need the following lemma which is stated below.

Lemma 4.1. *[13] Let τ be univalent in $\mathbb{D}$, ψ and ϕ be analytic in a domain D containing $\tau(\mathbb{D})$ with $\phi(\omega) \neq 0$ when $\omega \in \tau(\mathbb{D})$. Let $T(z) = z\tau'\phi(\tau(z))$ and $\kappa(z) = \psi(\tau(z)) + T(z)$ for $z \in \mathbb{D}$ and satisfy either T is starlike univalent in $\mathbb{D}$ or κ is convex univalent in $\mathbb{D}$. Also, assume that $\Re\left\{\frac{z\kappa'(z)}{T(z)}\right\} > 0, z \in \mathbb{D}$. If $p \in \mathcal{H}$ with $p(0) = \tau(0), p(\mathbb{D}) \subset D$, and*

$$\psi(p(z)) + zp'(z)\phi(p(z)) \prec \psi(\tau(z)) + z\tau'(z)\phi(\tau(z)),\ z \in \mathbb{D}$$

then $p \prec \tau$ and τ is the best dominant.

Theorem 4.1. *Let g be an analytic function with $g(0) = 1$ and let $0 < c \leq 2$. If g satisfies*

$$2z\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c \prec 1 + \frac{2z}{1-z^2},\ z \in \mathbb{D} \tag{4.2}$$

then

$$p(z) := (g(z))^2 \exp\{-2z\beta_c'(z)\} \prec L_0(z),\ z \in \mathbb{D}. \tag{4.3}$$

Proof. Let $\psi(\omega) = 1, \omega \in \mathbb{C}$ and $\phi(\omega) := \dfrac{1}{\omega}, \omega \in \mathbb{C} \setminus \{0\}$. Note that $L_0(\mathbb{D}) := \mathbb{C} \setminus \{0\}$. Thus,

$$T(z) = zL_0'(z)\phi(L_0(z)) = \frac{zL_0'(z)}{L_0(z)} = \frac{2z}{1-z^2} \tag{4.4}$$

is analytic and also well defined. Also, we have

$$\Re\left\{z\frac{T'(z)}{T(z)}\right\} = \Re\left\{\frac{1+z^2}{1-z^2}\right\} > 0,\ z \in \mathbb{D}. \tag{4.5}$$

This implies that T is a starlike univalent function. From this, for a function $\kappa(z) := \psi(L_0(z)) + T(z) = 1 + T(z), z \in \mathbb{D}$, we get

$$\Re\left\{z\frac{\kappa'(z)}{T(z)}\right\} = \Re\left\{z\frac{T'(z)}{T(z)}\right\} > 0,\ z \in \mathbb{D}.$$

From Lemma 4.1,

$$1 + z\frac{p'(z)}{p(z)} \prec 1 + z\frac{L_0'(z)}{L_0(z)} = 1 + \frac{2z}{1-z^2},\ z \in \mathbb{D},$$

which implies that $p \prec L_0$. Let us take the analytic function g with $g(0) = 1$ and $g(z) \neq 0$ for $z \in \mathbb{D}$ satisfying (4.2). For a function as in (4.3), we can notice that $p(0) = L_0(0) = 1, p(z) \neq 0$ for $z \in \mathbb{D}$ and p is analytic. The proof of Theorem 4.1 is completed by observing that

$$1 + z\frac{p'(z)}{p(z)} = 2z\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c,\ z \in \mathbb{D}.$$

Theorem 4.2. *Let* $g(z)$ *be an analytic function with* $g(0) = 1$ *and let* $0 < c \leq 2$. *If*

$$2z\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c \prec \frac{1+z}{1-z} + \frac{2z}{1-z^2},\ z \in \mathbb{D} \tag{4.6}$$

then

$$p(z) := z\,(g(z))^2 \exp\{-2z\beta_c'(z)\}\left(\int_0^z (g(\zeta))^2 \exp\{-2\zeta\beta_c'(\zeta)\}\,d\zeta\right)^{-1} \prec L_0(z),\ z \in \mathbb{D}. \tag{4.7}$$

Proof. Let $\psi(\omega) = \omega, \omega \in \mathbb{C}$ and $\phi(\omega) := \dfrac{1}{\omega}, \omega \in \mathbb{C} \setminus \{0\}$. Note that $L_0(\mathbb{D}) := \mathbb{C} \setminus \{0\}$ and ψ and ϕ are analytic in D. Thus, T defined by (4.4) is analytic and univalent and satisfies (4.5). Hence, $\kappa(z) = \psi(L_0(z)) + T(z) = L_0(z) + T(z),\ z \in \mathbb{D}$. By using (4.5) we get,

$$\Re\left\{z\frac{\kappa'(z)}{T(z)}\right\} = \Re\left\{z\frac{L_0'(z)}{T(z)}\right\} + \Re\left\{z\frac{T'(z)}{T(z)}\right\} = \Re\{L_0(z)\} + \Re\left\{z\frac{T'(z)}{T(z)}\right\} > 0, z \in \mathbb{D}.$$

Note that, p is also analytic with $p(0) = L_0(0) = 1$ and $p(z) \neq 0$ for $z \in \mathbb{D}$. From Lemma 4.1, we have

$$p(z) + z\frac{p'(z)}{p(z)} \prec L_0(z) + z\frac{L_0'(z)}{L_0(z)} = \frac{1+z}{1-z} + \frac{2z}{1-z^2},\ z \in \mathbb{D}.$$

This essentially gives us that $p \prec L_0$. Let us take the analytic function g with $g(0) = 1$ and $g(z) \neq 0$ for $z \in \mathbb{D}$ satisfying (4.6). For a function defined as in (4.7), we can observe that

$$\begin{aligned} p(0) &= \lim_{z\to 0} z\,(g(z))^2 \exp\{-2z\beta_c'(z)\}\left(\int_0^z (g(\zeta))^2 \exp\{-2\zeta\beta_c'(\zeta)\}\,d\zeta\right)^{-1} \\ &= (g(0))^2 \lim_{z\to 0}\left(\int_0^z (g(\zeta))^2 \exp\{-2\zeta\beta_c'(\zeta)\}\,d\zeta\right)^{-1} = 1 = L_0(0). \end{aligned}$$

Therefore, $p(z) \neq 0$ and analytic for all $z \in \mathbb{D}$. The proof of the theorem is then completed by noting that

$$p(z) + z\frac{p'(z)}{p(z)} = 2z\frac{g'(z)}{g(z)} + \left(\frac{1+z}{1-z}\right)^c,\ z \in \mathbb{D}.$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. D. Aharonov, M. Elin, D. Shoikhet, Spiral-like functions with respect to a boundary point, *J. Math. Anal. Appl.*, **280** (2003), 17–29. https://doi.org/10.1016/S0022-247X(02)00615-7
2. R. M. Ali, V. Ravichandran, N. Seenivasagan, Coefficient bounds for p-valent functions, *Appl. Math. Comput.*, **187** (2007), 35–46. https://doi.org/10.1016/j.amc.2006.08.100
3. Z. J. Jakubowski, On properties of the Pick function and some applications of them, *Acta Universitatis Purkynianae*, **42** (1999), 51–62.
4. Z. J. Jakubowski, A. Włodarczyk, On some classes of functions of Robertson type, *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, **59** (2005), 27–42.
5. F. R. Keogh, E. P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.*, **20** (1969), 8–12. https://doi.org/10.1090/S0002-9939-1969-0232926-9
6. A. Lecko, On the class of functions starlike with respect to a boundary point, *J. Math. Anal. Appl.*, **261** (2001), 649–664. https://doi.org/10.1006/jmaa.2001.7564
7. A. Lecko, A. Lyzzaik, A note on univalent functions starlike with respect to a boundary point, *J. Math. Anal. Appl.*, **282** (2003), 846–851. https://doi.org/10.1016/S0022-247X(03)00258-0
8. A. Lecko, On coefficient inequalities in the Carathéodory class of functions, *Ann. Polon. Math.*, **75** (2000), 59–67. https://doi.org/10.4064/ap-75-1-59-67
9. A. Lecko, *Some methods in the theory of univalent functions*, Oficyna Wdawnicza Politechniki Rzeszowskiej, 2005.
10. A. Lecko, δ-spirallike functions with respect to a boundary point, *Rocky Mountain J. Math.*, **38** (2008), 979–992. https://doi.org/10.1216/RMJ-2008-38-3-979
11. A. Lyzzaik, On a conjecture of M. S. Robertson, *Proc. Amer. Math. Soc.*, **91** (1984), 108– 110. http://doi.org/10.1090/s0002-9939-1984-0735575-7
12. W. C. Ma, D. Minda, A unified treatment of some special classes of univalent functions, In: *Proceedings of the conference on complex analysis*, International Press Inc., 1992.
13. S. S. Miller, P. T. Mocanu, *Differential subordinations: Theory and applications*, CRC Press, 2000. https://doi.org/10.1201/9781482289817
14. M. H. Mohd, M. Darus, Starlike function with respect to a boundary point defined by subordination, *Adv. Math. Sci. J.*, **1** (2012), 15–21.
15. D. Prokhorov, J. Szynal, Inverse coefficients for (α,β)-convex functions, *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, **35** (1981), 125–143.
16. M. S. Robertson, Univalent functions starlike with respect to a boundary point, *J. Math. Anal. Appl.*, **81** (1981), , 327–345. https://doi.org/10.1016/0022-247x(81)90067-6
17. H. Silverman, E. M. Silvia, Subclasses of univalent functions starlike with respect to a boundary point, *Houston J. Math.*, **16** (1990), 289–299.

18. S. Sivasubramanian, On a closely related class involving spirallike functions with respect to a boundary point, *Mediterr. J. Math.*, **17** (2020), 92. https://doi.org/10.1007/s00009-020-01529-z

19. D. Styer, On weakly starlike multivalent functions, *J. Analyse Math.*, **26** (1973), 217–233. https://doi.org/10.1007/bf02790430

20. P. G. Todorov, On the univalent functions starlike with respect to a boundary point, *Proc. Amer. Math. Soc.*, **97** (1986), 602–604. https://doi.org/10.1090/s0002-9939-1986-0845972-9