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Research article

Composition operators from harmonic $\mathcal{H}^\infty$ space into harmonic Zygmund space

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Abstract: This research paper sought to characterize the boundedness and compactness of composition operators from the space $\mathcal{H}^\infty$ of bounded harmonic mappings into harmonic Zygmund space $\mathcal{Z}_H$, on the open unit disk. Furthermore, we obtain an estimate of the essential norms of such an operator. These results extends the similar results that were proven for composition operators on analytic function spaces.

Keywords: harmonic Zygmund space; harmonic $\mathcal{H}^\infty$ space; composition operators; essential norm
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1. Introduction

Let $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the open unit disk in the complex plane. A *harmonic mapping* with domain $\mathbb{D}$ is a complex-valued function u such that

$$\Delta u := 4\frac{\partial^2 u}{\partial\zeta\partial\overline{\zeta}} \equiv 0.$$

In this paper, we denote $H(\mathbb{D})$ as the space consisting of analytic functions on the unit disk, $\mathcal{H}ar(\mathbb{D})$ as the space consisting of harmonic mappings.

The harmonic mapping u always a representation of the form $h + \overline{v}$, where h and v are analytic functions. Up to an additive constant, this representation is unique. Therefore, $u \in \mathcal{H}ar(\mathbb{D})$ if and only if $u = h + \overline{v}$, where $h, v \in H(\mathbb{D})$ and $v(0) = 0$.

For a general reference on the theory of harmonic functions, see [8]. Harmonic mappings appear regularly and play a fundamental role in math, physics and engineering; see e.g., [5], [6], [7], [15], and [22].

The composition operator C_φ induced by analytic or conjugate analytic self-maps of $\mathbb{D}$ is defined as the operator

$$C_\varphi u = u \circ \varphi, \quad \forall\, u \in \mathcal{H}ar(\mathbb{D}).$$

Obviously, such an operator preserves harmonicity.

Recall that, for any two normed linear spaces X and Y, the linear operator $T : X \longrightarrow Y$ is said to be bounded if there exists $C > 0$ such that $\|Tu\|_Y \leq C\|u\|_X, \forall u \in X$. Furthermore, a linear operator $T : X \longrightarrow Y$ is said to be compact if it maps every bounded set in X to a relatively compact set in Y (i.e., a set whose closure is compact).

The operator theory has been characterized for spaces of analytic functions in different settings and a significant number of related papers have appeared in the literature (see, for example, [9], [11], [14], and [18]). However, a similar investigation of the harmonic setting remains limited.

In [1], we have examined numerous characterizations of the weighted Bloch spaces and closed separable subspaces of harmonic mappings. We then presented the relationships between the weighted harmonic Bloch space and its Carleson measure. In [2], Aljuaid and Colonna studied the weighted Bloch space as the Banach space for harmonic mappings on an open unit disk. They showed that the mappings in weighted Bloch space can be characterized in terms of a Lipschitz condition relative to the metric and can also be thought of as the harmonic growth space. Besides, in [4] they studied the harmonic Zygmund spaces and their closed separable subspace called the little harmonic Zygmund space. In [13], Colonna introduced and studied Bloch harmonic mappings on $\mathbb{D}$ as Lipschitz maps from the hyperbolic disk into $\mathbb{C}$. In [20], Lusky investigated weighted spaces of harmonic functions on $\mathbb{D}$ and, in [21], isomorphism classes of weighted spaces of holomorphic and harmonic functions with a radial weight on $\mathbb{C}$ and on $\mathbb{D}$. In [23], Yoneda studied harmonic Bloch spaces and harmonic Besov spaces. Characterizations of the isometries between weighted spaces of harmonic functions were provided by Boyd and Rueda in [10]. In [17], Jordá and Zarco studied Banach spaces of harmonic functions and composition operators between weighted Banach spaces of pluriharmonic functions. Isomorphisms on weighted Banach spaces of harmonic and holomorphic functions were treated in [16].

Lately, studies on operator theory acting on spaces of harmonic mappings on the unit disk have been conducted. In [3], the composition operators were studied on the Banach spaces of harmonic mappings on $\mathbb{D}$, including the weighted Bloch spaces, the growth spaces, the Zygmund space, the analytic Besov spaces, and the space BMOA. Shao et al. in [12] studied composition operators in the space of bounded harmonic functions $\mathbb{D}$ and then provided criteria for determining the essential norm of the difference between two composition operators. In [19], Laitila and Tylli characterized the weak compactness of the composition operators on vector-valued harmonic Hardy spaces and on the spaces of vector-valued Cauchy transforms for reflexive Banach spaces.

Unlike what happens in the class of analytical functions which is closed under the customary composition, the usual composition product of two harmonic functions is not in general a harmonic function. This fact causes some problems which are studied for a long time in the space of analytical functions that do not make sense or are difficult to translate and treat on the set of complex harmonic functions with the tools of the complex variable. We give two typical examples: the theory of linear composition operators whose symbols are complex harmonic functions and the corresponding theory of iterations for complex harmonic functions.

In this work, we are concerned with the operator-theoretic properties of composition operators between distinct spaces of harmonic mappings in order to overcome these difficulties. Specifically, we investigate the composition operators from the space of bounded harmonic mappings $\mathcal{H}^\infty$ into the harmonic Zygmund space $\mathcal{Z}_H$.

The reason behind our study of the properties of composition operators between distinct spaces of harmonic mappings is the wide range of applications for different harmonic mappings, especially in operator theory.

We start with several preliminaries used to derive the main results of this work. Then, we focus on the boundedness and compactness of the composition operators from $\mathcal{H}^\infty$ space into the harmonic Zygmund space $\mathcal{Z}_H$. We conclude by approximating the essential norm.

The space of bounded harmonic mappings $\mathcal{H}^\infty$. First, we denote $\mathcal{H}^\infty = \mathcal{H}^\infty(\mathbb{D})$ as space consisting of all bounded harmonic mappings u on $\mathbb{D}$ equipped with the norm

$$\|u\|_\infty = \sup_{\zeta\in\mathbb{D}} |u(\zeta)|.$$

The harmonic Bloch space containing of all $u \in \mathcal{H}ar(\mathbb{D})$ is defined such that

$$\beta_u := \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\left(\left|\frac{\partial u(\zeta)}{\partial \zeta}\right|+\left|\frac{\partial u(\zeta)}{\partial \bar{\zeta}}\right|\right)< \infty. \tag{1.1}$$

If u is a harmonic Bloch mapping represented as $u = h + \bar{v}$, with $h, v \in H(\mathbb{D})$, the Bloch seminorm β_u can be characterized as

$$\beta_u = \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)(|h'(\zeta)| + |v'(\zeta)|) < \infty. \tag{1.2}$$

The quantity

$$\|u\|_{\mathcal{B}_H} := |u(0)| + \beta_u,$$

yields a Banach space structure on $\mathcal{B}_H$, see [2].

The harmonic Zygmund space containing of all $u \in \mathcal{H}ar(\mathbb{D})$ such that $\frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial \bar{\zeta}} \in \mathcal{B}_H$. Define

$$\|u\|_{\mathcal{Z}_H} := |u(0)| + \left|\frac{\partial u}{\partial \zeta}(0)\right|+\left|\frac{\partial u}{\partial \bar{\zeta}}(0)\right|+ \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\left(\left|\frac{\partial^2 u}{\partial \zeta^2}(\zeta)\right|+\left|\frac{\partial^2 u}{\partial \bar{\zeta}^2}(\zeta)\right|\right),$$

is a norm on $\mathcal{Z}_H$ and $\mathcal{Z}_H$ is a Banach space, see [4].

Remark 1.1. *When $u \in H(\mathbb{D})$, the mapping $\frac{\partial u}{\partial \zeta}$ reduces to u' and $\frac{\partial u}{\partial \bar{\zeta}} = \frac{\partial^2 u}{\partial \bar{\zeta}^2} = 0$. Thus, the collection of analytic functions in the space $\mathcal{Z}_H$ is the classical Zygmund space $\mathcal{Z}$ and both norms are identical.*

Throughout this paper, we use the notation $A \lesssim B$, which implies that there is a constant $C > 0$ such that $A \leq CB$. Therefore, when $B \lesssim A \lesssim B$, we use the notation $A \approx B$, meaning that A and B are equivalent. Moreover, if $A \approx B$ then $B < \infty \Longleftrightarrow A < \infty$.

2. Boundedness

Given $n \in \mathbb{N}$, and $u \in \mathcal{H}ar(\mathbb{D})$ be represented as $u = h + \bar{v}$, with $h, v \in H(\mathbb{D})$. Let us define

$$\beta^n_H(u) = \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)^n(|h^{(n)}(\zeta)| + |v^{(n)}(\zeta)|), \tag{2.1}$$

and

$$\beta_{H,0}^{n}(u) = \lim_{|\zeta|\to 1}(1 - |\zeta|^2)^n(|h^{(n)}(\zeta)| + |v^{(n)}(\zeta)|). \tag{2.2}$$

The following lemma as a result of Theorem 19 provided in [2] will help characterize the boundedness of the composition operators.

Lemma 2.1. *For $u \in \mathcal{H}ar(\mathbb{D})$ represented as $u = h + \bar{v}$, with $h, v \in H(\mathbb{D})$.*

(1) If $u \in \mathcal{H}^\infty$ then for any $n \in \mathbb{N}$, $\beta_H^n(u) \leq \|u\|_\infty$.
(2) $u \in \mathcal{B}_H \Longleftrightarrow \beta_H^n(u) < \infty$.
(3) $u \in \mathcal{B}_{H,0} \Longleftrightarrow \beta_{H,0}^n(u) = 0$.

Let $b \in \mathbb{D}$ be a fixed and let $1 \leq k \leq 3$. Then, for any $\zeta \in \mathbb{D}$, we consider a set of functions $h_{b,k}$ as follows:

$$h_{b,k}(\zeta) = \left(\frac{1 - |b|^2}{1 - \overline{b}\zeta}\right)^k + \left(\frac{1 - |b|^2}{1 - b\overline{\zeta}}\right)^k. \tag{2.3}$$

For every $k \in \mathbb{N}$, it can be demonstrated that $h_{b,k} \in \mathcal{H}^\infty$ and $\sup_{b\in\mathbb{D}} \|h_{b,k}\|_{\mathcal{H}^\infty} \leq 1$. Moreover, it is evident that $\lim_{|b|\to 1} h_{b,k} = 0$ uniformly on compact subsets $\overline{\mathbb{D}} \subset \mathbb{D}$. Recall the power series representations of $h_{b,k}$ are given as

$$h_{b,k}(\zeta) = (1 - |b|^2)^k \sum_{j=k-1}^{\infty} \binom{j}{k-1} \left\{(\overline{b}\zeta)^{j-k+1} + (b\overline{\zeta})^{j-k+1}\right\}. \tag{2.4}$$

For all $n \in \mathbb{N}$ and $1 \leq k \leq 3$, by direct calculation we know that

$$\begin{aligned} \frac{\partial^n h_{b,k}}{\partial \zeta^n}(\zeta) &= \frac{(n + k - 1)!}{(k - 1)!}\left[\frac{\overline{b}^n(1 - |b|^2)^k}{(1 - \overline{b}\zeta)^{k+n}}\right]; \\ \frac{\partial^n h_{b,k}}{\partial \overline{\zeta}^n}(\zeta) &= \frac{(n + k - 1)!}{(k - 1)!}\left[\frac{b^n(1 - |b|^2)^k}{(1 - b\overline{\zeta})^{k+n}}\right]. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \frac{\partial^n h_{b,k}}{\partial \zeta^n}(b) &= \frac{(n + k - 1)!}{(k - 1)!} \frac{\overline{b}^n}{(1 - |b|^2)^n}; \\ \frac{\partial^n h_{b,k}}{\partial \overline{\zeta}^n}(b) &= \frac{(n + k - 1)!}{(k - 1)!} \frac{b^n}{(1 - |b|^2)^n}. \end{aligned} \tag{2.5}$$

Now, we are prepared to show and prove our fundamental theorem in this section.

Theorem 2.1. *Let $\varphi \in S(\mathbb{D})$. Then, $C_\varphi : \mathcal{H}^\infty \to \mathcal{Z}_H$ is bounded if and only if*

$$\sup_{j\geq 0} \|\varphi^j + \overline{\varphi}^j\|_{\mathcal{Z}_H} < \infty. \tag{2.6}$$

Proof. Let the sequence $p_j(w) = w^j + \overline{w}^j$ for $w \in \mathbb{D}$ and when $j \geq 0$ is an integer. Since the sequence $\{p_j\}$ is bounded in the harmonic $\mathcal{H}^\infty$ space with $\|p_j\|_\infty \leq 1$, if $C_\varphi : \mathcal{H}^\infty \to \mathcal{Z}_H$ is bounded then for each $j \geq 0$ we have

$$\|\varphi^j + \overline{\varphi}^j\|_{\mathcal{Z}_H} = \|C_\varphi p_j\|_{\mathcal{Z}_H} \leq \|C_\varphi\|_\infty.$$

Therefore,

$$\sup_{j\geq 0}\|\varphi^j+\overline{\varphi}^j\|_{\mathcal{Z}_H}<\infty.$$

Conversely, suppose that (2.6) holds and set

$$L=\sup_{j\geq 0}\|\varphi^j+\overline{\varphi}^j\|_{\mathcal{Z}_H}<\infty.$$

Since the sequence $p_j(w)=w^j+\overline{w}^j$, $C_\varphi p_0=(\varphi)^0+(\overline{\varphi})^0=2\in\mathcal{Z}_H$ and $\|2\|_{\mathcal{Z}_H}=\|C_\varphi p_0\|_{\mathcal{Z}_H}\leq L$.

Note that for any $\zeta\in\mathbb{D}$ and $u\in\mathcal{H}ar(\mathbb{D})$ represented as $u=h+\bar{v}$, with $h,v\in H(\mathbb{D})$, $|(C_\varphi u)(0)|=|u(\varphi(0))|\leq\|u\|_\infty$. Therefore, because $|\varphi(0)|<1$ we note that

$$\begin{aligned}\left|\frac{\partial(C_\varphi u)}{\partial\zeta}(0)\right|+\left|\frac{\partial(C_\varphi u)}{\partial\overline{\zeta}}(0)\right| &= \left|\frac{\partial u(\varphi(0))}{\partial\zeta}\varphi'(0)\right|+\left|\frac{\partial u(\varphi(0))}{\partial\overline{\zeta}}\overline{\varphi'(0)}\right|\\ &= |h'(\varphi(0))\varphi'(0)|+|v'(\varphi(0))\overline{\varphi'(0)}|\\ &\leq \frac{|\varphi'(0)|}{(1-|\varphi(0)|^2)}\|u\|_\infty<\infty.\end{aligned}$$

On the other hand, for any $\zeta\in\mathbb{D}$ and $u\in\mathcal{H}ar(\mathbb{D})$,

$$\begin{aligned}\left|\frac{\partial^2(C_\varphi u)}{\partial\zeta^2}(\zeta)\right| &= \left|\frac{\partial^2 u(\varphi(\zeta))}{\partial\zeta^2}[\varphi'(\zeta)]^2+\frac{\partial u(\varphi(\zeta))}{\partial\zeta}\varphi''(\zeta)\right|\\ &\leq |\varphi'(\zeta)|^2\left|\frac{\partial^2 u(\varphi(\zeta))}{\partial\zeta^2}\right|+|\varphi''(\zeta)|\left|\frac{\partial u(\varphi(\zeta))}{\partial\zeta}\right|;\\ \left|\frac{\partial^2(C_\varphi u)}{\partial\overline{\zeta}^2}(\zeta)\right| &= \left|\frac{\partial^2 u(\varphi(\zeta))}{\partial\overline{\zeta}^2}[\overline{\varphi'(\zeta)}]^2+\frac{\partial u(\varphi(\zeta))}{\partial\overline{\zeta}}\overline{\varphi''(\zeta)}\right|\\ &\leq |\varphi'(\zeta)|^2\left|\frac{\partial^2 u(\varphi(\zeta))}{\partial\overline{\zeta}^2}\right|+|\varphi''(\zeta)|\left|\frac{\partial u(\varphi(\zeta))}{\partial\overline{\zeta}}\right|.\end{aligned}$$

By adding the above expressions and multiplying by $(1-|\zeta|^2)$ we obtain

$$\begin{aligned}(1-|\zeta|^2)\left(\left|\frac{\partial^2(C_\varphi u)}{\partial\zeta^2}(\zeta)\right|+\left|\frac{\partial^2(C_\varphi u)}{\partial\overline{\zeta}^2}(\zeta)\right|\right) &\leq (1-|\zeta|^2)|\varphi'(\zeta)|^2\left(\left|\frac{\partial^2 u(\varphi(\zeta))}{\partial\zeta^2}\right|+\left|\frac{\partial^2 u(\varphi(\zeta))}{\partial\overline{\zeta}^2}\right|\right)\\ &+ (1-|\zeta|^2)|\varphi''(\zeta)|\left(\left|\frac{\partial u(\varphi(\zeta))}{\partial\zeta}\right|+\left|\frac{\partial u(\varphi(\zeta))}{\partial\overline{\zeta}}\right|\right).\end{aligned}$$

Since $u\in\mathcal{H}ar(\mathbb{D})$ can be represented as $u=h+\bar{v}$, with $h,v\in H(\mathbb{D})$, by Lemma 2.1, we obtain

$$\begin{aligned}(1-|\zeta|^2)\left(\left|\frac{\partial^2(C_\varphi u)}{\partial\zeta^2}(\zeta)\right|+\left|\frac{\partial^2(C_\varphi u)}{\partial\overline{\zeta}^2}(\zeta)\right|\right) &\leq (1-|\zeta|^2)|\varphi'(\zeta)|^2\Big(|h''(\varphi(\zeta))|+|v''(\varphi(\zeta))|\Big)\\ &+ (1-|\zeta|^2)|\varphi''(\zeta)|\Big(|h'(\varphi(\zeta))+|v'(\varphi(\zeta))|\Big)\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(1-|\zeta|^2)|\varphi'(\zeta)|^2}{(1-|\varphi(\zeta)|^2)^2}\beta_H^2(u)+\frac{(1-|\zeta|^2)|\varphi''(\zeta)|}{1-|\varphi(\zeta)|^2}\beta_H(u)\\
&\leq (L_1+L_2)\|u\|_\infty,
\end{aligned}$$

where $L_1 = \frac{(1-|\zeta|^2)|\varphi'(\zeta)|^2}{(1-|\varphi(\zeta)|^2)^2}$ and $L_2 = \frac{(1-|\zeta|^2)|\varphi''(\zeta)|}{1-|\varphi(\zeta)|^2}$. To prove the boundedness, it suffices to show that the quantity $L_1 + L_2$ is finite. Since $C_\varphi p_1 = \varphi + \overline{\varphi}$, for $\zeta \in \mathbb{D}$, we have

$$\frac{\partial^2(C_\varphi p_1)}{\partial\zeta^2}(\zeta) = \frac{\partial^2(C_\varphi p_1)}{\partial\overline{\zeta}^2}(\zeta) = \varphi''(\zeta) + \overline{\varphi''(\zeta)}.$$

Then,

$$\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi''(\zeta)| \leq \frac{1}{4}\|C_\varphi p_1\|_{\mathcal{Z}_H} \leq \frac{L}{4}. \tag{2.7}$$

Moreover, since $p_j(w) = w^j + \overline{w}^j$ with $j \geq 0$ is an integer, we have $C_\varphi p_2 = (\varphi)^2 + (\overline{\varphi})^2$,

$$\begin{aligned}
\frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\zeta^2} &= 2(\varphi'(\zeta))^2 + 2(\overline{\varphi'(\zeta)})^2 + 2\varphi(\zeta)\varphi''(\zeta) + 2\overline{\varphi(\zeta)\varphi''(\zeta)},\\
\frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\overline{\zeta}^2} &= 2(\overline{\varphi'(\zeta)})^2 + 2(\varphi'(\zeta))^2 + 2\varphi(\zeta)\varphi''(\zeta) + 2\overline{\varphi(\zeta)\varphi''(\zeta)}.
\end{aligned}$$

Since $|\varphi(\zeta)| \leq 1$ for $\zeta \in \mathbb{D}$, we have

$$|\varphi'(\zeta)|^2 \leq \frac{1}{8}\left\{\left|\frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\zeta^2}\right| + \left|\frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\overline{\zeta}^2}\right|\right\} + |\varphi''(\zeta)|.$$

Thus,

$$\begin{aligned}
\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi'(\zeta)|^2 &\leq \frac{1}{8}\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\left(\left|\frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\zeta^2}\right| + \left|\frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\overline{\zeta}^2}\right|\right)\\
&\quad + \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi''(\zeta)|\\
&\leq \frac{1}{8}\|C_\varphi p_2\|_{\mathcal{Z}_H} + \frac{1}{4}\|C_\varphi p_1\|_{\mathcal{Z}_H} \leq \frac{3L}{8}.
\end{aligned} \tag{2.8}$$

On the other hand, by the linearity of the test function (2.4) for $k = 1, 2, 3$ and $\zeta \in \mathbb{D}$, we have

$$\|C_\varphi h_{\varphi(\zeta),k}\|_{\mathcal{Z}_H} \leq (1-|\varphi(\zeta)|^2)^k \sum_{j=k-1}^{\infty}\binom{j}{k-1}|\varphi(\zeta)|^{j-k+1}\|C_\varphi p_{j-k+1}\|_{\mathcal{Z}_H} \leq 2^k L. \tag{2.9}$$

From (2.5), for $k = 1, 2, 3$ and $\zeta \in \mathbb{D}$, we obtain

$$\frac{\partial^2[C_\varphi h_{\varphi(\zeta),k}(\zeta)]}{\partial\zeta^2} = k(k+1)\left(\frac{\varphi(\zeta)+\overline{\varphi(\zeta)}}{1-|\varphi(\zeta)|^2}\right)^2[\varphi'(\zeta)]^2 + k\left(\frac{\varphi(\zeta)+\overline{\varphi(\zeta)}}{1-|\varphi(\zeta)|^2}\right)\varphi''(\zeta),$$

$$\frac{\partial^2[C_\varphi h_{\varphi(\zeta),k}(\zeta)]}{\partial\overline{\zeta}^2} = k(k+1)\Big(\frac{\varphi(\zeta)+\overline{\varphi(\zeta)}}{1-|\varphi(\zeta)|^2}\Big)^2[\overline{\varphi'(\zeta)}]^2 + k\Big(\frac{\varphi(\zeta)+\overline{\varphi(\zeta)}}{1-|\varphi(\zeta)|^2}\Big)\overline{\varphi''(\zeta)}.$$

Thus, for $k = 1, 2, 3$, we let

$$\begin{aligned} Q_{\varphi(\zeta),k} &= \frac{\partial^2[C_\varphi h_{\varphi(\zeta),k}(\zeta)]}{\partial\zeta^2} + \frac{\partial^2[C_\varphi h_{\varphi(\zeta),k}(\zeta)]}{\partial\overline{\zeta}^2} \\ &= k(k+1)\Big(\frac{\varphi(\zeta)+\overline{\varphi(\zeta)}}{1-|\varphi(\zeta)|^2}\Big)^2\Big([\varphi'(\zeta)]^2+[\overline{\varphi'(\zeta)}]^2\Big) \\ &\quad + k\Big(\frac{\varphi(\zeta)+\overline{\varphi(\zeta)}}{1-|\varphi(\zeta)|^2}\Big)\Big(\varphi''(\zeta)+\overline{\varphi''(\zeta)}\Big). \end{aligned} \tag{2.10}$$

Using (2.10) by subtracting, we get

$$Q_{\varphi(\zeta),1} - 2Q_{\varphi(\zeta),2} + Q_{\varphi(\zeta),3} = 2\Big(\frac{\varphi(\zeta)+\overline{\varphi(\zeta)}}{1-|\varphi(\zeta)|^2}\Big)^2\Big((\varphi'(\zeta))^2+(\overline{\varphi'(\zeta)})^2\Big). \tag{2.11}$$

On the other hand, using (2.9) and (2.11) we obtain

$$\begin{aligned} \frac{(1-|\zeta|^2)|\varphi(\zeta)|^2|\varphi'(\zeta)|^2}{\left(1-|\varphi(\zeta)|^2\right)^2} &\le \frac{1}{18}\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\Big(|Q_{\varphi(\zeta),1}|+2|Q_{\varphi(\zeta),2}|+|Q_{\varphi(\zeta),3}|\Big) \\ &\le \frac{1}{18}\Big(\|C_\varphi h_{\varphi(\zeta),1}\|_{\mathcal{Z}_H}+2\|C_\varphi h_{\varphi(\zeta),2}\|_{\mathcal{Z}_H}+\|C_\varphi h_{\varphi(\zeta),3}\|_{\mathcal{Z}_H}\Big) \\ &\le L. \end{aligned} \tag{2.12}$$

Now, we let $0 < s < 1$. Then, if $|\varphi(\zeta)| > s$ in (2.12) we have

$$\frac{(1-|\zeta|^2)|\varphi'(\zeta)|^2}{\left(1-|\varphi(\zeta)|^2\right)^2} \le \frac{L}{s^2}. \tag{2.13}$$

Conversely, if we let $|\varphi(\zeta)| \le s$ in (2.8), we have

$$\frac{(1-|\zeta|^2)|\varphi'(\zeta)|^2}{\left(1-|\varphi(\zeta)|^2\right)^2} \le \frac{3L}{8(1-s^2)}. \tag{2.14}$$

From (2.13) and (2.14) it follows that the quantity L_2 is finite.

For the second time, we go back to (2.9) by subtracting we get

$$2Q_{\varphi(\zeta),2} - Q_{\varphi(\zeta),3} = \Big(\frac{\varphi(\zeta)+\overline{\varphi(\zeta)}}{1-|\varphi(\zeta)|^2}\Big)\Big(\varphi''(\zeta)+\overline{\varphi''(\zeta)}\Big), \tag{2.15}$$

which implies that

$$\frac{(1-|\zeta|^2)|\varphi(\zeta)||\varphi''(\zeta)|}{1-|\varphi(\zeta)|^2} \le \frac{1}{4}\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\{2|Q_{\varphi(\zeta),2}|+|Q_{\varphi(\zeta),3}|\}$$

$$\leq \frac{1}{4}\Big(2\|C_\varphi h_{\varphi(\zeta),2}\|_{\mathcal{Z}_H} + \|C_\varphi h_{\varphi(\zeta),3}\|_{\mathcal{Z}_H}\Big) \leq 4L. \tag{2.16}$$

If we instead let $0 < s < 1$, then if $|\varphi(\zeta)| > s$ in (2.16), we deduce

$$\frac{(1-|\zeta|^2)s|\varphi''(\zeta)|}{1-|\varphi(\zeta)|^2} \leq \frac{(1-|\zeta|^2)|\varphi(\zeta)||\varphi''(\zeta)|}{1-|\varphi(\zeta)|^2} \leq 4L.$$

Thus,

$$\frac{(1-|\zeta|^2)|\varphi''(\zeta)|}{1-|\varphi(\zeta)|^2} \leq \frac{4L}{s}. \tag{2.17}$$

If we instead let $|\varphi(\zeta)| \leq s$ in (2.7), we have

$$\frac{(1-|\zeta|^2)|\varphi''(\zeta)|}{1-|\varphi(\zeta)|^2} \leq \frac{L}{4(1-|\varphi(\zeta)|^2)} \leq \frac{L}{4(1-s^2)}. \tag{2.18}$$

Therefore, the quantity L_1 is finite and so is the quantity $L_1 + L_2$. The proof of Theorem 2.1 is complete.

3. Compactness

In this section, we focus on discussing the composition operators' compactness. We make use of the following lemma:

Lemma 3.1. *The bounded operator $T : \mathcal{H}^\infty \to \mathcal{Z}_H$ is compact if and only if $\|Tu_m\|_{\mathcal{Z}_H} \to 0$ as $m \to \infty$, for any bounded sequence $\{u_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}^\infty$ converges to zero uniformly on compact subsets $\overline{\mathbb{D}} \subset \mathbb{D}$.*

Proof. We focus on demonstrating the sufficiency. Suppose that $T : \mathcal{H}^\infty \to \mathcal{Z}_H$ is not compact. Then, there is a bounded sequence u_m in $\mathcal{H}^\infty$ such that $\{Tu_m\}$ has no convergent subsequence. However, we know that every bounded sequence in $\mathcal{H}^\infty$ has a subsequence that converges uniformly on compact subsets $\overline{\mathbb{D}} \subset \mathbb{D}$. Therefore, u_m has a subsequence u such that $u_m(w) \to u(w)$ for $w \in \mathbb{D}$, and because

$$\sup_{w\in\mathbb{D}} |u_m(w)| \leq |u(w)| \leq C \quad \forall\ m = 1, 2, 3, \ldots.$$

Therefore, $u \in \mathcal{H}^\infty$. The sequence $(u_m - u)$ is bounded in $\mathcal{H}^\infty$ and converges to zero uniformly on compact subsets $\overline{\mathbb{D}} \subset \mathbb{D}$. If we assume $\|T(u_m - u)\|_{\mathcal{Z}_H} \to 0$ as $n \to \infty$, then the subsequence Tu_m of Tu converges in $\mathcal{Z}_H$, which is a contradiction.

The following result indicates that the compactness of the composition operators can be characterized in terms of the sequence $\|C_\varphi p_j\|_{\mathcal{Z}_H}$, where $p_j(w) = w^j + \overline{w}^j$.

Theorem 3.1. *Let $\varphi \in S(\mathbb{D})$ and assume that the operator $C_\varphi : \mathcal{H}^\infty \to \mathcal{Z}_H$ is bounded. Then, $C_\varphi : \mathcal{H}^\infty \to \mathcal{Z}_H$ is compact if and only if*

$$\lim_{j\to\infty} \|\varphi^j + \overline{\varphi}^j\|_{\mathcal{Z}_H} = 0. \tag{3.1}$$

Proof. As in the proof of Theorem 2.1, we let the sequence $p_j(w) = w^j + \overline{w}^j$, where $w \in \mathbb{D}$ and $j \geq 0$ is an integer. Since the sequence $\{p_j\}$ is bounded in the harmonic space $\mathcal{H}^\infty$ and converges to zero uniformly on compact subsets $\overline{\mathbb{D}} \subset \mathbb{D}$, if $C_\varphi : \mathcal{H}^\infty \to \mathcal{Z}_H$ is compact then it is a bounded operator and (3.1) holds.

Conversely, assume the operator $C_\varphi : \mathcal{H}^\infty \to \mathcal{Z}_H$ is bounded and $\lim_{j\to\infty} \|\varphi^j + \overline{\varphi}^j\|_{\mathcal{Z}_H} = 0$.

Now, we define a sequence $\{h_j\}$ in the harmonic space $\mathcal{H}^\infty$ with $L_\infty = \sup_{j\in\mathbb{N}} \|h_j\|_\infty < \infty$ and $h_j \to 0$ uniformly on compact subsets $\overline{\mathbb{D}} \subset \mathbb{D}$, as $j \to \infty$.

To prove the compactness of $C_\varphi : \mathcal{H}^\infty \to \mathcal{Z}_H$, it suffices to show that $\lim_{j\to\infty} \|h_j\|_{\mathcal{Z}_H} = 0$.

Next, we suppose $\|C_\varphi p_j\|_{\mathcal{Z}_H} \leq L$ (L is an upper bound for $\|C_\varphi p_j\|_{\mathcal{Z}_H}$). Then, for $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\|\varphi^j + \overline{\varphi}^j\|_{\mathcal{Z}_H} = \|C_\varphi p_j\|_{\mathcal{Z}_H} < \varepsilon, \ \forall \ j \geq N.$$

By using the test function (2.4), for $k = 1, 2, 3$ and $\zeta \in \mathbb{D}$, we have

$$\begin{aligned} \|C_\varphi h_{\varphi(\zeta),k}\|_{\mathcal{Z}_H} &\leq (1 - |\varphi(\zeta)|^2)^k \left\{ \left[\sum_{j=k-1}^{k+N-2} + \sum_{j=k+N-1}^{\infty} \right] \binom{j}{k-1} |\varphi(\zeta)|^{j-k+1} \|C_\varphi p_{j-k+1}\|_{\mathcal{Z}_H} \right\} \\ &< (1 - |\varphi(\zeta)|^2)^k \binom{k+N-1}{N-1} L + 2^k \varepsilon. \end{aligned}$$

On the other hand, for any $\zeta \in \mathbb{D}$ let $0 < s < 1$ be sufficiently close to 1 such that $|\varphi(\zeta)| > s$. Thus,

$$\|C_\varphi h_{\varphi(\zeta),k}\|_{\mathcal{Z}_H} < 2^{k+1}\varepsilon, \text{ for } k = 1, 2, 3.$$

Since ε is arbitrary, for $k = 1, 2, 3$, it follows that

$$\lim_{|\varphi(\zeta)|\to 1} \|C_\varphi h_{\varphi(\zeta),k}\|_{\mathcal{Z}_H} = 0. \tag{3.2}$$

Going back to the proof of Theorem 2.1, from (2.12) and (2.16), we know

$$\begin{aligned} \frac{(1 - |\zeta|^2)|\varphi'(\zeta)|^2}{(1 - |\varphi(\zeta)|^2)^2} &\leq \frac{\|C_\varphi h_{\varphi(\zeta),1}\|_{\mathcal{Z}_H} + 2\|C_\varphi h_{\varphi(\zeta),2}\|_{\mathcal{Z}_H} + \|C_\varphi h_{\varphi(\zeta),3}\|_{\mathcal{Z}_H}}{18|\varphi(\zeta)|^2}, \\ \frac{(1 - |\zeta|^2)|\varphi''(\zeta)|}{1 - |\varphi(\zeta)|^2} &\leq \frac{2\|C_\varphi h_{\varphi(\zeta),2}\|_{\mathcal{Z}_H} + \|C_\varphi h_{\varphi(\zeta),3}\|_{\mathcal{Z}_H}}{4|\varphi(\zeta)|}. \end{aligned} \tag{3.3}$$

Using (3.3), we have

$$\lim_{|\varphi(\zeta)|\to 1} \frac{(1 - |\zeta|^2)|\varphi'(\zeta)|^2}{(1 - |\varphi(\zeta)|^2)^2} = 0, \quad \lim_{|\varphi(\zeta)|\to 1} \frac{(1 - |\zeta|^2)|\varphi''(\zeta)|}{1 - |\varphi(\zeta)|^2} = 0.$$

Thus, for any $0 < s < 1$ sufficiently close to 1 if $|\varphi(\zeta)| > s$. Then,

$$\frac{(1 - |\zeta|^2)|\varphi'(\zeta)|^2}{(1 - |\varphi(\zeta)|^2)^2} < \varepsilon, \text{ and } \frac{(1 - |\zeta|^2)|\varphi''(\zeta)|}{1 - |\varphi(\zeta)|^2} < \varepsilon. \tag{3.4}$$

By Lemma 2.1, if $h_m \in \mathcal{H}^\infty$, then $\beta_H^n(h_m) \leq \|h_m\|_\infty$, for any $m \in \mathbb{N}$. Thus, using (3.4), for $|\varphi(w)| > s$ we have

$$
\begin{aligned}
& (1-|\zeta|^2)\left(\left|\frac{\partial^2(C_\varphi h_m)}{\partial \zeta^2}(\zeta)\right|+\left|\frac{\partial^2(C_\varphi h_m)}{\partial \bar{\zeta}^2}(\zeta)\right|\right) \\
\leq\ & (1-|\zeta|^2)|\varphi'(\zeta)|^2\left(\left|\frac{\partial^2 h_m(\varphi(\zeta))}{\partial \zeta^2}\right|+\left|\frac{\partial^2 h_m(\varphi(\zeta))}{\partial \bar{\zeta}^2}\right|\right) \\
+\ & (1-|\zeta|^2)|\varphi''(\zeta)|\left(\left|\frac{\partial h_m(\varphi(\zeta))}{\partial \zeta}\right|+\left|\frac{\partial h_m(\varphi(\zeta))}{\partial \bar{\zeta}}\right|\right) \\
\leq\ & \|h_m\|_\infty\left(\frac{(1-|\zeta|^2)|\varphi'(\zeta)|^2}{(1-|\varphi(\zeta)|^2)^2}+\frac{(1-|\zeta|^2)|\varphi''(\zeta)|}{(1-|\varphi(\zeta)|^2)}\right) \\
\leq\ & \varepsilon L_\infty.
\end{aligned} \tag{3.5}
$$

Once again going back to the proof of Theorem 2.1, from (2.7) and (2.8), we know

$$
\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi''(\zeta)| \leq \frac{L}{4} \text{ and } \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi'(\zeta)|^2 \leq \frac{3L}{8}. \tag{3.6}
$$

We know by Cauchy's estimates that, the sequences $\{\frac{\partial h_m}{\partial \zeta}\}$, $\{\frac{\partial h_m}{\partial \bar{\zeta}}\}$, $\{\frac{\partial^2 h_m}{\partial \zeta^2}\}$ and $\{\frac{\partial^2 h_m}{\partial \bar{\zeta}^2}\}$ are convergent to zero on $\overline{\mathbb{D}}$. Thus, using (3.6), for any $0 < s < 1$ if $|\varphi(\zeta)| \leq s$, we obtain

$$
\begin{aligned}
& (1-|\zeta|^2)\left(\left|\frac{\partial^2(C_\varphi h_m)}{\partial \zeta^2}(\zeta)\right|+\left|\frac{\partial^2(C_\varphi h_m)}{\partial \bar{\zeta}^2}(\zeta)\right|\right) \\
\leq\ & \frac{3L}{8}\left(\left|\frac{\partial^2 h_m(\varphi(\zeta))}{\partial \zeta^2}\right|+\left|\frac{\partial^2 h_m(\varphi(\zeta))}{\partial \bar{\zeta}^2}\right|\right)+\frac{L}{4}\left(\left|\frac{\partial h_m(\varphi(\zeta))}{\partial \zeta}\right|+\left|\frac{\partial h_m(\varphi(\zeta))}{\partial \bar{\zeta}}\right|\right),
\end{aligned} \tag{3.7}
$$

which implies that

$$
\begin{aligned}
& \lim_{m\to\infty}(1-|\zeta|^2)\left(\left|\frac{\partial^2(C_\varphi h_m)}{\partial \zeta^2}(\zeta)\right|+\left|\frac{\partial^2(C_\varphi h_m)}{\partial \bar{\zeta}^2}(\zeta)\right|\right) \\
\leq\ & \lim_{m\to\infty}\left|\frac{\partial^2 h_m(\varphi(\zeta))}{\partial \zeta^2}\right|+\lim_{m\to\infty}\left|\frac{\partial^2 h_m(\varphi(\zeta))}{\partial \bar{\zeta}^2}\right|+\lim_{m\to\infty}\left|\frac{\partial h_m(\varphi(\zeta))}{\partial \zeta}\right|+\lim_{m\to\infty}\left|\frac{\partial h_m(\varphi(\zeta))}{\partial \bar{\zeta}}\right|=0.
\end{aligned} \tag{3.8}
$$

Therefore, $\lim_{m\to\infty}|C_\varphi h_m(0)| = 0$ and $\lim_{m\to\infty}\left|\frac{\partial[C_\varphi h_m](0)}{\partial \bar{\zeta}}\right| = 0$. Thus, we obtain

$$
\lim_{m\to\infty}\|C_\varphi h_m\|_{\mathcal{Z}_H} = 0. \tag{3.9}
$$

By Lemma 3.1, we verify that $C_\varphi : \mathcal{H}^\infty \to \mathcal{Z}_H$ is compact. The proof of the main theorem of this section is complete.

Our next goal of this paper is to provide an approximation of the essential norm.

4. Essential norm

In this section, we characterize the essential norms of the composition operators from $\mathcal{H}^\infty$ to $\mathcal{Z}_H$. We know that the essential norm $\|T\|_e$ of an operator T is its distance from the compact operators in the operator norm. Consider X and Y to be Banach spaces and let $T : X \to Y$ be a bounded linear operator. Then, the essential norm of T between X and Y is given by

$$\|T\|_{e,X\to Y} = \inf\{\|T - \mathcal{T}\|_{X\to Y} | \mathcal{T} : X \to Y \text{ is compact}\}.$$

Let $b \in \mathbb{D}$ be fixed and let $1 \leq k \leq 3$ in (2.3). Then, for any $\zeta \in \mathbb{D}$ we obtain

$$h_{b,k}(\zeta) = \left(\frac{1 - |b|^2}{1 - \overline{b}\zeta}\right)^k + \left(\frac{1 - |b|^2}{1 - b\overline{\zeta}}\right)^k.$$

Now, we define $B_1 = \limsup\limits_{|\varphi(w)|\to 1} \frac{(1-|w|^2)|\varphi''(w)|}{(1-|\varphi(w)|^2)}$ and $B_2 = \limsup\limits_{|\varphi(w)|\to 1} \frac{(1-|w|^2)|\varphi'(w)|^2}{(1-|\varphi(w)|^2)^2}$.

Theorem 4.1. *Let $\varphi \in S(\mathbb{D})$ and consider $C_\varphi : \mathcal{H}^\infty \to \mathcal{Z}_H$ is bounded. Then,*

$$\begin{aligned} \|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} &\approx \max_{1\leq k\leq 3}\{\limsup_{|b|\to 1} \|C_\varphi h_{b,k}(\zeta)\|_{\mathcal{Z}_H}\} \\ &\approx \max\{B_1, B_2\}. \end{aligned}$$

Proof. First, we prove that

$$\max_{1\leq k\leq 3}\{\limsup_{|b|\to 1} \|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}\} \leq \|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H}.$$

Fix $b \in \mathbb{D}$ since for all $1 \leq k \leq 3, h_{b,k} \in \mathcal{H}^\infty$ and $h_{b,k}$ converges uniformly to 0 on compact subsets $\overline{\mathbb{D}} \subset \mathbb{D}$. Then, for a compact operator $\mathcal{T} : \mathcal{H}^\infty \to \mathcal{Z}_H$ we have

$$\lim_{|b|\to 1} \|\mathcal{T} h_{b,k}\|_{\mathcal{Z}_H} = 0, \ \forall\, k = 1, 2, 3.$$

Thus,

$$\begin{aligned} \|C_\varphi - \mathcal{T}\|_{\mathcal{H}^\infty\to\mathcal{Z}_H} &\geq \limsup_{|b|\to 1} \|(C_\varphi - \mathcal{T})h_{b,k}\|_{\mathcal{Z}_H} \\ &\geq \limsup_{|b|\to 1} \|C_\varphi h_{b,k}\|_{\mathcal{Z}_H} - \limsup_{|b|\to 1} \|\mathcal{T} h_{b,k}\|_{\mathcal{Z}_H}. \end{aligned}$$

Hence, we obtain

$$\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} = \inf_{\mathcal{T}} \|C_\varphi - \mathcal{T}\| \geq \max_{1\leq k\leq 3}\{\limsup_{|b|\to 1} \|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}\}.$$

Next, to prove that $\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} \geq \max\{B_1, B_2\}$ we define the sequence $\{w_i\}$ such that $\lim\limits_{i\to\infty} |\varphi(w_i)| = 1$ for $w_i \in \mathbb{D}$ and $i \geq 0$ is an integer.

Moreover, we define

$$\begin{aligned} G_{i,1}(\zeta) &= h_{\varphi(w_i),1}(\zeta) - \frac{5}{3}h_{\varphi(w_i),2}(\zeta) + \frac{2}{3}h_{\varphi(w_i),3}(\zeta), \\ G_{i,2}(\zeta) &= h_{\varphi(w_i),1}(\zeta) - 2h_{\varphi(w_i),2}(\zeta) + h_{\varphi(w_i),3}(\zeta). \end{aligned}$$

For all $\zeta \in \mathbb{D}$, it can be proven that $G_{i,1}, G_{i,2} \in \mathcal{H}^\infty$ and $\lim_{|\varphi(w_i)|\to 1} G_{i,1} = \lim_{|\varphi(w_i)|\to 1} G_{i,2} = 0$ uniformly on compact subsets $\overline{\mathbb{D}} \subset \mathbb{D}$. By direct calculation, we see that $G_{i,1}(\varphi(w_i)) = G_{i,2}(\varphi(w_i)) = 0$.

By (2.5) we know that

$$\begin{aligned} \frac{\partial h_{\varphi(w_i),1}}{\partial \zeta}(\varphi(w_i)) &= \frac{\overline{\varphi(w_i)}}{(1-|\varphi(w_i)|^2)}; \quad \frac{\partial h_{\varphi(w_i),1}}{\partial \overline{\zeta}}(\varphi(w_i)) = \frac{\varphi(w_i)}{(1-|\varphi(w_i)|^2)}, \\ \frac{\partial h_{\varphi(w_i),2}}{\partial \zeta}(\varphi(w_i)) &= \frac{2\overline{\varphi(w_i)}}{(1-|\varphi(w_i)|^2)}; \quad \frac{\partial h_{\varphi(w_i),2}}{\partial \overline{\zeta}}(\varphi(w_i)) = \frac{2\varphi(w_i)}{(1-|\varphi(w_i)|^2)}, \\ \frac{\partial h_{\varphi(w_i),3}}{\partial \zeta}(\varphi(w_i)) &= \frac{3\overline{\varphi(w_i)}}{(1-|\varphi(w_i)|^2)}; \quad \frac{\partial h_{\varphi(w_i),3}}{\partial \overline{\zeta}}(\varphi(w_i)) = \frac{3\varphi(w_i)}{(1-|\varphi(w_i)|^2)}. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{\partial^2 h_{\varphi(w_i),1}}{\partial \zeta^2}(\varphi(w_i)) &= \frac{2(\overline{\varphi(w_i)})^2}{(1-|\varphi(w_i)|^2)^2}; \quad \frac{\partial^2 h_{\varphi(w_i),1}}{\partial \overline{\zeta}^2}(\varphi(w_i)) = \frac{2(\varphi(w_i))^2}{(1-|\varphi(w_i)|^2)^2}, \\ \frac{\partial^2 h_{\varphi(w_i),2}}{\partial \zeta^2}(\varphi(w_i)) &= \frac{6(\overline{\varphi(w_i)})^2}{(1-|\varphi(w_i)|^2)^2}; \quad \frac{\partial^2 h_{\varphi(w_i),2}}{\partial \overline{\zeta}^2}(\varphi(w_i)) = \frac{6(\varphi(w_i))^2}{(1-|\varphi(w_i)|^2)^2}, \\ \frac{\partial^2 h_{\varphi(w_i),3}}{\partial \zeta^2}(\varphi(w_i)) &= \frac{12(\overline{\varphi(w_i)})^2}{(1-|\varphi(w_i)|^2)^2}; \quad \frac{\partial^2 h_{\varphi(w_i),3}}{\partial \overline{\zeta}^2}(\varphi(w_i)) = \frac{12(\varphi(w_i))^2}{(1-|\varphi(w_i)|^2)^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \left|\frac{\partial(G_{i,1})}{\partial \zeta}(\varphi(w_i))\right| &= \left|\frac{\partial[h_{\varphi(w_i),1}(\zeta)]}{\partial \zeta} - \frac{5}{3}\frac{\partial[h_{\varphi(w_i),2}(\zeta)]}{\partial \zeta} + \frac{2}{3}\frac{\partial[h_{\varphi(w_i),3}(\zeta)]}{\partial \zeta}\right| \\ &= \frac{1}{3}\frac{|\varphi(w_i)|}{(1-|\varphi(w_i)|^2)}, \\ \frac{\partial^2(G_{i,1})}{\partial \zeta^2}(\varphi(w_i)) &= \frac{\partial^2 h_{\varphi(w_i),1}}{\partial \zeta^2}(\varphi(w_i)) - \frac{5}{3}\frac{\partial^2 h_{\varphi(w_i),2}}{\partial \zeta^2}(\varphi(w_i)) + \frac{2}{3}\frac{\partial^2 h_{\varphi(w_i),3}}{\partial \zeta^2}(\varphi(w_i)) = 0. \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \frac{\partial(G_{i,2})}{\partial \zeta}(\varphi(w_i)) &= \frac{\partial^2 h_{\varphi(w_i),1}}{\partial \zeta^2}(\varphi(w_i)) - 2\frac{\partial^2 h_{\varphi(w_i),2}}{\partial \zeta^2}(\varphi(w_i)) + \frac{\partial^2 h_{\varphi(w_i),1}}{\partial \zeta^2}(\varphi(w_i)) = 0, \\ \left|\frac{\partial^2(G_{i,2})}{\partial \zeta^2}(\varphi(w_i))\right| &= \left|\frac{\partial^2 h_{\varphi(w_i),1}}{\partial \zeta^2}(\varphi(w_i)) - 2\frac{\partial^2 h_{\varphi(w_i),2}}{\partial \zeta^2}(\varphi(w_i)) + \frac{\partial^2 h_{\varphi(w_i),3}}{\partial \zeta^2}(\varphi(w_i))\right| \\ &= \frac{2|\varphi(w_i)|^2}{(1-|\varphi(w_i)|^2)^2}. \end{aligned}$$

Similarly,

$$\begin{aligned}\left|\frac{\partial(G_{i,1})}{\partial\bar{\zeta}}(\varphi(w_i))\right| &= \frac{1}{3}\frac{|\varphi(w_i)|}{(1-|\varphi(w_i)|^2)}, \quad \frac{\partial^2(G_{i,1})}{\partial\bar{\zeta}^2}(\varphi(w_i)) = 0,\\ \left|\frac{\partial^2(G_{i,2})}{\partial\bar{\zeta}^2}(\varphi(w_i))\right| &= \frac{2|\varphi(w_i)|^2}{(1-|\varphi(w_i)|^2)^2}, \quad \frac{\partial(G_{i,2})}{\partial\bar{\zeta}}(\varphi(w_i)) = 0.\end{aligned}$$

Since $\mathcal{T}:\mathcal{H}^\infty\to\mathcal{Z}_H$ is a compact operator, by Lemma 3.1 we have

$$\begin{aligned}\|C_\varphi-\mathcal{T}\|_{\mathcal{H}^\infty\to\mathcal{Z}_H} &\geq \limsup_{i\to\infty}\|C_\varphi G_{i,1}\|_{\mathcal{Z}_H}-\limsup_{i\to\infty}\|\mathcal{T}G_{i,1}\|_{\mathcal{Z}_H}\\ &= \limsup_{i\to\infty}(1-|w_i|^2)\left\{\left|\frac{\partial^2(C_\varphi G_{i,1}(\zeta))}{\partial\zeta^2}\right|+\left|\frac{\partial^2(C_\varphi G_{i,1}(\zeta))}{\partial\bar{\zeta}^2}\right|\right\}\\ &= \limsup_{i\to\infty}(1-|w_i|^2)|\varphi'(w_i)|^2\left\{\left|\frac{\partial^2(G_{i,1})}{\partial\zeta^2}(\varphi(w_i))\right|+\left|\frac{\partial^2(G_{i,1})}{\partial\bar{\zeta}^2}(\varphi(w_i))\right|\right\}\\ &\quad+\limsup_{i\to\infty}(1-|w_i|^2)|\varphi''(w_i)|\left\{\left|\frac{\partial(G_{i,1})}{\partial\zeta}(\varphi(w_i))\right|+\left|\frac{\partial(G_{i,1})}{\partial\bar{\zeta}}(\varphi(w_i))\right|\right\}\\ &\geq \limsup_{i\to\infty}(1-|w_i|^2)\frac{|\varphi(w_i)||\varphi''(w_i)|}{(1-|\varphi(w_i)|^2)}.\end{aligned}$$

Thus,

$$\begin{aligned}\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} &= \inf_{\mathcal{T}}\|C_\varphi-\mathcal{T}\|\\ &\geq \limsup_{i\to\infty}(1-|w_i|^2)\frac{|\varphi(w_i)||\varphi''(w_i)|}{(1-|\varphi(w_i)|^2)}\\ &= \limsup_{|\varphi(w)|\to 1}\frac{(1-|w|^2)|\varphi''(w)|}{(1-|\varphi(w)|^2)} = B_1.\end{aligned}$$

In the same way, we have

$$\begin{aligned}\|C_\varphi-\mathcal{T}\|_{\mathcal{H}^\infty\to\mathcal{Z}_H} &\geq \limsup_{i\to\infty}\|C_\varphi G_{i,2}\|_{\mathcal{Z}_H}-\limsup_{i\to\infty}\|\mathcal{T}G_{i,2}\|_{\mathcal{Z}_H}\\ &= \limsup_{i\to\infty}(1-|w_i|^2)\left\{\left|\frac{\partial^2(C_\varphi G_{i,2}(\zeta))}{\partial\zeta^2}\right|+\left|\frac{\partial^2(C_\varphi G_{i,2}(\zeta))}{\partial\bar{\zeta}^2}\right|\right\}\\ &= \limsup_{i\to\infty}(1-|w_i|^2)|\varphi'(w_i)|^2\left\{\left|\frac{\partial^2(G_{i,2})}{\partial\zeta^2}(\varphi(w_i))\right|+\left|\frac{\partial^2(G_{i,2})}{\partial\bar{\zeta}^2}(\varphi(w_i))\right|\right\}\\ &\quad+\limsup_{i\to\infty}(1-|w_i|^2)|\varphi''(w_i)|\left\{\left|\frac{\partial(G_{i,2})}{\partial\zeta}(\varphi(w_i))\right|+\left|\frac{\partial(G_{i,2})}{\partial\bar{\zeta}}(\varphi(w_i))\right|\right\}\\ &\geq \limsup_{i\to\infty}(1-|w_i|^2)\frac{|\varphi(w_i)|^2|\varphi'(w_i)|^2}{(1-|\varphi(w_i)|^2)^2}.\end{aligned}$$

Thus,

$$\begin{aligned}\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} &= \inf_{\mathcal{T}}\|C_\varphi-\mathcal{T}\| \\ &\geq \limsup_{i\to\infty}(1-|w_i|^2)\frac{|\varphi(w_i)|^2|\varphi'(w_i)|^2}{(1-|\varphi(w_i)|^2)^2} \\ &= \limsup_{|\varphi(w)|\to1}\frac{(1-|w|^2)|\varphi'(w)|^2}{(1-|\varphi(w)|^2)^2}=B_2.\end{aligned}$$

Hence, we obtain

$$\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H}=\inf_{\mathcal{T}}\|C_\varphi-\mathcal{T}\|\geq\max\{B_1,B_2\}.$$

Next, we prove that

$$\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H}\leq\max_{1\leq k\leq3}\{\limsup_{|b|\to1}\|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}\}.$$

For any $0\leq\delta<1$, let the operator $\mathcal{T}_\delta:\mathcal{H}ar(\mathbb{D})\to\mathcal{H}ar(\mathbb{D})$ such that

$$(\mathcal{T}_\delta u)(w)=u_\delta(w)=u(\delta w),\quad u\in\mathcal{H}ar(\mathbb{D}).$$

Without a doubt, $u_\delta\to u$ uniform on compact subsets of the unit disk as $\delta\to1$. Moreover, $\mathcal{T}_\delta$ is a compact operator on $\mathcal{H}^\infty$ and $\|\mathcal{T}_\delta\|_{\mathcal{H}^\infty\to\mathcal{H}^\infty}\leq1$. For $\{\delta_i\}\subset(0,1)$ a sequence such that $\delta_i\to1$ as $i\to\infty$. Thus, for all positive integers i, we obtain $C_\varphi\mathcal{T}_{\delta_i}:\mathcal{H}^\infty\to\mathcal{Z}_H$ is a compact operator.

However, the definition of the essential norm indicates that

$$\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H}\leq\limsup_{i\to\infty}\|C_\varphi-C_\varphi\mathcal{T}_{\delta_i}\|_{\mathcal{H}^\infty\to\mathcal{Z}_H}. \tag{4.1}$$

Thus, we only need to demonstrate that

$$\limsup_{i\to\infty}\|(C_\varphi-C_\varphi\mathcal{T}_{\delta_i}\|_{\mathcal{H}^\infty\to\mathcal{Z}_H}\leq\max_{1\leq k\leq3}\{\limsup_{|b|\to1}\|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}\}.$$

Let $u\in\mathcal{H}^\infty$ such that $\|u\|_\infty\leq1$. Then,

$$\begin{aligned}\|(C_\varphi-C_\varphi\mathcal{T}_{\delta_i})u\|_{\mathcal{Z}_H} &= |u(\varphi(0))-u(\delta_i\varphi(0))| \\ &\quad+|\varphi'(0)|\left\{\left|\frac{\partial(u-u_{\delta_i})}{\partial\zeta}(\varphi(0))\right|+\left|\frac{\partial(u-u_{\delta_i})}{\partial\bar{\zeta}}(\varphi(0))\right|\right\} \\ &\quad+\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\left\{\left|\frac{\partial^2[(u-u_{\delta_i})\circ\varphi(\zeta)]}{\partial\zeta^2}\right|+\left|\frac{\partial^2[(u-u_{\delta_i})\circ\varphi(\zeta)]}{\partial\bar{\zeta}^2}\right|\right\}.\end{aligned} \tag{4.2}$$

Thus, we have that

$$\begin{aligned}\lim_{i\to\infty}|u(\varphi(0))-u(\delta_i\varphi(0))| &= \lim_{i\to\infty}\left|\frac{\partial(u-u_{\delta_i})}{\partial\zeta}(\varphi(0))\right||\varphi'(0)| \\ &= \lim_{i\to\infty}\left|\frac{\partial(u-u_{\delta_i})}{\partial\bar{\zeta}}(\varphi(0))\right||\varphi'(0)|=0.\end{aligned} \tag{4.3}$$

Moreover, we consider

$$
\begin{aligned}
&\lim_{i\to\infty}\sup(1-|\zeta|^2)\left\{\left|\frac{\partial^2[(u-u_{\delta_i})\circ\varphi(\zeta)]}{\partial\zeta^2}\right|+\left|\frac{\partial^2[(u-u_{\delta_i})\circ\varphi(\zeta)]}{\partial\overline{\zeta}^2}\right|\right\}\\
\le\;&\lim_{i\to\infty}\sup\sup_{|\varphi(\zeta)|\le\delta_N}(1-|\zeta|^2)\left\{\left|\frac{\partial^2[(u-u_{\delta_i})\circ\varphi(\zeta)]}{\partial\zeta^2}\right|+\left|\frac{\partial^2[(u-u_{\delta_i})\circ\varphi(\zeta)]}{\partial\overline{\zeta}^2}\right|\right\}\\
+\;&\lim_{i\to\infty}\sup\sup_{|\varphi(\zeta)|>\delta_N}(1-|\zeta|^2)\left\{\left|\frac{\partial^2[(u-u_{\delta_i})\circ\varphi(\zeta)]}{\partial\zeta^2}\right|+\left|\frac{\partial^2[(u-u_{\delta_i})\circ\varphi(\zeta)]}{\partial\overline{\zeta}^2}\right|\right\}\\
=\;&I_{\varphi,i}+J_{\varphi,i}.
\end{aligned}\tag{4.4}
$$

Now, let $N\in\mathbb{N}$ be large enough and $\delta_i\ge\frac{1}{2}$, for all $i\ge N$. Then,

$$
\begin{aligned}
I_{\varphi,i}\le\;&\lim_{i\to\infty}\sup\sup_{|\varphi(\zeta)|\le\delta_N}(1-|\zeta|^2)|\varphi''(\zeta)|\left\{\left|\frac{\partial[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\zeta}\right|+\left|\frac{\partial[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\overline{\zeta}}\right|\right\}\\
+\;&\lim_{i\to\infty}\sup\sup_{|\varphi(\zeta)|\le\delta_N}(1-|\zeta|^2)|\varphi'(\zeta)|^2\left\{\left|\frac{\partial^2[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\zeta^2}\right|+\left|\frac{\partial^2[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\overline{\zeta}^2}\right|\right\}.
\end{aligned}
$$

Since $C_\varphi:\mathcal{H}^\infty\to\mathcal{Z}_H$ is bounded, from Theorem 2.1 we see that

$$
\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi''(\zeta)|<\infty,\quad \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi'(\zeta)|^2<\infty.
$$

Moreover, since the following limits are uniform on compact subsets $\overline{\mathbb{D}}\subset\mathbb{D}$,

$$
\begin{aligned}
&\lim_{i\to\infty}\delta_i\frac{\partial u_{\delta_i}}{\partial\zeta}=\frac{\partial u}{\partial\zeta}, &&\lim_{i\to\infty}\delta_i\frac{\partial u_{\delta_i}}{\partial\overline{\zeta}}=\frac{\partial u}{\partial\overline{\zeta}},\\
&\lim_{i\to\infty}(\delta_i)^2\frac{\partial^2 u_{\delta_i}}{\partial\zeta^2}=\frac{\partial^2 u}{\partial\zeta^2}, &&\lim_{i\to\infty}(\delta_i)^2\frac{\partial^2 u_{\delta_i}}{\partial\overline{\zeta}^2}=\frac{\partial^2 u}{\partial\overline{\zeta}^2}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\lim_{i\to\infty}\sup\sup_{|w|\le\delta_N}\left\{\left|\frac{\partial u(w)}{\partial\zeta}-\frac{\partial u_{\delta_i}(w)}{\partial\zeta}\right|+\left|\frac{\partial u(w)}{\partial\overline{\zeta}}-\frac{\partial u_{\delta_i}(w)}{\partial\overline{\zeta}}\right|\right\}=0,\\
&\lim_{i\to\infty}\sup\sup_{|w|\le\delta_N}\left\{\left|\frac{\partial^2 u(w)}{\partial\zeta^2}-\frac{\partial^2 u_{\delta_i}(w)}{\partial\zeta^2}\right|+\left|\frac{\partial^2 u(w)}{\partial\overline{\zeta}^2}-\frac{\partial^2 u_{\delta_i}(w)}{\partial\overline{\zeta}^2}\right|\right\}=0.
\end{aligned}
$$

Hence, by the above equations we have

$$
I_{\varphi,i}=0.\tag{4.5}
$$

Next, considering $|\varphi(\zeta)| > \delta_N$, we obtain

$$
\begin{aligned}
J_{\varphi,i} &\leq \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)|\varphi''(\zeta)|\left\{\left|\frac{\partial[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\zeta}\right|+\left|\frac{\partial[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\overline{\zeta}}\right|\right\} \\
&+ \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)|\varphi'(\zeta)|^2\left\{\left|\frac{\partial^2[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\zeta^2}\right|+\left|\frac{\partial^2[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\overline{\zeta}^2}\right|\right\} \\
&\leq \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)|\varphi''(\zeta)|\left\{\left|\frac{\partial u(\varphi(\zeta))}{\partial\zeta}\right|+\left|\frac{\partial u(\varphi(\zeta))}{\partial\overline{\zeta}}\right|\right\} \\
&+ \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)|\varphi''(\zeta)|\delta_i\left\{\left|\frac{\partial u(\delta_i\varphi(\zeta))}{\partial\zeta}\right|+\left|\frac{\partial u(\delta_i\varphi(\zeta))}{\partial\overline{\zeta}}\right|\right\} \\
&+ \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)|\varphi'(\zeta)|^2\left\{\left|\frac{\partial^2 u(\varphi(\zeta))}{\partial\zeta^2}\right|+\left|\frac{\partial^2 u(\varphi(\zeta))}{\partial\overline{\zeta}^2}\right|\right\} \\
&+ \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)|\varphi'(\zeta)|^2(\delta_i)^2\left\{\left|\frac{\partial^2 u(\delta_i\varphi(\zeta))}{\partial\zeta^2}\right|+\left|\frac{\partial^2 u(\delta_i\varphi(\zeta))]}{\partial\overline{\zeta}^2}\right|\right\} \\
&= \limsup_{i\to\infty} \sum_{j=1}^{4} R_j.
\end{aligned}
$$

Now we estimate the quantities R_j, where $j = 1, 2, 3$. We define

$$
\begin{aligned}
G_{b,1}(\zeta) &= h_{b,1}(\zeta) - \frac{5}{3}h_{b,2}(\zeta) + \frac{2}{3}h_{b,3}(\zeta), \\
G_{b,2}(\zeta) &= h_{b,1}(\zeta) - 2h_{b,2}(\zeta) + h_{b,3}(\zeta).
\end{aligned}
$$

By Lemma 2.1, since $\beta_H(u) \leq \|u\|_\infty$ for all $u \in \mathcal{H}^\infty$. Because $\|u\|_\infty \leq 1$, we have

$$
\begin{aligned}
R_1 &= \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)|\varphi''(\zeta)|\left\{\left|\frac{\partial u(\varphi(\zeta))}{\partial\zeta}\right|+\left|\frac{\partial u(\varphi(\zeta))}{\partial\overline{\zeta}}\right|\right\}, \\
&\leq \frac{1}{\delta_N}\|u\|_\infty \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)|\varphi''(\zeta)|\frac{|\varphi(\zeta)|}{3(1-|\varphi(\zeta)|^2)} \\
&\leq \sup_{|b|>\delta_N} \|C_\varphi G_{b,1}\|_{\mathcal{Z}_H} \\
&\leq \sup_{|b|>\delta_N} \|C_\varphi h_{b,1}\|_{\mathcal{Z}_H} + \frac{5}{3}\sup_{|b|>\delta_N} \|C_\varphi h_{b,2}\|_{\mathcal{Z}_H} + \frac{2}{3}\sup_{|b|>\delta_N} \|C_\varphi h_{b,3}\|_{\mathcal{Z}_H}.
\end{aligned}
\tag{4.6}
$$

Consequently,

$$
\limsup_{i\to\infty} R_1 \leq \sum_{k=1}^{3} \limsup_{|b|\to 1} \|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}. \tag{4.7}
$$

Similarly, we see that

$$
\limsup_{i\to\infty} R_2 \leq \sum_{k=1}^{3} \limsup_{|b|\to 1} \|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}. \tag{4.8}
$$

By direct calculation, $\beta_H^2(u) \leq \|u\|_\infty$, for all $u \in \mathcal{H}^\infty$. Because $\|u\|_\infty \leq 1$,

$$\begin{aligned}
R_3 &= \sup_{|\varphi(\zeta)|>\delta_N}(1-|\zeta|^2)|\varphi'(\zeta)|^2\Big\{\Big|\frac{\partial^2 u(\varphi(\zeta))}{\partial\zeta^2}\Big|+\Big|\frac{\partial^2 u(\varphi(\zeta))}{\partial\overline{\zeta}^2}\Big|\Big\},\\
&\leq \|u\|_\infty \sup_{|\varphi(\zeta)|>\delta_N}(1-|\zeta|^2)|\varphi'(\zeta)|^2\frac{2|\varphi(\zeta)|^2}{3(1-|\varphi(\zeta)|^2)^2}\\
&\leq \sup_{|b|>\delta_N}\|C_\varphi G_{b,2}\|_{\mathcal{Z}_H}\\
&\leq \sup_{|b|>\delta_N}\|C_\varphi h_{b,1}\|_{\mathcal{Z}_H}+2\sup_{|b|>\delta_N}\|C_\varphi h_{b,2}\|_{\mathcal{Z}_H}+\sup_{|b|>\delta_N}\|C_\varphi h_{b,3}\|_{\mathcal{Z}_H}. \qquad (4.9)
\end{aligned}$$

Thus, we obtain

$$\limsup_{i\to\infty} R_3 \leq \sum_{k=1}^{3}\limsup_{|b|\to 1}\|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}. \qquad (4.10)$$

Similarly, we see that

$$\limsup_{i\to\infty} R_4 \leq \sum_{k=1}^{3}\limsup_{|b|\to 1}\|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}. \qquad (4.11)$$

By the inequalities (4.7)-(4.11), we obtain

$$J_{\varphi,i} \leq \max_{1\leq k\leq 3}\{\limsup_{|b|\to 1}\|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}\}. \qquad (4.12)$$

Hence, by applying (4.5) and (4.12) we determine that

$$\limsup_{i\to\infty}\|(C_\varphi - C_\varphi\mathcal{T}_{\delta_i}\|_{\mathcal{H}^\infty\to\mathcal{Z}_H} \leq \max_{1\leq k\leq 3}\{\limsup_{|b|\to 1}\|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}\}.$$

Finally, we prove that

$$\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} \leq \max\{B_1, B_2\}.$$

According to the definition of the essential norm, we only need to prove that

$$\limsup_{i\to\infty}\|C_\varphi - C_\varphi\mathcal{T}_{\delta_i}\|_{\mathcal{H}^\infty\to\mathcal{Z}_H} \leq \max\{B_1, B_2\}.$$

From (4.6), we see that

$$\limsup_{i\to\infty} R_1 \leq \limsup_{|\varphi(\zeta)|\to 1}(1-|\zeta|^2)|\varphi''(\zeta)|\frac{|\varphi(\zeta)|}{(1-|\varphi(\zeta)|^2)} = B_2. \qquad (4.13)$$

Similarly,

$$\limsup_{i\to\infty} R_2 \leq B_2. \qquad (4.14)$$

Moreover, for (4.9), we see that

$$\limsup_{i\to\infty} R_3 \quad \le \quad \limsup_{|\varphi(\zeta)|\to 1}(1-|\zeta|^2)|\varphi'(\zeta)|^2 \frac{2|\varphi(\zeta)|^2}{3(1-|\varphi(\zeta)|^2)^2} = B_1. \tag{4.15}$$

Similarly,

$$\limsup_{i\to\infty} R_4 \quad \le \quad B_1. \tag{4.16}$$

Hence, by the inequalities (4.13)-(4.16) we obtain

$$\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} \le \max\{B_1, B_2\}.$$

The proof is complete.

Theorem 4.2. *Let $\varphi \in S(\mathbb{D})$ such that $C_\varphi : \mathcal{H}^\infty \to \mathcal{Z}_H$ is bounded. Then,*

$$\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} \approx \limsup_{j\to\infty} \|\varphi^j + \overline{\varphi}^j\|_{\mathcal{Z}_H}.$$

Proof. First, we prove that

$$\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} \ge \limsup_{j\to\infty} \|\varphi^j + \overline{\varphi}^j\|_{\mathcal{Z}_H}.$$

Set the sequence $p_j(w) = w^j + \overline{w}^j$, for $w \in \mathbb{D}$ and when $j \ge 0$ is an integer. Then, $\|p_j\|_\infty = 1$ and p_j converges uniformly to 0 on compact subsets $\overline{\mathbb{D}} \subset \mathbb{D}$. Therefore, by Lemma 3.1 we see that

$$\lim_{j\to\infty} \|\mathcal{T} p_j\|_{\mathcal{Z}_H} = 0.$$

Hence,

$$\|C_\varphi - \mathcal{T}\|_{\mathcal{H}^\infty\to\mathcal{Z}_H} \ge \limsup_{j\to\infty} \|(C_\varphi - \mathcal{T})p_j\|_{\mathcal{Z}_H} \ge \limsup_{j\to\infty} \|C_\varphi p_j\|_{\mathcal{Z}_H}.$$

Therefore,

$$\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} \ge \limsup_{j\to\infty} \|C_\varphi p_j\|_{\mathcal{Z}_H} = \limsup_{j\to\infty} \|\varphi^j + \overline{\varphi}^j\|_{\mathcal{Z}_H}. \tag{4.17}$$

Next, we prove that

$$\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} \le \limsup_{j\to\infty} \|\varphi^j + \overline{\varphi}^j\|_{\mathcal{Z}_H}.$$

Since $C_\varphi : \mathcal{H}^\infty \to \mathcal{Z}_H$ is bounded, by Theorem 2.1

$$L := \sup_{j\ge 0} \|\varphi^j + \overline{\varphi}^j\|_{\mathcal{Z}_H} < \infty.$$

Now, consider the test function $h_{b,k}$ with $b \in \mathbb{D}$ in (2.4), for $k = 1, 2, 3$. By linearity of C_φ, for any fixed positive integer $n \ge 2$, we have

$$\begin{aligned}\|C_\varphi h_{b,1}\|_{\mathcal{Z}_H} &\le (1-|b|^2)\sum_{j=0}^{\infty}|b|^j\|C_\varphi p_j\|_{\mathcal{Z}_H}\\ &= (1-|b|^2)\Big[\Big\{\sum_{j=0}^{n-1}+\sum_{j=n}^{\infty}\Big\}|b|^j\|C_\varphi p_j\|_{\mathcal{Z}_H}\Big]\\ &\le nL(1-|b|^2)+2\sup_{j\ge n}\|\varphi^j+\overline{\varphi}^j\|_{\mathcal{Z}_H}.\end{aligned}$$

Letting $|b|\to 1$ in the above inequality leads to

$$\begin{aligned}\limsup_{|b|\to1}\|C_\varphi h_{b,1}\|_{\mathcal{Z}_H} &\le 2\sup_{j\ge n}\|\varphi^j+\overline{\varphi}^j\|_{\mathcal{Z}_H}\\ &\le \limsup_{j\to\infty}\|\varphi^j+\overline{\varphi}^j\|_{\mathcal{Z}_H}.\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}\limsup_{|b|\to1}\|C_\varphi h_{b,2}\|_{\mathcal{Z}_H} &\le \limsup_{j\to\infty}\|\varphi^j+\overline{\varphi}^j\|_{\mathcal{Z}_H},\\ \limsup_{|b|\to1}\|C_\varphi h_{b,3}\|_{\mathcal{Z}_H} &\le \limsup_{j\to\infty}\|\varphi^j+\overline{\varphi}^j\|_{\mathcal{Z}_H}.\end{aligned}$$

Hence,

$$\max_{1\le k\le3}\{\limsup_{|b|\to1}\|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}\} \le \limsup_{j\to\infty}\|\varphi^j+\overline{\varphi}^j\|_{\mathcal{Z}_H}.$$

By Theorem 4.1, we obtain

$$\|C_\varphi\|_{e,\mathcal{H}^\infty\to\mathcal{Z}_H} \le \max_{1\le k\le3}\{\limsup_{|b|\to1}\|C_\varphi h_{b,k}\|_{\mathcal{Z}_H}\}\le \sup_{j\to\infty}\|\varphi^j+\overline{\varphi}^j\|_{\mathcal{Z}_H}. \tag{4.18}$$

By (4.17) and (4.18), we have achieved the desired result.

5. Conclusions

In this work, an interesting result in harmonic mappings about the operator-theoretic properties of composition operators between $\mathcal{H}^\infty$ space and harmonic Zygmund space $\mathcal{Z}_H$ has been obtained. It is well known that the existing similar results in spaces of analytic functions have been applied many times to the composition operators between $\mathcal{H}^\infty$ and Zygmund space $\mathcal{Z}$. We hope that this study can attract people's attention to the operator theory on harmonic mappings.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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