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Research article

New Einstein-Randers metrics on certain homogeneous manifolds arising from the generalized Wallach spaces

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Abstract: In this article, we find several new non-Riemannian Einstein-Randers metrics on some homogeneous manifolds arising from the generalized Wallach spaces. We first prove the existence of Riemannian Einstein metrics on these homogeneous manifolds. Based on these metrics, we prove that there exist non-Riemannian Einstein-Randers metrics on these homogeneous manifolds.

Keywords: Riemannian Einstein metrics; Einstein-Randers metrics; Finsler manifolds; homogeneous manifolds; the generalized Wallach spaces
Mathematics Subject Classification: 53C25, 53C30

1. Introduction

Randers metrics were introduced by Randers in the context of general relativity, and later named after him by Ingarden. A Randers metric F is built from a Riemannian metric and a 1-form, i.e.,

$$F = \alpha + \beta$$

where α is a Riemannian metric and β is a 1-form and the length of β corresponding to the Riemannian metric α is less than 1 everywhere.

Then, F is Riemannian if and only if $F(x, y) = F(x, -y)$. But sometimes it is convenient to use the following definition of a Randers metric in [2], i.e.,

$$F(x, y) = \frac{\sqrt{[h(W, y)]^2 + h(y, y)\lambda}}{\lambda} - \frac{h(W, y)}{\lambda}. \tag{1.1}$$

Here, $\lambda = 1 - h(W, W) > 0$, and F is Riemannian if and only if $W = 0$. The pair (h, W) is called the navigation data of the corresponding Randers metric F.

The Ricci scalar $Ric(x, y)$ of a Finsler metric is defined by the sum of those $n-1$ flag curvatures $K(x, y, e_\nu)$, where $\{e_\nu : \nu = 1, 2, \cdots, n-1\}$ is any collection of $n-1$ orthonormal transverse edges perpendicular to the flagpole, i.e.,

$$Ric(x, y) = \sum_{\nu=1}^{n-1} R_{\nu\nu}. \tag{1.2}$$

The Ricci tensor is defined by

$$Ric_{ij} = (\frac{1}{2}F^2 Ric)_{y^i y^j}. \tag{1.3}$$

The Ricci scalar depends on the position x and the flagpole y (see [2, 3]), but does not depend on the specific $n-1$ flags with transverse edges orthogonal to y. It is well known that the Ricci scalar in Riemannian geometry only depends on x. Thus, it is quite interesting to study a Finsler manifold whose Ricci scalar does not depend on the flagpole y. Generally, a Finsler metric with such a property is called an Einstein metric, i.e.,

$$Ric(x, y) = (n-1)K(x) \tag{1.4}$$

for some function $K(x)$ on M. In particular, for a Randers manifold (M, F) with $\dim M \geq 3$, F is an Einstein metric if and only if $K(x)$ is a constant on M (see [2]). The following lemma is an important result on Einstein-Randers metrics.

Lemma 1.1 ([2]). *Suppose that (M, F) is a Randers space with the navigation data (h, W). Then, (M, F) is Einstein with Ricci scalar $Ric(x) = (n-1)K(x)$ if and only if there exists a constant σ satisfying the following conditions:*

1) h is Einstein with Ricci scalar $(n-1)K(x) + \frac{1}{16}\sigma^2$, and
2) W is an infinitesimal homothety of h, i.e., $\mathfrak{L}_W h = -\sigma h$.

Furthermore, σ must be zero whenever h is not Ricci-flat.

It is well known that $K(x)$ is a constant if (M, F) is a homogeneous Einstein Finsler manifold. Here, a Finsler manifold (M, F) is called homogeneous if its full group of isometry acts transitively on M. Based on Lemma 1.1, Deng-Hou obtained a characterization of homogeneous Einstein-Randers metrics.

Lemma 1.2 ([8]). *Let G be a connected Lie group and H a closed subgroup of G such that G/H is a reductive homogeneous space with a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Suppose that h is a G-invariant Riemannian metric on G/H and $W \in \mathfrak{m}$ is invariant under H with $h(W, W) < 1$. Let $\widetilde{W}$ be the corresponding G-invariant vector field on G/H with $\widetilde{W}|_o = W$. Then, the Randers metric F with the navigation data $(h, \widetilde{W})$ is Einstein with the Ricci constant K if and only if h is Einstein with the Ricci constant K and W satisfies*

$$\langle [W, X]_{\mathfrak{m}}, Y\rangle + \langle X, [W, Y]_{\mathfrak{m}}\rangle = 0, \quad \forall X, Y \in \mathfrak{m}, \tag{1.5}$$

where $\langle , \rangle$ is the restriction of h on $T_o(G/H) \simeq \mathfrak{m}$. In this case, $\widetilde{W}$ is a Killing vector field with respect to the Riemannian metric h.

Just as in the Riemannian case, it is a fundamental problem to classify homogeneous Einstein Finsler spaces. In particular, it is very important to know if a homogeneous manifold admits invariant Einstein

Finsler metrics. In fact, there are many studies on homogenous Einstein Randers manifolds [5,6,9–11, 14–17] and a little progress on homogeneous Einstein (α,β)-metrics [18].

The main goal of this paper is to find non-Riemannian Einstein-Randers metrics on homogeneous manifolds G/H for $G = E_7$ and $G = E_6$, which means we prove the following theorem.

Theorem 1.3. *1) There are at least six families of G-invariant non-Riemannian Einstein-Randers metrics on the homogeneous manifold* E_7/A_1.

2) There are at least four families of G-invariant non-Riemannian Einstein-Randers metrics on the homogeneous manifold E_7/A_5.

3) There are at least two families of G-invariant non-Riemannian Einstein-Randers metrics on the homogeneous manifold $\mathrm{E}_7/(A_1\times A_5)$.

4) There are at least ten families of G-invariant non-Riemannian Einstein-Randers metrics on the homogeneous manifold E_6/A_1.

5) There are at least four families of G-invariant non-Riemannian Einstein-Randers metrics on the homogeneous manifold E_6/A_3.

6) There are at least four families of G-invariant non-Riemannian Einstein-Randers metrics on the homogeneous manifold $\mathrm{E}_6/(A_1\times A_3)$.

7) There are at least two families of G-invariant non-Riemannian Einstein-Randers metrics on the homogeneous manifold $\mathrm{E}_6/(A_1\times A_1)$.

8) There is at least one family of G-invariant non-Riemannian Einstein-Randers metric on the homogeneous manifold $\mathrm{E}_6/(A_1\times A_1\times A_3)$.

2. The Ricci tensor for reductive homogeneous spaces

Let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$, K a connected closed subgroup of G with Lie algebra $\mathfrak{k}$. Through this paper, we denote by B the negative of the Killing form of $\mathfrak{g}$, which is positive definite because of the compactness of G. As a result, B can be treated as an inner product on $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$ be the reductive decomposition with respect to B such that $[\mathfrak{k},\mathfrak{m}]\subset\mathfrak{m}$, where $\mathfrak{m}$ is the tangent space of G/K. We assume that $\mathfrak{m}$ can be decomposed into mutually non-equivalent irreducible $\mathrm{Ad}(K)$-modules as follows:

$$\mathfrak{m} = \mathfrak{m}_1\oplus\cdots\oplus\mathfrak{m}_q.$$

We will write $\mathfrak{k} = \mathfrak{k}_0\oplus\mathfrak{k}_1\oplus\cdots\oplus\mathfrak{k}_p$, where $\mathfrak{k}_0 = Z(\mathfrak{k})$ is the center of $\mathfrak{k}$ and $\mathfrak{k}_i$ is the simple ideal for $i = 1,\cdots,p$. Let $G\times K$ act on G by $(g_1,g_2)g = g_1gg_2^{-1}$. Then, $G\times K$ acts almost effectively on G with isotropy group $\Delta(K) = \{(k,k)|k\in K\}$. As a result, G can be treated as the coset space $(G\times K)/\Delta(K)$ and we have $\mathfrak{g}\oplus\mathfrak{k} = \Delta(\mathfrak{k})\oplus\Omega$, where $\Omega\cong T_0((G\times K)/\Delta(K))\cong\mathfrak{g}$ via the linear map $(X,Y)\to(X-Y)\in\mathfrak{g}$, $(X,Y)\in\Omega$.

A direct conclusion is that there exists a 1-1 corresponding between all G-invariant metrics on the reductive homogeneous space G/K and $\mathrm{Ad}_G(K)$-invariant inner products on $\mathfrak{m}$. Now, we have a orthogonal decomposition of $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{k}_0\oplus\mathfrak{k}_1\oplus\cdots\oplus\mathfrak{k}_p\oplus\mathfrak{m}_1\oplus\cdots\oplus\mathfrak{m}_q = (\mathfrak{k}_0\oplus\mathfrak{k}_1\oplus\cdots\oplus\mathfrak{k}_p)\oplus(\mathfrak{k}_{p+1}\oplus\cdots\oplus\mathfrak{k}_{p+q})$, with $\mathfrak{m}_1\oplus\cdots\oplus\mathfrak{m}_q = \mathfrak{k}_{p+1}\oplus\cdots\oplus\mathfrak{k}_{p+q}$. In addition, we assume that $\dim_{\mathbb{R}}\mathfrak{k}_0\leq 1$ and the ideals $\mathfrak{k}_i$ are mutually non-isomorphic for $i = 1,\cdots,p$. Then, we consider all $\mathrm{Ad}(K)$-invariant metrics on G of the form:

$$\langle,\rangle = x_0\cdot B|_{\mathfrak{k}_0} + x_1\cdot B|_{\mathfrak{k}_1} + \cdots + x_{p+q}\cdot B|_{\mathfrak{k}_{p+q}}, \tag{2.1}$$

where $x_i \in \mathbb{R}^+$ for $i = 0, 1, \cdots, p+q$. And all $\mathrm{Ad}(K)$-invariant metrics on G/K of the form:

$$(,) = x_{p+1} \cdot B|_{\mathfrak{k}_{p+1}} + \cdots + x_{p+q} \cdot B|_{\mathfrak{k}_{p+q}}, \tag{2.2}$$

where $x_i \in \mathbb{R}^+$ for $i = p+1, \cdots, p+q$.

Set from now on $d_i = \dim_{\mathbb{R}} \mathfrak{k}_i$ and $\{e^i_\alpha\}_{\alpha=1}^{d_i}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{g}$ which means $e^i_\alpha \in \mathfrak{k}_i$ and α is the number of basis in $\mathfrak{k}_i$. Then, we consider the numbers $A^\gamma_{\alpha,\beta} = B([e^i_\alpha, e^j_\beta], e^k_\gamma)$ such that $[e^i_\alpha, e^j_\beta] = \sum_\gamma A^\gamma_{\alpha,\beta} e^k_\gamma$, and set

$$(ijk) := \left[\begin{array}{c} i \\ j\,k \end{array} \right] = \sum (A^\gamma_{\alpha,\beta})^2,$$

where the sum is taken over all indices α, β, γ with $e^i_\alpha \in \mathfrak{k}_i, e^j_\beta \in \mathfrak{k}_j, e^k_\gamma \in \mathfrak{k}_k$. Then, (ijk) is independent of the choice for the B-orthonormal basis of $\mathfrak{k}_i, \mathfrak{k}_j, \mathfrak{k}_k$, and symmetric for all three indices which means $(ijk) = (jik) = (jki)$.

In [1] and [13], the authors obtained the formulas for the components of the Ricci tensor with respect to the left-invariant metric given by (2.1), which can be described by the following lemma.

Lemma 2.1. *Let G be a compact connected semi-simple Lie group endowed with the left-invariant metric $\langle , \rangle$ given by (2.1). Then, the components $r_0, r_1, \cdots, r_{p+q}$ of the Ricci tensor associated to $\langle , \rangle$ are expressed as follows:*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \left[\begin{array}{c} k \\ j\,i \end{array} \right] - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \left[\begin{array}{c} j \\ k\,i \end{array} \right], \quad (k = 0, 1, \cdots, p+q).$$

Here, the sums are taken over all $i = 0, 1, \cdots, p+q$. In particular, for each k it holds that

$$\sum_{i,j}^{p+q} \left[\begin{array}{c} j \\ k\,i \end{array} \right] = \sum_{ij}^{p+q} (kij) = d_k.$$

For $k = p+1, \cdots, p+q$ and by considering the sums appearing in the expression of r_k only for i, j with $p+1 \le i, j \le p+q$, one obtains the components $\hat{r}_{p+1}, \cdots, \hat{r}_{p+q}$ of the Ricci tensor $\hat{r}$ of the G-invariant metric $(,)$ on G/K defined by (2.2).

3. Decompositions arising from generalized Wallach spaces

We recall the definition of generalized Wallach spaces. Let G/K be a reductive homogeneous space, where G is a semi-simple compact connected Lie group, K is a connected closed subgroup of G, $\mathfrak{g}$ and $\mathfrak{k}$ are the corresponding Lie algebras, respectively. If $\mathfrak{m}$, the tangent space of G/K at $o = \pi(e)$, can be decomposed into three $\mathrm{ad}(\mathfrak{k})$-invariant irreducible summands pairwise orthogonal with respect to B as:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

satisfying $[\mathfrak{m}_i, \mathfrak{m}_i] \in \mathfrak{k}$ for $i \in \{1, 2, 3\}$ and $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then we call G/K a generalized Wallach space.

In 2014, classification for generalized Wallach spaces arising from a compact simple Lie group has been obtained by Nikonorov [12] and Chen, Kang and Liang [7], in particular, Nikonorov investigated the semi-simple case and gave the classification in [12]. For convenience, in this article, we use the notations in [7].

According to [7], each kind of the classification corresponds to two commutative involutive automorphisms of $\mathfrak{g}$, the Lie algebra of the compact simple Lie group G. In [4], the authors calculated all the coefficients (ijk) in the expression for the components of Ricci tensor with respect to the metric of form (2.1). In [7], E_6-II and E_7-II have the following decomposition in the view of Lie algebra:

$$\begin{aligned}\mathrm{E}_6 &= \mathrm{T} \oplus A_1^1 \oplus A_1^2 \oplus A_3 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3,\\ \mathrm{E}_7 &= \mathrm{T} \oplus A_1 \oplus A_5 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3.\end{aligned}$$

By the notations in the above section, we give the following two lemmas in [4] for later use.

Lemma 3.1. *In the case of* E_7*-II, the possible non-zero coefficients in the expression for the components of Ricci tensor with respect to the metric of the form (2.1) are as follows:*

$$\begin{aligned}&(033) = \frac{4}{9}, (044) = \frac{5}{9}, (055) = 0, (345) = \frac{20}{3},\\ &(111) = \frac{1}{3}, (133) = 1, (144) = 0, (155) = \frac{5}{3},\\ &(222) = \frac{35}{3}, (233) = \frac{35}{9}, (244) = \frac{70}{9}, (255) = \frac{35}{3}.\end{aligned}$$

Lemma 3.2. *In the case of* E_6*-II, the possible non-zero coefficients in the expression for the components of Ricci tensor with respect to the metric of the form (2.1) are as follows:*

$$\begin{aligned}&(044) = 1/2, (055) = 1/2, (066) = 0, (456) = 4,\\ &(111) = 1/2, (144) = 0, (155) = 1, (166) = 3/2,\\ &(222) = 1/2, (244) = 1, (255) = 0, (266) = 3/2,\\ &(333) = 5, (344) = 5/2, (355) = 5/2, (366) = 5.\end{aligned}$$

4. Einstein metrics on homogenous manifolds

In this section, we will consider the Einstein metrics on some homogeneous manifolds which can be obtained from the generalized Wallach spaces, namely $\mathrm{E}_7/A_1, \mathrm{E}_7/A_5, \mathrm{E}_7/(A_1 \times A_5), \mathrm{E}_6/A_1, \mathrm{E}_6/A_3, \mathrm{E}_6/(A_1 \times A_3), \mathrm{E}_6/(A_1 \times A_1)$ and $\mathrm{E}_6/(A_1 \times A_1 \times A_3)$.

Case of E_7/A_1**.** Consider the homogeneous manifold E_7/A_1 with the decomposition

$$\mathfrak{m} = \mathrm{T} \oplus A_5 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3, \tag{4.1}$$

and $\mathrm{Ad}(A_1)$-invariant metrics which is also $\mathrm{Ad}(\mathrm{T} \oplus A_1 \oplus A_5)$-invariant on E_7/A_1 defined by

$$<\,,\,>= u_0 \cdot B|_{\mathrm{T}} + x_1 \cdot B|_{A_5} + x_2 \cdot B|_{\mathfrak{p}_1} + x_3 \cdot B|_{\mathfrak{p}_2} + x_4 \cdot B|_{\mathfrak{p}_3}, \tag{4.2}$$

where $u_0, x_1, x_2, x_3, x_4 \in \mathbb{R}^+$. By Lemmas 2.1 and 3.1, the components of Ricci tensor with respect to the metric (4.2) are as follows:

$$\begin{cases} r_{\mathrm{T}} = \dfrac{1}{4}\left(\dfrac{4u_0}{9x_2^2} + \dfrac{5u_0}{9x_3^2}\right), \\ r_{A_5} = \dfrac{1}{140}\left(\dfrac{35}{3x_1} + \dfrac{35x_1}{9x_2^2} + \dfrac{70x_1}{9x_3^2} + \dfrac{35x_1}{3x_4^2}\right), \\ r_{\mathfrak{p}_1} = \dfrac{1}{2x_2} + \dfrac{5}{36}\left(\dfrac{x_2}{x_3x_4} - \dfrac{x_3}{x_4x_2} - \dfrac{x_4}{x_2x_3}\right) - \dfrac{1}{48}\left(\dfrac{4u_0}{9x_2^2} + \dfrac{35x_1}{9x_2^2}\right), \\ r_{\mathfrak{p}_2} = \dfrac{1}{2x_3} + \dfrac{1}{9}\left(\dfrac{x_3}{x_4x_2} - \dfrac{x_4}{x_2x_3} - \dfrac{x_2}{x_3x_4}\right) - \dfrac{1}{60}\left(\dfrac{5u_0}{9x_3^2} + \dfrac{70x_1}{9x_3^2}\right), \\ r_{\mathfrak{p}_3} = \dfrac{1}{2x_4} + \dfrac{1}{12}\left(\dfrac{x_4}{x_2x_3} - \dfrac{x_2}{x_3x_4} - \dfrac{x_3}{x_4x_2}\right) - \dfrac{1}{80}\dfrac{35x_1}{3x_4^2}. \end{cases}$$

We consider the homogeneous Einstein equation as follows:

$$\{r_{\mathrm{T}} - r_{A_5} = 0, r_{A_5} - r_{\mathfrak{p}_1} = 0, r_{\mathfrak{p}_1} - r_{\mathfrak{p}_2} = 0, r_{\mathfrak{p}_2} - r_{\mathfrak{p}_3} = 0\}$$

and it turns out to be equivalent to the following system of equations (we normalize the metric by setting $u_0 = 1$).

$$\begin{cases} g_0 = -3\,x_1^2x_2^2x_3^2 - 2\,x_1^2x_2^2x_4^2 - x_1^2x_3^2x_4^2 - 3\,x_2^2x_3^2x_4^2 + 5\,x_1x_2^2x_4^2 + 4\,x_1x_3^2x_4^2 = 0, \\ g_1 = 36\,x_1^2x_2^2x_3^2 + 24\,x_1^2x_2^2x_4^2 + 47\,x_1^2x_3^2x_4^2 - 60\,x_1x_2^3x_3x_4 + 60\,x_1x_2x_3^3x_4 - 216\,x_1x_2x_3^2x_4^2 \\ \quad + 60\,x_1x_2x_3x_4^3 + 36\,x_2^2x_3^2x_4^2 + 4\,x_1x_3^2x_4^2 = 0, \\ g_2 = 56\,x_1x_2^2x_4 - 35\,x_1x_3^2x_4 + 108\,x_2^3x_3 - 216\,x_2^2x_3x_4 - 108\,x_2x_3^3 + 216\,x_2x_3^2x_4 - 12\,x_2x_3x_4^2 \\ \quad + 4\,x_2^2x_4 - 4\,x_3^2x_4 = 0, \\ g_3 = 63\,x_1x_2x_3^2 - 56\,x_1x_2x_4^2 - 12\,x_2^2x_3x_4 - 216\,x_2x_3^2x_4 + 216\,x_2x_3x_4^2 + 84\,x_3^3x_4 - 84\,x_3x_4^3 \\ \quad - 4\,x_2x_4^2 = 0. \end{cases}$$

We consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3, x_4]$ and an ideal I, generated by polynomials $\{g_0, g_1, g_2, g_3, zx_1x_2x_3x_4 - 1\}$. We take a lexicographic ordering $>$ with $z > x_1 > x_2 > x_3 > x_4$ for a monomial ordering on R. Then, with the aid of computer software MAPLE 15, we use the Gröbner basis for the ideal I to contain a polynomial of x_4 given by $(20\,x_4^2 - 72\,x_4 + 49) \cdot h(x_4)$, where $h(x_4)$ is a polynomial of degree 30 given by

$$
\begin{aligned}
h(x_4) = {} & 147263121465141201705316711269370210741339344914420477996120017678192287817728000\, x_4^{30} \\
& - 110257665020954175479217476835155952503081737443324371558377360379620992486727680 0\, x_4^{29} \\
& + 43926084466337156416422771934423038413566008105153129129561574879985877682234675 20\, x_4^{28} \\
& - 16381030403148760413861847066334150627671304859530037939042221945751621483816924672\, x_4^{27} \\
& + 44632927369562126482344261311667463916531381076564814811273737776429261123742947776\, x_4^{26} \\
& - 10514551913020119137437590738175112453218135015793517677482929730275565432111 9817568\, x_4^{25} \\
& + 21508705556035306866672118084047977611319044717494073132903343525125415187312786428 8\, x_4^{24} \\
& - 34451837402804811107041226967326977778824617925823990893979091038871106317536716178 4\, x_4^{23} \\
& + 44587940313331568187408504292362929483211692146090423443665425961683938752034817059 6\, x_4^{22} \\
& - 45400315073071285545752682186102790358057291331388888887855537595204059250354444708 0\, x_4^{21} \\
& + 30817176272958530663651801160600301432571661666722152092219915889272628820664241170 2\, x_4^{20} \\
& - 8892852090716697336069399313604278178603440624079207839399395350620691391128893564 8\, x_4^{19} \\
& - 4003443046354553391789019418735708747582529538191658608475366020463462695500043090 8\, x_4^{18} \\
& + 4451881542279682372690060005698966614633957940437282564172139476331302182653918329 4\, x_4^{17} \\
& - 1323112516388796056983223014462460032439960619096921248731446619508496698157467624 7\, x_4^{16} \\
& - 797439538169345403753233795452611566212570589018415600726606800661093425142878880\, x_4^{15} \\
& + 1511979559539474891852062236262989481312808910795807182871248144865160620831183632\, x_4^{14} \\
& - 379199106951137596458809929302879785793133656913117463400584039552474682406503552\, x_4^{13} \\
& + 37797140139775487751864398109781462786788335556812190904364736929666448564191712\, x_4^{12} \\
& - 1785948980557995395564396777431636788073937513566295662291234226016378838797312\, x_4^{11} \\
& + 36118558601423456958783378429381850110652729957442957944225856533578930373440 0\, x_4^{10} \\
& - 29814163618672798385666919644831689684183641996050223223894494904315006447104\, x_4^{9} \\
& + 507366481118078491288306527464444079080233387113779600841252619530160870 4\, x_4^{8} \\
& + 55049909220845662880254271319404494752671319946031114132616313964384415744\, x_4^{7} \\
& + 3753669713205988245353403631212112410702925290650176001229552989542586368\, x_4^{6} \\
& - 461850039539407240955273834215202137394483687360075268757733061237596160\, x_4^{5} \\
& + 14855590404155632214609581796329294787394060775182341751503634575953920\, x_4^{4} \\
& - 151882382457983307126403603474158205457944387452892184268979018137600\, x_4^{3} \\
& + 99570041038140589135861869530740578778470729364919643027537920000 0\, x_4^{2} \\
& - 134173366770676378641150457892018840456467759007178689819443200 00\, x_4 \\
& + 568400491719774288687807984496995610757827345058316786073600000.
\end{aligned}
$$

Solving $h(x_4) = 0$ numerically, we see that there exist four positive solutions, which are given approximately by $x_4 \approx 0.4025066142, x_4 \approx 1.032699036, x_4 \approx 1.438454385, x_4 \approx 4.052215439$. Further, we remark that x_1, x_2 and x_3 can be expressed by polynomials of x_4, thus the corresponding solutions of the system $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, h(x_4) = 0\}$ with $x_1 x_2 x_3 x_4 \neq 0$ are as follows:

$$\{x_1 \approx 0.4492704820, x_2 \approx 0.7040380158, x_3 \approx 0.7236008036, x_4 \approx 0.4025066142\},$$
$$\{x_1 \approx 0.2625192922, x_2 \approx 1.149055866, x_3 \approx 0.6923507482, x_4 \approx 1.032699036\},$$
$$\{x_1 \approx 0.3393694928, x_2 \approx 0.7028774603, x_3 \approx 1.499308774, x_4 \approx 1.438454385\},$$
$$\{x_1 \approx 1.476432886, x_2 \approx 1.076444066, x_3 \approx 4.066468725, x_4 \approx 4.052215439\}.$$

For $20\,{x_4}^2 - 72\,x_4 + 49 = 0$, we have $x_4 \approx 0.9111805582$ and $x_4 \approx 2.688819441$, whose corresponding solutions of the system $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, 20\,{x_4}^2 - 72\,x_4 + 49 = 0\}$ with $x_1x_2x_3x_4 \neq 0$ are as follows:

$$\{x_1 = x_3 = 1, x_2 = x_4 \approx 0.9111805582\},$$
$$\{x_1 = x_3 = 1, x_2 = x_4 \approx 2.688819441\}.$$

In conclusion, we find six different G-invariant Einstein metrics on homogeneous manifold E_7/A_1.

Case of E_7/A_5**.** Consider the homogeneous manifold E_7/A_5 with the decomposition

$$\mathfrak{m} = \mathrm{T} \oplus A_1 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3, \tag{4.3}$$

and $\mathrm{Ad}(A_5)$-invariant metrics which is also $\mathrm{Ad}(\mathrm{T} \oplus A_1 \oplus A_5)$-invariant on E_7/A_5 defined by

$$< , >= u_0 \cdot B|_{\mathrm{T}} + x_1 \cdot B|_{A_1} + x_2 \cdot B|_{\mathfrak{p}_1} + x_3 \cdot B|_{\mathfrak{p}_2} + x_4 \cdot B|_{\mathfrak{p}_3}, \tag{4.4}$$

where $u_0, x_1, x_2, x_3, x_4 \in \mathbb{R}^+$. By Lemmas 2.1 and 3.1, the components of Ricci tensor with respect to the metric (4.4) are as follows:

$$\begin{cases} r_{\mathrm{T}} = \dfrac{1}{4}\left(\dfrac{4u_0}{9x_2^2} + \dfrac{5u_0}{9x_3^2}\right), \\ r_{A_1} = \dfrac{1}{12}\left(\dfrac{1}{3x_1} + \dfrac{x_1}{x_2^2} + \dfrac{5x_1}{3x_4^2}\right), \\ r_{\mathfrak{p}_1} = \dfrac{1}{2x_2} + \dfrac{5}{36}\left(\dfrac{x_2}{x_3x_4} - \dfrac{x_3}{x_4x_2} - \dfrac{x_4}{x_2x_3}\right) - \dfrac{1}{48}\left(\dfrac{4u_0}{9x_2^2} + \dfrac{x_1}{x_2^2}\right), \\ r_{\mathfrak{p}_2} = \dfrac{1}{2x_3} + \dfrac{1}{9}\left(\dfrac{x_3}{x_4x_2} - \dfrac{x_4}{x_2x_3} - \dfrac{x_2}{x_3x_4}\right) - \dfrac{1}{60}\dfrac{5u_0}{9x_3^2}, \\ r_{\mathfrak{p}_3} = \dfrac{1}{2x_4} + \dfrac{1}{12}\left(\dfrac{x_4}{x_2x_3} - \dfrac{x_2}{x_3x_4} - \dfrac{x_3}{x_4x_2}\right) - \dfrac{1}{80}\dfrac{5x_1}{3x_4^2}. \end{cases}$$

Moreover, the homogeneous Einstein equation is given by

$$\{r_{\mathrm{T}} - r_{A_1} = 0, r_{A_1} - r_{\mathfrak{p}_1} = 0, r_{\mathfrak{p}_1} - r_{\mathfrak{p}_2} = 0, r_{\mathfrak{p}_2} - r_{\mathfrak{p}_3} = 0\},$$

which is equivalent to the following system of equations by setting $u_0 = 1$:

$$\begin{cases} g_0 = -5\,{x_1}^2{x_2}^2{x_3}^2 - 3\,{x_1}^2{x_3}^2{x_4}^2 - {x_2}^2{x_3}^2{x_4}^2 + 5\,x_1{x_2}^2{x_4}^2 + 4\,x_1{x_3}^2{x_4}^2 = 0, \\ g_1 = 60\,{x_1}^2{x_2}^2x_3 + 45\,{x_1}^2x_3{x_4}^2 - 60\,x_1{x_2}^3x_4 + 60\,x_1x_2{x_3}^2x_4 - 216\,x_1x_2x_3{x_4}^2 + 60\,x_1x_2{x_4}^3 \\ \quad + 12\,{x_2}^2x_3{x_4}^2 + 4\,x_1x_3{x_4}^2 = 0, \\ g_2 = -9\,x_1{x_3}^2x_4 + 108\,{x_2}^3x_3 - 216\,{x_2}^2x_3x_4 - 108\,x_2{x_3}^3 + 216\,x_2{x_3}^2x_4 - 12\,x_2x_3{x_4}^2 + 4\,{x_2}^2x_4 \\ \quad - 4\,{x_3}^2x_4 = 0, \\ g_3 = 9\,x_1x_2{x_3}^2 - 12\,{x_2}^2x_3x_4 - 216\,x_2{x_3}^2x_4 + 216\,x_2x_3{x_4}^2 + 84\,{x_3}^3x_4 - 84\,x_3{x_4}^3 - 4\,x_2{x_4}^2 = 0. \end{cases}$$

We consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3, x_4]$ and an ideal I, generated by polynomials $\{g_0, g_1, g_2, g_3, zx_1x_2x_3x_4 - 1\}$. We take a lexicographic ordering $>$ with $z > x_1 > x_2 > x_3 > x_4$ for a monomial ordering on R. Then, with the aid of computer software MAPLE 15, we use the Gröbner basis for the ideal I to contain a polynomial $h(x_4)$ of degree 24 given by

$$
\begin{aligned}
h(x_4) =& 743926266335759400800157293103178343319505630645046435156250000\, {x_4}^{24} \\
&- 437814457854303901136135838506108156916167440067538307831687500\, {x_4}^{23} \\
&+ 110049032268802097039560630575593014565915258358887938713698025000\, {x_4}^{22} \\
&- 157283437018312012527511818740548319115197251415637333728437548000\, {x_4}^{21} \\
&+ 150162449581176203303029956936619600973807605121615956989744708900\, {x_4}^{20} \\
&- 119943087819267070011016662616323904614977879472081817202050332000\, {x_4}^{19} \\
&+ 9738633984043261058161384571494874932415716842246643813641906732\, {x_4}^{18} \\
&- 6775088500386513521204014691159118392435727285614732468339912598\, {x_4}^{17} \\
&+ 2706243877713720652207013644363017005932182610630242895498110351\, {x_4}^{16} \\
&- 1200015385861936836241292092591650758176038018711970332471811 2\, {x_4}^{15} \\
&- 4708528956125922005834776100335084599725469730382962250906136 48\, {x_4}^{14} \\
&+ 1586573690992713261375888791215592580447532696150196285898251 52\, {x_4}^{13} \\
&- 4309426841561170090469256176045397413256930487964567974681056\, {x_4}^{12} \\
&- 6417570511115254453560753629391603456981811382551849470944256\, {x_4}^{11} \\
&+ 1526468649104808736601756484180315699890352358674705978019200\, {x_4}^{10} \\
&- 181418867068729727750349886730055784794192965200077070182912\, {x_4}^{9} \\
&+ 13350908794995611352389703910041292729770053075493309180672\, {x_4}^{8} \\
&- 6329961433036003721112288649338367023748698445801967247 36\, {x_4}^{7} \\
&+ 182585400294338395168838280106547438292112296472129720 32\, {x_4}^{6} \\
&- 229558007445965630066366475684636804625527678725234688\, {x_4}^{5} \\
&- 2954197894924039316661797095646508811861752372301824\, {x_4}^{4} \\
&+ 12132205987273583166791906816786597168291755484774 4\, {x_4}^{3} \\
&+ 543967044145299748038471088930322206082300116992\, {x_4}^{2} \\
&- 6901642973943651961491325730372486415117189120 0\, x_4 \\
&+ 84960625990872668149132453749913531822637056 0.
\end{aligned}
$$

By solving $h(x_4) = 0$ numerically, we get four different positive solutions, which are given approximately by $x_4 \approx 0.6449885281, x_4 \approx 0.7443959787, x_4 \approx 1.060416256, x_4 \approx 1.501487029$. Further, we remark that x_1, x_2, x_3 can be expressed by x_4. Therefore, the real solutions of the system $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, h(x_4) = 0\}$ with $x_1x_2x_3x_4 \neq 0$ are as follows:

$$\{x_1 \approx 1.173286641, x_2 \approx 0.9076818373, x_3 \approx 0.5898460162, x_4 \approx 0.6449885281\},$$
$$\{x_1 \approx 0.05521141092, x_2 \approx 0.9874672655, x_3 \approx 0.5836473658, x_4 \approx 0.7443959787\},$$
$$\{x_1 \approx 0.07379520877, x_2 \approx 0.6077356344, x_3 \approx 1.170652643, x_4 \approx 1.060416256\},$$
$$\{x_1 \approx 1.140144619, x_2 \approx 0.6881276139, x_3 \approx 1.513843081, x_4 \approx 1.501487029\}.$$

To conclude, we find four different G-invariant Einstein metrics on homogeneous manifold E_7/A_5.

Case of $\mathrm{E}_7/(A_1 \times A_5)$**.** Consider the homogeneous manifold $\mathrm{E}_7/(A_1 \times A_5)$ with the decomposition

$$\mathfrak{m} = \mathrm{T} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3, \tag{4.5}$$

and $\mathrm{Ad}(A_1 \times A_5)$-invariant metrics which is also $\mathrm{Ad}(\mathrm{T} \oplus A_1 \oplus A_5)$-invariant on $\mathrm{E}_7/(A_1 \times A_5)$ defined by

$$< \, , \, >= u_0 \cdot B|_{\mathrm{T}} + x_1 \cdot B|_{\mathfrak{p}_1} + x_2 \cdot B|_{\mathfrak{p}_2} + x_3 \cdot B|_{\mathfrak{p}_3}, \tag{4.6}$$

where $u_0, x_1, x_2, x_3 \in \mathbb{R}^+$. By Lemmas 2.1 and 3.1, the components of Ricci tensor with respect to the metric (4.6) are as follows:

$$\begin{cases} r_{\mathrm{T}} = \dfrac{1}{4}\left(\dfrac{4u_0}{9x_1^2} + \dfrac{5u_0}{9x_2^2}\right), \\ r_{\mathfrak{p}_1} = \dfrac{1}{2x_1} + \dfrac{5}{36}\left(\dfrac{x_1}{x_2x_3} - \dfrac{x_2}{x_3x_1} - \dfrac{x_3}{x_1x_2}\right) - \dfrac{1}{48}\dfrac{4u_0}{9x_1^2}, \\ r_{\mathfrak{p}_2} = \dfrac{1}{2x_2} + \dfrac{1}{9}\left(\dfrac{x_2}{x_3x_1} - \dfrac{x_3}{x_1x_2} - \dfrac{x_1}{x_2x_3}\right) - \dfrac{1}{60}\dfrac{5u_0}{9x_2^2}, \\ r_{\mathfrak{p}_3} = \dfrac{1}{2x_3} + \dfrac{1}{12}\left(\dfrac{x_3}{x_1x_2} - \dfrac{x_1}{x_2x_3} - \dfrac{x_2}{x_3x_1}\right). \end{cases}$$

The homogeneous Einstein equation is equivalent to the following system of equations (we normalize the metric by setting $u_0 = 1$):

$$\begin{cases} g_0 = -15\,{x_1}^3x_2 + 15\,x_1{x_2}^3 - 54\,x_1{x_2}^2x_3 + 15\,x_1x_2{x_3}^2 + 15\,{x_1}^2x_3 + 13\,{x_2}^2x_3 = 0, \\ g_1 = 27\,{x_1}^3x_2 - 54\,{x_1}^2x_2x_3 - 27\,x_1{x_2}^3 + 54\,x_1{x_2}^2x_3 - 3\,x_1x_2{x_3}^2 + {x_1}^2x_3 - {x_2}^2x_3 = 0, \\ g_2 = -3\,{x_1}^2x_2 - 54\,x_1{x_2}^2 + 54\,x_1x_2x_3 + 21\,{x_2}^3 - 21\,x_2{x_3}^2 - x_1x_3 = 0. \end{cases}$$

We consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3]$ and an ideal I, generated by polynomials $\{g_0, g_1, g_2, zx_1x_2x_3 - 1\}$. We take a lexicographic ordering $>$ with $z > x_1 > x_2 > x_3$ for a monomial ordering on R. Then, with the aid of computer software MAPLE 15, we use the Gröbner basis for the ideal I to contain a polynomial $h(x_3)$ of degree 8 given by

$$\begin{aligned} h(x_3) = {} & 126500890017116160000\,{x_3}^8 - 443159701524243843840\,{x_3}^7 \\ & + 626866608280069521864\,{x_3}^6 - 439964490275093593980\,{x_3}^5 \\ & + 146470811201043075009\,{x_3}^4 - 18042589809134073603\,{x_3}^3 \\ & + 1095420294616803246\,{x_3}^2 - 33677044349391072\,x_3 \\ & + 438386090662912. \end{aligned}$$

By solving $h(x_3) = 0$ numerically, there are two positive solutions, which are given approximately by $x_3 \approx 0.7485101873$ and $x_3 \approx 1.034311056$. Moreover, we remark that x_1 and x_2 can be expressed by x_3. As a result, the real solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, h(x_3) = 0\}$ are

$$\{x_1 \approx 0.9908972484, x_2 \approx 0.5833878061, x_3 \approx 0.7485101873\},$$
$$\{x_1 \approx 0.6019040936, x_2 \approx 1.154991982, x_3 \approx 1.034311056\}.$$

In conclusion, we find two different G-invariant Einstein metrics on the homogeneous manifold $\mathrm{E}_7/(A_1 \times A_5)$.

Case of E_6/A_1**.** Consider the homogeneous manifold E_6/A_1^1 with the decomposition

$$\mathfrak{m} = \mathrm{T} \oplus A_1^2 \oplus A_3 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3, \tag{4.7}$$

and $\mathrm{Ad}(A_1^1)$-invariant metrics which is also $\mathrm{Ad}(\mathrm{T} \oplus A_1^1 \oplus A_1^2 \oplus A_3)$-invariant on E_6/A_1^1 defined by

$$< , >= u_0 \cdot B|_{\mathrm{T}} + x_1 \cdot B|_{A_1^2} + x_2 \cdot B|_{A_3} + x_3 \cdot B|_{\mathfrak{p}_1} + x_4 \cdot B|_{\mathfrak{p}_2} + x_5 \cdot B|_{\mathfrak{p}_3}, \tag{4.8}$$

where $u_0, x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}^+$. By Lemmas 2.1 and 3.2, the components of Ricci tensor with respect to the metric (4.8) are as follows:

$$\begin{cases}
r_{\mathrm{T}} = \dfrac{1}{4}\left(\dfrac{u_0}{2x_3^2} + \dfrac{u_0}{2x_4^2}\right), \\
r_{A_1^2} = \dfrac{1}{12}\left(\dfrac{1}{2x_1} + \dfrac{x_1}{x_3^2} + \dfrac{3x_1}{2x_5^2}\right), \\
r_{A_3} = \dfrac{1}{60}\left(\dfrac{5}{x_2} + \dfrac{5x_2}{2x_3^2} + \dfrac{5x_2}{2x_4^2} + \dfrac{5x_2}{x_5^2}\right), \\
r_{\mathfrak{p}_1} = \dfrac{1}{2x_3} + \dfrac{1}{8}\left(\dfrac{x_3}{x_4x_5} - \dfrac{x_4}{x_5x_3} - \dfrac{x_5}{x_3x_4}\right) - \dfrac{1}{32}\left(\dfrac{u_0}{2x_3^2} + \dfrac{x_1}{x_3^2} + \dfrac{5x_2}{2x_3^2}\right), \\
r_{\mathfrak{p}_2} = \dfrac{1}{2x_4} + \dfrac{1}{8}\left(\dfrac{x_4}{x_5x_3} - \dfrac{x_5}{x_3x_4} - \dfrac{x_3}{x_4x_5}\right) - \dfrac{1}{32}\left(\dfrac{u_0}{2x_4^2} + \dfrac{5x_2}{2x_4^2}\right), \\
r_{\mathfrak{p}_3} = \dfrac{1}{2x_5} + \dfrac{1}{12}\left(\dfrac{x_5}{x_3x_4} - \dfrac{x_3}{x_4x_5} - \dfrac{x_4}{x_5x_3}\right) - \dfrac{1}{48}\left(\dfrac{3x_1}{2x_5^2} + \dfrac{5x_2}{x_5^2}\right).
\end{cases}$$

We study the following homogeneous Einstein equation

$$\{r_{\mathrm{T}} - r_{A_1^2} = 0, r_{A_1^2} - r_{A_3} = 0, r_{A_3} - r_{\mathfrak{p}_1} = 0, r_{\mathfrak{p}_1} - r_{\mathfrak{p}_2} = 0, r_{\mathfrak{p}_2} - r_{\mathfrak{p}_3} = 0\},$$

which is equivalent to the following system of equations by setting $u_0 = 1$:

$$\begin{cases}
g_0 = -3\,{x_1}^2{x_3}^2{x_4}^2 - 2\,{x_1}^2{x_4}^2{x_5}^2 - {x_3}^2{x_4}^2{x_5}^2 + 3\,x_1{x_3}^2{x_5}^2 + 3\,x_1{x_4}^2{x_5}^2 = 0, \\
g_1 = 3\,{x_1}^2x_2{x_3}^2{x_4}^2 + 2\,{x_1}^2x_2{x_4}^2{x_5}^2 - 2\,x_1{x_2}^2{x_3}^2{x_4}^2 - x_1{x_2}^2{x_3}^2{x_5}^2 - x_1{x_2}^2{x_4}^2{x_5}^2 - 2\,x_1{x_3}^2{x_4}^2{x_5}^2 \\
\quad + x_2{x_3}^2{x_4}^2{x_5}^2 = 0, \\
g_2 = 6\,x_1x_2{x_4}^2{x_5}^2 + 16\,{x_2}^2{x_3}^2{x_4}^2 + 8\,{x_2}^2{x_3}^2{x_5}^2 + 23\,{x_2}^2{x_4}^2{x_5}^2 - 24\,x_2{x_3}^3x_4x_5 + 24\,x_2x_3{x_4}^3x_5 \\
\quad - 96\,x_2x_3{x_4}^2{x_5}^2 + 24\,x_2x_3x_4{x_5}^3 + 16\,{x_3}^2{x_4}^2{x_5}^2 + 3\,x_2{x_4}^2{x_5}^2 = 0, \\
g_3 = -2\,x_1{x_4}^2x_5 + 5\,x_2{x_3}^2x_5 - 5\,x_2{x_4}^2x_5 + 16\,{x_3}^3x_4 - 32\,{x_3}^2x_4x_5 - 16\,x_3{x_4}^3 + 32\,x_3{x_4}^2x_5 + {x_3}^2x_5 \\
\quad - {x_4}^2x_5 = 0, \\
g_4 = 6\,x_1x_3{x_4}^2 + 20\,x_2x_3{x_4}^2 - 15\,x_2x_3{x_5}^2 - 8\,{x_3}^2x_4x_5 - 96\,x_3{x_4}^2x_5 + 96\,x_3x_4{x_5}^2 + 40\,{x_4}^3x_5 \\
\quad - 40\,x_4{x_5}^3 - 3\,x_3{x_5}^2 = 0.
\end{cases}$$

We consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3, x_4, x_5]$ and an ideal I, generated by polynomials $\{g_0, g_1, g_2, g_3, g_4, zx_1x_2x_3x_4x_5 - 1\}$. We take a lexicographic ordering $>$ with $z > x_1 > x_2 > x_3 > x_4 > x_5$ for a monomial ordering on R. Then, with the aid of computer software MAPLE 15, we use the Gröbner basis for the ideal I to contain a polynomial $(4\,x_5{}^2 - 16\,x_5 + 11) \cdot h(x_5)$, where $h(x_5)$ is a polynomial of degree 74. Since the length of the polynomial $h(x_5)$ may affect the readers read, we put it in Appendix.

By solving $h(x_5) = 0$, there exist eight different positive solutions, which are given approximately by $x_5 \approx 0.4678383774, x_5 \approx 0.8564644408, x_5 \approx 1.195992694, x_5 \approx 1.331593277, x_5 \approx 2.323059800, x_5 \approx 2.650131702, x_5 \approx 3.078478984, x_5 \approx 3.248554868$. Moreover, we remark that x_1, x_2, x_3, x_4 can be written into polynomials of x_5. As a result, we get all solutions of the system $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, h(x_5) = 0\}$ with $x_1x_2x_3x_4x_5 \neq 0$, which are given below:

$$
\begin{aligned}
&\{x_1 \approx 0.5079142344, x_2 \approx 0.5564563075, x_3 \approx 0.7674520218, x_4 \approx 0.7344343505, x_5 \approx 0.4678383774\},\\
&\{x_1 \approx 1.391791299, x_2 \approx 0.2653010181, x_3 \approx 1.024348994, x_4 \approx 0.6953615240, x_5 \approx 0.8564644408\},\\
&\{x_1 \approx 0.1314059305, x_2 \approx 0.2920839602, x_3 \approx 1.311645073, x_4 \approx 0.6903614938, x_5 \approx 1.195992694\},\\
&\{x_1 \approx 0.1535432666, x_2 \approx 0.3214000505, x_3 \approx 0.7137178306, x_4 \approx 1.420816234, x_5 \approx 1.331593277\},\\
&\{x_1 \approx 3.641968638, x_2 \approx 1.362098346, x_3 \approx 2.353687875, x_4 \approx 0.9881377660, x_5 \approx 2.323059800\},\\
&\{x_1 \approx 1.083726408, x_2 \approx 0.7345275471, x_3 \approx 0.9316800862, x_4 \approx 2.652053038, x_5 \approx 2.650131702\},\\
&\{x_1 \approx 0.3244500133, x_2 \approx 1.435139205, x_3 \approx 3.101828021, x_4 \approx 1.010069757, x_5 \approx 3.078478984\},\\
&\{x_1 \approx 0.4746512742, x_2 \approx 1.288027846, x_3 \approx 1.023041915, x_4 \approx 3.259446111, x_5 \approx 3.248554868\}.
\end{aligned}
$$

For $4\,x_5{}^2 - 16\,x_5 + 11 = 0$, we have two different positive solutions given approximately by $x_5 \approx 0.8819660112$ and $x_5 \approx 3.118033988$. Thus, the solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, 4\,x_5{}^2 - 16\,x_5 + 11 = 0\}$ with $x_1x_2x_3x_4x_5 \neq 0$ are as follows:

$$
\begin{aligned}
&\{x_1 = x_2 = x_3 = 1, x_4 = x_5 \approx 0.8819660112\},\\
&\{x_1 = x_2 = x_3 = 1, x_4 = x_5 \approx 3.118033988\}.
\end{aligned}
$$

In conclusion, we find ten different G-invariant Einstein metrics on homogeneous manifold E_6/A_1.

Remark 4.1. One can consider the homogeneous manifold E_6/A_1^2, but the Einstein metrics on which is the same as the above up to isometry.

Case of E_6/A_3**.** Consider the homogeneous manifold E_6/A_3 with the decomposition

$$\mathfrak{m} = \mathrm{T} \oplus A_1^1 \oplus A_1^2 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3, \tag{4.9}$$

and Ad(A_3)-invariant metrics which is also Ad$(\mathrm{T} \oplus A_1^1 \oplus A_1^2 \oplus A_3)$-invariant on E_6/A_3 defined by

$$<,>= u_0 \cdot B|_{\mathrm{T}} + x_1 \cdot B|_{A_1^1} + x_2 \cdot B|_{A_1^2} + x_3 \cdot B|_{\mathfrak{p}_1} + x_4 \cdot B|_{\mathfrak{p}_2} + x_5 \cdot B|_{\mathfrak{p}_3}, \tag{4.10}$$

where $u_0, x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}^+$. By Lemmas 2.1 and 3.2, the components of Ricci tensor with respect to the metric (4.10) are as follows:

$$\begin{cases} r_{\mathrm{T}} = \dfrac{1}{4}\left(\dfrac{u_0}{2x_3^2} + \dfrac{u_0}{2x_4^2}\right), \\ r_{A_1^1} = \dfrac{1}{12}\left(\dfrac{1}{2x_1} + \dfrac{x_1}{x_4^2} + \dfrac{3x_1}{2x_5^2}\right), \\ r_{A_1^2} = \dfrac{1}{12}\left(\dfrac{1}{2x_2} + \dfrac{x_2}{x_3^2} + \dfrac{3x_2}{2x_5^2}\right), \\ r_{\mathfrak{p}_1} = \dfrac{1}{2x_3} + \dfrac{1}{8}\left(\dfrac{x_3}{x_4x_5} - \dfrac{x_4}{x_5x_3} - \dfrac{x_5}{x_3x_4}\right) - \dfrac{1}{32}\left(\dfrac{u_0}{2x_3^2} + \dfrac{x_2}{x_3^2}\right), \\ r_{\mathfrak{p}_2} = \dfrac{1}{2x_4} + \dfrac{1}{8}\left(\dfrac{x_4}{x_5x_3} - \dfrac{x_5}{x_3x_4} - \dfrac{x_3}{x_4x_5}\right) - \dfrac{1}{32}\left(\dfrac{u_0}{2x_4^2} + \dfrac{x_1}{x_4^2}\right), \\ r_{\mathfrak{p}_3} = \dfrac{1}{2x_5} + \dfrac{1}{12}\left(\dfrac{x_5}{x_3x_4} - \dfrac{x_3}{x_4x_5} - \dfrac{x_4}{x_5x_3}\right) - \dfrac{1}{48}\left(\dfrac{3x_1}{2x_5^2} + \dfrac{3x_2}{2x_5^2}\right). \end{cases}$$

Further, the homogeneous Einstein equation is

$$\{r_{\mathrm{T}} - r_{A_1^1} = 0, r_{A_1^1} - r_{A_1^2} = 0, r_{A_1^2} - r_{\mathfrak{p}_1} = 0, r_{\mathfrak{p}_1} - r_{\mathfrak{p}_2} = 0, r_{\mathfrak{p}_2} - r_{\mathfrak{p}_3} = 0\},$$

which is equivalent to the following system of equations by setting $u_0 = 1$:

$$\begin{cases} g_0 = -3\,{x_1}^2{x_3}^2{x_4}^2 - 2\,{x_1}^2{x_3}^2{x_5}^2 - {x_3}^2{x_4}^2{x_5}^2 + 3\,x_1{x_3}^2{x_5}^2 + 3\,x_1{x_4}^2{x_5}^2 = 0, \\ g_1 = 3\,{x_1}^2x_2{x_3}^2{x_4}^2 + 2\,{x_1}^2x_2{x_3}^2{x_5}^2 - 3\,x_1{x_2}^2{x_3}^2{x_4}^2 - 2\,x_1{x_2}^2{x_4}^2{x_5}^2 - x_1{x_3}^2{x_4}^2{x_5}^2 \\ \quad + x_2{x_3}^2{x_4}^2{x_5}^2 = 0, \\ g_2 = 24\,{x_2}^2{x_3}^2x_4 + 22\,{x_2}^2x_4{x_5}^2 - 24\,x_2{x_3}^3x_5 + 24\,x_2x_3{x_4}^2x_5 - 96\,x_2x_3x_4{x_5}^2 + 24\,x_2x_3{x_5}^3 \\ \quad + 8\,{x_3}^2x_4{x_5}^2 + 3\,x_2x_4{x_5}^2 = 0, \\ g_3 = 2\,x_1{x_3}^2x_5 - 2\,x_2{x_4}^2x_5 + 16\,{x_3}^3x_4 - 32\,{x_3}^2x_4x_5 - 16\,x_3{x_4}^3 + 32\,x_3{x_4}^2x_5 + {x_3}^2x_5 - {x_4}^2x_5 = 0, \\ g_4 = 6\,x_1x_3{x_4}^2 - 6\,x_1x_3{x_5}^2 + 6\,x_2x_3{x_4}^2 - 8\,{x_3}^2x_4x_5 - 96\,x_3{x_4}^2x_5 + 96\,x_3x_4{x_5}^2 + 40\,{x_4}^3x_5 \\ \quad - 40\,x_4{x_5}^3 - 3\,x_3{x_5}^2 = 0. \end{cases}$$

We consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3, x_4, x_5]$ and an ideal I, generated by polynomials $\{g_0, g_1, g_2, g_3, g_4, zx_1x_2x_3x_4x_5 - 1\}$. We take a lexicographic ordering $>$ with $z > x_1 > x_2 > x_3 > x_4 > x_5$ for a monomial ordering on R. Then, with the aid of computer software MAPLE 15, we use the Gröbner basis for the ideal I to contain a polynomial $h(x_5)$ of degree 31 given by

$$
\begin{aligned}
h(x_5) = {} & 12611081656164228013442029815950198323190201163588773290369351680 00\,{x_5}^{31} \\
& - 2151788432405784864427357292265755888622754231825843818424015257600\,{x_5}^{30} \\
& - 2774119470594771842109777615792789059591610394561214417063934689280\,{x_5}^{29} \\
& + 5270344831923001751708303816759088216591749544446664977785609519104\,{x_5}^{28} \\
& - 267759863524378249157490847814955848518971246169904017730950922240\,{x_5}^{27} \\
& + 3382539429618210066981097554617654004425041655004361517303374807040\,{x_5}^{26} \\
& - 9157476976548260444114456636484244081952190409782190142886449250304\,{x_5}^{25} \\
& + 3265956979395568431463257163604494142186261853585696957494114123776\,{x_5}^{24} \\
& + 1113697025312385709566702871382152926625644868347096267275321556992\,{x_5}^{23} \\
& + 2112333421951376168597459647362124180457453099511088975109229199360\,{x_5}^{22} \\
& - 1631290981926813934052954101478705657981981503958339738878928916480\,{x_5}^{21} \\
& - 1395050115461884401753191844047758010380519323029179088958100982784\,{x_5}^{20} \\
& + 863710166438884070851640864653697900991831475370745010492288656384\,{x_5}^{19} \\
& + 241554806478645820779502711256075967575496587804599790265232872960\,{x_5}^{18} \\
& - 132746306295681893821384103002486474701425792860693217422446582528\,{x_5}^{17} \\
& - 32676027659284781509206156012974576068380158453057436146834110 08\,{x_5}^{16} \\
& + 88929891321020036502481458506864580753686928690173001848979050 88\,{x_5}^{15} \\
& - 13282010111910362924140370601406308162746205516298105873762104 00\,{x_5}^{14} \\
& - 98351649199565867328202679801373812742043192868226187056696624\,{x_5}^{13} \\
& + 57195416139228700916611387885001217660977148332951265434464548\,{x_5}^{12} \\
& - 8584293427926321735849799193107683389353503523382291011168608\,{x_5}^{11} \\
& + 658987123591863795218834332702519196588690183339615796158964\,{x_5}^{10} \\
& - 21514437280227836839597312688258324479163856784281755956140\,{x_5}^{9} \\
& - 562803146421214665893608417569171948144549496211000931459\,{x_5}^{8} \\
& + 69880133370536372146966764193862531901488645811504750528\,{x_5}^{7} \\
& - 6425711687436454624078281417734621909397832006130146 80\,{x_5}^{6} \\
& - 141173037619723104390974578987052013640153011100310880\,{x_5}^{5} \\
& + 4922149886224089807286830168142173254755229906374200\,{x_5}^{4} \\
& + 92795089337275364876653655416844512848233792832000\,{x_5}^{3} \\
& - 608452918033176063631016486369162856333252666 0000\,{x_5}^{2} \\
& - 5129216598301377143307946637505032203229340 0000\,x_5 \\
& + 458728260058419435605736700591266310794525 0000.
\end{aligned}
$$

By solving $h(x_5) = 0$ numerically, we have four positive and one negative solutions, which can be given approximately by $x_5 \approx 0.7601951682, x_5 \approx 0.9753459906, x_5 \approx 1.141009718, x_5 \approx 1.523722394, x_5 \approx -1.755049707$. With aid of computer we get all the solutions of system of equations

$\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, h(x_5) = 0\}$ with $x_1x_2x_3x_4x_5 \neq 0$ are as follows:

$\{x_1 \approx 0.1044773890, x_2 \approx 1.350603440, x_3 \approx 0.9684903881, x_4 \approx 0.6352925758, x_5 \approx 0.7601951682\}$,
$\{x_1 \approx 1.350603440, x_2 \approx 0.1044773890, x_3 \approx 0.6352925758, x_4 \approx 0.9684903881, x_5 \approx 0.7601951682\}$,
$\{x_1 \approx 0.1100292802, x_2 \approx 0.1051859906, x_3 \approx 1.141500322, x_4 \approx 0.6242772748, x_5 \approx 0.9753459906\}$,
$\{x_1 \approx 0.1051859906, x_2 \approx 0.1100292802, x_3 \approx 0.6242772748, x_4 \approx 1.141500322, x_5 \approx 0.9753459906\}$,
$\{x_1 \approx 1.126287414, x_2 \approx 1.862577839, x_3 \approx 1.181080704, x_4 \approx 0.7489214139, x_5 \approx 1.141009718\}$,
$\{x_1 \approx 1.862577839, x_2 \approx 1.126287414, x_3 \approx 0.7489214139, x_4 \approx 1.181080704, x_5 \approx 1.141009718\}$,
$\{x_1 \approx 1.191985212, x_2 \approx 0.1562720393, x_3 \approx 1.546458511, x_4 \approx 0.7400787612, x_5 \approx 1.523722394\}$,
$\{x_1 \approx 0.1562720393, x_2 \approx 1.191985212, x_3 \approx 0.7400787612, x_4 \approx 1.546458511, x_5 \approx 1.523722394\}$.

We remark that among these solutions, the ones with x_5 equal induce the same metrics up to isometry. As a result, there are four different G-invariant Einstein metrics on homogeneous manifold E_6/A_3.

Case of $\mathrm{E}_6/(A_1 \times A_3)$**.** Consider the homogeneous manifold $\mathrm{E}_6/(A_1^1 \times A_3)$ with the decomposition

$$\mathfrak{m} = \mathrm{T} \oplus A_1^2 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3, \tag{4.11}$$

and $\mathrm{Ad}(A_1^1 \times A_3)$-invariant metrics which is also $\mathrm{Ad}(\mathrm{T} \oplus A_1^1 \oplus A_1^2 \oplus A_3)$-invariant on $\mathrm{E}_6/(A_1^1 \times A_3)$ defined by

$$<,>= u_0 \cdot B|_{\mathrm{T}} + x_1 \cdot B|_{A_1^2} + x_2 \cdot B|_{\mathfrak{p}_1} + x_3 \cdot B|_{\mathfrak{p}_2} + x_4 \cdot B|_{\mathfrak{p}_3}, \tag{4.12}$$

where $u_0, x_1, x_2, x_3, x_4 \in \mathbb{R}^+$. By Lemmas 2.1 and 3.2, the components of Ricci tensor with respect to the metric (4.12) are as follows:

$$\begin{cases} r_{\mathrm{T}} = \dfrac{1}{4}\left(\dfrac{u_0}{2x_2^2} + \dfrac{u_0}{2x_3^2}\right), \\ r_{A_1^2} = \dfrac{1}{12}\left(\dfrac{1}{2x_1} + \dfrac{x_1}{x_2^2} + \dfrac{3x_1}{2x_4^2}\right), \\ r_{\mathfrak{p}_1} = \dfrac{1}{2x_2} + \dfrac{1}{8}\left(\dfrac{x_2}{x_3x_4} - \dfrac{x_3}{x_4x_2} - \dfrac{x_4}{x_2x_3}\right) - \dfrac{1}{32}\left(\dfrac{u_0}{2x_2^2} + \dfrac{x_1}{x_2^2}\right), \\ r_{\mathfrak{p}_2} = \dfrac{1}{2x_3} + \dfrac{1}{8}\left(\dfrac{x_3}{x_4x_2} - \dfrac{x_4}{x_2x_3} - \dfrac{x_2}{x_3x_4}\right) - \dfrac{1}{32}\dfrac{u_0}{2x_3^2}, \\ r_{\mathfrak{p}_3} = \dfrac{1}{2x_4} + \dfrac{1}{12}\left(\dfrac{x_4}{x_2x_3} - \dfrac{x_2}{x_3x_4} - \dfrac{x_3}{x_4x_2}\right) - \dfrac{1}{48}\dfrac{3x_1}{2x_4^2}. \end{cases}$$

Moreover, the homogeneous Einstein equation is

$$\{r_{\mathrm{T}} - r_{A_1^2} = 0, r_{A_1^2} - r_{\mathfrak{p}_1} = 0, r_{\mathfrak{p}_1} - r_{\mathfrak{p}_2} = 0, r_{\mathfrak{p}_2} - r_{\mathfrak{p}_3} = 0\},$$

which is equivalent to the following system of equations by setting $u_0 = 1$:

$$\begin{cases} g_0 = -3\,x_1^2x_2^2x_3^2 - 2\,x_1^2x_3^2x_4^2 - x_2^2x_3^2x_4^2 + 3\,x_1x_2^2x_4^2 + 3\,x_1x_3^2x_4^2 = 0, \\ g_1 = 24\,x_1^2x_2^2x_3 + 22\,x_1^2x_3x_4^2 - 24\,x_1x_2^3x_4 + 24\,x_1x_2x_3^2x_4 - 96\,x_1x_2x_3x_4^2 + 24\,x_1x_2x_4^3 \\ \quad + 8\,x_2^2x_3x_4^2 + 3\,x_1x_3x_4^2 = 0, \\ g_2 = -2\,x_1x_3^2x_4 + 16\,x_2^3x_3 - 32\,x_2^2x_3x_4 - 16\,x_2x_3^3 + 32\,x_2x_3^2x_4 + x_2^2x_4 - x_3^2x_4 = 0, \\ g_3 = 6\,x_1x_2x_3^2 - 8\,x_2^2x_3x_4 - 96\,x_2x_3^2x_4 + 96\,x_2x_3x_4^2 + 40\,x_3^3x_4 - 40\,x_3x_4^3 - 3\,x_2x_4^2 = 0. \end{cases}$$

We consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3, x_4]$ and an ideal I, generated by polynomials $\{g_0, g_1, g_2, g_3, zx_1x_2x_3x_4 - 1\}$. We take a lexicographic ordering $>$ with $z > x_1 > x_2 > x_3 > x_4$ for a monomial ordering on R. Then, with the aid of computer software MAPLE 15, we use the Gröbner basis for the ideal I to contain a polynomial $h(x_4)$ of degree 24 given by

$$\begin{aligned}
h(x_4) = {} & 72476006270718119441917460051459511141772221197844480000\,{x_4}^{24} \\
& - 45495066171698588865729456978767723694512098317120307200000\,{x_4}^{23} \\
& + 124079956270742842408317239310011418292880685126930923520000\,{x_4}^{22} \\
& - 1970081603696809915606955263099866285017189995656259829760\,{x_4}^{21} \\
& + 2133680174837814991281651924028555250280554719591199145984\,{x_4}^{20} \\
& - 1888122403913080816870939349773401896543897381362091006361\,{x_4}^{19} \\
& + 1580772871423545862182954895415322209635732819122337926348\,{x_4}^{18} \\
& - 1133934222137230298195154492478669981839433680181520865689\,{x_4}^{17} \\
& + 511251877284473517380175022259633105209997475809448647782\,{x_4}^{16} \\
& - 403396001117807932930095402077870112549798991179251580928\,{x_4}^{15} \\
& - 903609223572125644190343758413680417593338129263631754880\,{x_4}^{14} \\
& + 461550807018117050045055022022782250907255388226385775616\,{x_4}^{13} \\
& - 65891887023479142966230863046964022312825636044010288900\,{x_4}^{12} \\
& - 136754288831803450953013382264878381535329655366923182720\,{x_4}^{11} \\
& + 7404379070352221664927980007974819413355565861485929092\,{x_4}^{10} \\
& - 14839973678661123782277700618684935535502937184031561280\,{x_4}^{9} \\
& + 1752680243658642869963069054942874873140221266664126710\,{x_4}^{8} \\
& - 1298323375637342743747567164772691897345919970590787200\,{x_4}^{7} \\
& + 557098428728836786107911315677130125808601844881240\,{x_4}^{6} \\
& - 7614872308049576020573960694434807958692132824768\,{x_4}^{5} \\
& - 444217606400638301563490761267824391591116546648\,{x_4}^{4} \\
& + 16454149288639736436126283769710330401564494976\,{x_4}^{3} \\
& + 5178497514306536645205178876689360151036319040\,{x_4}^{2} \\
& - 416494521994260824005204961108166680962905600\,x_4 \\
& + 743025277331758306683208760331078413656560.
\end{aligned}$$

By solving $h(x_4) = 0$ numerically, we have four positive solutions, which can be given approximately by $x_4 \approx 0.7380550950, x_4 \approx 0.9342575736, x_4 \approx 0.9881689149$ and $x_4 \approx 1.544605922$. We remark that x_1, x_2, x_3 can be written into polynomials of x_4. As a result, we obtain the solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, h(x_4) = 0\}$ with $x_1x_2x_3x_4 \neq 0$ as follows:

$$\{x_1 \approx 1.330255613, x_2 \approx 0.9639605550, x_3 \approx 0.6236782194, x_4 \approx 0.7380550950\},$$
$$\{x_1 \approx 0.1012164462, x_2 \approx 1.121509362, x_3 \approx 0.6122385007, x_4 \approx 0.9342575736\},$$
$$\{x_1 \approx 0.1102891330, x_2 \approx 0.6235977518, x_3 \approx 1.152510591, x_4 \approx 0.9881689149\},$$
$$\{x_1 \approx 1.194390595, x_2 \approx 0.7396962365, x_3 \approx 1.566827089, x_4 \approx 1.544605922\}.$$

In conclusion, we find four different G-invariant Einstein metrics on homogeneous manifold $\mathrm{E}_6/(A_1 \times A_3)$.

Remark 4.2. One can consider Einstein metrics on $\mathrm{E}_6/(A_1^2 \times A_3)$, which is the same as the above results up to isometry.

Case of $\mathrm{E}_6/(A_1 \times A_1)$**.** Consider the homogeneous manifold $\mathrm{E}_6/(A_1^1 \times A_1^2)$ with the decomposition

$$\mathfrak{m} = \mathrm{T} \oplus A_3 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3, \tag{4.13}$$

and $\mathrm{Ad}(A_1^1 \times A_1^2)$-invariant metrics which is also $\mathrm{Ad}(\mathrm{T} \oplus A_1^1 \oplus A_1^2 \oplus A_3)$-invariant on $\mathrm{E}_6/(A_1^1 \times A_1^2)$ defined by

$$<,>= u_0 \cdot B|_{\mathrm{T}} + x_1 \cdot B|_{A_3} + x_2 \cdot B|_{\mathfrak{p}_1} + x_3 \cdot B|_{\mathfrak{p}_2} + x_4 \cdot B|_{\mathfrak{p}_3}, \tag{4.14}$$

where $u_0, x_1, x_2, x_3, x_4 \in \mathbb{R}^+$. By Lemmas 2.1 and 3.2, the components of Ricci tensor with respect to the metric (4.14) are as follows:

$$\begin{cases} r_{\mathrm{T}} = \dfrac{1}{4}\left(\dfrac{u_0}{2x_2^2} + \dfrac{u_0}{2x_3^2}\right), \\ r_{A_3} = \dfrac{1}{60}\left(\dfrac{5}{x_1} + \dfrac{5x_1}{2x_2^2} + \dfrac{5x_1}{2x_3^2} + \dfrac{5x_1}{x_4^2}\right), \\ r_{\mathfrak{p}_1} = \dfrac{1}{2x_2} + \dfrac{1}{8}\left(\dfrac{x_2}{x_3x_4} - \dfrac{x_3}{x_4x_2} - \dfrac{x_4}{x_2x_3}\right) - \dfrac{1}{32}\left(\dfrac{u_0}{2x_2^2} + \dfrac{5x_1}{2x_2^2}\right), \\ r_{\mathfrak{p}_2} = \dfrac{1}{2x_3} + \dfrac{1}{8}\left(\dfrac{x_3}{x_4x_2} - \dfrac{x_4}{x_2x_3} - \dfrac{x_2}{x_3x_4}\right) - \dfrac{1}{32}\left(\dfrac{u_0}{2x_3^2} + \dfrac{5x_1}{2x_3^2}\right), \\ r_{\mathfrak{p}_3} = \dfrac{1}{2x_4} + \dfrac{1}{12}\left(\dfrac{x_4}{x_2x_3} - \dfrac{x_2}{x_3x_4} - \dfrac{x_3}{x_4x_2}\right) - \dfrac{1}{48}\dfrac{5x_1}{x_4^2}. \end{cases}$$

Moreover, the homogeneous Einstein equation is

$$\{r_{\mathrm{T}} - r_{A_3} = 0, r_{A_3} - r_{\mathfrak{p}_1} = 0, r_{\mathfrak{p}_1} - r_{\mathfrak{p}_2} = 0, r_{\mathfrak{p}_2} - r_{\mathfrak{p}_3} = 0\},$$

which is equivalent to the following system of equations by setting $u_0 = 1$:

$$\begin{cases} g_0 = -2\,x_1^2x_2^2x_3^2 - x_1^2x_2^2x_4^2 - x_1^2x_3^2x_4^2 - 2\,x_2^2x_3^2x_4^2 + 3\,x_1x_2^2x_4^2 + 3\,x_1x_3^2x_4^2 = 0, \\ g_1 = 16\,x_1^2x_2^2x_3^2 + 8\,x_1^2x_2^2x_4^2 + 23\,x_1^2x_3^2x_4^2 - 24\,x_1x_2^3x_3x_4 + 24\,x_1x_2x_3^3x_4 - 96\,x_1x_2x_3^2x_4^2 \\ \quad + 24\,x_1x_2x_3x_4^3 + 16\,x_2^2x_3^2x_4^2 + 3\,x_1x_3^2x_4^2 = 0, \\ g_2 = 5\,x_1x_2^2x_4 - 5\,x_1x_3^2x_4 + 16\,x_2^3x_3 - 32\,x_2^2x_3x_4 - 16\,x_2x_3^3 + 32\,x_2x_3^2x_4 + x_2^2x_4 - x_3^2x_4 = 0, \\ g_3 = 20\,x_1x_2x_3^2 - 15\,x_1x_2x_4^2 - 8\,x_2^2x_3x_4 - 96\,x_2x_3^2x_4 + 96\,x_2x_3x_4^2 + 40\,x_3^3x_4 - 40\,x_3x_4^3 \\ \quad - 3\,x_2x_4^2 = 0. \end{cases}$$

We consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3, x_4]$ and an ideal I, generated by polynomials $\{g_0, g_1, g_2, g_3, z x_1 x_2 x_3 x_4 - 1\}$. We take a lexicographic ordering $>$ with $z > x_1 > x_2 > x_3 > x_4$ for a monomial ordering on R. Then, with the aid of computer software MAPLE 15, we use the Gröbner basis for the ideal I to contain a polynomial $h(x_4)$ of degree 17 given by

$$\begin{aligned} h(x_4) = {} & 110023745349636492444277742390625\, {x_4}^{17} - 633649771052061357670508068621875\, {x_4}^{16} \\ & + 1693219483961674637778856732365000\, {x_4}^{15} - 4316833335694592632675706273589750\, {x_4}^{14} \\ & + 8843142281870361254769815434556400\, {x_4}^{13} - 13636914380079309535418557047150830\, {x_4}^{12} \\ & + 17268089388308771236690983095499516\, {x_4}^{11} - 16542332326263296024484260888236418\, {x_4}^{10} \\ & + 9539252746514719228613145099164214\, {x_4}^{9} - 1216363759832722486352208197180776\, {x_4}^{8} \\ & - 2038138365512054114906312824023776\, {x_4}^{7} + 1194675635182956573455161439625750\, {x_4}^{6} \\ & - 183551938256104631960037855371472\, {x_4}^{5} - 208039301356602713300293199477 86\, {x_4}^{4} \\ & + 6531836600529448404797287589404\, {x_4}^{3} + 791935107278639477469266691 4\, {x_4}^{2} \\ & - 645358448199893697863925155 11\, x_4 + 37281575459782690699730491 71. \end{aligned}$$

By solving $h(x_4) = 0$ numerically, we have two positive and one negative solutions, which can be given approximately by $x_4 \approx 1.216390732, x_4 \approx 3.124923887$ and $x_4 \approx -0.4548317660$. With the aid of a computer, we obtain the solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, h(x_4) = 0\}$ with $x_1 x_2 x_3 x_4 \neq 0$ as follows:

$$\begin{aligned} &\{x_1 \approx 0.2932652315, x_2 \approx 1.330112486, x_3 \approx 0.6899374515, x_4 \approx 1.216390732\}, \\ &\{x_1 \approx 0.2932652315, x_2 \approx 0.6899374515, x_3 \approx 1.330112486, x_4 \approx 1.216390732\}, \\ &\{x_1 \approx 1.438513860, x_2 \approx 3.147955135, x_3 \approx 1.011105585, x_4 \approx 3.124923887\}, \\ &\{x_1 \approx 1.438513860, x_2 \approx 1.011105585, x_3 \approx 3.147955135, x_4 \approx 3.124923887\}. \end{aligned}$$

It is easy to see that the first two solutions induce the same metric up to isometry, and so do the later two solutions. In conclusion, we find two different G-invariant Einstein metrics on homogeneous manifold $\mathrm{E}_6/(A_1 \times A_1)$.

Case of $\mathrm{E}_6/(A_1 \times A_1 \times A_3)$**.** Consider the homogeneous manifold $\mathrm{E}_6/(A_1 \times A_1 \times A_3)$ with the decomposition

$$\mathfrak{m} = \mathrm{T} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3, \tag{4.15}$$

and $\mathrm{Ad}(A_1 \times A_1 \times A_3)$-invariant metrics which is also $\mathrm{Ad}(\mathrm{T} \oplus A_1 \oplus A_1 \oplus A_3)$-invariant on $\mathrm{E}_6/(A_1 \times A_1 \times A_3)$ defined by

$$< \,,\, >= u_0 \cdot B|_{\mathrm{T}} + x_1 \cdot B|_{\mathfrak{p}_1} + x_2 \cdot B|_{\mathfrak{p}_2} + x_3 \cdot B|_{\mathfrak{p}_3}, \tag{4.16}$$

where $u_0, x_1, x_2, x_3 \in \mathbb{R}^+$. By Lemmas 2.1 and 3.2, the components of Ricci tensor with respect to the metric (4.16) are as follows:

$$\begin{cases} r_{\mathrm{T}} = \frac{1}{4}\left(\frac{u_0}{2x_1^2} + \frac{u_0}{2x_2^2}\right), \\ r_{\mathfrak{p}_1} = \frac{1}{2x_1} + \frac{1}{8}\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_3x_1} - \frac{x_3}{x_1x_2}\right) - \frac{1}{32}\frac{u_0}{2x_1^2}, \\ r_{\mathfrak{p}_2} = \frac{1}{2x_2} + \frac{1}{8}\left(\frac{x_2}{x_3x_1} - \frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3}\right) - \frac{1}{32}\frac{u_0}{2x_2^2}, \\ r_{\mathfrak{p}_3} = \frac{1}{2x_3} + \frac{1}{12}\left(\frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3} - \frac{x_2}{x_3x_1}\right). \end{cases}$$

The homogeneous Einstein equation is

$$\{r_{\mathrm{T}} - r_{\mathfrak{p}_1} = 0, r_{\mathfrak{p}_1} - r_{\mathfrak{p}_2} = 0, r_{\mathfrak{p}_2} - r_{\mathfrak{p}_3} = 0\},$$

which is equivalent to the following system of equations by setting $u_0 = 1$:

$$\begin{cases} g_0 = -8\,{x_1}^3x_2 + 8\,x_1{x_2}^3 - 32\,x_1{x_2}^2x_3 + 8\,x_1x_2{x_3}^2 + 8\,{x_1}^2x_3 + 9\,{x_2}^2x_3 = 0, \\ g_1 = 16\,{x_1}^3x_2 - 32\,{x_1}^2x_2x_3 - 16\,x_1{x_2}^3 + 32\,x_1{x_2}^2x_3 + {x_1}^2x_3 - {x_2}^2x_3 = 0, \\ g_2 = -8\,{x_1}^2x_2 - 96\,x_1{x_2}^2 + 96\,x_1x_2x_3 + 40\,{x_2}^3 - 40\,x_2{x_3}^2 - 3\,x_1x_3 = 0. \end{cases}$$

We consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3]$ and an ideal I, generated by polynomials $\{g_0, g_1, g_2, zx_1x_2x_3 - 1\}$. We take a lexicographic ordering $>$ with $z > x_1 > x_2 > x_3$ for a monomial ordering on R. Then, with the aid of computer software MAPLE 15, we use the Gröbner basis for the ideal I to contain a polynomial $h(x_3)$ of degree 5 given by

$$h(x_3) = 2359296000\,{x_3}^5 - 6217728000\,{x_3}^4 + 6426104320\,{x_3}^3 - 2832337088\,{x_3}^2 + 316436504\,x_3 - 12778713.$$

By solving $h(x_3) = 0$ numerically, we have only one solution, which can be given approximately by $x_3 \approx 0.9460130230$. With the aid of a computer, we obtain the solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, h(x_3) = 0\}$ with $x_1x_2x_3 \neq 0$ as follows:

$$\{x_1 \approx 1.131543385, x_2 \approx 0.6115481021, x_3 \approx 0.9460130230\},$$
$$\{x_1 \approx 0.6115481021, x_2 \approx 1.131543385, x_3 \approx 0.9460130230\}.$$

It is easy to see that these two solutions induce the same metric up to isometry. In conclusion, we find one G-invariant Einstein metric on homogeneous manifold $\mathrm{E}_6/(A_1 \times A_1 \times A_3)$.

To summarize the above conclusions, we write the results in the table below:

Table 1. Number of G-invariant Einstein metrics on homogeneous manifolds arising from E_7 and E_6.

G	K_1	Decomposition	No.
E_7	A_1	$\mathfrak{m} = \mathrm{T} \oplus A_5 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$	6
	A_5	$\mathfrak{m} = \mathrm{T} \oplus A_1 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$	4
	$A_1 \times A_5$	$\mathfrak{m} = \mathrm{T} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$	2
E_6	A_1	$\mathfrak{m} = \mathrm{T} \oplus A_1 \oplus A_3 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$	10
	A_3	$\mathfrak{m} = \mathrm{T} \oplus A_1 \oplus A_1 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$	4
	$A_1 \times A_3$	$\mathfrak{m} = \mathrm{T} \oplus A_1 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$	4
	$A_1 \times A_1$	$\mathfrak{m} = \mathrm{T} \oplus A_3 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$	2
	$A_1 \times A_1 \times A_3$	$\mathfrak{m} = \mathrm{T} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$	1

5. Einstein-Randers metrics on some homogeneous manifolds

Consider $\mathrm{Ad}(K)$-invariant Einstein metrics on homogenous spaces G/K_1 given in the above section. By the equivalence of the adjoint representation and the isotropy representation of K_1 on $\mathfrak{m}$, the vector field

$$\tilde{W}|_{gK} = d(\tau(g))|_o(W), \quad \forall g \in G, W \in T$$

is well-defined, and it is G-invariant (see [2]). Furthermore, one can easily verify the equation

$$\langle [W, X]_{\mathfrak{m}}, Y\rangle + \langle X, [W, Y]_{\mathfrak{m}}\rangle = 0$$

holds for any $W \in T$ and $X, Y \in \mathfrak{m}$, using the facts that $\mathfrak{k}_0 \subset \mathfrak{k}$ and the metric is $\mathrm{Ad}(K)$-invariant. Then, by Lemma 1.2 the homogeneous metric

$$F(x, y) = \frac{\sqrt{[\langle W, y\rangle]^2 + \langle y, y\rangle\lambda}}{\lambda} - \frac{\langle W, y\rangle}{\lambda} \tag{5.1}$$

is a G-invariant Einstein-Randers metric on G/K_1 when $\langle W, W\rangle < 1$, and F is Riemannian if and only if $W = 0$. We proved Theorem 1.3.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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Appendix

$h(x_5) = 28673579100848574881047069574381084050781898649580724543209211321664729662678456497786169904210920939212951086389$7

$7554187676031867891021155890330861568000000000\,{x_5}^{74} +$

$2796702360491023511399046660536637056382478262600265755362699196643628199738907627007066161079220950613040640661232590966817$

$902565200918959678967922032640000000000\,{x_5}^{73} +$

$1291875369012442867211801075880067073381597480213557796617426583241242991214981592367762230991657789990452362121863422117755$9

$602942186611596383456289882112000000000\,{x_5}^{72} -$

$4983801537702228227925113026474382151720503216092876104910840001391075434202484157530847326670319544091020993901548182918350$3

$434877344571206868885121584332800000000\,{x_5}^{71} +$

$1721652153985630409857169329659502226032975341583317887710677089625662087322185442771778165512080219045660865452433056851075$5

$6800528549679786916627031827415040000000\,{x_5}^{70} -$

$4713490690364353104364268012175780148134566470808210642093395381439372113698779779813112767060313878656834968322148874587775$11

$317621496263659948975851938250752000000\,{x_5}^{69} +$

$1169343846793204543092251374952124574251593365613123778383782906902283447294895731557854147898548813765865576192956880617933$23

$4971285486629650950675675825255219200000\,{x_5}^{68} -$

$2643927715076808289168643976430859266446396231607558663997980719475089388472359495230413902199169436604363399484771025971361$64

$0228584260666639009674010445888880640000\,{x_5}^{67} +$

$4967707215536637075073765547761992189830320484048713754383770772979214594444223866619966981773745523930681536037861734118161$73

$3043595688975911067026104259430055936000\,{x_5}^{66} -$

$8768815363581668942409014591942797489319117509226473338484136537216345758210677435602510009558577388380891316336947583915030$95

$3578702037915493399568382530846038425600\,{x_5}^{65} +$

$1353996391599382735671148619891202772655823606889067093509028978601542176282751608583473339088348526857141341056905085655107$75

$23600251050430681892199159199283722321920\,{x_5}^{64} -$

$1679108547475437185660568212994645164643772461725891123158580966363405293001006874370650732383155195645634456790427758748319$89

$08599580765759349331228838708047291351040\,{x_5}^{63} +$

$2024377183626606775678510053831418719122245342425676478204228441602122708192528859546345061829970886879234512221697201074948$55

$95698219914491417172065621136135390494720\,{x_5}^{62} -$

$1643266925862135091118830990819637801462544715335474944082855405559362175761251892455585186736185234872856223681752840248061$70

$32265887475986153212766817543104516587520\,{x_5}^{61} +$

$7355439130809836845486384767020365332424998276903652724643631547557827905497452088087366669229720211353024333558239807474065$33

$4409090939090003560818136973056650772480\,{x_5}^{60} -$

$2977912903772032962612143499080289905511015530392193726337013667312534785808895250827591501651610554227219348629292917908109$36

$3314778934461429411559172876397864550400\,{x_5}^{59} -$

$9500253998611392484428919654195939879912597271205772735136817499598404335972099711536014296582951451150742367434863151527569$03

$0198079167959501373882539172166170050560\,{x_5}^{58} -$

$2087360513131713395807768087997422192583002098898462330491896155081139258048023286553351371173142016094678880454731682468958$79

$7458408485343768596289824887345690705920\,{x_5}^{57} +$

357820038513677601517671405104585167085413667252626307765376472526401324862847003444795713388616879476776345376853922923052932
1482266896205869758729238704226207531008 x_5^{56} −
663158529960431853656971893878302396064221608535144330731009251904451372008535280554899539322377095054343549855001605098015181
6502058416666705061851236622042973339648 x_5^{55} +
129242530385167128481743588157231297614638175401572620766575234711521316359776854211452524730076304278514688819944292487745435
9366514969302226386317264964998252409651 2 x_5^{54} −
172429464393321153901442571547631261975664984790646982622831580576997137931578313268799977159882950866560188283114344290363347
4323613088095256972691064488758338597683 2 x_5^{53} +
165327076363460249103451029975828727406508825987543171322369652086644826683112450572072351036337887941442239573091514025440074
1226368260454197177805071741458523776614 4 x_5^{52} −
157641322489197222974791760894139639287167405524049995112204311728514369131747209549168434857906361754329989939844856317929648
2188329590638742824302343770886621259366 4 x_5^{51} +
111515718774216562668698476201485733792099203307133355990409551170489102120550685358628480599783405172754004888814621548232269
9943019867115442669000380695365772561612 8 x_5^{50} −
251284127105075570014905106736797228201656729304022954145781236710503499951371272596452015227235790668846535701424226691495345
931625031668974938832815014874488347033 6 x_5^{49} −
500677076164051058426443385400134248987615816137846857500109109415745747151204297709666521875803117629403380212254167517987363
160583621714473963960185791024061284352 x_5^{48} +
163101638875856546751259251885942683965746990016893375296240776310201730956981584069173297908931810073096408358782797532258040
787102890744575767565167998301571226009 6 x_5^{47} −
421370843551487087941615573592545372772562641380654830378663535697334040971945605805801195876506191929102342776852116782159153
650085270373278127544852977937167741747 2 x_5^{46} +
123834109038089934671294472137246930069523060031714250844706619429291169662133105896189791403231667938090032771493056909574046
470625950132214674044712216058727183155 2 x_5^{45} +
182020970274658793719058657381437633418879215571814116696924003282242176870141717350730229351112909089925971481867309468099964
219602394875113115845281469488696902451 2 x_5^{44} +
400743055041300892963443539080044602830884292424436315228840509766618435766290960441960859284046170976587040945767175985967410
209841081365781438099257427726663221248 x_5^{43} −
424978255370012183621237732817242493573701195971726919570061852259609413271867163640427683432675518196930740559924270812831375
563611884253337782941503899872012306432 x_5^{42} −
173095529689168945586540432091991868245292417145936250123555793382238071295393560910168427565624671717695384495055692485081314
375502916279364328815214744547040514867 2 x_5^{41} +
281373438899048481105271134347921199661036418844255240056878070519516179425882512854013882329799066439692129594076722311464918 7
772292545158771089021123883697826216 96 x_5^{40} +
147860981274795965977669487707253191502190185421129845689285783532348539485837558077958714373430206086616628899513397648239882 7
757332314718512078892601607805397073920 x_5^{39} −
353018880768188340208723906896764823806245308584346903157358410373682214392816464843496880561727205226398332829802172155717634 8
612007639862156790451843288467630650 24 x_5^{38} −
643733485008734767675328103272134976486121021695730058350986749622659224208821096802945223125730240168450387063503989568983536 4
351598390705458126071217891607738675 20 x_5^{37} +
215158505162867417400087397227696169834462704214967345128389320686323310871231094476638097614256584644025418919038002419489109 9
477237230465300483508023623372721722 40 x_5^{36} +
157139206347850440302668117150049760898288856292903484323891770776275945637520026690277685911052100101092549578922322531170922 9

$6664056494911917083994077017516536947 2 x_5^{35} -$

$69700871983381428926314196377252976820264780609761818837335684357162073632953389789290290410638378043739831012380569958473337914855629458083803826009955084443825108 x_5^{34} -$

$21039536883924173758959469827482480642664709584650301287428288924287450818493830499812376134423795770455639033556720904039955038834734820067394966023586825729448032 x_5^{33} +$

$13196510398565918039953215966820938134210903881369471160767073704071969324789847237233804111369018360889339015526638778885819765501542665365346335641906308193200195 x_5^{32} +$

$12015927238786502969857849085583767487262154782165714699779363605396416822417516831465217855383079440269866240237073339560267724340259733871122374461510131986 03456 x_5^{31} -$

$15190727379527882880064220579100419411506945614527112277097548201427960617418216579306221938876053443298053719118704101876187422303546997109833686596137142438035 48 x_5^{30} +$

$46015023280184733821114426954466902067929876435869940637089284609485321653133664727011819586988699888407602886801894697871713071441713839568058060774361029056640 x_5^{29} +$

$10720373082405198253985509542640788041149224883563812647331512343212434416423320156313654507374804197761269691397792208107097790197501472675959421454435956784480 6 x_5^{28} -$

$12050078032081738994266788254546939998188636283797500610800299543318247300674388611288364518537336496998410075243321389996667910512463244554688058984458230168128 x_5^{27} -$

$45749191950588582905005363738959315943052394432361531880949863521269088105372493340419551210031010462141048993518161279540005613211044343328470820161356294 02216 x_5^{26} +$

$84676800557305017284983097958973273857366395571757210838756891274645118729573994360428242451319922082370092343269071879121476562287411287742314771414680735721 6 x_5^{25} +$

$11186402339311131838603010415394409811145116207857981647648446030702115966223332474897448256427151352851155554464418786536157390770109811963440115580417690736 5 x_5^{24} -$

$32304987931852937897402782574766290717397529575290959182975693649169591846522880220103519679166700954287385786744036306347961011129382412439897351900300621440 x_5^{23} -$

$11322403073508214449556774670920097300956221470400859796528999735961553158138961183555963992041135818134731335333163746870129483827568707597522893370731565 12 x_5^{22} +$

$73557433153755055561152302609325396801314563204267245579243762949000410255773648651598396357492947339655423905598797301427071621003094533743218371052366464 0 x_5^{21} -$

$14999532336243140460342507387232125203348694737868913721453851277144842529797407891993368294596675915038357527317300784842628919315680630275508211504721620 x_5^{20} -$

$93790187790092835366473254108550177825093842806627180539728701342500001022541298438005304377037775458088974138729565215126308440125853397233242540841666 56 x_5^{19} +$

$61524765582830949345537739715932283186189283706176529904251975080237084640266090986959482304132978373987264874488891690394328585181360163019593673628406 0 x_5^{18} +$

$34343640670702821254421560281933514675990169038961700875217207515201072175355348136123839352646898118015258668131588986722717059036401414884124685763040 x_5^{17} -$

$58278384971267019869328354966661338701659813093850280114469939845291630442213678817578421281270056399525398673069991662048879174300599521297323250258 11 x_5^{16} +$

$83984630739141066803315107707259211575562707197245602797299649625579793248385576232506725546616903847757553815834311900647411381754645075862666698950 4 x_5^{15} -$

$53565419416942921070058965050263538472851290931429804195241243197580383155402731310241372080185894444822751063727734696006788463098071336194295739100 x_5^{14} -$

1318377828150277777127336891641150250427350777659622675682615021476138295395231637196585132855588210523219410313968075431468830
7874888360639240646784 x_5^{13} +
1902077237100881325382190450900412331681260283403499858270746211313757599203601019544182069933187598481473443984343691055578661
6671742208899107923 74 x_5^{12} +
4511046190804774740008184941508686054929624817963028231582886217622724959404025149854302776138821203068788152179522042646665949
1107595548644366784 x_5^{11} −
1991829869529178252598639317542226777282549769238367714945744943637238303240545879220782478883955081401802987445060563657819254 7
5758421301210424256 x_5^{10} +
5525463302414569174625187350837831177167198568108845685272504608607381315336088049551088675517171587469973529053983408338810511
35329509909803360 x_5^{9} +
9262206884852009570179257703195743746202785342339468538701102654638957726165560723071303596803573478877546605001672549466872282
3336970481722075 x_5^{8} −
5672574475164801873135086854241321214182423482250684291875985672216895556293522790017523397206426241300113918610058494279352311
055797710041600 x_5^{7} −
1381908406226132452950875948252498427689518360441836766458835743894878227061669556619556797528744757532621857543694410597505098
45805856559400 x_5^{6} +
1526539052715504800597854929279754784690651844341401771106162829507870153113840725219008300417233514689735638232300926938731937
1263010656000 x_5^{5} +
1867003566556869050458512424751716750175295547371196312368667351875534823826528942209107253593189949525377797225073796702198937
19635025000 x_5^{4} −
2590040660497044940143711257052597650563737993192612085149492257367139346125068573373223873355302574449971615120713521428248841
2067200000 x_5^{3} −
2882564710234246038459307854969088478058847577006313927236505206307145638217985855809201639015509595615572799535357578656824734
18500000 x_5^{2} +
2224563290664978215283948523395639332992581815936524068816279095684157347469924868471458760599935821864409765646204754501435500
0000000 x_5 +
5321955484562939896106776126356300860859539556887924777311505619930765156630074075061463851472474466548206633328265696065429687
50000.