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Research article

Global stability of a multi-group delayed epidemic model with logistic growth

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Abstract: In this work, we aim to investigate the mechanism of a multi-group epidemic model taking into account the influences of logistic growth and delay time distribution. Despite the importance of the logistic growth effect in such models, its consideration remains rare. We show that $\mathcal{R}_0$ has a crusher role in the global stability of a disease-free and endemic equilibria. That is, if $\mathcal{R}_0$ is less than or equal to one, then the disease-free equilibrium is globally asymptotically stable, whereas, if $\mathcal{R}_0$ is greater than one, then a unique endemic equilibrium exists and is globally asymptotically stable. In addition, we construct suitable Lyapunov functions to investigate the global stability of disease-free and endemic equilibria. Finally, we introduce numerical simulations of the model.

Keywords: multi-group epidemic model; delay distribution; global stability; Lyapunov functions; logistic growth
Mathematics Subject Classification: 37N25, 92C42

1. Introduction

Mathematical models provide powerful tools for investigating the behavior and mechanism of infectious disease transmission. There are several types of epidemic models. specifically, the SIR model, which was developed to include births and deaths, the SIS model, which does not include immunity after infection, the SIRS model, which has a short period of immunity, the SEIS and SEIR models, which have a latent period of the disease to represent a person who is not yet infectious, and the MSIR model, which considers that children are born with immunity [1–3].

In more realistic models, we take into account the time delay effect in studying epidemiological diseases since the dynamic behavior of disease transmission at time t is affected by both this time and the previous time. Moreover, time delays are classified into two types: discrete delay and distributed delay [4]. Several authors have studied epidemic models with time delays (see, [5–8]). Beretta and

Takeuchi [7] analyzed the global stability of disease-free and endemic equilibria of the SIR model with distributed delays. McCluskey [8] used a Lyapunov function to prove the global stability of the endemic equilibrium in an SIR model with distributed delay and discrete delay. Tipsri and Chinviriyasit [6] investigated the local stability of an endemic equilibrium and a disease-free equilibrium in an SEIR epidemic model with a nonlinear incidence rate and time delay; they also studied the bifurcation of periodic solutions.

Furthermore, natural populations may grow exponentially for some time, but eventually are limited by the availability of resources which is called the carrying capacity. Some examples of applications of logistic growth. In [9] the authors used a logistic growth model to forecast COVID-19 in Egypt and Oman. Zhao et al. [10] applied logistic growth to study the new mutants of the SARS CoV-2 Alpha (i.e., B.1.1.7) variants in England. Many studies have included epidemiological models with logistic growth (see, for example, [11–13]). Wang et al. [11] evaluated an SIR model with a susceptible population that includes a logistic growth trait and bilinear incidence rate. They examined the global stability of the disease-free equilibrium and the local stability of the endemic equilibrium. Also, they derived the conditions for Hopf bifurcation to occur. Xu et al. [12] discussed a delayed SEIS model with logistic growth and saturation incidence. They proved the local and global stability of the disease-free and endemic equilibria. Also, they established the conditions of Hopf bifurcations at the endemic equilibrium. Perez et al. [13] investigated an SIR epidemic model with a saturated treatment rate, logistic growth and a nonlinear incidence rate; they also studied the local stability and various types of bifurcations.

Recently, multi-group epidemiological models have received a significant amount of interest as a way to depict the prevalence of a variety of infectious diseases in a heterogeneous population, such as chickenpox, measles, gonorrhea and mumps. The host population can be divided into several groups based on different communication or transmission patterns or geographical distributions. Examples of studies of multi-group epidemic models can be found in [14–21]. Lajmanovich and Yorke [14] investigated global asymptotic behavior in one of the earliest multi-group models and studied the gonorrhea model in a heterogeneous population. The global stability of the endemic equilibrium is one of the major challenges in the analysis of multi-group epidemic models [15, 22, 23]. Guo et al. [24] provided a comprehensive resolution to this issue by using graph theory; the authors investigated a class of multi-group SEIR models and demonstrated the global stability of the endemic equilibrium. Li et al. [16] discussed a multi-group epidemic model with distributed delays and proved that the global stability of disease-free and endemic equilibria for different values of $\mathcal{R}_0$. They used applied Lyapunov functions in a graph-theoretical approach; the proof and the model discussed have the following form:

$$\begin{aligned}
\frac{dS_k}{dt} &= \Lambda_k - \sum_{j=1}^{n} \beta_{kj} S_k \int_0^{+\infty} h_j(s) E_j(t-s) ds - d_k^S S_k, \\
\frac{dE_k}{dt} &= \sum_{j=1}^{n} \beta_{kj} S_k \int_0^{+\infty} h_j(s) E_j(t-s) ds - (d_k^E + \epsilon_k) E_k, \\
\frac{dI_k}{dt} &= \epsilon_k E_k - (d_k^I + \gamma_k) I_k, \\
\frac{dR_k}{dt} &= \gamma_k I_k - d_k^R R_k.
\end{aligned} \tag{1.1}$$

In this paper, we will use the same method as in [16] to demonstrate global stability of the disease-

free and endemic equilibria, but we will assume that the susceptible population in the kth group satisfies the logistic growth equation.

This paper is structured as follows. In Section 2 we will introduce the model; we prove tha global stability of the disease-free equilibrium P_0 and we prove tha global stability of the endemic equilibrium P^*. Section 4 introduces figures from the simulation to explain our main result. Section 5 contains the conclusions.

2. Model of multi-group epidemics with distributed delays and logistic growth

We assume that the susceptible population in the k-th group satisfies the logistic growth equation; then system (1.1) becomes

$$\begin{aligned}
\frac{dS_k}{dt} &= r_k S_k\left(1-\frac{S_k}{M_k}\right)-\sum_{j=1}^{n}\beta_{kj}S_k\int_0^{+\infty}h_j(s)E_j(t-s)ds-d_k^S S_k,\\
\frac{dE_k}{dt} &= \sum_{j=1}^{n}\beta_{kj}S_k\int_0^{+\infty}h_j(s)E_j(t-s)ds-(d_k^E+\epsilon_k)E_k,\\
\frac{dI_k}{dt} &= \epsilon_k E_k-(d_k^I+\gamma_k)I_k,\\
\frac{dR_k}{dt} &= \gamma_k I_k-d_k^R R_k.
\end{aligned} \tag{2.1}$$

Here $k = 1, ..., n$. S_k denotes the number of individuals in the susceptible population, E_k denotes the number of individuals in the infected but non-infectious population, I_k denotes the number of individuals in the infectious population and R_k denotes the number of individuals in the recovered population in the kth group. The nonnegative function $h_j(s)$ satisfies $\int_0^{+\infty}h_j(s)ds = 1$.

The model's parameters are all non-negative constants, which are as follows:

r_k	the intrinsic growth rate of the $S_k(t)$ class in the kth group,
M_k	the carrying capacity of the $S_k(t)$ class in the kth group,
β_{kj}	coefficient of transmission between compartments S_k and E_j,
d_k^S	the natural death rates of $S_k(t)$ in the kth group,
d_k^E	the natural death rates of $E_k(t)$ in the kth group,
d_k^I	the natural death rates of $I_k(t)$ in the kth group,
d_k^R	the natural death rates of $R_k(t)$ in the kth group,
ϵ_k	the rate of infection after the latent period in the kth group,
γ_k	the recovery rate of infectious individuals in the kth group.

We explain the model (2.1) by the transfer diagram, (see Figure 1),

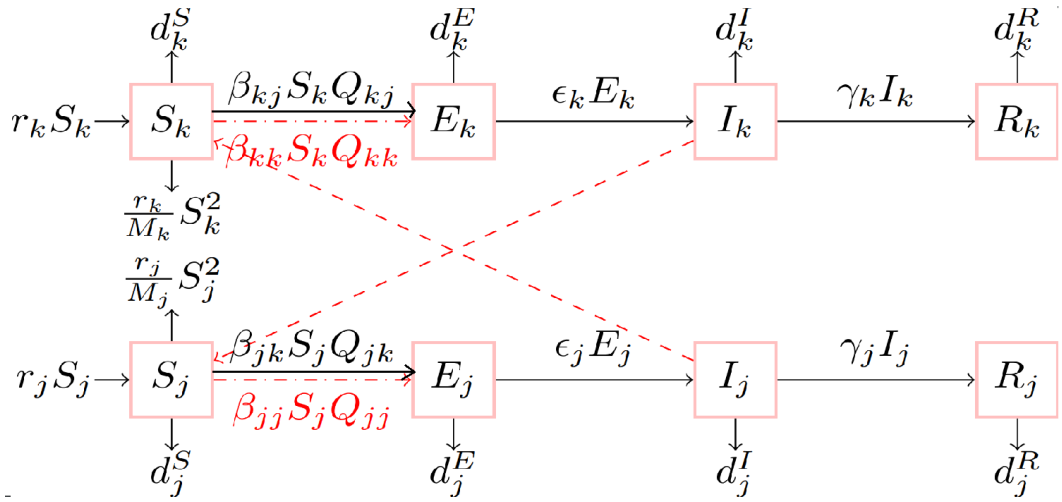

Figure 1. Transfer diagram for model (2.1).

Here $Q_{jk} = \int_0^\infty h_j(s)E_k(t-s)ds$.

Since the first two equations in the model (2.1) do not contain the I_k and R_k, then the model (2.1) can be reduced to

$$\begin{aligned} \frac{dS_k}{dt} &= r_k S_k\left(1-\frac{S_k}{M_k}\right) - \sum_{j=1}^{n}\beta_{kj}S_k\int_0^{+\infty} h_j(s)E_j(t-s)ds - d_k^S S_k, \\ \frac{dE_k}{dt} &= \sum_{j=1}^{n}\beta_{kj}S_k\int_0^{+\infty} h_j(s)E_j(t-s)ds - (d_k^E+\epsilon_k)E_k, \end{aligned} \tag{2.2}$$

with the initial conditions,

$$S_k(0) > 0, \quad E_k(\theta) = \phi_k \in BC((-\infty,0],(0,+\infty)),$$

where $BC((-\infty,0],(0,+\infty))$ is the space of functions that are both bounded and continuous from $(-\infty,0]$ to $(0,+\infty)$ with the norm $\|\phi\| = \sup_{\theta\le 0}|\phi(\theta)|$. From the first equation of (2.2), we obtain that $\frac{dS_k}{dt} \le r_k S_k\left(1-\frac{S_k}{M_k}\right) - d_k^S S_k$; hence, $\lim\limits_{t\longrightarrow\infty}[\sup S_k(t)] \le S_k^0$, where $S_k^0 = \frac{M_k}{r_k}(r_k - d_k^S)$, and $r_k > d_k^S$.

Moreover, adding up the two equations in (2.2) yields

$$\frac{d\left(S_k(t)+E_k(0)\right)}{dt} = \psi_k(S_k(t)) - (d_k^E+\epsilon_k)E_k(0),$$

where

$$\begin{aligned} \psi_k(S_k(t)) &= r_k S_k\left(1-\frac{S_k}{M_k}\right) - d_k^S S_k, \\ \frac{d\left(S_k(t)+E_k(0)\right)}{dt} &\le m_k - (d_k^E+\epsilon_k)E_k(0), \\ \frac{d\left(S_k(t)+E_k(0)\right)}{dt} &\le 2m_k - \frac{m_k}{S_k^0}S_k(t) - (d_k^E+\epsilon_k)E_k(0), \\ \frac{d\left(S_k(t)+E_k(0)\right)}{dt} &\le 2m_k - \bar{d}_k\left(S_k(t)+E_k(0)\right), \end{aligned}$$

where $m_k = \sup_{S_k \in [0, S_k^0]} \psi_k(S_k(t))$ and $\bar{d}_k = min\left\{\frac{m_k}{S_k^0}, d_k^E, \epsilon_k\right\}$; hence, $\lim_{t \longrightarrow \infty} \sup [S_k(t) + E_k(0)] \leq \frac{2m_k}{\bar{d}_k}$.

Therefore, for system (2.2), the following region is positively invariant.

$$\Theta = \left\{(S_1, E_1, ..., S_n, E_n) \in \mathbb{R}_+^{2n} : 0 \leq S_k(t) \leq S_k^0, 0 \leq S_k(t) + E_k(0) \leq \frac{2m_k}{\bar{d}_k}, E_k(s) \geq 0, s \in (-\infty, 0], k = 1, 2, ..., n\right\},$$

$$\Theta^o = \left\{(S_1, E_1, ..., S_n, E_n) \in \mathbb{R}_+^{2n} : 0 < S_k(t) < S_k^0, 0 < S_k(t) + E_k(0) < \frac{2m_k}{\bar{d}_k}, E_k(s) > 0, s \in (-\infty, 0], k = 1, 2, ..., n\right\},$$

where Θ^o is the interior of Θ.

There are two possible equilibria in system (2.2), i.e., the disease free-equilibrium $P_0 = (\frac{M_1}{r_1}(r_1 - d_1^S), 0, ..., \frac{M_n}{r_n}(r_n - d_n^S), 0)$ and the endemic equilibrium $P^* = (S_1^*, E_1^*, ..., S_n^*, E_n^*)$ where $S_k^*, E_k^* > 0$ satisfy the equations

$$r_k S_k^* = \frac{r_k}{M_k}(S_k^*)^2 + \sum_{j=1}^{n} \beta_{kj} S_k^* E_j^* + d_k^S S_k^*, \tag{2.3}$$

$$\sum_{j=1}^{n} \beta_{kj} S_k^* E_j^* = \left(d_k^E + \epsilon_k\right) E_k^*. \tag{2.4}$$

The basic reproduction number $\mathcal{R}_0$ is a critical threshold parameter for determining disease extinction and persistence in epidemic models. It is defined as the predicted number of secondary infectious cases produced in a completely susceptible population by a typical infected individual during their entire period of infectiousness. We will use the next generation matrix method [25] to determine the basic reproduction number $\mathcal{R}_0$ as follows:

$$\mathcal{F} = \left(\beta_{kj} S_k E_j\right), \quad \mathcal{V} = \left(\left(d_k^E + \epsilon_k\right) E_k\right),$$

$$F = \frac{\partial \mathcal{F}}{\partial E_j}|_{\text{free equilibrium}} = \begin{pmatrix} \beta_{11} S_1^0 & \beta_{12} S_1^0 & \cdots & \beta_{1n} S_1^0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} S_n^0 & \beta_{n2} S_n^0 & \cdots & \beta_{nn} S_n^0 \end{pmatrix},$$

$$V = \frac{\partial \mathcal{V}}{\partial E_k}|_{\text{free equilibrium}} = \begin{pmatrix} d_1^E + \epsilon_1 & 0 & \cdots & 0 \\ 0 & d_2^E + \epsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^E + \epsilon_n \end{pmatrix}.$$

The basic reproduction number is the spectral radius of FV^{-1} , where

$$V^{-1} = diag\left(\frac{1}{d_k^E + \epsilon_k}\right)_{n\times n} \text{ and } \quad \rho\left(FV^{-1}\right) = \rho\left(V^{-1}F\right).$$

That is

$$\mathcal{R}_0 = \rho\begin{pmatrix} \frac{\beta_{11}S_1^0}{d_1^E+\epsilon_1} & \frac{\beta_{12}S_1^0}{d_1^E+\epsilon_1} & \cdots & \frac{\beta_{1n}S_1^0}{d_1^E+\epsilon_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_{n1}S_n^0}{d_n^E+\epsilon_n} & \cdots & \cdots & \frac{\beta_{nn}S_n^0}{d_n^E+\epsilon_n} \end{pmatrix},$$

$$\mathcal{R}_0 = \rho\left(H_0\right) = \rho\left(\frac{\beta_{kj}S_k^0}{d_k^E + \epsilon_k}\right)_{n\times n} = \quad \rho\left(\frac{\beta_{kj}M_k(r_k - d_k^S)}{(d_k^E + \epsilon_k)r_k}\right)_{n\times n}$$

such that ρ is the spectral radius of the matrix [25].

3. Preliminaries

We make the following assumptions for the intrinsic growth rate of susceptible individuals in the kth group based on biological factors.

(A1) $\exists \quad S_k^0 > 0$ such that $\psi_k(S_k^0) = 0$ and

$$\left[\psi_k(S_k) - \psi_k(S_k^0)\right]\left(S_k - S_k^0\right) < 0 \text{ for } S_k \neq S_k^0,\ k = 1, 2, ..., n.$$

(A2) $\frac{S_k}{S_k^0} + \frac{S_k^0}{S_k} \geq 2$ for $S_k = S_k^0$.

(A3) $\left[\psi_k(S_k) - \psi_k(S_k^*)\right]\left(S_k - S_k^*\right) < 0$ for $S_k \neq S_k^*,\ S_k \in [0, S_k^0],\ k = 1, 2, ..., n.$

(A4) $\frac{S_k}{S_k^*} + \frac{S_k^*}{S_k} \geq 2$ for $S_k = S_k^*$.

Theorem 3.1. *Assume that B=(β_{kj}) is irreducible; then, the following holds:*

(1) If $\mathcal{R}_0 \leq 1$, (A1) and (A2) establishes that the disease-free equilibrium P_0 of system (2.2) is globally asymptotically stable in Θ.

(2) If $\mathcal{R}_0 > 1$, (A3) and (A4) establishes that the endemic equilibrium P^* of system (2.2) is unique and globally asymptotically stable in Θ^o.

Proof. Global stability of the disease-free equilibrium P_0.

We will prove that, for case 1 of the previous theorem, the disease-free equilibrium P_0 is globally asymptotically stable in Θ. Since B is irreducible, we know that the matrix $H_0 = \left(\frac{\beta_{kj}S_k^0}{d_k^E+\epsilon_k}\right)_{n\times n}$ and has a positive left eigenvector $(\Omega_1, \Omega_2, ..., \Omega_n)$ corresponding to the spectral radius $\rho(H_0) > 0$ (see [26–28]).

Put $a_k = \frac{\Omega_k}{d_k^E + \epsilon_k} > 0$ and consider the following Lyapunov function:

$$V(S_1, E_1, ..., S_n, E_n) = \sum_{k=1}^{n} a_k \left[\left(S_k - S_k^0 - S_k^0 \ln \frac{S_k}{S_k^0} \right) + E_k + \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_0^{\infty} \int_{t-s}^{t} h_j(s) E_j(\tau) d\tau ds \right].$$

Note that $\frac{r_k}{M_k} = \frac{(r_k - d_k^S)}{S_k^0}$ and $\frac{S_k}{S_k^0} + \frac{S_k^0}{S_k} \geq 2$ if and only if $S_k = S_k^0$.

$$\begin{aligned}
V' &= \sum_{k=1}^{n} a_k \left[\left(1 - S_k^0 \left(\frac{S_k^0}{S_k} \right) \left(\frac{1}{S_k^0} \right) \right) \frac{dS_k}{dt} + \frac{dE_k}{dt} + \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_0^{\infty} h_j(s) E_j(\tau) ds |_{t-s}^{t} \right] \\
V' &= \sum_{k=1}^{n} a_k \left[\left(1 - \frac{S_k^0}{S_k} \right) \left(r_k S_k \left(1 - \frac{S_k}{M_k} \right) - \sum_{j=1}^{n} \beta_{kj} S_k \int_0^{+\infty} h_j(s) E_j(t-s) ds - d_k^S S_k \right) \right. \\
&\quad + \sum_{j=1}^{n} \beta_{kj} S_k \int_0^{+\infty} h_j(s) E_j(t-s) ds - (d_k^E + \epsilon_k) E_k + \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_0^{+\infty} h_j(s) E_j(t) ds \\
&\quad \left. - \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_0^{\infty} h_j(s) E_j(t-s) ds \right] \qquad (3.1) \\
V' &= \sum_{k=1}^{n} \left[r_k S_k - \frac{r_k}{M_k} S_k^2 - \sum_{j=1}^{n} \beta_{kj} S_k \int_0^{\infty} h_j(s) E_j(t-s) ds - d_k^S S_k - r_k S_k^0 + \frac{r_k}{M_k} S_k^0 S_k \right. \\
&\quad + \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_0^{\infty} h_j(s) E_j(t-s) ds + d_k^S S_k^0 + \sum_{j=1}^{n} \beta_{kj} S_k \int_0^{\infty} h_j(s) E_j(t-s) ds - \left(d_k^E + \epsilon_k \right) E_k \\
&\quad \left. + \sum_{j=1}^{n} \beta_{kj} S_k^0 E_j(t) - \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_0^{\infty} h_j(s) E_j(t-s) ds \right] \\
V' &= \sum_{k=1}^{n} a_k \left[r_k S_k - \frac{r_k}{M_k} S_k^2 - d_k^S S_k - r_k S_k^0 + \frac{r_k}{M_k} S_k^0 S_k + d_k^S S_k^0 - \left(d_k^E + \epsilon_k \right) E_k + \sum_{j=1}^{n} \beta_{kj} S_k^0 E_j(t) \right];
\end{aligned}$$

since $\frac{r_k}{M_k} = \frac{(r_k - d_k^S)}{S_k^0}$, then

$$V' = \sum_{k=1}^{n} a_k \left[r_k S_k - \left(r_k - d_k^S\right) \frac{S_k^2}{S_k^0} - d_k^S S_k - r_k S_k^0 + \left(r_k - d_k^S\right) S_k + d_k^S S_k^0 - \left(d_k^E + \epsilon_k\right) E_k \right.$$

$$\left. + \sum_{j=1}^{n} \beta_{kj} S_k^0 E_j(t) \right],$$

$$V' = \sum_{k=1}^{n} a_k \left[\left(r_k - d_k^S\right) S_k \left(2 - \frac{S_k}{S_k^0} - \frac{S_k^0}{S_k}\right) + \sum_{j=1}^{n} \beta_{kj} S_k^0 E_j(t) - \left(d_k^E + \epsilon_k\right) E_k \right];$$

given $S_k(t) \equiv S_k^0$, $a_k = \frac{\Omega_k}{d_k^E + \epsilon_k} > 0$ and $2 - \frac{S_k}{S_k^0} - \frac{S_k^0}{S_k} = 0$, we have that

$$\begin{aligned} V' &\leq \sum_{k=1}^{n} \frac{\Omega_k}{d_k^E + \epsilon_k} \left(\sum_{j=1}^{n} \beta_{kj} S_k^0 E_j - \left(d_k^E + \epsilon_k\right) E_k \right) \\ &= (\Omega_1, \Omega_2, ..., \Omega_n) (H_0 E - E) \\ &= (\rho(H_0) - 1) (\Omega_1, \Omega_2, ..., \Omega_n) E \\ &= (\mathcal{R}_0 - 1) (\Omega_1, \Omega_2, ..., \Omega_n) E \leq 0, \quad if \quad \mathcal{R}_0 \leq 1 \end{aligned}$$

Here, $E(t) = (E_1(t), E_2(t), ..., E_n(t))^T$.

Consider $X = \{S_1, E_1(.), ..., S_n, E_n(.) \in \Theta | V' = 0\}$, and $Y = \{P_0\}$ represents the most compact invariant set in X; from $2 - \frac{S_k}{S_k^0} - \frac{S_k^0}{S_k} = 0$ and $S_k(t) \equiv S_k^0 = \frac{M_k}{r_k}(r_k - d_k^S)$, we obtain that $V' = 0$. Hence, from the first equation of system (2.2), we get

$$0 = \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_0^{\infty} h_j(s) E_j(t - s) ds.$$

By the irreducibility of β_{kj} and $S_k^0 > 0$, it follows that

$$0 = \int_0^{\infty} h_j(s) E_j(t - s) ds.$$

That is the disease-free equilibrium P_0 of system (2.2) is globally asymptotically stable in Θ . □

Proof. Global stability of the endemic equilibrium P^*.

We will prove that, for case 2 of the previous theorem, the endemic equilibrium P^* is globally asymptotically stable and unique in Θ^o. Choose $\bar{\beta}_{kj} = \beta_{kj} S_k^* E_j^*$, $1 \leq k \leq n$, $n \geq 2$ and

$$\bar{B} = \begin{pmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{pmatrix}.$$

It should be noted that $\bar{B}$ is the matrix Laplacian form of $(\bar{\beta}_{kj})_{n\times n}$. Since $(\beta_{kj})_{n\times n}$, matrix $(\bar{\beta}_{kj})_{n\times n}$ and $\bar{B}$ are irreducible. We know that the solution to the system $\bar{B}v = 0$ is positive (see [26–28]).

$$v = (v_1, v_2, ..., v_n) = (C_{11}, C_{22}, ..., C_{nn}),$$

since C_{kk} denotes the kth diagonal entry cofactor of $\bar{B}$ for k=1,2,...,n and $v_k = C_{kk} > 0$.

Consider the following Lyapunov function

$$V = \sum_{k=1}^{n} v_k \left[\left(S_k - S_k^* - S_k^* \ln \frac{S_k}{S_k^*} \right) + \left(E_k - E_k^* - E_k^* \ln \frac{E_k}{E_k^*} \right) + \left(\sum_{j=1}^{n} \beta_{kj} S_k^* \int_0^{\infty} \int_{t-s}^{t} h_j(s) E_j(\tau) d\tau ds \right) \right].$$

Note that (2.3), (2.4) yield that $2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k} \leq 0$, $\int_0^{\infty} h_j(s)ds = 1$ and $\sum_{j=1}^{n} \beta_{kj} S_k^* E_j^* = \sum_{j=1}^{n} \bar{\beta}_{kj}$. Calculating the time derivative of V, we obtain that

$$V' = \sum_{k=1}^{n} v_k \left[\left(1 - \frac{S_k^*}{S_k} \right) \frac{dS_k}{dt} + \left(1 - \frac{E_k^*}{E_k} \right) \frac{dE_k}{dt} + \sum_{j=1}^{n} \beta_{kj} S_k^* \int_0^{\infty} h_j(s) E_j(\tau) ds \Big|_{t-s}^{t} \right]$$

$$V' = \sum_{k=1}^{n} v_k \left[\left(1 - \frac{S_k^*}{S_k} \right) \left(r_k S_k - \frac{r_k}{M_k} S_k^2 - \sum_{j=1}^{n} \beta_{kj} S_k \int_0^{+\infty} h_j(s) E_j(t-s) ds - d_k^S S_k \right) \right.$$

$$+ \left(1 - \frac{E_k^*}{E_k} \right) \left(\sum_{j=1}^{n} \beta_{kj} S_k \int_0^{+\infty} h_j(s) E_j(t-s) ds - (d_k^E + \epsilon_k) E_k \right) + \sum_{j=1}^{n} \beta_{kj} S_k^* \int_0^{\infty} h_j(s) E_j(t) ds$$

$$\left. - \sum_{j=1}^{n} \beta_{kj} S_k^* \int_0^{\infty} h_j(s) E_j(t-s) ds \right];$$

since $\int_0^{\infty} h_j(s)ds = 1$, then

$$V' = \sum_{k=1}^{n} v_k \Big[r_k S_k - \frac{r_k}{M_k} S_k^2 - \sum_{j=1}^{n} \beta_{kj} S_k \int_0^{\infty} h_j(s) E_j(t-s) ds - d_k^S S_k - r_k S_k^* + \frac{r_k}{M_k} S_k^* S_k$$

$$+ \sum_{j=1}^{n} \beta_{kj} S_k^* \int_0^{\infty} h_j(s) E_j(t-s) ds + d_k^S S_k^* + \sum_{j=1}^{n} \beta_{kj} S_k \int_0^{\infty} h_j(s) E_j(t-s) ds - \left(d_k^E + \epsilon_k \right) E_k$$

$$- \sum_{j=1}^{n} \beta_{kj} S_k \frac{E_k^*}{E_k} \int_0^{\infty} h_j(s) E_j(t-s) ds + \left(d_k^E + \epsilon_k \right) E_k^* + \sum_{j=1}^{n} \beta_{kj} S_k^* E_j$$

$$- \sum_{j=1}^{n} \beta_{kj} S_k^* \int_0^{\infty} h_j(s) E_j(t-s) ds \Big],$$

$$V' = \sum_{k=1}^{n} v_k \Big[r_k S_k^* \frac{S_k}{S_k^*} - \frac{r_k}{M_k} S_k^2 - d_k^S S_k - r_k S_k^* + \frac{r_k}{M_k} S_k^* S_k + d_k^S S_k^* - (d_k^E + \epsilon_k) E_k^* \frac{E_k}{E_k^*}$$

$$- \sum_{j=1}^{n} \beta_{kj} S_k^* E_j^* \frac{S_k}{S_k^*} \frac{E_k^*}{E_k} \int_0^{\infty} h_j(s) \frac{E_j(t-s)}{E_j^*} ds + (d_k^E + \epsilon_k) E_k^* + \sum_{j=1}^{n} \beta_{kj} S_k^* E_j^* \frac{E_j}{E_j^*} \Big]; \tag{3.2}$$

from (2.3), (2.4) and $\sum_{j=1}^{n}\beta_{kj}S_k^*E_j^* = \sum_{j=1}^{n}\bar{\beta}_{kj}$, substituting in (3.2) we have

$$\begin{aligned}
V' = &\sum_{k=1}^{n} v_k\Bigg\{\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k}{S_k^*} + \frac{r_k}{M_k}S_k^*S_k + d_k^S S_k - \frac{r_k}{M_k}S_k^2 - d_k^S S_k - \sum_{j=1}^{n}\bar{\beta}_{kj}\\
&- \frac{r_k}{M_k}(S_k^*)^2 - d_k^S S_k^* + \frac{r_k}{M_k}S_k^*S_k + d_k^S S_k^* - \sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_k}{E_k^*}\\
&- \sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k}{S_k^*}\frac{E_k^*}{E_k}\int_0^{\infty} h_j(s)\frac{E_j(t-s)}{E_j^*}ds + \sum_{j=1}^{n}\bar{\beta}_{kj} + \sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_j}{E_j^*}\Bigg\},\\
V' = &\sum_{k=1}^{n} v_k\Bigg\{\frac{r_k}{M_k}S_kS_k^*\left(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k}\right) + \sum_{j=1}^{n}\bar{\beta}_{kj}\left(\frac{S_k}{S_k^*} - \frac{S_k}{S_k^*}\frac{E_k^*}{E_k}\int_0^{\infty} h_j(s)\frac{E_j(t-s)}{E_j^*}ds\right)\Bigg\}\\
&+ \sum_{k=1}^{n}\sum_{j=1}^{n} v_k\bar{\beta}_{kj}\left(\frac{E_j}{E_j^*} - \frac{E_k}{E_k^*}\right);
\end{aligned}$$

given $S_k = S_k^*$, that is $2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k} = 0$, t follows that

$$V' \le \sum_{k=1}^{n}\sum_{j=1}^{n} v_k\bar{\beta}_{kj}\left(\frac{S_k}{S_k^*} - \frac{S_k}{S_k^*}\frac{E_k^*}{E_k}\int_0^{\infty} h_j(s)\frac{E_j(t-s)}{E_j^*}ds\right) + \sum_{k=1}^{n}\sum_{j=1}^{n} v_k\bar{\beta}_{kj}\left(\frac{E_j}{E_j^*} - \frac{E_k}{E_k^*}\right) \tag{3.3}$$

from the classical results of graph theory [28], we have

$$\sum_{k=1}^{n}\sum_{j=1}^{n} v_k\bar{\beta}_{kj}\left(\frac{E_j}{E_j^*} - \frac{E_k}{E_k^*}\right) = 0$$

then (3.3) becomes

$$V' \le \sum_{k=1}^{n}\sum_{j=1}^{n} v_k\bar{\beta}_{kj}\left(\frac{S_k}{S_k^*} - \frac{S_k}{S_k^*}\frac{E_k^*}{E_k}\int_0^{\infty} h_j(s)\frac{E_j(t-s)}{E_j^*}ds\right),$$

and, given $a - 1 - \ln a \ge 0$ for any $a > 0$, where $a = \frac{S_k}{S_k^*}\frac{E_k^*}{E_k}\int_0^{\infty} h_j(s)\frac{E_j(t-s)}{E_j^*}ds$ then we have

$$\begin{aligned}
\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k}{S_k^*}\frac{E_k^*}{E_k}\int_0^{\infty} h_j(s)\frac{E_j(t-s)}{E_j^*}ds &\ge \sum_{j=1}^{n}\bar{\beta}_{kj}\left(1 + \ln\frac{S_k}{S_k^*}\frac{E_k^*}{E_k}\int_0^{\infty} h_j(s)\frac{E_j(t-s)}{E_j^*}ds\right)\\
&= \sum_{j=1}^{n}\bar{\beta}_{kj}\left(1 + \ln\frac{S_k}{S_k^*} + \ln\frac{E_k^*}{E_k} + \ln\int_0^{\infty} h_j(s)\frac{E_j(t-s)}{E_j^*}ds\right).
\end{aligned}$$

That is V' becomes

$$\begin{aligned}
V' &\le \sum_{k=1}^{n}\sum_{j=1}^{n} v_k\bar{\beta}_{kj}\left(\frac{S_k}{S_k^*} - \left(1 + \ln\frac{S_k}{S_k^*} + \ln\frac{E_k^*}{E_k} + \ln\int_0^{\infty} h_j(s)\frac{E_j(t-s)}{E_j^*}ds\right)\right)\\
V' &\le -\sum_{k=1}^{n}\sum_{j=1}^{n} v_k\bar{\beta}_{kj}\left(1 - \frac{S_k}{S_k^*} + \ln\frac{S_k}{S_k^*} + \ln\frac{E_k^*}{E_k} + \ln\int_0^{\infty} h_j(s)\frac{E_j(t-s)}{E_j^*}ds\right).
\end{aligned}$$

We can prove that $V' \leq 0$ and $V' = 0$ if and only if

$$S_k = S_k^*, E_k = E_k^*, \text{and } E_j(t-s) = E_j^*. \tag{3.4}$$

By replacing (3.4) into the first equation of system (2.2), we get

$$0 = r_k S_k^* - \frac{r_k}{M_k}(S_k^*)^2 - \sum_{j=1}^{n} \beta_{kj} S_k^* E_j^* - d_k^S S_k^*. \tag{3.5}$$

Therefore, the largest compact invariant subset is

$$\{(S_1, E_1, ..., S_n, E_n) \in \Theta^o | V' = 0\}.$$

We already know that (3.5) at P^* holds by (2.3). As a result, by LaSalle's invariance principle (see [29], Theorem 5.3.1,): the singleton $\{P^*\}$ is the only compact invariant subset of the set where $V' = 0$.

That is, the endemic equilibrium P^* of system (2.2) is unique and globally asymptotically stable in Θ^o. □

4. Numerical simulations

Euler's method from [30] is used to validate the analytical results. For convenience, set $h_j(s) = e^{(-s)}$, $s \geq 0$ and the initial condition is $E_j(\theta) = v_j e^{\theta}, \theta \leq 0$, where $v_j > 0$. Then Eq (2.1) becomes

$$\begin{aligned}
S_k^{(i+1)} &= S_k^{(i)} + \left[r_k S_k^{(i)} - \frac{r_k}{M_k} S_k^{2(i)} - \frac{e^{-i\triangle t}}{2} \sum_{j=1}^{n} \beta_{kj} v_j S_k^{(i)} - e^{-i\triangle t} \sum_{j=1}^{2} \beta_{kj} S_k^{(i)} \sum_{\tau=1}^{i} e^{\tau \triangle t} E_j^{(\tau)}(s) \triangle t - d_k^S S_k^{(i)} \right] \triangle t, \\
E_k^{(i+1)} &= E_k^{(i)} + \left[\frac{e^{-i\triangle t}}{2} \sum_{j=1}^{n} \beta_{kj} v_j S_k^{(i)} + e^{-i\triangle t} \sum_{j=1}^{2} \beta_{kj} S_k^{(i)} \sum_{\tau=1}^{i} e^{\tau \triangle t} E_j^{(\tau)}(s) \triangle t - (d_k^E + \epsilon_k) E_k^{(i)} \right] \triangle t, \\
I_k^{(i+1)} &= I_k^{(i)} + \left[\epsilon_k E_k^{(i)} - (\gamma_k + d_k^I) I_k^{(i)} \right] \triangle t, \\
I_k^{(i+1)} &= I_k^{(i)} + \left[\gamma_k I_k^{(i)} - d_k^R R_k^{(i)} \right] \triangle t.
\end{aligned} \tag{4.1}$$

Here $i = 1, ..., n$, $j = 1, 2, 3, 4$, $k = 1, 2$ and for simplicity, we assume that $d_k^S = d_k^E = d_k^I = d_k^R = d_k$.

Here S_k, E_k, I_k, R_k and v_j have units of "population"; β_{kj} has units of 1/(population × time); $d_k^s, d_k^E, d_k^I, d_k^R, r_k, M_k, \epsilon_k$ and γ_k have units of 1/time. We chose these parameters from [31] for the deterministic model, but we replaced the exponential growth with logistic growth and assume the values of r_k and M_k.

From Figures 2 and 3, it can be seen that the disease-free equilibrium P_0 is globally asymptotically stable and unique for $\mathcal{R}_0 < 1$. That is, the delayed class (green solid line of group one and green dashed line of group two), infectious class (red solid line of group one and red dashed line of group two) and recovery class (yellow solid line of group one and yellow dashed line of group two) are almost close to zero, as well as all of the trajectories with varying initial conditions ending at the disease-free equilibrium P_0. To obtain the results we choose the values $v_1 = 2.5$, $v_2 = 3.5$, $r_1 = 4.6$, $r_2 = 5.75$, $M_1 = 40$, $M_2 = 50$, $\beta_{11} = 0.001$, $\beta_{12} = 0.002$, $\beta_{21} = 0.009$, $\beta_{22} = 0.0003$, $\epsilon_1 = 0.9$, $\epsilon_2 = 0.9$, $\gamma_1 = 0.8$,

$\gamma_2 = 1.75$, $d_1 = 0.4$, $d_2 = 0.5$, $\triangle t = 0.01$, $S_1(0) = 4$, $S_2(0) = 2.5$, $I_1(0) = 1$, $I_2(0) = 0.5$, $R_1(0) = 2$, $R_2(0) = 1.5$.

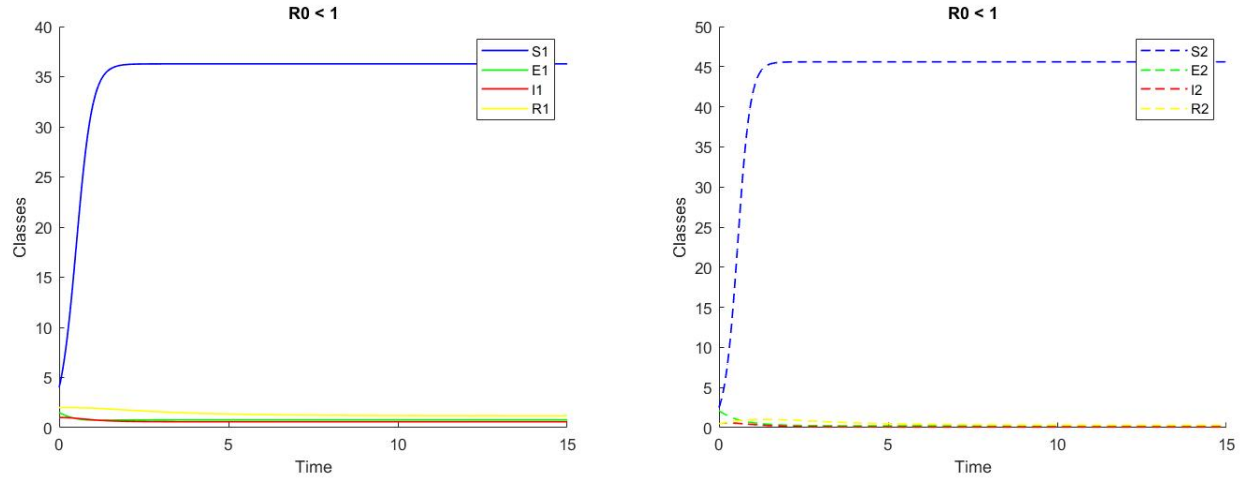

Figure 2. Solution of system (4.1) for $v_1 = 2.5$, $v_2 = 3.5$, $r_1 = 4.6$, $r_2 = 5.75$, $M_1 = 40$, $M_2 = 50$, $\beta_{11} = 0.001$, $\beta_{12} = 0.002$, $\beta_{21} = 0.009$, $\beta_{22} = 0.0003$, $\epsilon_1 = 0.9$, $\epsilon_2 = 0.9$, $\gamma_1 = 0.8$, $\gamma_2 = 1.75$, $d_1 = 0.4$, $d_2 = 0.5$, $j = 1, 2, 3, 4$, $k = 1, 2$, $\triangle t = 0.01$, $S_1(0) = 4$, $S_2(0) = 2.5$, $I_1(0) = 1$, $I_2(0) = 0.5$, $R_1(0) = 2$, $R_2(0) = 1.5$.

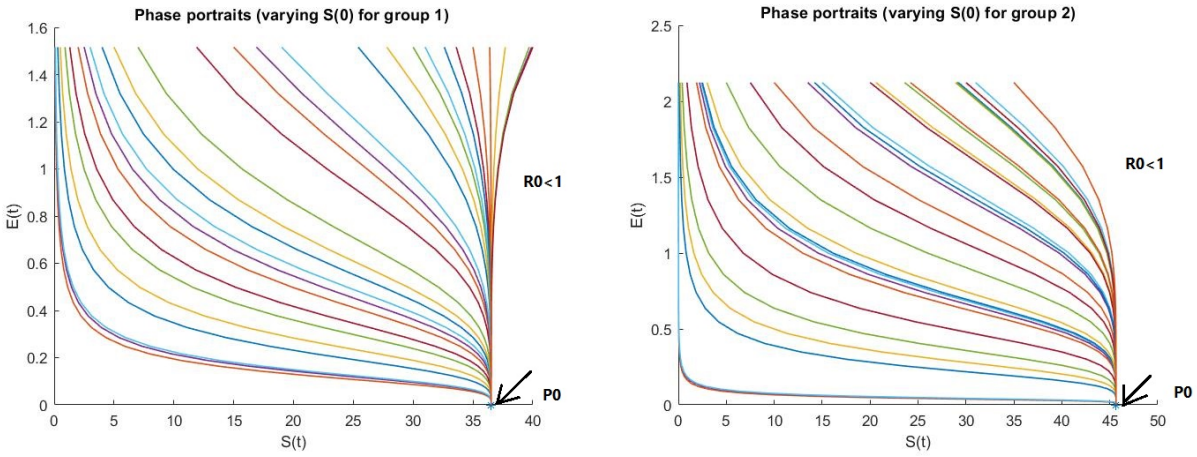

Figure 3. The phase portraits for the disease-free equilibrium of system (4.1), displaying the convergence of the proportions of susceptible and infected individuals towards a stable equilibrium point in the phase plane.

From Figures 4 and 5, it can be seen that the endemic equilibrium P^* is globally asymptotically stable and unique for $\mathcal{R}_0 > 1$. That is, the susceptible class (blue solid line of group one and blue dashed line of group two) and recovery class (yellow solid line of group one and yellow dashed line of group two) are greater than the delayed class (green solid line of group one and green dashed line of group two) and infectious class (red solid line of group one and red dashed line of group two), which means that the disease is endemic and controlled. This is true or all trajectories with varying initial conditions ending at the endemic equilibrium P^*. To obtain the results, we chose the values $v_1 = 2.5$,

$v_2 = 3.5$, $r_1 = 4.6$, $r_2 = 5.75$, $M_1 = 40$, $M_2 = 50$, $\beta_{11} = 0.135$, $\beta_{12} = 0.1$, $\beta_{21} = 0.0889$, $\beta_{22} = 0.24$, $\epsilon_1 = 0.9$, $\epsilon_2 = 0.9$, $\gamma_1 = 0.8$, $\gamma_2 = 1.75$, $d_1 = 0.4$, $d_2 = 0.5$, $\triangle t = 0.01$, $S_1(0) = 4$, $S_2(0) = 2.5$, $I_1(0) = 1$, $I_2(0) = 0.5$, $R_1(0) = 2$, $R_2(0) = 1.5$.

In Figures 2 and 4, the unit of "population" is thousands of people and the unit of "time" is days. The only difference between Figures 2 and 4 is the values of $\mathcal{R}_0$. In Figure 2 $\mathcal{R}_0 < 1$, and we can see that the disease will die out after some period of time in both group one on the 2 left-hand side and group two on the right-hand side. In Figure 4, $\mathcal{R}_0 > 1$, and we can see that the disease will prevail and be endemic in group one on the 4 left-hand side and group two on the right-hand side.

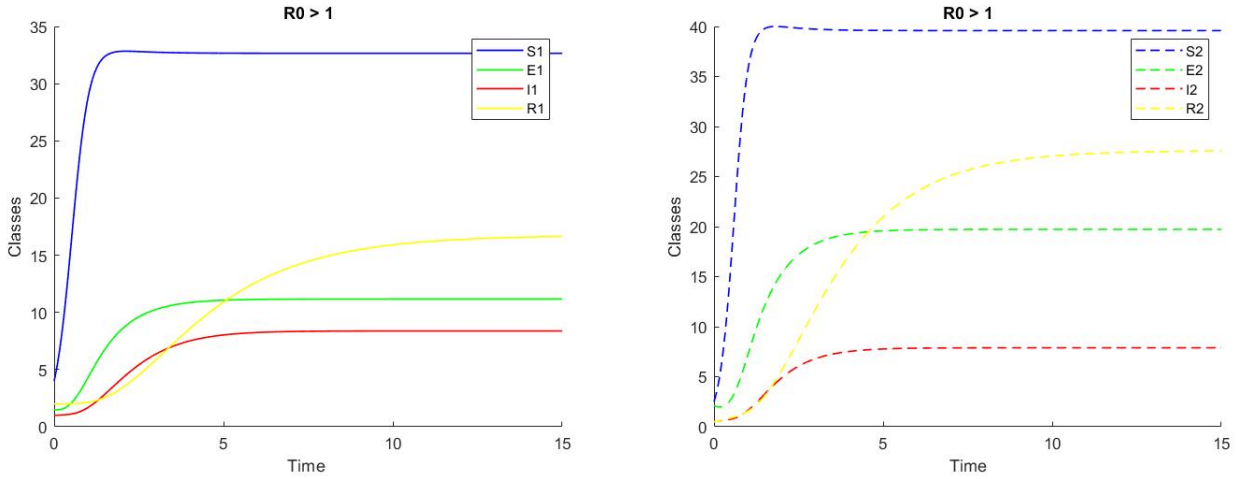

Figure 4. Solution of system (4.1) for $v_1 = 2.5$, $v_2 = 3.5$, $r_1 = 4.6$, $r_2 = 5.75$, $M_1 = 40$, $M_2 = 50$, $\beta_{11} = 0.135$, $\beta_{12} = 0.1$, $\beta_{21} = 0.0889$, $\beta_{22} = 0.24$, $\epsilon_1 = 0.9$, $\epsilon_2 = 0.9$, $\gamma_1 = 0.8$, $\gamma_2 = 1.75$, $d_1 = 0.4$, $d_2 = 0.5$, $j = 1, 2, 3, 4$, $k = 1, 2$, $\triangle t = 0.01$, $S_1(0) = 4$, $S_2(0) = 2.5$, $I_1(0) = 1$, $I_2(0) = 0.5$, $R_1(0) = 2$, $R_2(0) = 1.5$.

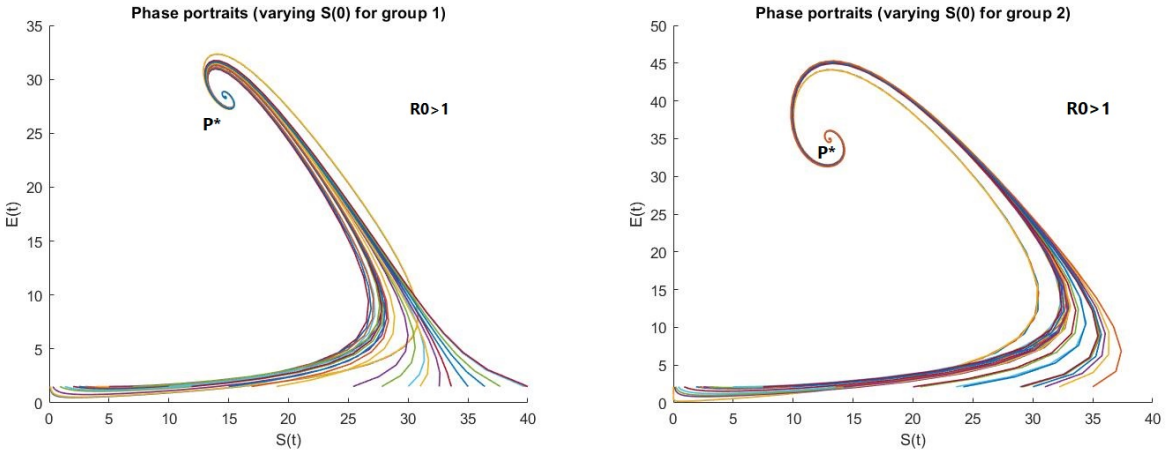

Figure 5. The phase portraits for the endemic equilibrium of system (4.1), displaying the convergence of the proportions of susceptible and infected individuals towards a stable equilibrium point in the phase plane.

5. Conclusions

We have proposed and investigated a multi-group SEIR epidemic model with infinite delays and logistic growth. We proved the global stability for both disease-free and endemic equilibria analytically and numerically. We found that the disease-free equilibrium P_0 is globally asymptotically stable when $\mathcal{R}_0$ is less than or equal to 1 ($\mathcal{R}_0 \leq 1$). Moreover, we proved that the endemic equilibrium P^* is unique and globally asymptotically stable if the parameter $\mathcal{R}_0 > 1$ In addition, we have constructed appropriate Lyapunov functions to show the global stability of the disease-free and endemic equilibria analytically. For the numerical solutions, we used Euler's method; hence, we proved the convergence of the solutions toward one point with different initial conditions, where we noticed that, if $\mathcal{R}_0$ is less than one, then all of the solution paths lead to the same disease-free equilibrium P_0 (see Figure 3), whereas, if $\mathcal{R}_0$ is greater than one, then all of the solution paths lead to the endemic equilibrium P^* (see Figure 5); this confirms the validity of the theorem. We compared our results with the results of other reviews [16, 24] and got almost the same results. However, adding the logistic growth is more realistic because the logistic growth takes the carrying capacity into account. It tells us that the population has a limit because it considers environmental limits, i.e., density, food abundance, resting place, sickness, parasites, etc. Finally, our study helps to improve the epidemic models by adding the logistic growth.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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