Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 23032–23045.
DOI: 10.3934/math.20231172
Received: 24 May 2023
Revised: 02 July 2023
Accepted: 06 July 2023
Published: 19 July 2023

Research article

On positive solutions of fractional pantograph equations within function-dependent kernel Caputo derivatives

Ridha Dida[1], **Hamid Boulares**[2], **Bahaaeldin Abdalla**[3], **Manar A. Alqudah**[4] and **Thabet Abdeljawad**[3,5,6,*]

[1] Depatement of Mathematics, Faculty of Sciences,University Badji Mokhtar Annaba, P.O. Box 12, Annaba, 23000, Algeria

[2] Laboratory of Analysis and Control of Differential Equations "ACED", University of 8 May 1945 Guelma, P.O. Box 401, 24000 Guelma, Algeria

[3] Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia

[4] Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

[5] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

[6] Department of Mathematics, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Korea

* **Correspondence:** Email: tabdeljawad@psu.edu.sa.

Abstract: Our main interest in this manuscript is to explore the main positive solutions (PS) and the first implications of their existence and uniqueness for a type of fractional pantograph differential equation using Caputo fractional derivatives with a kernel depending on a strictly increasing function Ψ (shortly Ψ-Caputo). Such function-dependent kernel fractional operators unify and generalize several types of fractional operators such as Riemann-Liouvile, Caputo and Hadamard etc. Hence, our investigated qualitative concepts in this work generalise and unify several existing results in literature. Using Schauder's fixed point theorem (SFPT), we prove the existence of PS to this equation with the addition of the upper and lower solution method (ULS). Furthermore using the Banach fixed point theorem (BFPT), we are able to prove the existence of a unique PS. Finally, we conclude our work and give a numerical example to explain our theoretical results.

Keywords: fractional differential equations; fixed point theorem; positive solution; Ψ-Caputo fractional derivative
Mathematics Subject Classification: 26A33, 34B10, 34B15

1. Introduction

The choice of certain fractional operator in modeling real world problems is always of interest to many authors working the filed of fractional calculus and its applications. The authors in [1], have shown how suitable fractional formulations are really extensions of the integer order definitions currently used in signal processing. The recent works [2–9] reflect several theoretical aspects and applications about types of fractional operators and their possible applying. For example, the reference [5] deals with the Ψ-Hilfer case and the articles [7, 8] deal with the Ψ-Caputo and Caputo operators, respectively.

The investigation of positive solutions of differential equations is vital due to their presence naturally when we model some real applications appearing in economics, physics, engineering and biology. As a generalization to the ordinary derivative calculus, the theory of fractional calculus appeared and started to develop since 1695 (see [10–12]). Since that date, many authors have authored several works to investigate qualitatively the positive attributes of FDE solutions [13, 14]. Part of the studies have been conducted exclusively on investigating the existence of solutions of problems using Caputo and generalized Caputo fractional derivatives (CFD). Indeed, Ψ-Caputo derivatives (Ψ-CFD) have been considered. Of special interest is the logarithmic case kernel which is called Hadamard (see [15, 16, 20, 25, 33]).

A pantograph is a mechanical linkage system consisting of four bars of equal length hinged at their ends. The equations that describe the motion of a pantograph are based on the principle of similar triangles. These equations determine the scaling factor or the relationship between the lengths of the bars and the size of the image produced. The equations are usually derived using trigonometry and vector algebra and they take into account the angles formed between the bars and the lengths of the individual bars. The goal of the equations is to determine the path of the stylus (or the end of the fourth bar) given the movement of the original object being traced, see for instance [17–19, 21–23] and the references therein.

Exclusively, the authors in [24], for $1 < \alpha \le 2$, and ${}^{C}D^{\alpha}$ is the usual CFD, examine the uniqueness and existence of the PS of the following FDE

$$\begin{cases} {}^{C}D^{\alpha}z(t) = f(t, z(t)),\ 0 < t \le 1, \\ z(0) = 0,\ z'(0) = \theta > 0, \end{cases}$$

where $f \in C([0,1] \times [0,\infty), [0,\infty))$. By utilizing ULS technique and FPTs, the authors in [25] got positive results. The novelty in this work is to generalize the results in [25] by utilizing the so-called Ψ-CFD. The Caputo Hadamard fractional derivatives fall within this class of operators by taking $\Psi(t) = \ln t$. Therefore, the idea of the pantograph is to be considered more generally. In fact, our real concern in this paper is to deal with the problems of PS to pantograph FDEs. It is worth to mentioning that the above works have been motivated and inspired by the papers [26–37].

Our main concern in this work is to deal with the PS of the below pantograph FDE:

$$\begin{cases} \mathcal{D}_{\ell}^{\alpha;\Psi}\phi(\eta) = \mathcal{F}(\eta, \phi(\eta), \phi(\ell + \vartheta\eta)) + \mathcal{D}_{\ell}^{\alpha-1;\Psi}\mathcal{G}(\eta, \phi(\ell + \vartheta\eta)),\ \eta \in [\ell, \mathcal{T}], \\ \phi(\ell) = \mu_1 > 0\ \phi'(\ell) = \mu_2 > 0, \end{cases} \tag{1.1}$$

where $\vartheta \in \left(0, \frac{\mathcal{T}-\ell}{\mathcal{T}}\right)$, $\phi(\ell(1+\vartheta)) = \phi_0 > 0$, $\mathcal{D}_{\ell}^{\alpha;\Psi}$ is Ψ-CFD of order $1 < \alpha \le 2$, $\mathcal{G}, \mathcal{F} : [\ell, \mathcal{T}] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions (CFs), $\mathcal{G}$ is non-decreasing on ϕ and $\mu_2 > \mathcal{G}(1, \phi_0)$.

By applying the Riemann-Liouville fractional integral with respect to the function Ψ to (1.1), we transform it to an equivalent integral equation on which we utilize ULS and SFPT, BFPT to prove the existence and uniqueness of the PS.

Our manuscript is to be divided as follows. Section 2 includes some key concepts, definitions, lemmas and theories that will be used in proving the main results. Section 3 will be devoted to our theoretical main results. Our main results generale those obtained in [24, 25]. In Section 4, we shall give an illustrative example. Section 5 includes our conclusions.

2. Essential preliminaries

The basic tools to be presented in this section can be recalled from [10, 12–14, 30–33, 38, 39], where more details can be found.

Let $\Psi : [\ell_1, \ell_2] \to \mathbb{R}$ be an increasing with $\Psi'(\eta) > 0$, $\forall\eta$. The symbol $Y = C([\ell, \mathcal{T}], \mathbb{R})$ represents the Banach space of CFs $\phi : [\ell, \mathcal{T}] \to \mathbb{R}$ by norm $\|\phi\| = \sup\{|\phi(\eta)| : \eta \in [\ell, \mathcal{T}]\}$.

We define $\mathcal{A} = \{\phi \in Y : \phi(\eta) \geq 0,\ \eta \in [\ell, \mathcal{T}]\}$ subset of Y consisting of all positive functions in Y. Suppose $\hbar_1, \hbar_2 \in \mathbb{R}^+$ with $\hbar_2 > \hbar_1$. For any $\phi, \chi \in [\hbar_1, \hbar_2]$, we associate the lower-control function

$$L(\eta, \phi, \chi) = \inf\{\mathcal{F}(\eta, \upsilon, \mu) : \phi \leq \upsilon \leq \hbar_2, \chi \leq \mu \leq \hbar_2\},$$

and the upper-control function

$$U(\eta, \phi, \chi) = \sup\{\mathcal{F}(\eta, \upsilon, \mu) : \hbar_1 \leq \upsilon \leq \phi, \hbar_1 \leq \mu \leq \chi\}.$$

The function $\mathcal{F}$ was defined above in Section 1. On the arguments ϕ, χ, L and U are monotonous non-decreasing and

$$L(\eta, \phi, \chi) \leq \mathcal{F}(\eta, \phi, \chi) \leq U(\eta, \phi, \chi).$$

Definition 2.1. *[10, 12, 39] For a function $\phi : [0, +\infty) \to \mathbb{R}$, the Riemann-Liouville fractional integral (RLFI) of order $\alpha > 0$ is defined as*

$$I^\alpha \phi(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} \phi(s) ds$$

where the Euler gamma function Γ is given by

$$\Gamma(\alpha) = \int_0^\infty e^{-\eta} \eta^{\alpha-1} d\eta.$$

Definition 2.2. *[10, 12, 39] The Ψ-RLFI of order $\alpha > 0$ for a CF $\phi : [\ell, \mathcal{T}] \to \mathbb{R}$ is defined as*

$$\mathcal{I}_\ell^{\alpha;\Psi} \phi(\eta) = \int_\ell^\eta \frac{(\Psi(\eta) - \Psi(s))^{\alpha-1}}{\Gamma(\alpha)} \Psi'(s) \phi(s) ds.$$

Definition 2.3. *[10, 12, 39] The CFD of order $\alpha > 0$ for a $\phi : [0, +\infty) \to \mathbb{R}$ is intended by*

$$D^\alpha \phi(\eta) = \frac{1}{\Gamma(n - \alpha)} \int_0^\eta (\eta - s)^{n-\alpha-1} \phi^{(n)}(s) ds,\ n - 1 < \alpha < n,\ n \in \mathbb{N}.$$

Definition 2.4. *[10, 12, 39] The Ψ-CFD of order* $\alpha > 0$ *for a* $\phi : [\ell, \mathcal{T}] \to \mathbb{R}$ *is defined by*

$$\mathcal{D}_{\ell}^{\alpha;\Psi}\phi(\eta) = \int_{\ell}^{\eta} \frac{(\Psi(\eta) - \Psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} \Psi'(s)\, \partial_{\Psi}^{n}\phi(s)ds,\ \eta > \ell,\ n-1 < \alpha < n$$

where $\partial_{\Psi}^{n} = \left(\frac{1}{\Psi'(\eta)}\frac{d}{d\eta}\right)^{n}, n \in \mathbb{N}$.

Lemma 2.1. *[10] Suppose* $q, \ell > 0$ *and* $\phi \in C([\ell, \hbar], \mathbb{R})$. *Then,* $\forall \eta \in [\ell, \hbar]$ *and by assuming* $F_{\ell}(\eta) = \Psi(\eta) - \Psi(\ell)$, *we have*

- $\mathcal{D}_{\ell}^{q;\Psi}\mathcal{I}_{\ell}^{q;\Psi}\phi(t) = \phi(t)$,
- $\mathcal{I}_{\ell}^{q;\Psi}(F_{\ell}(\eta))^{\ell-1} = \frac{\Gamma(\ell)}{\Gamma(\ell+q)}(F_{\ell}(\eta))^{\ell+q-1}$,
- $\mathcal{D}_{\ell}^{q;\Psi}(F_{\ell}(\eta))^{\ell-1} = \frac{\Gamma(\ell)}{\Gamma(\ell-q)}(F_{\ell}(\eta))^{\ell-q-1}$,
- $\mathcal{D}_{\ell}^{q;\Psi}(F_{\ell}(\eta))^{k} = 0,\ k \in \{0, \ldots, n-1\},\ n \in \mathbb{N},\ q \in (n-1, n]$.

Lemma 2.2. *[10, 32] Let* $n - 1 < \alpha_1 \leq n, \alpha_2 > 0,\ \ell > 0,\ \phi \in \mathbb{L}(\ell, \mathcal{T}),\ \mathcal{D}_{\ell_1}^{\alpha_1;\Psi}\phi \in \mathbb{L}(\ell, \mathcal{T})$. *Then, the differential equation*

$$\mathcal{D}_{\ell}^{\alpha_1;\Psi}\phi = 0$$

has the unique solution

$$\phi(\eta) = w_0 + w_1(\Psi(\eta) - \Psi(\ell)) + w_2(\Psi(\eta) - \Psi(\ell))^2 + \cdots + w_{n-1}(\Psi(\eta) - \Psi(\ell))^{n-1}$$

and

$$\mathcal{I}_{\ell}^{\alpha_1;\Psi}\mathcal{D}_{\ell}^{\alpha_1;\Psi}\phi(\eta) = \phi(\eta) + w_0 + w_1(\Psi(\eta) - \Psi(\ell)) + w_2(\Psi(\eta) - \Psi(\ell))^2 + \cdots + w_{n-1}(\Psi(\eta) - \Psi(\ell))^{n-1}$$

with $w_{\ell} \in \mathbb{R},\ \ell = 0, 1, \ldots, n-1$. *Furthermore,*

$$\mathcal{D}_{\ell}^{\alpha_2;\Psi}\mathcal{I}_{\ell}^{\alpha_1;\Psi}\phi(\eta) = \phi(\eta)$$

and

$$\mathcal{I}_{\ell}^{\alpha_1;\Psi}\mathcal{I}_{\ell}^{\alpha_2;\Psi}\phi(\eta) = \mathcal{I}_{\ell}^{\alpha_2;\Psi}\mathcal{I}_{\ell}^{\alpha_1;\Psi}\phi(\eta) = \mathcal{I}_{\ell}^{\alpha_1+\alpha_2;\Psi}\phi(\eta).$$

Lemma 2.3. *Let* $\phi \in C^1([\ell, \mathcal{T}])$, $\phi^{(2)}$ *and* $\frac{\partial \mathcal{G}}{\partial \eta}$ *exist. Then,* ϕ *is a solution of (1.1) if and only if*

$$\phi(\eta) = \mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell)) + \int_{\ell}^{\eta} \mathcal{G}(\mathfrak{s}, \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\, d\mathfrak{s} + \frac{1}{\Gamma(\alpha)} \int_{\ell}^{\eta} (\Psi(\eta) - \Psi(s))^{\alpha-1} \mathcal{F}(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\, d\mathfrak{s}. \tag{2.1}$$

Proof. Let ϕ be a solution of (1.1). Then, we have

$$\mathcal{I}_{\ell}^{\alpha;\Psi}\mathcal{D}_{\ell}^{\alpha;\Psi}\phi(\eta) = \mathcal{I}_{\ell}^{\alpha;\Psi}\left(\mathcal{F}(\eta, \phi(\eta), \phi(\ell + \vartheta\eta)) + \mathcal{D}_{\ell}^{\alpha-1;\Psi}\mathcal{G}(\eta, \phi(\ell + \vartheta\eta))\right),\ \ell < \eta \leq \mathcal{T}. \tag{2.2}$$

From Lemma 2.2, we got

$$
\begin{aligned}
&\phi(\eta) - \phi(\ell) - \phi'(\ell)(\Psi(\eta) - \Psi(\ell)) \\
&= \mathcal{I}_{\ell}^{\alpha;\Psi}\mathcal{D}_{\ell}^{\alpha-1;\Psi}\mathcal{G}(\eta, \phi(\ell + \vartheta\eta)) + \mathcal{I}_{\ell}^{\alpha;\Psi}\mathcal{F}(\eta, \phi(\eta), \phi(\ell + \vartheta\eta)) \\
&= \mathcal{I}_{\ell}\mathcal{I}_{\ell}^{\alpha-1;\Psi}\mathcal{D}_{\ell}^{\alpha-1;\Psi}\mathcal{G}(\eta, \phi(\ell + \vartheta\eta)) + \mathcal{I}_{\ell}^{\alpha;\Psi}\mathcal{F}(\eta, \phi(\eta), \phi(\ell + \vartheta\eta)) \\
&= \mathcal{I}_{\ell}(\mathcal{G}(\eta, \phi(\ell + \vartheta\eta)) - \mathcal{G}(\ell, \phi_0)) + \mathcal{I}_{\ell}^{\alpha;\Psi}\mathcal{F}(\eta, \phi(\eta), \phi(\ell + \vartheta\eta)) \\
&= \int_{\ell}^{\eta} \mathcal{G}(\mathfrak{s}, \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} - \mathcal{G}(\ell, \phi_0)(\Psi(\eta) - \Psi(\ell)) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{\ell}^{\eta}(\Psi(\eta) - \Psi(s))^{\alpha-1}\mathcal{F}(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s},
\end{aligned}
\tag{2.3}
$$

The converse can be proven straightforward as well. □

In what follows, we recall the FPTs that will be used to prove the uniqueness and existence of PS for Eq (1.1).

Definition 2.5. *Let $(Y, \|.\|)$ be a Banach space. Then, a mapping $\Theta : Y \to Y$ is called contraction. If there is a $l \in (0, 1)$ such that for every $\phi, \chi \in Y$, Θ we have*

$$
\|\Theta\phi - \Theta\chi\| \leq l\,\|\phi - \chi\|\,.
$$

Theorem 2.1. *(BFPT [38]) Let $\Omega \neq \emptyset$ be a closed convex subset of a Banach space Y and $\Theta : \Omega \to \Omega$ be a contraction mapping. Then, there is a unique $\phi \in \Omega$ with $\Theta\phi = \phi$.*

Theorem 2.2. *(BFPT [38]) Let $\Omega \neq \emptyset$ be a closed convex subset of a Banach space Y and $\Theta : \Omega \to \Omega$ be a continuous compact operator. So, Θ has a fixed point in Ω.*

3. Main results

In this section, we present the results of the existence of FDE (1.1). We also provide the necessary hypotheses for the uniqueness of (1.1).

We set the operator $\Theta : \mathcal{A} \longrightarrow Y$ by inversion Eq (2.1) and then apply SFPT

$$
\begin{aligned}
(\Theta\phi)(\eta) &= \mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell)) + \int_{\ell}^{\eta}\mathcal{G}(\mathfrak{s}, \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{\ell}^{\eta}(\Psi(\eta) - \Psi(s))^{\alpha-1}\mathcal{F}(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s},\ \eta \in [\ell, \mathcal{T}]
\end{aligned}
\tag{3.1}
$$

where the fixed point is needed to fulfill the identity operator equation $\Theta\phi = \phi$.

For the next step of our main results, the following forms are adopted.

(Σ1) Let $\phi_*, \phi^* \in \mathcal{A}$, as well as $\hbar_1 \leq \phi_*(\eta) \leq \phi^*(\eta) \leq \hbar_2$,

$$
\begin{cases}
\mathcal{D}_{\ell}^{\alpha;\Psi}\phi^*(\eta) - \mathcal{D}_{\ell}^{\alpha-1;\Psi}\mathcal{G}(\eta, \phi^*(\ell + \vartheta\eta)) \geq U(\eta, \phi^*(\eta), \phi^*(\ell + \vartheta\eta)), \\
\mathcal{D}_{\ell}^{\alpha;\Psi}\phi_*(\eta) - \mathcal{D}_{\ell}^{\alpha-1;\Psi}\mathcal{G}(\eta, \phi_*(\ell + \vartheta\eta)) \leq L(\eta, \phi_*(\eta), \phi_*(\ell + \vartheta\eta)),
\end{cases}
\tag{3.2}
$$

for any $\eta \in [\ell, \mathcal{T}]$.

(Σ2) For $\eta \in [\ell, \mathcal{T}]$ and $\phi_1, \phi_2, \chi_1, \chi_2 \in Y$, there exist $\beta_1, \beta_2, \beta_3 > 0$ such that

$$
\begin{aligned}
|\mathcal{G}(\eta, \chi_1) - \mathcal{G}(\eta, \phi_1)| &\leq \beta_1\,\|\chi_1 - \phi_1\|\,, \\
|\mathcal{F}(\eta, \chi_1, \chi_2) - \mathcal{F}(\eta, \phi_1, \phi_2)| &\leq \beta_2\,\|\chi_1 - \phi_1\| + \beta_3\,\|\chi_2 - \phi_2\|\,.
\end{aligned}
\tag{3.3}
$$

For (1.1), the functions ϕ^* and ϕ_* are known as the ULS.

Theorem 3.1. *If* $(\Sigma 1)$ *is satisfied, FDE (1.1) posses at least one solution* $\phi \in Y$ *and fulfills* $\phi_*(\eta) \le \phi(\eta) \le \phi^*(\eta), \eta \in [\ell, \mathcal{T}]$.

Proof. Set $\Omega = \{\phi \in \mathcal{A} : \phi_*(\eta) \le \phi(\eta) \le \phi^*(\eta),\ \eta \in [\ell, \mathcal{T}]\}$. If we use the norm $\|\phi\| = \max_{\eta\in[\ell,\mathcal{T}]} |\phi(\eta)|$, we see that $\|\phi\| \le \hbar_2$. So, we deduce that Ω is a convex and closed, bounded subset of Y. Moreover, the functions $\mathcal{G}$ and $\mathcal{F}$ being CF implies that Θ, marked by (3.1), is a CF on Ω . If $\phi \in \Omega$, there exist $c_{\mathcal{F}}, c_{\mathcal{G}} > 0$ constants as well as

$$\max\{\mathcal{F}(\eta, \phi(\eta), \phi(\ell + \vartheta\eta)) : \eta \in [\ell, \mathcal{T}],\ \phi(\eta), \phi(\ell + \vartheta\eta) \le \hbar_2\} < c_{\mathcal{F}} \tag{3.4}$$

and

$$\max\{\mathcal{G}(\eta, \phi(\ell + \vartheta\eta)) : \eta \in [\ell, \mathcal{T}],\ \phi(\ell + \vartheta\eta) \le \hbar_2\} < c_{\mathcal{G}}. \tag{3.5}$$

Then,

$$\begin{aligned} |(\Theta\phi)(\eta)| &\le |\mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell))| + \int_{\ell}^{\eta} |\mathcal{G}(\mathfrak{s}, \phi(\ell + \vartheta\mathfrak{s}))| \Psi'(\mathfrak{s})\, d\mathfrak{s} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{\ell}^{\eta} (\Psi(\eta) - \Psi(\mathfrak{s}))^{\alpha-1} |\mathcal{F}(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))| \Psi'(\mathfrak{s})\, d\mathfrak{s} \\ &\le \mu_1 + (\mu_2 + c_0 + c_{\mathcal{G}})(\Psi(\mathcal{T}) - \Psi(\ell)) + \frac{c_{\mathcal{F}}(\Psi(\mathcal{T}) - \Psi(\ell))^{\alpha}}{\Gamma(\alpha + 1)} \end{aligned} \tag{3.6}$$

where $|\mathcal{G}(\ell, \phi_0)| = c_0$. Thus,

$$\|\Theta\phi\| \le \mu_1 + (\mu_2 + c_0 + c_{\mathcal{G}})(\Psi(\mathcal{T}) - \Psi(\ell)) + \frac{c_{\mathcal{F}}(\Psi(\mathcal{T}) - \Psi(\ell))^{\alpha}}{\Gamma(\alpha + 1)}. \tag{3.7}$$

From which it follows that $\Theta(\Omega)$ is uniformly bounded. The equicontinuity of $\Theta(\Omega)$ is then can be handled. Let $\phi \in \Omega$ and $\ell \le \eta_1 < \eta_2 \le \mathcal{T}$. Then,

$$\begin{aligned} |(\Theta\phi)(\eta_1) - (\Theta\phi)(\eta_2)| &\le (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta_2) - \Psi(\eta_1)) \\ &\quad + \left| \int_{\ell}^{\eta_1} \mathcal{G}(\mathfrak{s}, \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\, d\mathfrak{s} - \int_{\ell}^{\eta_2} \mathcal{G}(\mathfrak{s}, \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\, d\mathfrak{s} \right| \\ &\quad + \left| \frac{1}{\Gamma(\alpha)} \int_{\ell}^{\eta_1} (\Psi(\eta_1) - \Psi(\mathfrak{s}))^{\alpha-1} \mathcal{F}(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\, d\mathfrak{s} \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_{\ell}^{\eta_2} (\Psi(\eta_2) - \Psi(\mathfrak{s}))^{\alpha-1} \mathcal{F}(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\, d\mathfrak{s} \right| \\ &\le (\mu_2 + c_0)(\Psi(\eta_2) - \Psi(\eta_1)) + \left| \int_{\eta_1}^{\eta_2} \mathcal{G}(\mathfrak{s}, \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\, d\mathfrak{s} \right| \\ &\quad + \left| \frac{1}{\Gamma(\alpha)} \int_{\ell}^{\eta_1} \left((\Psi(\eta_1) - \Psi(\mathfrak{s}))^{\alpha-1} - (\Psi(\eta_2) - \Psi(\mathfrak{s}))^{\alpha-1} \right) \right. \\ &\quad \left. \times \mathcal{F}(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\, d\mathfrak{s} \right| \\ &\quad + \left| \frac{1}{\Gamma(\alpha)} \int_{\eta_1}^{\eta_2} (\Psi(\eta_2) - \Psi(\mathfrak{s}))^{\alpha-1} \mathcal{F}(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\, d\mathfrak{s} \right| \\ &\le (\mu_2 + c_0 + c_{\mathcal{G}})(\Psi(\eta_2) - \Psi(\eta_1)) + \frac{c_{\mathcal{F}}}{\Gamma(\alpha + 1)} \left[(\Psi(\eta_2) - \Psi(\ell))^{\alpha} - (\Psi(\eta_1) - \Psi(\ell))^{\alpha} \right]. \end{aligned} \tag{3.8}$$

The right-hand side of above inequality approaches to zero as $\eta_1 \to \eta_2$. As a consequence, $\Theta(\Omega)$ is equicontinuous. Therefore, the compactness of $\Theta : \Omega \longrightarrow Y$ follows by Arzelè-Ascoli theorem. Finally, in order to employ SFPT, we need to prove that $\Theta(\Omega) \subseteq \Omega$. Let $\phi \in \Omega$. Then,

$$\begin{aligned}
(\Theta\phi)(\eta) &= \mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \mathcal{G}(\mathfrak{s}, \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_\ell^\eta (\Psi(\eta) - \Psi(\mathfrak{s}))^{\alpha-1}\mathcal{F}(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\le \mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \mathcal{G}(\mathfrak{s}, \phi^*(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\quad \frac{1}{\Gamma(\alpha)}\int_\ell^\eta (\Psi(\eta) - \Psi(\mathfrak{s}))^{\alpha-1} U(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\le \mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \mathcal{G}(\mathfrak{s}, \phi^*(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_\ell^\eta (\Psi(\eta) - \Psi(\mathfrak{s}))^{\alpha-1} U(\mathfrak{s}, \phi^*(\mathfrak{s}), \phi^*(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\le \phi^*(\eta),
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
(\Theta\phi)(\eta) &= \mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \mathcal{G}(\mathfrak{s}, \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_\ell^\eta (\Psi(\eta) - \Psi(\mathfrak{s}))^{\alpha-1}\mathcal{F}(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\ge \mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \mathcal{G}(\mathfrak{s}, \mathrm{x}_*(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_\ell^\eta (\Psi(\eta) - \Psi(\mathfrak{s}))^{\alpha-1} L(\eta, \phi(\mathfrak{s}), \phi(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\ge \mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \mathcal{G}(\mathfrak{s}, \mathrm{x}_*(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_\ell^\eta (\Psi(\eta) - \Psi(\mathfrak{s}))^{\alpha-1} L(\eta, \phi_*(\mathfrak{s}), \phi_*(\ell + \vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s} \\
&\ge \phi_*(\eta).
\end{aligned} \tag{3.10}$$

Consequently, $\phi_*(\eta) \le (\Theta\phi)(\eta) \le \phi^*(\eta)$, $\eta \in [\ell, \mathcal{T}]$, that is, $\Theta(\Omega) \subseteq \Omega$. Hence,SFPT asserts that the mapping Θ has at least one fixed point $\phi \in \Omega$. This means that FDE (1.1) admits at least one PS $\phi \in Y$ and $\phi_*(\eta) \le \phi(\eta) \le \phi^*(\eta)$, $\eta \in [\ell, \mathcal{T}]$. □

Next, we adopt and offer a different set of uses for the above theorem.

Corollary 3.1. *Suppose that CFs φ_1, φ_2, φ_3 and φ_4 exist. So,*

$$\begin{aligned}
&0 < \varphi_1(\eta) \le \mathcal{G}(\eta, \phi(\ell + \vartheta\eta)) \le \varphi_2(\eta) < \infty, \ (\eta, \phi(\ell + \vartheta\eta)) \in [\ell, \mathcal{T}] \times [0, +\infty), \\
&\mu_2 \ge \varphi_1(\ell), \ \mu_2 \ge \varphi_2(\ell),
\end{aligned} \tag{3.11}$$

and

$$0 < \varphi_3(\eta) \leq \mathcal{F}(\eta, \phi(\eta), \phi(\ell + \vartheta\eta)) \leq \varphi_4(\eta) < \infty,\ (\eta, \phi(\eta), \phi(\ell + \vartheta\eta)) \in [\ell, \mathcal{T}] \times ([0, +\infty))^2. \quad (3.12)$$

Then, the FDE (1.1) must have at least one PS $\phi \in Y$. Also,

$$\begin{aligned}\mu_1 + (\mu_2 - \varphi_1(\ell))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \varphi_1(\mathfrak{s})\Psi'(\mathfrak{s})\, d\mathfrak{s} + \mathcal{I}_\ell^{\alpha;\Psi}\varphi_3(\eta) \leq \phi(\eta) \\ \leq \mu_1 + (\mu_2 - \varphi_2(\ell))(\Psi(\eta) - \Psi(\ell)) \\ + \int_\ell^\eta \varphi_2(\mathfrak{s})\Psi'(\mathfrak{s})\, d\mathfrak{s} + \mathcal{I}_\ell^{\alpha;\Psi}\varphi_4(\eta).\end{aligned} \quad (3.13)$$

Proof. Starting from formula (3.12) and control function, we reach $\varphi_3(\eta) \leq L(\eta, \phi, \chi) \leq U(\eta, \phi, \chi) \leq \varphi_4(\eta)$, $(\eta, \phi(\eta), \chi(\eta)) \in [\ell, \mathcal{T}] \times [\hbar_1, \hbar_2] \times [\hbar_1, \hbar_2]$. We consider the equations

$$\begin{cases} \mathcal{D}_\ell^{\alpha;\Psi}\phi(\eta) = \varphi_3(\eta) + \mathcal{D}_\ell^{\alpha-1;\Psi}\varphi_1(\eta),\ \phi(\ell) = \mu_1,\ \phi'(\ell) = \mu_2, \\ \mathcal{D}_\ell^{\alpha;\Psi}\phi(\eta) = \varphi_4(\eta) + \mathcal{D}_\ell^{\alpha-1;\Psi}\varphi_2(\eta),\ \phi(\ell) = \mu_1,\ \phi'(\ell) = \mu_2. \end{cases} \quad (3.14)$$

Equation (3.14) is evidently equivalent to

$$\begin{aligned}\phi(\eta) &= \mu_1 + (\mu_2 - \varphi_1(\ell))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \varphi_1(\mathfrak{s})\Psi'(\mathfrak{s})\, d\mathfrak{s} + \mathcal{I}_\ell^{\alpha;\Psi}\varphi_3(\eta), \\ \phi(\eta) &= \mu_1 + (\mu_2 - \varphi_2(\ell))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \varphi_2(\mathfrak{s})\Psi'(\mathfrak{s})\, d\mathfrak{s} + \mathcal{I}_\ell^{\alpha;\Psi}\varphi_4(\eta).\end{aligned}$$

So, the first part of (3.14) involves

$$\phi(\eta) - \mu_1 - (\mu_2 - \varphi_1(\ell))(\Psi(\eta) - \Psi(\ell)) - \int_\ell^\eta \varphi_1(\mathfrak{s})\Psi'(\mathfrak{s})\, d\mathfrak{s} = \mathcal{I}_\ell^{\alpha;\Psi}\varphi_3(\eta) \leq \mathcal{I}_\ell^{\alpha;\Psi}(L(\eta, \phi(\eta), \phi(\ell + \vartheta\eta))) \quad (3.15)$$

and the second part of (3.14) suggests

$$\phi(\eta) - \mu_1 - (\mu_2 - \varphi_2(\ell))(\Psi(\eta) - \Psi(\ell)) - \int_\ell^\eta \varphi_2(\mathfrak{s})\Psi'(\mathfrak{s})\, d\mathfrak{s} = \mathcal{I}_\ell^{\alpha;\Psi}\varphi_4(\eta) \geq \mathcal{I}_\ell^{\alpha;\Psi}(U(\eta, \phi(\eta), \phi(\ell + \vartheta\eta))). \quad (3.16)$$

Hence, both equations in (3.14) have ULS. Therefore, the FDE (1.1) has at least one solution $\phi \in Y$ fulfilling (3.13) when Theorem 3.1 is conducted. □

Corollary 3.2. *Assume (3.11) is satisfied and* $0 < \sigma < \varphi(\eta) = \lim_{\phi,\chi\to\infty} \mathcal{F}(\eta, \phi, \chi) < \infty$ *for* $\eta \in [\ell, \mathcal{T}]$. *The FDE (1.1) must posses at least one PS* $\phi, \chi \in Y$.

Proof. If $\phi, \chi > \rho > 0$ then $0 \leq |\mathcal{F}(\eta, \phi, \chi) - \varphi(\eta)| < \sigma$ for any $\eta \in [\ell, \mathcal{T}]$. Hence, $0 < \varphi(\eta) - \sigma \leq \mathcal{F}(\eta, \phi, \chi) \leq \varphi(\eta) + \sigma$ for $\eta \in [\ell, \mathcal{T}]$ and $\rho < \phi, \chi < +\infty$. If $\max\{\mathcal{F}(\eta, \phi, \chi) : \eta \in [\ell, \mathcal{T}],\ \phi, \chi \leq \rho\} \leq \nu$ then $\varphi(\eta) - \sigma \leq \mathcal{F}(\eta, \phi, \chi) \leq \varphi(\eta) + \sigma + \nu$ for $\eta \in [\ell, \mathcal{T}]$ and $0 < \phi, \chi < +\infty$. By Corollary 3.1, the FDE (1.1) has at least one PS $\phi \in Y$ with

$$\mu_1 + (\mu_2 - \varphi_1(\ell))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \varphi_1(\mathfrak{s})\Psi'(\mathfrak{s})\, d\mathfrak{s} + \mathcal{I}_\ell^{\alpha;\Psi}\varphi(\eta) - \frac{\sigma(\Psi(\eta) - \Psi(\ell))^\alpha}{\Gamma(\alpha + 1)} \leq \phi(\eta)$$

$$\leq \mu_1 + (\mu_2 - \varphi_2(\ell))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \varphi_2(\mathfrak{s})\Psi'(\mathfrak{s})\,d\mathfrak{s} + \mathcal{I}_\ell^{\alpha;\Psi}\varphi(\eta)$$
$$+ \frac{(\sigma+\nu)(\Psi(\eta)-\Psi(\ell))^\alpha}{\Gamma(\alpha+1)}. \tag{3.17}$$

□

Corollary 3.3. *If we assume that* $0 < \sigma < \mathcal{F}(\eta, \phi(\eta), \phi(\ell + \vartheta\eta)) \leq \gamma_1\phi(\eta) + \gamma_2\phi(\ell + \vartheta\eta) + \eta < \infty$ *for* $\eta \in [\ell, \mathcal{T}]$ *and* σ, η, γ_1 *and* $\gamma_2 > 0$ *constants. Then, FDE (1.1) admits at least one PS* $\phi \in C([\ell, \delta])$ *where* $\delta > \ell$.

Proof. Consider the equation

$$\begin{cases} \mathcal{D}_\ell^{\alpha;\Psi}\phi(\eta) - \mathcal{D}_\ell^{\alpha-1;\Psi}\mathcal{G}(\eta, \phi(\ell+\vartheta\eta)) = \gamma_1\phi(\eta) + \gamma_2\phi(\ell+\vartheta\eta) + \eta, \ \ell < \eta \leq \mathcal{T}, \\ \phi(\ell) = \mu_1 > 0, \ \phi'(\ell) = \mu_2 > 0, \end{cases} \tag{3.18}$$

where $\phi(\ell(1+\vartheta)) = \phi_0 > 0$. Equation (3.18) has a solution of the form:

$$\phi(\eta) = \mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \mathcal{G}(\mathfrak{s}, \phi(\ell+\vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s}$$
$$+ \frac{1}{\Gamma(\alpha)}\int_\ell^\eta (\Psi(\eta) - \Psi(\mathfrak{s}))^{\alpha-1}(\gamma_1\phi(\eta) + \gamma_2\phi(\ell+\vartheta\eta))\Psi'(\mathfrak{s})\,d\mathfrak{s}$$
$$= \mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \mathcal{G}(\mathfrak{s}, \phi(\mathfrak{s}), \phi(\ell+\vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s}$$
$$+ \frac{\eta(\Psi(\eta)-\Psi(\ell))^\alpha}{\Gamma(\alpha+1)} + \frac{\gamma_1}{\Gamma(\alpha)}\int_\ell^\eta (\Psi(\eta)-\Psi(\mathfrak{s}))^{\alpha-1}\phi(\mathfrak{s})\Psi'(\mathfrak{s})\,d\mathfrak{s}$$
$$+ \frac{\gamma_2}{\Gamma(\alpha)}\int_\ell^\eta (\Psi(\eta)-\Psi(\mathfrak{s}))^{\alpha-1}\phi(\ell+\vartheta\mathfrak{s})\Psi'(\mathfrak{s})\,d\mathfrak{s}. \tag{3.19}$$

For ω a positive constant, and $\varpi \in (0,1)$, there exists $\delta > \ell$ such that $0 < \frac{(\gamma_1+\gamma_2)(\Psi(\delta)-\Psi(\ell))^\alpha}{\Gamma(\alpha+1)} < \varpi < 1$ and

$$\omega > (1-\varpi)^{-1}\left(\mu_1 + (\mu_2 + c_0 + c_{\mathcal{G}})(\Psi(\delta) - \Psi(\ell)) + \frac{\eta(\Psi(\delta)-\Psi(\ell))^\alpha}{\Gamma(\alpha+1)}\right). \tag{3.20}$$

Then, if $\ell \leq \eta \leq \delta$, the set $\mathfrak{B}_\omega = \{\phi \in Y : |\phi(\eta)| \leq \omega, \ \ell \leq \eta \leq \delta\}$ is a closed, convex and bounded subset of $C([\ell, \delta])$. The mapping $\Theta : \mathfrak{B}_\omega \longrightarrow \mathfrak{B}_\omega$ reported as

$$(\Theta\phi)(\eta) = \mu_1 + (\mu_2 - \mathcal{G}(\ell, \phi_0))(\Psi(\eta) - \Psi(\ell)) + \int_\ell^\eta \mathcal{G}(\mathfrak{s}, \phi(\ell+\vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s}$$
$$+ \frac{\eta(\Psi(\eta)-\Psi(\ell))^\alpha}{\Gamma(\alpha+1)} + \frac{\gamma_1}{\Gamma(\alpha)}\int_\ell^\eta (\Psi(\eta)-\Psi(\mathfrak{s}))^{\alpha-1}\phi(\mathfrak{s})\Psi'(\mathfrak{s})\,d\mathfrak{s}$$
$$+ \frac{\gamma_2}{\Gamma(\alpha)}\int_\ell^\eta (\Psi(\eta)-\Psi(\mathfrak{s}))^{\alpha-1}\phi(\ell+\vartheta\mathfrak{s})\Psi'(\mathfrak{s})\,d\mathfrak{s}, \tag{3.21}$$

is compact with the same approach as in the proof of Theorem 3.1. With same way,

$$|(\Theta\phi)(\eta)| \leq \mu_1 + (\mu_2 + c_0 + c_{\mathcal{G}})(\Psi(\mathcal{T}) - \Psi(\ell)) + \frac{\eta(\Psi(\mathcal{T})-\Psi(\ell))^\alpha}{\Gamma(\alpha+1)} + \frac{(\gamma_1+\gamma_2)(\Psi(\mathcal{T})-\Psi(\ell))^\alpha}{\Gamma(\alpha+1)}\|\phi\|. \tag{3.22}$$

Now, for $\phi \in \mathfrak{B}_\omega$ we have

$$|(\Theta\phi)(\eta)| \leq (1-\varpi)\,\omega + \varpi\omega = \omega$$

and hence $\|\Theta\phi\| \leq \omega$. We conclude, the SFPT emphasize that Θ posses at least one fixed point in $\mathfrak{B}_\omega$, and so Eq (3.18) has at least one PS $\phi^*(\eta)$ where $\ell < \eta < \delta$. Hence, if $\eta \in [\ell, \mathcal{T}]$ one can claim that

$$\begin{aligned}\phi^*(\eta) &= \mu_1 + (\mu_2 - \mathcal{G}(\ell,\phi_0))(\Psi(\eta)-\Psi(\ell)) + \int_\ell^\eta \mathcal{G}(\mathfrak{s},\phi^*(\ell+\vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s}\\&\quad+\frac{\eta(\Psi(\eta)-\Psi(\ell))^\alpha}{\Gamma(\alpha+1)}+\frac{\gamma_1}{\Gamma(\alpha)}\int_\ell^\eta(\Psi(\eta)-\Psi(\mathfrak{s}))^{\alpha-1}\phi^*(\mathfrak{s})\Psi'(\mathfrak{s})\,d\mathfrak{s}\\&\quad+\frac{\gamma_2}{\Gamma(\alpha)}\int_\ell^\eta(\Psi(\eta)-\Psi(\mathfrak{s}))^{\alpha-1}\phi^*(\ell+\vartheta\mathfrak{s})\Psi'(\mathfrak{s})\,d\mathfrak{s}.\end{aligned}\tag{3.23}$$

The term control function denotes

$$\begin{aligned}U(\eta,\phi^*(\eta),\phi^*(\ell+\vartheta\eta)) &\leq \gamma_1\phi^*(\eta)+\gamma_2\phi^*(\ell+\vartheta\eta)+\eta\\&= \mathcal{D}_\ell^{\alpha;\Psi}\phi^*(\eta)-\mathcal{D}_\ell^{\alpha-1;\Psi}\mathcal{G}(\eta,\phi^*(\ell+\vartheta\eta))\end{aligned}\tag{3.24}$$

hence ϕ^* is an upper PS of FDE (1.1). Secondly, one can take

$$\begin{aligned}\phi_*(\eta) &= \mu_1 + (\mu_2 - \mathcal{G}(\ell,\phi_0))(\Psi(\eta)-\Psi(\ell)) + \int_\ell^\eta \mathcal{G}(\mathfrak{s},\phi_*(\ell+\vartheta\mathfrak{s}))\Psi'(\mathfrak{s})\,d\mathfrak{s}\\&\quad+\frac{\sigma(\Psi(\eta)-\Psi(\ell))^\alpha}{\Gamma(\alpha+1)},\end{aligned}\tag{3.25}$$

as a lower PS of (1.1). By Theorem 3.1, the FDE (1.1) has at least one PS $\phi \in C([\ell,\delta])$ where $\delta > \ell$ and $\phi_*(\eta) \leq \phi(\eta) \leq \phi^*(\eta)$. □

The final result is the uniqueness of PS to (1.1) by adopting the Theorem 2.1.

Theorem 3.2. *Under the satisfaction of the investigators* $(\Sigma 1)$ *and* $(\Sigma 2)$ *and that*

$$\beta_1(\Psi(\mathcal{T})-\Psi(\ell))+\frac{(\beta_2+\beta_3)(\Psi(\mathcal{T})-\Psi(\ell))^\alpha}{\Gamma(\alpha+1)} < 1,\tag{3.26}$$

the FDE (1.1) has a unique PS $\phi \in \Omega$.

Proof. The FDE (1.1) posses at least one PS in Ω by Theorem 3.1. The mapping specified in (3.1) is a contraction on Y. Indeed, for any $\phi, \chi \in Y$ we get

$$\begin{aligned}|(\Theta\phi)(\eta)-(\Theta\chi)(\eta)| &\leq \int_\ell^\eta |\mathcal{G}(\mathfrak{s},\phi(\ell+\vartheta\mathfrak{s}))-\mathcal{G}(\mathfrak{s},\chi(\ell+\vartheta\mathfrak{s}))|\Psi'(\mathfrak{s})\,d\mathfrak{s}\\&\quad+\frac{1}{\Gamma(\alpha)}\int_\ell^\eta(\Psi(\eta)-\Psi(\mathfrak{s}))^{\alpha-1}|\mathcal{F}(\mathfrak{s},\phi(\mathfrak{s}),\phi(\ell+\vartheta\mathfrak{s}))-\mathcal{F}(\mathfrak{s},\chi(\mathfrak{s}),\chi(\ell+\vartheta\mathfrak{s}))|\Psi'(\mathfrak{s})\,d\mathfrak{s}\\&\leq\left(\beta_1(\Psi(\mathcal{T})-\Psi(\ell))+\frac{(\beta_2+\beta_3)(\Psi(\mathcal{T})-\Psi(\ell))^\alpha}{\Gamma(\alpha+1)}\right)\|\phi-\chi\|.\end{aligned}\tag{3.27}$$

On the light of (3.26), the mapping Θ is contraction and hence the FDE (1.1) has a unique PS $\phi \in \Omega$. □

4. Example

Let $\Psi(\eta) = \ln\eta$. We explore the pantograph FDE in this case (Caputo Hadamard fractional derivative).

$$\begin{cases} \mathcal{D}_1^{\frac{6}{5};\Psi}\phi(\eta) - \mathcal{D}_1^{\frac{1}{5};\Psi}\frac{\pi+\arctan(\phi(\eta))}{5} = \frac{1}{1+e+\eta}\left(1+e+\eta\sin\left(\phi(\eta)+\chi(\eta)\right)\right), \ 1 < \eta \le e, \\ \phi(\ell) = 1, \ \phi'(\ell) = \mu_2 \ge 1, \end{cases} \tag{4.1}$$

where $\mu_1 = 1$, $\ell = 1$, $\phi(1+\vartheta) = \phi_0 > 0$, $\mathcal{T} = e$, $\mathcal{G}(\eta,\phi) = \pi + \arctan(\phi)$ and $\mathcal{F}(\eta,\phi,\chi) = \frac{1}{1+e+\eta}(1+e+\eta\sin(\phi+\chi))$. As $\mathcal{G}$ is non-decreasing on ϕ,

$$\lim_{\phi\longrightarrow\infty}\frac{\pi+\arctan(\phi)}{5} = \frac{3\pi}{10}$$

and

$$\frac{\pi}{10} \le \mathcal{G}(\eta,\phi) \le \frac{3\pi}{10},$$
$$\frac{1}{1+2e} \le \mathcal{F}(\eta,\phi,\chi) \le 1.$$

Therefore, we conclude that Eq (4.1) has PS proportional to all the natural results mentioned above. We have

$$\beta_1\left(\Psi(\mathcal{T}) - \Psi(\ell)\right) + \frac{(\beta_2+\beta_3)\left(\Psi(\mathcal{T}) - \Psi(\ell)\right)^{\alpha}}{\Gamma(\alpha+1)} \simeq 0.966\,60 < 1.$$

Therefore, by making use of Theorem 3.2, we infer that Eq (4.1) posses a unique PS.

5. Conclusions

We have investigated and verified the existence and uniqueness of positive solutions of the fractional differential pantograph Eq (1.1) in Ψ-Caputo sense. We have followed the method of upper and lower solutions by imposing some of the necessary conditions to show the existence and uniqueness of our positive solution. Further, we have used and applied SFPT and BFPT to gain a positive solution for (1.1).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

Manar A. Alqudah: Princess Nourah bint Abdul- rahman University Researchers Supporting Project number (PNURSP2023R14), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. The authors B. Abdalla, and T. Abdeljawad would like to thank Prince Sultan University for the support through the TAS research lab.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. M. Ortigueira, J. Machado, Which derivative? *Fractal Fract.*, **1** (2017), 3. https://doi.org/10.3390/fractalfract1010003
2. S. Hussain, M. Sarwar, N. Mlaiki, F. Azmi, Existence and controllability of fractional semilinear mixed Volterra-Fredholm integro differential equation, *Alex. Eng. J.*, **73** (2023), 259–267. https://doi.org/10.1016/j.aej.2023.04.029
3. Kamran, S. U. Khan, S. Haque, N. Mlaiki, On the approximation of fractional-order differential equations using Laplace transform and weeks method, *Symmetry*, **15** (2023), 1214. https://doi.org/10.3390/sym15061214
4. A. Ali, K. J. Ansari, H. Alrabaiah, A. Aloqaily, N. Mlaiki, Coupled system of fractional impulsive problem involving power-law kernel with piecewise order, *Fractal Fract.*, **7** (2023), 436. https://doi.org/10.3390/fractalfract7060436
5. Asma, J. F. G. Aguilar, G. U. Rahman, M. Javed, Stability analysis for fractional order implicit Ψ-Hilfer differential equations, *Math. Method. Appl. Sci.*, **45** (2021), 2701–2712. https://doi.org/10.1002/mma.7948
6. S. A. David, C. A. Valentim, Fractional Euler-lagrange equations applied to oscillatory system, *Mathematics*, **3** (2015), 258–272. https://doi.org/10.3390/math3020258
7. M. S. Abdo, W. Shammakh, H. Z. Alzumi, N. Alghamd, M. D. Albalwi, Nonlinear piecewise Caputo fractional pantograph system with respect to another function, *Fractal Fract.*, **7** (2023), 162. https://doi.org/10.3390/fractalfract7020162
8. F. Evirgen, Transmission of Nipah virus dynamics under Caputo fractional derivative, *J. Comput. Appl. Math.*, **418** (2023), 114654. https://doi.org/10.1016/j.cam.2022.114654
9. S. A. David, C. A. Valentim, A. Debbouche, Fractional modeling applied to the dynamics of the action potential in cardiac tissue, *Fractal Fract.*, **6** (2022), 149. https://doi.org/10.3390/fractalfract6030149
10. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.
11. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Wiley, 1993.
12. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.
13. Z. Bai, H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311** (2005), 495–505. https://doi.org/10.1016/j.jmaa.2005.02.052
14. E. Kaufmann, E. Mboumi, Positive solutions of a boundary value problem for a nonlinear fractional differential equation, *Electron. J. Qual. Theo.*, **3** (2008), 1–11.

15. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci.*, **44** (2017), 460–481. https://doi.org/10.1016/j.cnsns.2016.09.006

16. S. Zhang, L. Hu, The existence and uniqueness result of solutions to initial value problems of nonlinear diffusion equations involving with the conformable variable, *Azerbaijan J. Math.*, **9** (2019), 22–45. https://doi.org/10.1007/s13398-018-0572-2

17. L. Fox, D. F. Mayers, J. R. Ockendon, A. B. Tayler, On a function differential equation, *IMA J. Appl. Math.*, **8** (1971), 271–307. https://doi.org/10.1093/imamat/8.3.271

18. A. Iserles, On the generalized pantograph functional-differential equation, *Eur. J. Appl. Math.*, **4** (1992), 1–38. https://doi.org/10.1017/S0956792500000966

19. J. Hale, *Theory of functional differential equations*, New York: Springer-Verlag, 1977. https://doi.org/10.1007/978-1-4612-9892-2

20. F. Jarad, T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Discrete Cont. Dyn.-S*, **13** (2020), 709–722. https://doi.org/10.3934/dcdss.2020039

21. T. Kato, *Asymptotic behavior of solutions of the functional differential equation* $y'(x) = \ell y(\vartheta x) + \hbar y(x)$, Academic Press, 1972.

22. K. Mahler, On a special functional equation, *J. Lond. Math. Soc.*, **15** (1940), 115–123. https://doi.org/10.1112/jlms/s1-15.2.115

23. J. R. Ockendon, A. B. Tayler, The dynamics of a current collection system for an electric locomotive, *Proc. Roy. Soc. London Ser. A*, **322** (1971), 447–468. https://doi.org/10.1098/rspa.1971.0078

24. M. Matar, On existence of positive solution for initial value problem of nonlinear fractional differential equations of order $1 < \alpha \leq 2$, *Acta Math. Univ. Comen.*, **84** (2015), 51–57. https://doi.org/10.14232/ejqtde.2015.1.84

25. H. Boulares, A. Benchaabane, N. Pakkaranang, R. Shafqat, B. Panyanak, Qualitative properties of positive solutions of a kind for fractional pantograph problems using technique fixed point theory, *Fractal Fract.*, **6** (2022), 593. https://doi.org/10.3390/fractalfract6100593

26. A. Ardjouni, H. Boulares, Y. Laskri, Stability in higher-order nonlinear fractional differential equations, *Acta Comment. Univ. Ta.*, **22** (2018), 37–47. https://doi.org/10.12697/ACUTM.2018.22.04 118

27. A. Hallaci, H. Boulares, A. Ardjouni, A. Chaoui, On the study of fractional differential equations in a weighted Sobolev space, *Bull. Int. Math. Virtual Inst.*, **9** (2019), 333–343. https://doi.org/10.7251/BIMVI1902333H 120

28. A. Ardjouni, H. Boulares, A. Djoudi, Stability of nonlinear neutral nabla fractional difference equations, *Commun. Optim. Theory*, **121** (2018), 1–10. https://doi.org/10.23952/cot.2018.8.

29. A. Hallaci, H. Boulares, M. Kurulay, On the study of nonlinear fractional differential equations on unbounded interval, *Gen. Lett. Math.*, **5** (2018), 111–117. https://doi.org/10.31559/glm2018.5.3.1

30. C. Kou, H. Zhou, Y. Yan, Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis, *Nonlinear Anal.*, **74** (2011), 5975–5986. https://doi.org/10.1016/j.na.2011.05.074

31. R. P. Agarwal, Y. Zhou, Y. He, Existence of fractional functional differential equations, *Comput. Math. Appl.*, **59** (2010), 1095–1100. https://doi.org/10.1016/j.camwa.2009.05.010

32. R. Almeida, A. B. Malinowska, M.Teresa, T. Monteiro, Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, *Math. Method. Appl. Sci.*, **41** (2018), 336–352. https://doi.org/10.1002/mma.4617

33. Z. B. Bai, T. T. Qiu, Existence of positive solution for singular fractional differential equation, *Appl. Math. Comput.*, **215** (2009), 2761–2767. https://doi.org/10.1016/j.amc.2009.09.017

34. T. A. Burton, *Stability and periodic solutions of ordinary and functional differential equations*, Florida: Academic Press, 1985.

35. D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **204** (1996), 609–625.

36. C. Wang, R. Wang, S. Wang, C. Yang, Positive solution of singular boundary value problem for a nonlinear fractional differential equation, *Bound. Value Probl.* **2011** (2011), 297026. https://doi.org/10.1155/2011/297026

37. S. Zhang, The existence of a positive solution for a fractional differential equation, *J. Math. Anal. Appl.*, **252** (2000), 804–812. https://doi.org/10.1016/j.aml.2010.04.035

38. D. R. Smart, *Fixed point theorems*, Cambridge: Cambridge University Press, 1980.

39. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach Science Publishers, 1993.