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Research article

The solutions of two classes of dual matrix equations

Yinlan Chen*, **Min Zeng, Ranran Fan and Yongxin Yuan**

School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, China

* **Correspondence:** Email: cylfzg@hbnu.edu.cn.

Abstract: The solvability conditions for the dual matrix equation $AXB = D$ and a pair of dual matrix equations $AX = C$ and $XB = D$ are deduced by applying the singular value decomposition, and the expressions of the general solutions to these dual matrix equations are provided. Furthermore, the minimum-norm solutions of these dual matrix equations are provided. Finally, two numerical experiments are given to validate the accuracy of the results obtained.

Keywords: dual matrix equation; singular value decomposition; general solution; minimum-norm solution
Mathematics Subject Classification: 15A24, 65F05

1. Introduction

We will adopt the following terminology. $\mathbb{R}^{m\times n}$ and $\mathbb{OR}^{n\times n}$ denote the sets of all $m \times n$ real matrices and $n \times n$ orthogonal matrices, respectively. I_n denotes the identity matrix of size n. $A^\top, A^\dagger, \mathrm{tr}(A)$ and $\|A\|_{\mathrm{F}}$ represent the transpose, the Moore-Penrose inverse, the trace and the Frobenius norm of the matrix A, respectively. We use $A \otimes B$ to represent the Kronecker product of A and B, and $\mathrm{vec}(\cdot)$ to represent the vec operator, that is, $\mathrm{vec}(A) = [a_1^\top, \cdots, a_n^\top]^\top$, where $A = [a_1, \cdots, a_n] \in \mathbb{R}^{m\times n}, a_i \in \mathbb{R}^m, i = 1, \cdots, n$. Also, the symbols E_A and F_A stand for the two orthogonal projectors $E_A = I_m - AA^\dagger,\ F_A = I_n - A^\dagger A$ induced by $A \in \mathbb{R}^{m\times n}$.

As we know, the linear matrix equation

$$AXB = D, \tag{1}$$

and the linear matrix equations

$$AX = C, XB = D \tag{2}$$

have been considered in real and complex matrix spaces by many scholars. For example, Penrose [1] acquired the consistency condition and the general solution of Eq (1) using the generalized inverse.

Dai [2] obtained the solvability conditions and the general symmetric solution to Eq (1) by utilizing the generalized singular value decomposition. Deng et al. [3] considered the symmetric, skew-symmetric and symmetric positive semidefinite solutions of Eq (1) by means of the quotient singular value decomposition. For Eq (2), Mitra [4] gave the solvability condition and the expression of the general common solution of Eq (2) by applying the generalized inverse. Dajić et al. [5] studied the positive solutions to Eq (2) for Hilbert space operators. Khatri et al. [6] provided the general expressions about the Hermitian solutions and Hermitian nonnegative definite solutions of Eqs (1) and (2).

In 1873, Clifford [7] introduced dual numbers to form dual quaternions for studying non-Euclidean geometry. Study [8] defined dual numbers as dual angles to specify the relation between two lines in Euclidean space. Due to the wide application in many engineering fields, the dual numbers and their algebra have attracted a large number of scholars in the past 30 years. For example, McAulay [9] used dual quaternions to describe finite displacement of rigid and deformable bodies. Dimentberg [10] pioneered kinematic analysis of spatial mechanisms through dual numbers. Pennock and Yang [11,12], Dooley and McCarthy [13], Ravani and Ge [14] studied the kinematics, dynamics and calibration of open-chain robot manipulators by applying dual numbers. The set of the dual numbers is usually denoted by

$$\mathbb{D} = \{a = a_1 + \varepsilon a_2 | a_1, a_2 \in \mathbb{R}, \varepsilon \neq 0, \varepsilon^2 = 0\}.$$

For any two dual numbers $a = a_1 + \varepsilon a_2$ and $b = b_1 + \varepsilon b_2$, the arithmetic operations for dual numbers are as follows:

(1) Equality : $a = b \Leftrightarrow a_1 = b_1, a_2 = b_2$;
(2) Addition : $a + b = (a_1 + b_1) + \varepsilon(a_2 + b_2)$;
(3) Multiplication : $ab = a_1 b_1 + \varepsilon(a_1 b_2 + a_2 b_1)$.

A matrix whose elements are dual numbers is called a dual matrix, namely, the set of all $m \times n$ real dual matrices is

$$\mathbb{D}^{m\times n} = \{A = A_1 + \varepsilon A_2 | A_1, A_2 \in \mathbb{R}^{m\times n}\}.$$

The operational rules for dual matrices are similar to those of dual numbers. Dual matrices also have important applications in kinematic analysis and robotics [15–17]. The solution of systems of linear dual equations is a task often required in synthesis problems and sensor calibration problems. For instance, Condurache and Burlacu [18] solved $AX = XB$ sensor calibration problems by applying the orthogonal dual tensor method. Condurache and Ciureanu [19] solved $AX = YB$ sensor calibration problems by applying dual algebra. Udwadiae [20] dealt with properties of dual generalized inverses and then used them to solve the dual matrix equation $Ax = b$.

Inspired by the works of papers [18–20], in this paper, we will consider the general solutions and the minimum-norm solutions of matrix equations (1) and (2) in the dual matrix space, which can be formulated as follows:

Problem I(a). Given dual matrices $A = A_1 + \varepsilon A_2$, $B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times p}$, $B_i \in \mathbb{R}^{q\times n}$ and $D_i \in \mathbb{R}^{m\times n}$ $(i = 1, 2)$. Find a p–by–q dual matrix $X = X_1 + \varepsilon X_2$ such that the dual matrix equation (1) is satisfied.

Problem I(b). Let S_1 be the solution set of Problem I(a). Find $\hat{X} = \hat{X}_1 + \varepsilon \hat{X}_2 \in S_1$ such that $||\hat{X}||_{\mathrm{D}} = \sqrt{||\hat{X}_1||_{\mathrm{F}}^2 + ||\hat{X}_2||_{\mathrm{F}}^2}$ =min.

Problem II(a). Given dual matrices $A = A_1 + \varepsilon A_2,\ B = B_1 + \varepsilon B_2,\ C = C_1 + \varepsilon C_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times p},\ B_i \in \mathbb{R}^{q\times n},\ C_i \in \mathbb{R}^{m\times q}$ and $D_i \in \mathbb{R}^{p\times n}$ $(i = 1, 2)$. Find a p-by-q dual matrix $X = X_1+\varepsilon X_2$ such that the dual matrix equations (2) are satisfied.
Problem II(b). Supposed that S_2 is the solution set of Problem II(a). Find $\hat{X} = \hat{X}_1 + \varepsilon\hat{X}_2 \in S_2$ such that $\|\hat{X}\|_{\mathrm{D}} = \sqrt{\|\hat{X}_1\|_{\mathrm{F}}^2 + \|\hat{X}_2\|_{\mathrm{F}}^2}$ =min.

In order to solve Problems I and II, we first split the dual matrix equations (1) and (2) into the two real-valued matrix equations. Then, by applying the singular value decomposition (SVD), we obtain the solvability conditions and the expressions of the general solutions for Problems I(a) and II(a). Furthermore, we deduce the unique minimum-norm solutions of Problems I(b) and II(b) by using the Kronecker product and stretching function. Finally, we give two numerical experiments to validate the accuracy of the results.

2. The solutions to Problems I(a) and I(b)

To begin with, we introduce the following lemmas.

Lemma 2.1. *[21] If $A \in \mathbb{R}^{m\times p}, B \in \mathbb{R}^{q\times n}$ and $D \in \mathbb{R}^{m\times n}$. Then the matrix equation* (1) *has a solution $X \in \mathbb{R}^{p\times q}$ if and only if $AA^\dagger DB^\dagger B = D$. In this case, the general solution is $X = A^\dagger DB^\dagger + F_A V_1 + V_2 E_B$, where V_1, V_2 are arbitrary matrices.*

Lemma 2.2. *[22] Suppose that A, B are two real matrices, and X is an unknown variable matrix. Then*

$$\frac{\partial \mathrm{tr}(BX)}{\partial X} = B^\top,\ \frac{\partial \mathrm{tr}\,(X^\top B^\top)}{\partial X} = B^\top,\ \frac{\partial \mathrm{tr}(AXBX)}{\partial X} = (BXA + AXB)^\top,$$
$$\frac{\partial \mathrm{tr}\,(AX^\top BX^\top)}{\partial X} = BX^\top A + AX^\top B,\ \frac{\partial \mathrm{tr}\,(AXBX^\top)}{\partial X} = AXB + A^\top XB^\top.$$

Lemma 2.3. *[23] Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times l}, C \in \mathbb{R}^{l\times s}$. Then* $\mathrm{vec}(ABC) = (C^\top \otimes A)\,\mathrm{vec}(B)$.

By separating the dual equation of (1) into the real part and the dual part leads to the following two equations:

$$A_1X_1B_1 = D_1,\ \ A_1X_2B_1 + A_1X_1B_2 + A_2X_1B_1 = D_2. \tag{3}$$

Suppose that the SVDs of the matrices A_1 and B_1 are

$$A_1 = P\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}Q^\top,\ B_1 = U\begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix}V^\top, \tag{4}$$

where $\Sigma = \mathrm{diag}(\gamma_1, \cdots, \gamma_{r_1}) > 0, r_1 = \mathrm{rank}(A_1), P = [P_1, P_2] \in \mathbb{OR}^{m\times m}, Q = [Q_1, Q_2] \in \mathbb{OR}^{p\times p}, \Omega = \mathrm{diag}(\beta_1, \cdots, \beta_{r_2}) > 0, r_2 = \mathrm{rank}(B_1), U = [U_1, U_2] \in \mathbb{OR}^{q\times q}, V = [V_1, V_2] \in \mathbb{OR}^{n\times n}$ with $P_1 \in \mathbb{R}^{m\times r_1},\ Q_1 \in \mathbb{R}^{p\times r_1},\ U_1 \in \mathbb{R}^{q\times r_2}$ and $V_1 \in \mathbb{R}^{n\times r_2}$. Let

$$Q^\top X_1U = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix},\ Q^\top X_2U = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix},\ P^\top D_1V = \begin{bmatrix} D_{11} & D_{12} \\ D_{13} & D_{14} \end{bmatrix}, \tag{5}$$

$$U^\top B_2V = \begin{bmatrix} B_{21} & B_{22} \\ B_{23} & B_{24} \end{bmatrix},\ P^\top A_2Q = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix},\ P^\top D_2V = \begin{bmatrix} D_{21} & D_{22} \\ D_{23} & D_{24} \end{bmatrix}. \tag{6}$$

Then the matrix equations (3) are equivalent to

$$\begin{bmatrix} \Sigma Y_1\Omega & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{13} & D_{14} \end{bmatrix}, \tag{7}$$

$$\begin{bmatrix} \Sigma Z_1\Omega + \Sigma Y_1 B_{21} + \Sigma Y_2 B_{23} + A_{21}Y_1\Omega + A_{22}Y_3\Omega & \Sigma Y_1 B_{22} + \Sigma Y_2 B_{24} \\ A_{23}Y_1\Omega + A_{24}Y_3\Omega & 0 \end{bmatrix} = \begin{bmatrix} D_{21} & D_{22} \\ D_{23} & D_{24} \end{bmatrix}. \tag{8}$$

It follows from the equations of (7) and (8) that

$$D_{12} = 0,\ D_{13} = 0,\ D_{14} = 0,\ D_{24} = 0, \tag{9}$$

$$Y_1 = \Sigma^{-1}D_{11}\Omega^{-1}, \tag{10}$$

$$\Sigma Y_1 B_{22} + \Sigma Y_2 B_{24} = D_{22},\ A_{23}Y_1\Omega + A_{24}Y_3\Omega = D_{23}, \tag{11}$$

$$\Sigma Z_1\Omega + \Sigma Y_1 B_{21} + \Sigma Y_2 B_{23} + A_{21}Y_1\Omega + A_{22}Y_3\Omega = D_{21}. \tag{12}$$

We note that

$$D_{13} = 0, D_{14} = 0 \Leftrightarrow P_2^\top D_1 V = 0 \Leftrightarrow P_2 P_2^\top D_1 = 0 \Leftrightarrow E_{A_1} D_1 = 0;$$
$$D_{12} = 0 \Leftrightarrow P_1^\top D_1 V_2 = 0 \Leftrightarrow P_1 P_1^\top D_1 V_2 V_2^\top = 0 \Leftrightarrow A_1 A_1^\dagger D_1 F_{B_1} = 0 \Leftrightarrow D_1 F_{B_1} = 0;$$
$$D_{24} = 0 \Leftrightarrow P_2^\top D_2 V_2 = 0 \Leftrightarrow P_2 P_2^\top D_2 V_2 V_2^\top = 0 \Leftrightarrow E_{A_1} D_2 F_{B_1} = 0.$$

Plugging (10) into (11), we have

$$Y_2 B_{24} = J_1, \tag{13}$$

$$A_{24} Y_3 = J_2, \tag{14}$$

where $J_1 = \Sigma^{-1}D_{22} - \Sigma^{-1}D_{11}\Omega^{-1}B_{22}, J_2 = D_{23}\Omega^{-1} - A_{23}\Sigma^{-1}D_{11}\Omega^{-1}$. By Lemma 1, Eqs (13) and (14) with respect to Y_2 and Y_3 have solutions if and only if $J_1 F_{B_{24}} = 0,\ E_{A_{24}} J_2 = 0$, and the general solutions are

$$Y_2 = J_1 B_{24}^\dagger + W_1 E_{B_{24}},\ Y_3 = A_{24}^\dagger J_2 + F_{A_{24}} W_2, \tag{15}$$

where W_1 and W_2 are arbitrary matrices. Inserting (10) and (15) into (12) yields

$$Z_1 = J_3 - W_1 E_{B_{24}} B_{23}\Omega^{-1} - \Sigma^{-1} A_{22} F_{A_{24}} W_2, \tag{16}$$

where $J_3 = \Sigma^{-1}(D_{21} - D_{11}\Omega^{-1}B_{21} - A_{21}\Sigma^{-1}D_{11})\Omega^{-1} - J_1 B_{24}^\dagger B_{23}\Omega^{-1} - \Sigma^{-1}A_{22}A_{24}^\dagger J_2$.

In summary, we have proven the following theorem.

Theorem 2.1. *Given dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times p}, B_i \in \mathbb{R}^{q\times n}$ and $D_i \in \mathbb{R}^{m\times n}$ $(i = 1, 2)$. Let the SVDs of A_1, B_1 be given by* (4) *and the partitions of the matrices $U^\top B_2 V, P^\top D_1 V, P^\top D_2 V$ and $P^\top A_2 Q$ be given by* (5) *and* (6)*. Then Eq* (1) *is solvable if and only if*

$$J_1 F_{B_{24}} = 0,\ E_{A_{24}} J_2 = 0,\ E_{A_1} D_1 = 0,\ D_1 F_{B_1} = 0,\ E_{A_1} D_2 F_{B_1} = 0, \tag{17}$$

where $J_1 = \Sigma^{-1}D_{22} - \Sigma^{-1}D_{11}\Omega^{-1}B_{22}, J_2 = D_{23}\Omega^{-1} - A_{23}\Sigma^{-1}D_{11}\Omega^{-1}$. In this case, the solution set of Problem I(a) can be expressed as $S_1 = \{X = X_1 + \varepsilon X_2 | X_1, X_2 \in \mathbb{R}^{p\times q}\}$, where

$$X_1 = Q \begin{bmatrix} \Sigma^{-1}D_{11}\Omega^{-1} & J_1 B_{24}^\dagger + W_1 E_{B_{24}} \\ A_{24}^\dagger J_2 + F_{A_{24}} W_2 & Y_4 \end{bmatrix} U^\top, \tag{18}$$

$$X_2 = Q\begin{bmatrix} J_3 - W_1 E_{B_{24}} B_{23}\Omega^{-1} - \Sigma^{-1} A_{22} F_{A_{24}} W_2 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} U^\top, \tag{19}$$

with

$$J_1 = \Sigma^{-1} D_{22} - \Sigma^{-1} D_{11}\Omega^{-1} B_{22},\ J_2 = D_{23}\Omega^{-1} - A_{23}\Sigma^{-1} D_{11}\Omega^{-1},$$
$$J_3 = \Sigma^{-1}(D_{21} - D_{11}\Omega^{-1}B_{21} - A_{21}\Sigma^{-1}D_{11})\Omega^{-1} - J_1 B_{24}^{\dagger} B_{23}\Omega^{-1} - \Sigma^{-1} A_{22} A_{24}^{\dagger} J_2,$$

and $W_1, W_2, Y_4, Z_2, Z_3, Z_4$ *are arbitrary matrices.*

Now, for $\forall\ X \in S_1$, we obtain

$$\begin{aligned} \|X\|_{\mathrm{D}}^2 &= \|X_1\|_{\mathrm{F}}^2 + \|X_2\|_{\mathrm{F}}^2 = \|Q^\top X_1 U\|_{\mathrm{F}}^2 + \|Q^\top X_2 U\|_{\mathrm{F}}^2 \\ &= \|\Sigma^{-1} D_{11}\Omega^{-1}\|_{\mathrm{F}}^2 + \|Y_4\|_{\mathrm{F}}^2 + \|Z_2\|_{\mathrm{F}}^2 + \|Z_3\|_{\mathrm{F}}^2 + \|Z_4\|_{\mathrm{F}}^2 + \|J_1 B_{24}^{\dagger} + W_1 E_{B_{24}}\|_{\mathrm{F}}^2 \\ &\quad + \|A_{24}^{\dagger} J_2 + F_{A_{24}} W_2\|_{\mathrm{F}}^2 + \|J_3 - W_1 M_2^\top B_{23}\Omega^{-1} - \Sigma^{-1} A_{22} T_2 W_2\|_{\mathrm{F}}^2. \end{aligned}$$

Then $\|X\|_{\mathrm{D}}$ is minimized if and only if

$$Y_4 = 0,\ Z_2 = 0,\ Z_3 = 0,\ Z_4 = 0, \tag{20}$$

$$\begin{aligned} \Phi(W_1, W_{12}) = \|J_1 B_{24}^{\dagger} + W_1 E_{B_{24}}\|_{\mathrm{F}}^2 + \|A_{24}^{\dagger} J_2 + F_{A_{24}} W_2\|_{\mathrm{F}}^2 \\ + \|J_3 - W_1 E_{B_{24}} B_{23}\Omega^{-1} - \Sigma^{-1} A_{22} F_{A_{24}} W_2\|_{\mathrm{F}}^2 = \min. \end{aligned} \tag{21}$$

Clearly, the minimization problem (21) is equivalent to

$$\begin{aligned} \Phi(W_1, W_2) = \operatorname{tr}[&(B_{24}^{\dagger})^\top J_1^\top J_1 B_{24}^{\dagger} + E_{B_{24}} W_1^\top W_1 E_{B_{24}} + W_2^\top F_{A_{24}} W_2 + W_2^\top F_{A_{24}} A_{22}^\top \Sigma^{-1}\Sigma^{-1} A_{22} F_{A_{24}} W_2 \\ &+ J_2^\top (A_{24}^{\dagger})^\top A_{24}^{\dagger} J_2 + J_3^\top J_3 + \Omega^{-1} B_{23}^\top E_{B_{24}} W_1^\top W_1 E_{B_{24}} B_{23}\Omega^{-1}] + 2\operatorname{tr}(-J_3^\top W_1 E_{B_{24}} B_{23}\Omega^{-1} \\ &+ \Omega^{-1} B_{23}^\top E_{B_{24}} W_1^\top \Sigma^{-1} A_{22} F_{A_{24}} W_2 - J_3^\top \Sigma^{-1} A_{22} F_{A_{24}} W_2) = \min. \end{aligned}$$

Therefore, $\Phi(W_1, W_2)$ is minimized if and only if $\frac{\partial \Phi(W_1,W_2)}{\partial W_1} = 0$ and $\frac{\partial \Phi(W_1,W_2)}{\partial W_2} = 0$, which implies that

$$W_1(E_{B_{24}} + L_1^\top L_1) + L_2 W_2 L_1 = J_3 L_1, \tag{22}$$

$$L_2^\top W_1 L_1^\top + (F_{A_{24}} + L_2^\top L_2) W_2 = L_2^\top J_3, \tag{23}$$

where $L_1 = \Omega^{-1} B_{23}^\top E_{B_{24}},\ L_2 = \Sigma^{-1} A_{22} F_{A_{24}}$. By applying the Kronecker product and stretching function, the matrix equations (22) and (23) can be equivalently written as

$$\Delta \begin{bmatrix} \operatorname{vec}(W_1) \\ \operatorname{vec}(W_2) \end{bmatrix} = \Gamma, \tag{24}$$

where

$$\Delta = \begin{bmatrix} (E_{B_{24}} + L_1^\top L_1) \otimes I_{r_1} & L_1^\top \otimes L_2 \\ L_1 \otimes L_2^\top & I_{r_2} \otimes (F_{A_{24}} + L_2^\top L_2) \end{bmatrix}, \Gamma = \begin{bmatrix} \operatorname{vec}(J_3 L_1) \\ \operatorname{vec}(L_2^\top J_3) \end{bmatrix}. \tag{25}$$

Then, W_1 and W_2 are determined by solving the unique solution of the Eq (24). Inserting (20), W_1 and W_2 into (18)–(19), we have the following result.

Theorem 2.2. *If the conditions* (17) *are satisfied, then Problem I(b) has the unique solution $\hat{X}$, and $\hat{X}$ admits the following representation: $\hat{X} = \hat{X}_1 + \varepsilon\hat{X}_2$, where*

$$\hat{X}_1 = Q\begin{bmatrix} \Sigma^{-1}D_{11}\Omega^{-1} & J_1B_{24}^{\dagger} + W_1E_{B_{24}} \\ A_{24}^{\dagger}J_2 + F_{A_{24}}W_2 & 0 \end{bmatrix}U^{\top}, \tag{26}$$

$$\hat{X}_2 = Q\begin{bmatrix} J_3 - W_1E_{B_{24}}B_{23}\Omega^{-1} - \Sigma^{-1}A_{22}F_{A_{24}}W_2 & 0 \\ 0 & 0 \end{bmatrix}U^{\top}, \tag{27}$$

with $W_1,\ W_2$ being determined by (24).

Based on Theorems 2.1 and 2.2, we can describe an algorithm to solve Problems I(a) and I(b) as follows.

Algorithm 1

1: Input matrices A_i, B_i and $D_i, i = 1, 2$.
2: Compute the SVDs of the matrices A_1, B_1 as in (4).
3: Compute $J_1 = \Sigma^{-1}D_{22} - \Sigma^{-1}D_{11}\Omega^{-1}B_{22},\ J_2 = D_{23}\Omega^{-1} - A_{23}\Sigma^{-1}D_{11}\Omega^{-1}$,

$$J_3 = \Sigma^{-1}(D_{21} - D_{11}\Omega^{-1}B_{21} - A_{21}\Sigma^{-1}D_{11})\Omega^{-1} - J_1B_{24}^{\dagger}B_{23}\Omega^{-1} - \Sigma^{-1}A_{22}A_{24}^{\dagger}J_2.$$

4: Compute the partitions of the matrices $U^{\top}B_2V, P^{\top}D_1V, P^{\top}D_2V$ and $P^{\top}A_2Q$ as in (5) and (6).
5: If the conditions (17) are satisfied, then continue, otherwise, Problem I(a) has no solution, and stop.
6: Calculate Y_1 according to (10).
7: Calculate Y_4, Z_2, Z_3, Z_4 on the basis of (20).
8: Calculate $L_1 = \Omega^{-1}B_{23}^{\top}E_{B_{24}},\ L_2 = \Sigma^{-1}A_{22}F_{A_{24}}$.
9: Compute Δ and Γ by (25).
10: Solve the equation of (24).
11: Compute Y_2, Y_3, Z_1 by (15) and (16).
12: Compute $\hat{X}_1, \hat{X}_2$ according to (26) and (27), then, compute the unique solution $\hat{X} = \hat{X}_1 + \varepsilon\hat{X}_2$ of Problem I(b).

Example 2.1. *Let $m = 8, n = 9, p = 10, q = 9$, and the matrices $A_1, A_2, B_1, B_2, D_1, D_2$ be given by*

$$A_1 = \begin{bmatrix} 0.1301 & 0.5962 & 0.7341 & 0.9667 & 1.3414 & 0.2427 & 0.6568 & 0.7459 & 1.0132 & 0.7080 \\ 0.0702 & 0.3974 & 0.5911 & 0.6660 & 0.8767 & 0.1408 & 0.2451 & 0.4824 & 0.5419 & 0.5388 \\ 0.0695 & 0.3726 & 0.5316 & 0.6196 & 0.8258 & 0.1366 & 0.2725 & 0.4556 & 0.5377 & 0.4903 \\ 0.0589 & 0.3715 & 0.5941 & 0.6315 & 0.8125 & 0.1230 & 0.1506 & 0.4450 & 0.4523 & 0.5310 \\ 0.1363 & 0.6145 & 0.7427 & 0.9935 & 1.3850 & 0.2530 & 0.7034 & 0.7709 & 1.0626 & 0.7206 \\ 0.1340 & 0.7068 & 0.9949 & 1.1725 & 1.5689 & 0.2620 & 0.5427 & 0.8662 & 1.0378 & 0.9212 \\ 0.0665 & 0.2921 & 0.3424 & 0.4699 & 0.6601 & 0.1224 & 0.3544 & 0.3679 & 0.5189 & 0.3355 \\ 0.0769 & 0.3061 & 0.3146 & 0.4831 & 0.6993 & 0.1374 & 0.4551 & 0.3920 & 0.6018 & 0.3226 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.0933 & 0.8244 & 0.6818 & 1.0802 & 0.8676 & 0.7711 & 0.8561 & 0.3433 & 1.1049 & 0.4056 \\ 0.6499 & 0.5789 & 0.3073 & 0.6518 & 0.6119 & 0.4878 & 0.6703 & 0.1769 & 0.7701 & 0.4037 \\ 0.8323 & 0.9439 & 0.5413 & 0.7775 & 0.9802 & 0.9786 & 0.9816 & 0.3260 & 1.0896 & 0.5079 \\ 0.6000 & 0.6703 & 0.4696 & 0.5449 & 0.6914 & 0.7547 & 0.6443 & 0.2677 & 0.7441 & 0.2778 \\ 1.0163 & 1.0320 & 0.5939 & 0.9790 & 1.0786 & 1.0003 & 1.1115 & 0.3479 & 1.2602 & 0.6043 \\ 0.5387 & 0.5719 & 0.3400 & 0.5104 & 0.5955 & 0.5792 & 0.5998 & 0.1997 & 0.6778 & 0.3120 \\ 0.9388 & 1.1907 & 0.8139 & 0.8177 & 1.2209 & 1.4099 & 1.1104 & 0.4779 & 1.2467 & 0.4531 \\ 1.2762 & 1.1933 & 0.7768 & 1.2358 & 1.2484 & 1.1587 & 1.2636 & 0.4326 & 1.4860 & 0.6551 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.8675 & 0.4523 & 0.8875 & 1.1213 & 0.9407 & 0.5982 & 0.6560 & 0.4344 & 0.3021 \\ 1.3819 & 0.7358 & 1.4222 & 1.7959 & 1.5165 & 0.9688 & 1.0587 & 0.6925 & 0.4697 \\ 1.3975 & 0.6758 & 1.2704 & 1.6948 & 1.3475 & 0.5510 & 1.0278 & 0.8306 & 0.6958 \\ 1.1300 & 0.5506 & 1.0451 & 1.3823 & 1.1071 & 0.4925 & 0.8326 & 0.6559 & 0.5393 \\ 1.2322 & 0.6571 & 1.2424 & 1.5862 & 1.3321 & 0.7931 & 0.9486 & 0.6440 & 0.4518 \\ 1.2011 & 0.7269 & 1.0570 & 1.4796 & 1.2359 & 0.3112 & 1.0301 & 0.8344 & 0.6341 \\ 1.1470 & 0.7370 & 1.1095 & 1.4859 & 1.2928 & 0.5518 & 1.0113 & 0.7204 & 0.4738 \\ 0.5748 & 0.0891 & 0.6922 & 0.7423 & 0.5551 & 0.7782 & 0.2157 & 0.0595 & 0.0786 \\ 1.6950 & 0.8346 & 1.5239 & 2.0498 & 1.6313 & 0.6149 & 1.2635 & 1.0333 & 0.8650 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.2446 & 0.6215 & 0.3151 & 0.5083 & 0.1282 & 0.4160 & 0.3089 & 0.8484 & 0.3954 \\ 0.1970 & 0.5539 & 0.2235 & 0.2395 & 0.1191 & 0.4638 & 0.1467 & 0.6858 & 0.1861 \\ 0.1373 & 0.3891 & 0.1541 & 0.1577 & 0.0839 & 0.3304 & 0.0966 & 0.4783 & 0.1225 \\ 0.1673 & 0.4592 & 0.1961 & 0.2388 & 0.0978 & 0.3669 & 0.1458 & 0.5818 & 0.1856 \\ 0.1324 & 0.3678 & 0.1528 & 0.1754 & 0.0787 & 0.3009 & 0.1072 & 0.4608 & 0.1363 \\ 0.0769 & 0.1862 & 0.1044 & 0.1894 & 0.0376 & 0.1084 & 0.1149 & 0.2664 & 0.1474 \\ 0.1138 & 0.3381 & 0.1188 & 0.0807 & 0.0742 & 0.3116 & 0.0500 & 0.3971 & 0.0626 \\ 0.3495 & 0.9051 & 0.4406 & 0.6721 & 0.1883 & 0.6355 & 0.4089 & 1.2131 & 0.5227 \\ 0.2584 & 0.7188 & 0.2976 & 0.3390 & 0.1539 & 0.5896 & 0.2073 & 0.8993 & 0.2634 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 37.2772 & 19.5497 & 35.8150 & 46.8226 & 38.6175 & 19.2825 & 28.6267 & 21.0905 & 15.9790 \\ 23.9336 & 12.5476 & 22.9702 & 30.0461 & 24.7711 & 12.3149 & 18.3790 & 13.5635 & 10.2912 \\ 22.6423 & 11.8716 & 21.7366 & 28.4287 & 23.4399 & 11.6655 & 17.3875 & 12.8265 & 9.7286 \\ 22.0040 & 11.5342 & 21.1080 & 27.6170 & 22.7643 & 11.2944 & 16.8970 & 12.4795 & 9.4750 \\ 38.5481 & 20.2168 & 37.0394 & 48.4211 & 39.9373 & 19.9489 & 29.6028 & 21.8065 & 16.5194 \\ 43.0737 & 22.5845 & 41.3541 & 54.0838 & 44.5943 & 22.2009 & 33.0774 & 24.3976 & 18.5028 \\ 18.4160 & 9.6589 & 17.6978 & 23.1345 & 19.0821 & 9.5372 & 14.1426 & 10.4156 & 7.8887 \\ 19.6970 & 10.3325 & 18.9394 & 24.7505 & 20.4193 & 10.2287 & 15.1265 & 11.1304 & 8.4236 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 84.6062 & 55.8530 & 82.8008 & 107.4806 & 84.9843 & 53.8552 & 65.0872 & 63.1407 & 39.8291 \\ 55.2478 & 36.3652 & 54.0591 & 70.1724 & 55.5620 & 35.0690 & 42.5217 & 41.0474 & 25.9362 \\ 67.2474 & 42.1992 & 65.6044 & 85.2283 & 68.0915 & 41.1427 & 51.6625 & 47.2077 & 30.8779 \\ 54.5082 & 35.3414 & 53.2854 & 69.1744 & 54.9224 & 34.2257 & 41.9065 & 39.7918 & 25.4096 \\ 91.4598 & 59.8473 & 89.4671 & 116.1522 & 92.0215 & 57.8224 & 70.3578 & 67.5163 & 42.8523 \\ 75.5141 & 52.9552 & 74.2156 & 96.2120 & 75.1667 & 50.4517 & 58.2216 & 60.4825 & 36.5619 \\ 70.9283 & 42.7535 & 69.0287 & 89.7357 & 72.1844 & 42.0976 & 54.3926 & 47.4731 & 32.0002 \\ 78.0985 & 46.9256 & 75.9816 & 98.7998 & 79.5347 & 46.1771 & 59.9163 & 52.0616 & 35.1900 \end{bmatrix}.$$

It is easy to verity that the conditions (17) hold:

$$\|J_1F_{B_{24}}\|_F = 1.6016\times 10^{-14},\quad \|E_{A_{24}}J_2\|_F = 1.4097\times 10^{-14},$$
$$\|E_{A_1}D_1\|_F = 5.0053\times 10^{-14},\quad \|D_1F_{B_1}\|_F = 3.0245\times 10^{-14},\quad \|E_{A_1}D_2F_{B_1}\|_F = 4.6871\times 10^{-14}.$$

By using Algorithm 1, we can obtain the unique solution $\hat{X} = \hat{X}_1 + \varepsilon\hat{X}_2$ of Problem I(b) as follows:

$$\hat{X}_1 = \begin{bmatrix} 0.3763 & 0.5895 & 0.3469 & 0.3112 & 0.4847 & 0.2216 & 0.3545 & 0.4573 & 0.3974 \\ 0.4878 & 0.7288 & 0.4641 & 0.4338 & 0.5808 & 0.4012 & 0.6001 & 0.4085 & 0.6289 \\ 0.3002 & 0.4251 & 0.4591 & 0.4264 & 0.3011 & 0.2937 & 0.4327 & 0.2866 & 0.7339 \\ 0.4595 & 0.6465 & 0.4887 & 0.4648 & 0.4980 & 0.4628 & 0.6505 & 0.2843 & 0.7430 \\ 0.5646 & 0.7673 & 0.5588 & 0.5374 & 0.5982 & 0.6037 & 0.8105 & 0.2761 & 0.8571 \\ 0.4124 & 0.6306 & 0.4015 & 0.3593 & 0.5174 & 0.2495 & 0.3785 & 0.5124 & 0.4744 \\ 0.4466 & 0.6045 & 0.2402 & 0.2273 & 0.5337 & 0.3848 & 0.4518 & 0.2738 & 0.2252 \\ 0.1781 & 0.1980 & 0.2181 & 0.2092 & 0.1482 & 0.2368 & 0.2677 & 0.0571 & 0.3676 \\ 0.5410 & 0.7325 & 0.4143 & 0.3931 & 0.6124 & 0.5088 & 0.6364 & 0.3260 & 0.5358 \\ 0.4019 & 0.6090 & 0.3849 & 0.3814 & 0.4576 & 0.4343 & 0.6872 & 0.1325 & 0.6204 \end{bmatrix},$$

$$\hat{X}_2 = \begin{bmatrix} 0.0594 & 0.0944 & 0.0812 & 0.0674 & 0.0811 & 0.0546 & 0.0604 & 0.0676 & 0.0954 \\ 0.2742 & 0.4398 & 0.3499 & 0.2935 & 0.3760 & 0.2580 & 0.3026 & 0.2770 & 0.4117 \\ 0.3404 & 0.5512 & 0.4009 & 0.3405 & 0.4687 & 0.3283 & 0.4070 & 0.2970 & 0.4726 \\ 0.4452 & 0.7152 & 0.5610 & 0.4714 & 0.6109 & 0.4206 & 0.4980 & 0.4398 & 0.6602 \\ 0.6164 & 0.9878 & 0.7924 & 0.6639 & 0.8450 & 0.5786 & 0.6749 & 0.6307 & 0.9321 \\ 0.1110 & 0.1770 & 0.1486 & 0.1238 & 0.1519 & 0.1028 & 0.1160 & 0.1218 & 0.1747 \\ 0.2968 & 0.4662 & 0.4423 & 0.3629 & 0.4033 & 0.2640 & 0.2681 & 0.3885 & 0.5185 \\ 0.3426 & 0.5488 & 0.4422 & 0.3702 & 0.4696 & 0.3212 & 0.3735 & 0.3529 & 0.5200 \\ 0.4623 & 0.7347 & 0.6340 & 0.5261 & 0.6314 & 0.4244 & 0.4690 & 0.5284 & 0.7446 \\ 0.3274 & 0.5287 & 0.3959 & 0.3348 & 0.4503 & 0.3134 & 0.3820 & 0.3000 & 0.4663 \end{bmatrix}.$$

The absolute errors are estimated by

$$\|A_1\hat{X}_1B_1 - D_1\|_F = 5.6258\times 10^{-14},$$
$$\|A_1\hat{X}_2B_1 + A_1\hat{X}_1B_2 + A_2\hat{X}_1B_1 - D_2\|_F = 1.7614\times 10^{-13},$$

which implies that $\hat{X}$ is the solution of Problem I(b).

3. The solutions to Problems II(a) and II(b)

We first introduce the following lemma.

Lemma 3.1. *[24] If $A\in\mathbb{R}^{m\times p}, B\in\mathbb{R}^{q\times n}, C\in\mathbb{R}^{m\times q}$ and $D\in\mathbb{R}^{p\times n}$, then the matrix equations* (2) *have a common solution $X\in\mathbb{R}^{p\times q}$ if and only if $E_AC = 0, DF_B = 0, AD = CB$. In this case, the general solution is $X = A^\dagger C + F_ADB^\dagger + F_AVE_B$, where V is an arbitrary matrix.*

Obviously, Eq (2) can be equivalently expressed as

$$A_1X_1 = C_1, X_1B_1 = D_1, A_2X_1 + A_1X_2 = C_2, X_1B_2 + X_2B_1 = D_2. \tag{28}$$

Let SVDs of the matrices A_1 and B_1 be given by (4), and let

$$P^{\top}C_1U = \begin{bmatrix} C_{11} & C_{12} \\ C_{13} & C_{14} \end{bmatrix}, P^{\top}C_2U = \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & C_{24} \end{bmatrix}, \tag{29}$$

$$Q^{\top}D_1V = \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{13} & \tilde{D}_{14} \end{bmatrix}, Q^{\top}D_2V = \begin{bmatrix} \tilde{D}_{21} & \tilde{D}_{22} \\ \tilde{D}_{23} & \tilde{D}_{24} \end{bmatrix}. \tag{30}$$

Then, the matrix equations (28) are equivalent to

$$\begin{bmatrix} \Sigma Y_1 & \Sigma Y_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{13} & C_{14} \end{bmatrix}, \begin{bmatrix} Y_1\Omega & 0 \\ Y_3\Omega & 0 \end{bmatrix} = \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{13} & \tilde{D}_{14} \end{bmatrix}, \tag{31}$$

$$\begin{bmatrix} A_{21}Y_1 + A_{22}Y_3 + \Sigma Z_1 & A_{21}Y_2 + A_{22}Y_4 + \Sigma Z_2 \\ A_{23}Y_1 + A_{24}Y_3 & A_{23}Y_2 + A_{24}Y_4 \end{bmatrix} = \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & C_{24} \end{bmatrix}, \tag{32}$$

$$\begin{bmatrix} Y_1B_{21} + Y_2B_{23} + Z_1\Omega & Y_1B_{22} + Y_2B_{24} \\ Y_3B_{21} + Y_4B_{23} + Z_3\Omega & Y_3B_{22} + Y_4B_{24} \end{bmatrix} = \begin{bmatrix} \tilde{D}_{21} & \tilde{D}_{22} \\ \tilde{D}_{23} & \tilde{D}_{24} \end{bmatrix}. \tag{33}$$

It follows from (31), (32) and (33) that

$$C_{13} = 0,\ C_{14} = 0,\ \tilde{D}_{12} = 0,\ \tilde{D}_{14} = 0, \tag{34}$$

$$Y_1 = \Sigma^{-1}C_{11},\ Y_2 = \Sigma^{-1}C_{12},\ Y_3 = \tilde{D}_{13}\Omega^{-1}, \tag{35}$$

$$A_{21}Y_1 + A_{22}Y_3 + \Sigma Z_1 = C_{21}, \tag{36}$$

$$A_{23}Y_2 + A_{24}Y_4 = C_{24},\ Y_3B_{22} + Y_4B_{24} = \tilde{D}_{24}, \tag{37}$$

$$A_{21}Y_2 + A_{22}Y_4 + \Sigma Z_2 = C_{22},\ Y_3B_{21} + Y_4B_{23} + Z_3\Omega = \tilde{D}_{23}, \tag{38}$$

$$Y_1\Omega = \tilde{D}_{11}, A_{23}Y_1 + A_{24}Y_3 = C_{23}, Y_1B_{22} + Y_2B_{24} = \tilde{D}_{22}, Y_1B_{21} + Y_2B_{23} + Z_1\Omega = \tilde{D}_{21}. \tag{39}$$

Plugging (35) into (36), we obtain

$$Z_1 = \Sigma^{-1}(C_{21} - A_{21}\Sigma^{-1}C_{11} - A_{22}\tilde{D}_{13}\Omega^{-1}). \tag{40}$$

Substituting (35) into (37) yields

$$A_{24}Y_4 = C_{24} - A_{23}\Sigma^{-1}C_{12},\ Y_4B_{24} = \tilde{D}_{24} - \tilde{D}_{13}\Omega^{-1}B_{22}. \tag{41}$$

By Lemma 3.1, Eq (41) with respect to Y_4 has a solution if and only if

$$\begin{aligned} &E_{A_{24}}(C_{24} - A_{23}\Sigma^{-1}C_{12}) = 0, (\tilde{D}_{24} - \tilde{D}_{13}\Omega^{-1}B_{22})F_{B_{24}} = 0, \\ &A_{24}(\tilde{D}_{24} - \tilde{D}_{13}\Omega^{-1}B_{22}) = (C_{24} - A_{23}\Sigma^{-1}C_{12})B_{24}, \end{aligned} \tag{42}$$

in this case, the general solution is

$$Y_4 = J_4 + F_{A_{24}}W_3E_{B_{24}}, \tag{43}$$

where W_3 is an arbitrary matrix and

$$J_4 = A_{24}^{\dagger}(C_{24} - A_{23}\Sigma^{-1}C_{12}) + F_{A_{24}}(\tilde{D}_{24} - \tilde{D}_{13}\Omega^{-1}B_{22})B_{24}^{\dagger}. \tag{44}$$

Inserting (35) and (43) into (38) yields

$$Z_2 = J_5 - \Sigma^{-1}A_{22}F_{A_{24}}W_3E_{B_{24}},\ Z_3 = J_6 - F_{A_{24}}W_3E_{B_{24}}B_{23}\Omega^{-1}, \tag{45}$$

where

$$J_5 = \Sigma^{-1}C_{22} - \Sigma^{-1}A_{21}\Sigma^{-1}C_{12} - \Sigma^{-1}A_{22}J_4, J_6 = \tilde{D}_{23}\Omega^{-1} - \tilde{D}_{13}\Omega^{-1}B_{21}\Omega^{-1} - J_4B_{23}\Omega^{-1}. \tag{46}$$

Plugging (35) and (40) into (39), we arrive at

$$C_{11}\Omega = \Sigma\tilde{D}_{11},\ A_{23}\Sigma^{-1}C_{11}\Omega + A_{24}\tilde{D}_{13} = C_{23}\Omega, C_{11}B_{22} + C_{12}B_{24} = \Sigma\tilde{D}_{22}, \tag{47}$$

$$C_{11}B_{21} + C_{12}B_{23} + (C_{21} - A_{21}\Sigma^{-1}C_{11} - A_{22}\tilde{D}_{13}\Omega^{-1})\Omega = \Sigma\tilde{D}_{21}. \tag{48}$$

We observe from (34), (47) and (48) that

$$\begin{aligned}
&C_{13} = 0, C_{14} = 0 \Leftrightarrow P_2^\top C_1 U = 0 \Leftrightarrow P_2P_2^\top C_1 = 0 \Leftrightarrow E_{A_1}C_1 = 0,\\
&\tilde{D}_{12} = 0, \tilde{D}_{14} = 0 \Leftrightarrow Q^\top D_1 V_2 = 0 \Leftrightarrow D_1V_2V_2^\top = 0 \Leftrightarrow D_1F_{B_1} = 0,\\
&C_{11}\Omega = \Sigma\tilde{D}_{11} \Leftrightarrow P_1^\top C_1U_1\Omega = \Sigma Q_1^\top D_1V_1 \Leftrightarrow P_1P_1^\top C_1U_1\Omega V_1^\top = P_1\Sigma Q_1^\top D_1V_1V_1^\top \Leftrightarrow C_1B_1 = A_1D_1,\\
&A_{23}\Sigma^{-1}C_{11}\Omega + A_{24}\tilde{D}_{13} = C_{23}\Omega \Leftrightarrow P_2^\top A_2Q_1\Sigma^{-1}P_1^\top C_1U_1\Omega + P_2^\top A_2Q_2Q_2^\top D_1V_1 = P_2^\top C_2U_1\Omega\\
\Leftrightarrow\ & P_2P_2^\top A_2Q_1\Sigma^{-1}P_1^\top C_1U_1\Omega V_1^\top + P_2P_2^\top A_2Q_2Q_2^\top D_1V_1V_1^\top = P_2P_2^\top C_2U_1\Omega V_1^\top\\
\Leftrightarrow\ & E_{A_1}(A_2D_1 - C_2B_1) = 0,\\
&C_{11}B_{22} + C_{12}B_{24} = \Sigma\tilde{D}_{22} \Leftrightarrow P_1^\top C_1U_1U_1^\top B_2V_2 + P_1^\top C_1U_2U_2^\top B_2V_2 = \Sigma Q_1^\top D_2V_2\\
\Leftrightarrow\ & P_1P_1^\top C_1U_1U_1^\top B_2V_2V_2^\top + P_1P_1^\top C_1U_2U_2^\top B_2V_2V_2^\top = P_1\Sigma Q_1^\top D_2V_2V_2^\top \Leftrightarrow (A_1D_2 - C_1B_2)F_{B_1} = 0,\\
&C_{11}B_{21} + C_{12}B_{23} + (C_{21} - A_{21}\Sigma^{-1}C_{11} - A_{22}\tilde{D}_{13}\Omega^{-1})\Omega = \Sigma\tilde{D}_{21}\\
\Leftrightarrow\ & P_1^\top C_1U_1U_1^\top B_2V_1 + P_1^\top C_1U_2U_2^\top B_2V_1 + P_1^\top C_2U_1\Omega - P_1^\top A_2Q_2Q_2^\top D_1V_1 + P_1^\top A_2Q_1\Sigma^{-1}P_1^\top C_1U_1\Omega\\
&\quad = \Sigma Q_1^\top D_2V_1\\
\Leftrightarrow\ & P_1P_1^\top C_1U_1U_1^\top B_2V_1V_1^\top + P_1P_1^\top C_1U_2U_2^\top B_2V_1V_1^\top + P_1P_1^\top C_2U_1\Omega V_1^\top - P_1P_1^\top A_2Q_2Q_2^\top D_1V_1V_1^\top\\
&\quad + P_1P_1^\top A_2Q_1\Sigma^{-1}P_1^\top C_1U_1\Omega V_1^\top = P_1\Sigma Q_1^\top D_2V_1V_1^\top \Leftrightarrow C_2B_1 - A_2D_1 = A_1D_2 - C_1B_2,\\
&A_{24}(\tilde{D}_{24} - \tilde{D}_{13}\Omega^{-1}B_{22}) = (C_{24} - A_{23}\Sigma^{-1}C_{12})B_{24}\\
\Leftrightarrow\ & P_2^\top A_2Q_2(Q_2^\top D_2V_2 - Q_2^\top D_1V_1\Omega^{-1}U_1^\top B_2V_2) = (P_2^\top C_2U_2 - P_2^\top A_2Q_1\Sigma^{-1}P_1^\top C_1U_2+)U_2^\top B_2V_2\\
\Leftrightarrow\ & P_2P_2^\top A_2Q_2(Q_2^\top D_2V_2 - Q_2^\top D_1V_1\Omega^{-1}U_1^\top B_2V_2)V_2^\top = P_2(P_2^\top C_2U_2 - P_2^\top A_2Q_1\Sigma^{-1}P_1^\top C_1U_2)U_2^\top B_2V_2V_2^\top\\
\Leftrightarrow\ & E_{A_1}(C_2B_2 - A_2D_2)F_{B_1} = 0.
\end{aligned}$$

Summing up the discussion above, we can reach a result as follows.

Theorem 3.1. *Assume that dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2, C = C_1 + \varepsilon C_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times p}, B_i \in \mathbb{R}^{q\times n}, C_i \in \mathbb{R}^{m\times q}, D_i \in \mathbb{R}^{p\times n}$ $(i = 1, 2)$. Let the SVDs of A_1, B_1 be given by* (4) *and the partitions of the matrices $U^\top B_2V, P^\top A_2Q, P^\top C_1U, P^\top C_2U, Q^\top D_1V$ and $Q^\top D_2V$ be given by* (6), (29) *and* (30). *Then Eq* (2) *is solvable if and only if*

$$\begin{aligned}
&E_{A_{24}}(C_{24} - A_{23}\Sigma^{-1}C_{12}) = 0, (\tilde{D}_{24} - \tilde{D}_{13}\Omega^{-1}B_{22})F_{B_{24}} = 0, E_{A_1}(C_2B_2 - A_2D_2)F_{B_1} = 0,\\
&E_{A_1}C_1 = 0,\ D_1F_{B_1} = 0,\ C_1B_1 = A_1D_1, C_2B_1 - A_2D_1 = A_1D_2 - C_1B_2,\\
&E_{A_1}(C_2B_1 - A_2D_1) = 0, (C_1B_2 - A_1D_2)F_{B_1} = 0.
\end{aligned} \tag{49}$$

In this case, the solution set of Problem II(a) can be expressed as

$$X_1 = Q\begin{bmatrix} \Sigma^{-1}C_{11} & \Sigma^{-1}C_{12}\\ \tilde{D}_{13}\Omega^{-1} & J_4 + F_{A_{24}}W_3E_{B_{24}} \end{bmatrix}U^\top, \tag{50}$$

$$X_2 = Q\begin{bmatrix} \Sigma^{-1}(C_{21} - A_{21}\Sigma^{-1}C_{11} - A_{22}\tilde{D}_{13}\Omega^{-1}) & J_5 - \Sigma^{-1}A_{22}F_{A_{24}}W_3E_{B_{24}} \\ J_6 - F_{A_{24}}W_3E_{B_{24}}B_{23}\Omega^{-1} & Z_4 \end{bmatrix}U^\top, \tag{51}$$

with J_4, J_5, J_6 being given by (44) *and* (46)*, and W_3, Z_4 are arbitrary matrices.*

Now, for $\forall\ X \in S_2$, we have

$$\begin{aligned} \|X\|_D^2 &= \|X_1\|_F^2 + \|X_2\|_F^2 = \|Q^\top X_1 U\|_F^2 + \|Q^\top X_2 U\|_F^2 \\ &= \|\Sigma^{-1}C_{12}\|_F^2 + \|\tilde{D}_{13}\Omega^{-1}\|_F^2 + \|J_4 + F_{A_{24}}W_3E_{B_{24}}\|_F^2 + \|\Sigma^{-1}(C_{21} - A_{21}\Sigma^{-1}C_{11} - A_{22}\tilde{D}_{13}\Omega^{-1})\|_F^2 \\ &\quad + \|\Sigma^{-1}C_{11}\|_F^2 + \|Z_4\|_F^2 + \|J_5 - \Sigma^{-1}A_{22}F_{A_{24}}W_3E_{B_{24}}\|_F^2 + \|J_6 - F_{A_{24}}W_3E_{B_{24}}B_{23}\Omega^{-1}\|_F^2. \end{aligned}$$

It is easily seen that $\|X\|_D$ is minimized if and only if $Z_4 = 0$, and

$$\Psi(W_3) = \|J_4 + F_{A_{24}}W_3E_{B_{24}}\|_F^2 + \|J_5 - \Sigma^{-1}A_{22}F_{A_{24}}W_3E_{B_{24}}\|_F^2 + \|J_6 - F_{A_{24}}W_3E_{B_{24}}B_{23}\Omega^{-1}\|_F^2 = \min. \tag{52}$$

The minimization problem (52) is equivalent to

$$\begin{aligned} \Psi(W_3) = \mathrm{tr}[&J_4^\top J_4 + E_{B_{24}}W_3^\top F_{A_{24}}W_3E_{B_{24}} + J_6^\top J_6 + J_5^\top J_5 + 2J_4^\top F_{A_{24}}W_3E_{B_{24}} \\ &+ \Omega^{-1}B_{23}^\top E_{B_{24}}W_3^\top F_{A_{24}}W_3E_{B_{24}}B_{23}\Omega^{-1} - 2J_5^\top\Sigma^{-1}A_{22}F_{A_{24}}W_3E_{B_{24}} \\ &+ E_{B_{24}}W_3^\top F_{A_{24}}A_{22}^\top\Sigma^{-1}\Sigma^{-1}A_{22}F_{A_{24}}W_3E_{B_{24}} - 2J_6^\top F_{A_{24}}W_3E_{B_{24}}B_{23}\Omega^{-1}] = \min. \end{aligned}$$

Therefore, $\Psi(W_3)$ is minimized if and only if $\frac{\partial\Psi(W_3)}{\partial W_3} = 0$, which implies that

$$(F_{A_{24}} + 2L_2^\top L_2)W_3E_{B_{24}} + F_{A_{24}}W_3(E_{B_{24}} + 2L_1^\top L_1) = J_7, \tag{53}$$

where

$$J_7 = 2F_{A_{24}}A_{22}^\top\Sigma^{-1}J_5E_{B_{24}} - 2F_{A_{24}}J_4E_{B_{24}} + 2F_{A_{24}}J_6\Omega^{-1}B_{23}^\top E_{B_{24}}. \tag{54}$$

By applying the Kronecker product and stretching function, it follows from (53) that

$$\tilde{\Pi}\mathrm{vec}(W_3) = \mathrm{vec}(J_7), \tag{55}$$

where

$$\tilde{\Pi} = E_{B_{24}} \otimes (F_{A_{24}} + 2L_2^\top L_2) + (E_{B_{24}} + 2L_1^\top L_1) \otimes F_{A_{24}}. \tag{56}$$

Thus, by solving the equation of (55), we can get the solution of W_3, and then, substituting $Z_4 = 0$ into (50) and (51), we have the following result.

Theorem 3.2. *If the conditions* (49) *are satisfied, then the solution of Problem II(b) is given by*

$$\hat{X} = \hat{X}_1 + \varepsilon\hat{X}_2,$$

where

$$\hat{X}_1 = Q\begin{bmatrix} \Sigma^{-1}C_{11} & \Sigma^{-1}C_{12} \\ \tilde{D}_{13}\Omega^{-1} & J_4 + F_{A_{24}}W_3E_{B_{24}} \end{bmatrix}U^\top, \tag{57}$$

$$\hat{X}_2 = Q\begin{bmatrix} \Sigma^{-1}J_4 & J_5 - \Sigma^{-1}A_{22}F_{A_{24}}W_3E_{B_{24}} \\ J_6 - F_{A_{24}}W_3E_{B_{24}}B_{23}\Omega^{-1} & 0 \end{bmatrix}U^\top, \tag{58}$$

with W_3 being given by (55).

Based on Theorems 3.1 and 3.2, we can describe an algorithm to solve Problems II(a) and II(b) as follows.

Algorithm 2

1: Input matrices A_i, B_i, C_i and $D_i, i = 1, 2$.
2: Compute the SVDs of the matrices $A_1,\ B_1$ on the basis of (4).
3: Compute the partitions of the matrices $U^\top B_2 V, P^\top A_2 Q, P^\top C_1 U, P^\top C_2 U, Q^\top D_1 V$ and $Q^\top D_2 V$ as in (6), (29) and (30).
4: If the conditions (49) are satisfied, then continue, otherwise, Problem II(a) has no solution, and stop.
5: Calculate Y_1, Y_2, Y_3, Z_1 according to (35) and (40).
6: Compute J_4, J_5, J_6 in the light of (44) and (46), respectively.
7: Compute $L_1 = \Omega^{-1} B_{23}^\top E_{B_{24}}, L_2 = \Sigma^{-1} A_{22} F_{A_{24}}$ and compute $J_7, \tilde{\Pi}$ on the basis of (54) and (56). Then, calculate W_3 by the equation of (55).
8: Compute $Y_4,\ Z_2,\ Z_3$ by (43) and (45), respectively.
9: Compute $\hat{X}_1, \hat{X}_2$ as in (57) and (58), and compute the unique solution $\hat{X} = \hat{X}_1 + \varepsilon \hat{X}_2$ of Problem II(b).

Example 3.1. *Let $m = 8,\ n = 9,\ p = 6,\ q = 7$, and the matrices $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ be given by*

$$A_1 = \begin{bmatrix} 1.5526 & 0.5212 & 0.3530 & 0.8404 & 0.8114 & 0.9951 \\ 0.9989 & 0.4602 & 0.2003 & 0.7210 & 0.6974 & 0.8205 \\ 0.8074 & 0.2321 & 0.1919 & 0.3809 & 0.3673 & 0.4613 \\ 0.8202 & 0.4030 & 0.1591 & 0.6283 & 0.6080 & 0.7100 \\ 1.3044 & 0.4626 & 0.2913 & 0.7418 & 0.7164 & 0.8717 \\ 0.3896 & 0.1243 & 0.0900 & 0.2016 & 0.1945 & 0.2404 \\ 0.5914 & 0.1352 & 0.1481 & 0.2287 & 0.2201 & 0.2877 \\ 1.1147 & 0.3696 & 0.2544 & 0.5968 & 0.5761 & 0.7078 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.3349 & 1.3005 & 1.2315 & 1.6795 & 0.9943 & 0.6848 \\ 0.2255 & 0.2339 & 0.1988 & 0.3186 & 0.2004 & 0.1629 \\ 1.1621 & 0.8659 & 0.7576 & 1.2040 & 0.6569 & 0.5779 \\ 0.8167 & 0.5498 & 0.5099 & 0.7510 & 0.3773 & 0.3187 \\ 1.0789 & 1.1427 & 1.0635 & 1.4789 & 0.9083 & 0.6311 \\ 0.8222 & 0.8648 & 0.8445 & 1.0876 & 0.6537 & 0.4073 \\ 0.8829 & 0.9214 & 0.8787 & 1.1768 & 0.7123 & 0.4716 \\ 0.7993 & 0.8774 & 0.8067 & 1.1399 & 0.7115 & 0.5010 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.0446 & 0.5327 & 0.2550 & 0.3699 & 0.1876 & 0.4685 & 0.3517 & 0.3635 & 0.4113 \\ 0.0686 & 0.8480 & 0.3537 & 0.5734 & 0.3122 & 0.7693 & 0.5629 & 0.5937 & 0.6660 \\ 0.0622 & 0.5497 & 0.6125 & 0.4845 & 0.1037 & 0.3269 & 0.3427 & 0.2741 & 0.3489 \\ 0.0768 & 0.8412 & 0.5404 & 0.6246 & 0.2609 & 0.6782 & 0.5475 & 0.5342 & 0.6198 \\ 0.0674 & 0.7740 & 0.4264 & 0.5538 & 0.2583 & 0.6557 & 0.5078 & 0.5120 & 0.5855 \\ 0.0836 & 0.7374 & 0.8254 & 0.6511 & 0.1381 & 0.4369 & 0.4596 & 0.3666 & 0.4673 \\ 0.0751 & 0.7531 & 0.6204 & 0.5994 & 0.1984 & 0.5459 & 0.4822 & 0.4387 & 0.5253 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1.5570 & 1.3803 & 0.9873 & 1.2107 & 1.4629 & 0.7456 & 1.7241 & 1.0553 & 1.4645 \\ 0.6003 & 0.4898 & 0.6809 & 0.6524 & 0.7224 & 0.5833 & 0.8351 & 0.7540 & 0.7619 \\ 0.4270 & 0.3812 & 0.7336 & 0.8245 & 0.7820 & 0.6548 & 0.7463 & 0.7206 & 0.7354 \\ 0.9920 & 0.7537 & 0.7935 & 0.5618 & 0.8125 & 0.6460 & 1.1754 & 1.0193 & 0.9963 \\ 1.1495 & 0.8863 & 1.1764 & 0.9592 & 1.1759 & 0.9975 & 1.5157 & 1.4030 & 1.3448 \\ 0.8428 & 0.7449 & 1.0424 & 1.1799 & 1.1993 & 0.8982 & 1.2326 & 1.0525 & 1.1568 \\ 0.3822 & 0.3760 & 0.3597 & 0.5329 & 0.5291 & 0.2940 & 0.4984 & 0.3125 & 0.4601 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 3.0365 & 2.3514 & 2.9403 & 2.3560 & 2.5632 & 3.2087 & 1.7223 \\ 2.2789 & 1.9450 & 2.1991 & 1.9504 & 1.9141 & 2.3853 & 1.3086 \\ 1.4777 & 1.0882 & 1.4332 & 1.0897 & 1.2503 & 1.5686 & 0.8332 \\ 1.9367 & 1.6841 & 1.8676 & 1.6890 & 1.6250 & 2.0232 & 1.1148 \\ 2.6154 & 2.0611 & 2.5310 & 2.0654 & 2.2058 & 2.7592 & 1.4866 \\ 0.7451 & 0.5677 & 0.7219 & 0.5687 & 0.6295 & 0.7886 & 0.4218 \\ 0.9917 & 0.6766 & 0.9641 & 0.6771 & 0.8420 & 1.0596 & 0.5544 \\ 2.1681 & 1.6724 & 2.0997 & 1.6755 & 1.8305 & 2.2919 & 1.2292 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 6.1864 & 6.6590 & 6.3910 & 6.6562 & 7.0540 & 4.9153 & 5.8256 \\ 2.6188 & 3.1458 & 2.1945 & 2.9235 & 2.8930 & 1.7231 & 2.9235 \\ 4.0004 & 4.0163 & 4.1155 & 3.9904 & 4.3817 & 3.3262 & 3.4919 \\ 3.4300 & 3.6096 & 3.2798 & 3.4675 & 3.7839 & 2.6145 & 3.3363 \\ 5.3391 & 5.8266 & 5.5391 & 5.8421 & 6.0971 & 4.2667 & 5.0249 \\ 2.9339 & 2.9822 & 3.4557 & 3.1650 & 3.4627 & 2.6087 & 2.4604 \\ 3.3261 & 3.4272 & 3.7918 & 3.5852 & 3.8690 & 2.8896 & 2.8395 \\ 4.2214 & 4.6594 & 4.3246 & 4.6529 & 4.8070 & 3.3348 & 4.0138 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.2502 & 2.5297 & 2.0412 & 2.0009 & 0.6774 & 1.8530 & 1.6223 & 1.4861 & 1.7738 \\ 0.2940 & 3.1995 & 2.0962 & 2.3877 & 0.9817 & 2.5612 & 2.0798 & 2.0200 & 2.3483 \\ 0.2053 & 2.0830 & 1.6636 & 1.6426 & 0.5622 & 1.5335 & 1.3368 & 1.2287 & 1.4643 \\ 0.2183 & 2.3651 & 1.5697 & 1.7710 & 0.7205 & 1.8842 & 1.5363 & 1.4874 & 1.7316 \\ 0.2591 & 2.6380 & 2.0893 & 2.0750 & 0.7165 & 1.9500 & 1.6940 & 1.5611 & 1.8583 \\ 0.2240 & 2.4095 & 1.6345 & 1.8146 & 0.7249 & 1.9038 & 1.5631 & 1.5051 & 1.7564 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 3.7771 & 6.1417 & 5.4631 & 6.0497 & 5.0201 & 5.2884 & 6.5640 & 5.5281 & 6.3668 \\ 4.2620 & 6.1961 & 5.8150 & 5.9103 & 5.1259 & 5.3279 & 6.9569 & 5.9395 & 6.5989 \\ 2.6236 & 3.9042 & 3.8013 & 3.9615 & 3.5098 & 3.8863 & 4.6084 & 4.2401 & 4.4804 \\ 2.7051 & 4.4944 & 3.9692 & 4.2023 & 3.5063 & 4.0012 & 4.8152 & 4.2480 & 4.6845 \\ 3.2666 & 4.7381 & 4.5846 & 4.7936 & 4.2062 & 4.3934 & 5.4957 & 4.8244 & 5.2769 \\ 3.3398 & 4.9858 & 4.5069 & 4.7309 & 4.0859 & 4.2395 & 5.4946 & 4.6454 & 5.2375 \end{bmatrix}.$$

It is easy to verity that the conditions (49) *hold:*

$$\|E_{A_{24}}(C_{24} - A_{23}\Sigma^{-1}C_{12})\|_{\mathrm{F}} = 1.9816 \times 10^{-15}, \|(\tilde{D}_{24} - \tilde{D}_{13}\Omega^{-1}B_{22})F_{B_{24}}\|_{\mathrm{F}} = 1.9246 \times 10^{-15},$$
$$\|E_{A_1}C_1\|_{\mathrm{F}} = 1.8511 \times 10^{-15}, \|D_1F_{B_1}\|_{\mathrm{F}} = 1.8127 \times 10^{-15}, \|C_1B_1 - A_1D_1\|_{\mathrm{F}} = 5.2525 \times 10^{-15},$$
$$\|E_{A_1}(C_2B_2 - A_2D_2)F_{B_1}\|_{\mathrm{F}} = 1.8738 \times 10^{-14}, \|E_{A_1}(C_2B_1 - A_2D_1)\|_{\mathrm{F}} = 1.0763 \times 10^{-14},$$
$$\|(C_1B_2 - A_1D_2)F_{B_1}\|_{\mathrm{F}} = 1.2767 \times 10^{-14}, \|C_2B_1 - A_2D_1 - A_1D_2 + C_1B_2\|_{\mathrm{F}} = 2.0019 \times 10^{-14}.$$

According to Algorithm 2, we can obtain the unique solution $\hat{X} = \hat{X}_1 + \varepsilon\hat{X}_2$ of Problem II(b) as follows:

$$\hat{X}_1 = \begin{bmatrix} 0.9156 & 0.1339 & 0.7289 & 0.0326 & 0.6173 & 0.8750 & 0.5012 \\ 0.7731 & 0.7486 & 0.7637 & 0.7232 & 0.8637 & 0.1447 & 0.4641 \\ 0.0944 & 0.1119 & 0.8647 & 0.4961 & 0.8063 & 0.5116 & 0.0583 \\ 0.2780 & 0.6347 & 0.4317 & 0.4945 & 0.5737 & 0.0871 & 0.6869 \\ 0.3531 & 0.4732 & 0.8058 & 0.9207 & 0.1796 & 0.8899 & 0.0477 \\ 0.6618 & 0.8004 & 0.0890 & 0.5937 & 0.2432 & 0.8028 & 0.0660 \end{bmatrix},$$

$$\hat{X}_2 = \begin{bmatrix} 0.6273 & 0.7506 & 0.4251 & 0.6879 & 0.6432 & 0.2920 & 0.8319 \\ 0.4378 & 0.5937 & 0.5012 & 0.6175 & 0.5620 & 0.5559 & 0.6663 \\ 0.4188 & 0.6570 & 0.0717 & 0.4749 & 0.4988 & 0.0348 & 0.3547 \\ 0.5093 & 0.6965 & 0.2578 & 0.5517 & 0.5655 & 0.1585 & 0.5424 \\ 0.4335 & 0.5651 & 0.2867 & 0.4791 & 0.4770 & 0.2045 & 0.5139 \\ 0.4439 & 0.5362 & 0.3786 & 0.5012 & 0.4774 & 0.2950 & 0.6010 \end{bmatrix}.$$

The absolute errors are estimated by

$$\|A_1\hat{X}_1 - C_1\|_F = 4.8343 \times 10^{-15},\ \|A_2\hat{X}_1 + A_1\hat{X}_2 - C_2\|_F = 6.7203 \times 10^{-15},$$
$$\|\hat{X}_1B_1 - D_1\|_F = 1.0333 \times 10^{-14},\ \|\hat{X}_1B_2 + \hat{X}_2B_1 - D_2\|_F = 2.8801 \times 10^{-14},$$

which implies that $\hat{X}$ is the solution of Problem II(b).

4. Conclusions

Solving dual matrix equations is often required in kinematic analysis and sensor calibration. In the present paper, the dual matrix equations (1) and (2) are first factorized as two real matrix equations by separating them into the real parts and dual parts. By applying the SVDs of A_1 and B_1, we have obtained the solvability conditions and the general solutions of Problems I(a) and II(a) (see Theorems 2.1 and 3.1). Further, based on the results of Theorems 2.1 and 3.1, the unique minimum-norm solutions $\hat{X} = \hat{X}_1 + \varepsilon\hat{X}_2$ of Problems I(b) and II(b) have achieved (see Theorems 2.2 and 3.2).

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Use of AI tools declaration

The authors declare that we have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interest.

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