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Research article

h-stability for stochastic functional differential equation driven by time-changed Lévy process

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Abstract: In this paper, we investigate a class of stochastic functional differential equations driven by the time-changed Lévy process. Using the Lyapunov technique, we obtain some sufficient conditions to ensure that the solutions of the considered equations are h-stable in p-th moment sense. Subsequently, using time-changed Itô formula and a proof by reduction ad absurdum, we capture some new criteria for the h-stability in mean square of the considered equations. In the end, we analyze some illustrative examples to show the interest and usefulness of the major results.

Keywords: h-stability; time-changed Lévy process; Lyapunov method; time-changed Itô formula
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1. Introduction

Recently, the time-changed semimartingale has attracted a lot of attention because of its extensive applications in cell biology, hydrology, physics, economics and finance (see Umarov et al. [20]). Since Kobayashi [11] developed stochastic calculus on the time-changed semimartingale, and derived the associated Itô formula while the time-changed process is the initial hitting time of the stable subordinator with the index being located to $(0, 1)$, many papers were devoted to the investigation of stochastic differential equations (SDEs) driven by the time-changed Brownian motion or Lévy process. For instance, [7, 10, 14] for the numerical approximation scheme; [2, 19] for the averaging principle.

In particular, there has been increasing interest for the stability in different senses for all kinds of SDEs with the time-changed semimartingale. Wu [21, 22] considered the stability in probability and moment exponential stability for several classes of SDEs driven by the time-changed Brownian motion. In [17, 18], Nane and Ni established the Itô formula for time-changed Lévy noise and studied the probability stability, moment stability and path stability. Zhang and Yuan [27] established some Razumikhin theorems on the exponential stability for the time-changed stochastic functional differential equations with Markovian switching. In [28], some sufficient conditions were presented to ensure the asymptotic stability of solutions to the time-changed SDEs with Markovian switching. In [26], Yin et al. provided some sufficient conditions to guarantee the solutions to nonlinear SDEs driven by the time-changed Lévy noise with impulsive effects are stable in different senses, and also considered some unstable time changed SDEs with impulses. Zhu et al. [29] considered the exponential stability for the time-changed stochastic differential equations. Li et al. [13] considered the global attracting sets and exponential stability of stochastic functional differential equations driven by the time-changed Brownian motion.

On the other hand, the notions of h-stability is a worthwhile broadening of some well-known stability types, such as polynomial stability, exponential stability and logarithmical stability etc. The h-stability of the deterministic systems has evoked more and more attention since it spreads a new perception in the long time behavior of the solution. For instance, Choi et al. [3] for a class of linear dynamic equations; Damak et al. [6] for the boundedness and h-stability of a class of perturbed equations; Ghanmi [8] for the practical h-stability; Xu and Liu [24] and Xu et al. [25] for h-stability for the numerical solutions of a class of pantograph equations; Damak et al. [5] for the converse theorem on practical h-stability of nonlinear differential equations; Damak [4] and Mihiţ [16] studied h-stability of some evolution equations in Banach spaces by using some Gronwall-type inequalities.

However, there is rare literature on the h-stability for stochastic systems and there is no result on the h-stability for the time-changed stochastic systems. By the Lyapunov's method, Caraballo et al. [1] explored the h-stability for stochastic neutral pantograph differential equations with Lévy noise; Li et al. [12] studied the h-stability for stochastic Volterra-Levin equations; Hu et al. [9] considered the h-stability for a class of unbounded delay neutral stochastic differential equations with general decay rate; Wu et al. [23] investigated the h-stability nonlinear neutral stochastic functional differential equations with infinite delay. In this work, we will attempt to explore the h-stability of the following functional SDEs with time-changed Lévy process:

$$\begin{cases} dx(t) = u(t, E_t, x_{t-})dt + f(t, E_t, x_{t-})dE_t + g(t, E_t, x_{t-})dB_{E_t} + \int_{|y|<c} b(t, E_t, x_{t-}, y)\tilde{N}(dE_t, dy), \\ x_0 = \xi \in D^b_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^d), \end{cases} \tag{1.1}$$

where E_t is designated as the inverse of the β-stable subordinator with index $\beta \in (0, 1)$; the constant $c > 0$ is the maximum allowable jump size; u, f, g and b are all progressively measurable with further conditions specified in the sequel. The first main aim of this paper is to give some conditions ensuring that the solution of (1.1) is h-stable in p-th moment sense by using some Lyapunov's technique.

It is worth noting that almost all literatures on h-stability for stochastic systems are based on Lyapunov function methods. Actually, it is not easy to seek for the Lyapunov's function for the time-changed stochastic systems with Lévy noise. On the other hand, the stability conditions captured by using the Lyapunov's function method are usually given on the basis of differential inequalities, matrix inequalities and so on, which are often implicit and not easy to be examined. The second aim of this work is to explore some new explicit conditions ensuring that the solution of (1.1) is the h-stability in the mean square under some generalized assumptions. Our approach is based on a comparison principle and a proof by contradiction.

The rest of this paper is organized as follows: In Section 2, we introduce some necessary notations and preliminaries. In Section 3, we state and prove our major results. In Section 4, we give some examples to elaborate the effectiveness of our theory.

2. Preliminary

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $\{D(t), t \geq 0\}$ be a left limit and right continuous non-decreasing Lévy process, which is named a subordinator that begins with 0. In particular, $D(t)$ is called the β-stable subordinator denoted by $D_\beta(t)$ if it is a strictly increasing with the following Laplace transform:

$$\mathbb{E}(e^{-\lambda D_\beta(t)}) = e^{-t\lambda^\beta}, \quad \lambda > 0,\ \beta \in (0, 1).$$

Define the generalized inverse of an adapted β-stable subordinator $D_\beta(t)$ as

$$E_t := E_t^\beta = \inf\{s > 0 : D_\beta(s) > t\},$$

which is known as the initial hitting time process. The time change process E_t is nondecreasing and continuous. Define the special filtration as

$$\mathcal{F}_t = \bigcap_{s>t}\{\sigma(B_r : 0 \leq r \leq s) \vee \sigma(E_r : r \geq 0)\},$$

where B_r is the standard Brownian motion and the notation $\sigma_1 \vee \sigma_2$ denotes the σ-algebra generated by the union of σ-algebras σ_1 and σ_2. By [15] we can conclude that B_{E_t} is the square integrable martingale with regard to the filtration $\mathcal{G}_t = \mathcal{F}_{E_t}$.

Given a Lévy measure ν, namely, ν is a σ-finite measure on the Borel measurable space $\mathbb{R}^d\backslash 0$ satisfying

$$\int_{\mathbb{R}^d\backslash\{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty,$$

we assume that the associated $\{\mathcal{F}_t\}$-Poisson random measure N on $\mathbb{R}^+ \times (\mathbb{R}^d\backslash\{0\})$ is independent of $B(t)$. The associated compensated martingale measure is denoted by

$$\widetilde{N}(dt, dy) := N(dt, dy) - \nu(dy)dt.$$

Let $r > 0$. Recall that a function $f\colon [-r, 0] \to \mathbb{R}^d$ is called càdlág if it is right continuous and has finite left-hand limits. Denote by $D([-r, 0]; \mathbb{R}^d)$ the space of all $\mathbb{R}^d$-valued càdlág functions defined on $[-r, 0]$, equipped with the uniform norm

$$\|\phi\|_D := \sup_{-r\leq s\leq 0} |\phi(s)|, \quad \phi \in D([-r, 0]; \mathbb{R}^d).$$

Let $D^b_{\mathcal{F}_t}([-r, 0]; \mathbb{R}^d)(D^b_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^d))$ denote the family of all bounded $\mathcal{F}_t(\mathcal{F}_0)$-measurable, $D([-r, 0]; \mathbb{R}^d)$-value random variables. Following [26], we present the following hypotheses for ensuring the existence and uniqueness of solution to (1.1):

(H1) $u, f\colon \mathbb{R}_+ \times \mathbb{R}_+ \times D \to \mathbb{R}^d$ are measurable functions and there is a positive constant L such that for any $t_1, t_2 \geq 0$ and $x, y \in D$

$$|f(t_1, t_2, x) - f(t_1, t_2, y)| \vee |u(t_1, t_2, x) - u(t_1, t_2, y)| \leq L\|x - y\|_C.$$

(H2) $g\colon \mathbb{R}_+ \times \mathbb{R}_+ \times D \to \mathbb{R}^{d\times k}$ and $b\colon \mathbb{R}_+ \times \mathbb{R}_+ \times D \times \mathbb{R}^d \backslash \{0\} \to \mathbb{R}^{d\times k}$ are measurable functions and there is a positive constant L such that for any $t_1, t_2 \geq 0$ and $x_1, x_2 \in D$

$$\|g(t_1, t_2, x_1) - g(t_1, t_2, x_2)\| \vee \int_{|y|<c} \|b(t_1, t_2, x_1, y) - b(t_1, t_2, x_1, y)\| dy \leq L\|x_1 - x_2\|_D.$$

(H3) If $X(t)$ is a right continuous with left limit and $\mathcal{G}_t$-adapted process, then

$$u(t, E_t, x_t), f(t, E_t, x_t), g(t, E_t, x_t), b(t, E_t, x_t, y) \in \mathcal{L}(\mathcal{G}_t),$$

where $\mathcal{L}(\mathcal{G}_t)$ denotes the class of left continuous with right limit.

For the sake of stability, we also assume that for any $t_1, t_2 \geq 0$

$$u(t_1, t_2, 0) \equiv 0, \quad f(t_1, t_2, 0) \equiv 0, \quad g(t_1, t_2, 0) \equiv 0, \quad b(t_1, t_2, 0, y) \equiv 0. \tag{2.1}$$

According to [26], we deduce that (1.1) has an unique $\mathcal{G}_t$-adapted solution process $x(t)$ under the assumptions (H1)–(H3). Moreover, the Eq (1.1) has a trivial solution when the initial value $\xi \equiv 0$.

Definition 2.1. ([1]) The function $h\colon \mathbb{R}_+ \to (0, +\infty)$ is called to be the h-type function if the following assumptions hold:

(i) It is nondecreasing and continuously differentiable in $\mathbb{R}_+$.

(ii) $h(0) = 1$, $\lim_{t\to\infty} h(t) = \infty$ and $J = \sup_{t>0} \left|\frac{h'(t)}{h(t)}\right| < \infty$.

(iii) For any $t \geq 0$, $s \geq 0$, $h(t + s) \leq h(t)h(s)$.

Definition 2.2. ([27]) The solution $x(t, \xi)$ of (1.1) is said to be h-stable p-th moment sense, if for any initial value ξ, there exists a pair of constants $\delta > 0$ and $K > 0$, such that for all $t \geq 0$

$$\mathbb{E}|x(t, \xi)|^p \leq K\mathbb{E}\|\xi\|_C h^{-\delta}(t). \tag{2.2}$$

In particular, if $p = 2$, this is called the h-stable in mean square.

Remark 2.1. We remark that h-stability coincides with some known stability types when h are some special cases functions. Actually, if

$$h(t) = e^t, \quad t \geq 0,$$

then h-stability is coincident with exponential stability; if

$$h(t) = 1 + t, \quad t \geq 0,$$

then h-stability is coincident with polynomial stability and if

$$h(t) = \ln(e + t), \quad t \geq 0,$$

then h-stability is coincident with logarithmical stability.

Remark 2.2. We remark that there is the h-type function which is faster than the speed at which e^t tends to infinity. For example,

$$h(t) = (1 + t)e^t$$

is an h-type function and

$$\lim_{t\to+\infty} \frac{(1 + t)e^t}{e^t} = +\infty.$$

3. Main results

Let

$$V \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}),$$
$$V_{t_1}(t_1, t_2, x) = \frac{\partial V(t_1, t_2, x)}{\partial t_1},$$
$$V_{t_2}(t_1, t_2, x) = \frac{\partial V(t_1, t_2, x)}{\partial t_2},$$
$$V_x(t_1, t_2, x) = \Big(\frac{\partial V(t_1, t_2, x)}{\partial x_1}, \cdots, \frac{\partial V(t_1, t_2, x)}{\partial x_d}\Big)$$

and

$$V_{xx}(t_1, t_2, x) = \Big(\frac{\partial^2 V(t_1, t_2, x)}{\partial x_i \partial x_j}\Big)_{d\times d}$$

are continuous for all

$$(t_1, t_2, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d.$$

By Itô formula (see [18]) for (1.1) we have

$$\begin{aligned} dV(t_1, t_2, x) =& J_1 V(t_1, t_2, x)dt + J_2 V(t_1, t_2, x)dE_t + V_x(t_1, t_2, x)g(t_1, t_2, x)dB_{E_t} \\ &+ \int_{|y|<c} V_x(t_1, t_2, x)b(t, E_t, x_{t-}, y)\tilde{N}(dE_t, dy), \end{aligned} \tag{3.1}$$

where

$$J_1 V(t_1, t_2, x) = V_{t_1}(t_1, t_2, x) + V_x(t_1, t_2, x)u(t_1, t_2, y_t), \tag{3.2}$$

and

$$\begin{aligned} J_2 V(t_1, t_2, x) =& V_{t_2}(t_1, t_2, x) + V_x(t_1, t_2, x)f(t_1, t_2, x) + \frac{1}{2}Tr[g^T(t_1, t_2, x)V_{xx}(t_1, t_2, x)g(t_1, t_2, x)] \\ &+ \int_{|y|<c} [V(t_1, t_2, x + b(t_1, t_2, x, y)) - V(t_1, t_2, x) - V_x(t_1, t_2, x)b(t_1, t_2, x, y)]\nu(dy). \end{aligned} \tag{3.3}$$

Theorem 3.1. Let the assumptions (H1)–(H3) hold. Assume that there is $V \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+)$ such that for any $(t, E_t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d$

(i) $c_1|x(t)|^p \le V(t, E_t, x(t)) \le c_2|x(t)|^p$,

(ii) $J_1 V(t, E_t, x(t)) \le -\lambda V(t, E_t, x(t))$,

(iii) $J_2 V(t, E_t, x(t)) \le 0$,

hold for the solution $x(t)$ of (1.1), where p, c_1, c_2 and λ are some positive constants. Then for $\delta \in (0, \lambda/J)$

$$\mathbb{E}(|x(t)|^p) \le \frac{c_2}{c_1} h^{-\delta}(t)\mathbb{E}(\|\xi\|_D^p).$$

This is, the trivial solution of (1.1) is h-stable in mean square.

Proof. Let $\delta \in (0, \lambda/J)$. Applying Itô formula to $h^\delta(t)V(t_1, t_2, x(t))$, we obtain for any $t \geq 0$

$$\begin{aligned} h^\delta(t)V(t_1, t_2, x(t)) =& V(0, 0, x(0)) + \int_0^t h^\delta(s)\Big[\delta\frac{h'(s)}{h(s)}V(s, E_s, x(s)) + V_{t_1}(s, E_s, x(s)) + V_x(s, E_s, x(s))f(s, E_s, x_{s-})\Big]ds \\ &+ \int_0^t h^\delta(s)\Big[V_{t_2}(s, E_s, x(s)) + V_x(s, E_s, x(s))u(s, E_s, x_{s-}) \\ &+ \frac{1}{2}Tr[g^T(s, E_s, x_{s-})V_{xx}(s, E_s, x(s))g(s, E_s, x_{s-})]\Big]dE_s \\ &+ \int_0^t \int_{|y|<c} [V(t_1, t_2, x + b(s, E_s, x_{s-}, y)) - V(t_1, t_2, x) - V_x(t_1, t_2, x)b(s, E_s, x_{s-}, y)]\nu(dy)dE_s \\ &+ \int_0^t h^\delta(s)V_x(s, E_s, x(s))g(s, E_s, x_{s-})dB_{E_s} + \int_0^t \int_{|y|<c} V_x(s, E_s, x)b(s, E_s, x_{s-}, y)\tilde{N}(dE_s, dy). \end{aligned} \tag{3.4}$$

Notice that

$$\mathbb{E}\int_0^t h^\delta(s)V_x(s, E_s, y(s))g(s, E_s, x_{s-})dB_{E_s} = 0$$

and

$$\mathbb{E}\int_0^t \int_{|y|<c} V_x(s, E_s, x)b(s, E_s, x_{s-}, y)\tilde{N}(dE_s, dy) = 0.$$

Then, taking expectation on both two sides of (3.4) we have

$$\begin{aligned} \mathbb{E}[h^\delta(t)V(t, E_t, x(t))] =& \mathbb{E}[V(0, 0, x(0))] + \int_0^t h^\delta(s)\Big[\delta\frac{h'(s)}{h(s)}V(s, E_s, x(s)) + J_1V(s, E_s, x(s))\Big]ds \\ &+ \int_0^t h^\delta(s)J_2V(s, E_s, x(s))dE_s. \end{aligned}$$

By using (ii) in the Definition 2.1 we get

$$\begin{aligned} \mathbb{E}[h^\delta(t)V(t, E_t, x(t))] \leq& \mathbb{E}[V(0, 0, x(0))] + \int_0^t h^\delta(s)\Big[\delta JV(s, E_s, x(s)) + J_1V(s, E_s, x(s))\Big]ds \\ &+ \int_0^t h^\delta(s)J_2V(s, E_s, x(s))dE_s. \end{aligned}$$

Since $\delta \in (0, \lambda/J)$, by conditions (ii) and (iii), we have

$$\mathbb{E}[h^\delta(t)V(t, E_t, x(s))] \leq \mathbb{E}[V(0, 0, x(0))].$$

Furthermore, we can obtain from the condition (i)

$$c_1\mathbb{E}[h^\delta(t)|x(t)|^p] \leq \mathbb{E}[h^\delta(t)V(t, E_t, x(t))] \leq \mathbb{E}[V(0, 0, x(0))] \leq c_2\mathbb{E}[||\xi||_D^p].$$

Hence

$$\mathbb{E}[|x(t)|^p] \leq \frac{c_2}{c_1}h^{-\delta}(t)\mathbb{E}[||\xi||_D^p].$$

The proof is complete. □

Remark 3.1. In [29], the authors proved that the solution of (1.1) without time delay is the p-th moment exponentially stable under (H1)–(H3) when the conditions (i)–(iii) hold. So, by the Remark 2.1 know that our Theorem 3.1 generalizes the Theorem 4.1 of [29].

Remark 3.2. In [27], the authors proved that the solution of (1.1) with Markovian switching is the p-th moment exponentially stable under (H1)–(H3) when the conditions (i)–(iii) hold. We remark that we can immediately show that the solution of (1.1) with Markovian switching is the h-stable in p-th moment sense under (H1)–(H3) when the corresponding conditions (i)–(iii) hold, which means that our Theorem 3.1 expands the Theorem 3.1 of [27] by using the Remark 2.1.

Now, we use the Theorem 3.1 to establish a useful corollary.

Corollary 3.1. Let the assumptions (H1)–(H3) hold. If there is a positive constant $\lambda > 0$ such that for all $t \geq 0$ and the solution $x(t)$ of (1.1)

$$\langle x(t), u(t, E_t, x_{t-})\rangle \leq -\lambda|x(t)|^2 \tag{3.5}$$

and

$$\begin{aligned} &2\langle x(t), f(t, E_t, x_{t-})\rangle + Tr[g^T(t, E_t, x_{t-})g(t, E_t, x_{t-})] \\ &+ \int_{|y|<c} Tr[b^T(t, E_t, x_{t-}, y)b(t, E_t, x_{t-}, y)]\nu(dy) \leq 0. \end{aligned} \tag{3.6}$$

Then, the trivial solution of (1.1) is the h-stable in mean square.

Proof. Let

$$V(t, E_t, x(t)) = |x(t)|^2.$$

We can immediately know that condition (i) in the Theorem 3.1 holds for $p = 2$, $c_1 = c_2 = 1$ and

$$J_1V(t, E_t, x(t)) = 2\langle x(t), f(t, E_t, x_{t-})\rangle,$$

$$J_2V(t, E_t, x(t)) = 2\langle x(t), u(t, E_t, x_{t-})\rangle + Tr[g^T(t, E_t, x_{t-})g(t, E_t, x_{t-})].$$

Thus, (3.5) and (3.6) imply the conditions (ii) and (iii) hold, respectively. The proof is complete. □

Nextly, we intend to explore some new conditions ensuring the h-stability for (1.1). To this end, we need to define some functions. Let

$$\eta(t, s) : \mathbb{R}_+ \times [-r, 0] \to \mathbb{R}$$

be increasing in s for each $t \in \mathbb{R}_+$. Furthermore, $\eta(t, s)$ is normalized to be continuous from the left in s on $[-r, 0]$. Assume that for each $\phi \in D([-r, 0]; \mathbb{R})$

$$L(t, \phi) := \int_{-r}^{0} \phi(s)d[\eta(t, s)] \tag{3.7}$$

is a locally bounded Borel-measurable function in t. Here, the integral in (3.7) is the Riemann-Stieltjes integral.

Theorem 3.2. Assume that there exits a locally bounded Borel-measurable function $\gamma(\cdot)$: $\mathbb{R}_+ \to \mathbb{R}$ such that for any $t \in \mathbb{R}_+$, $x \in D([-r, 0]; \mathbb{R}^d)$

$$2\langle x(t), u(t, E_t, x_{t-})\rangle \leq \gamma(t)|x(t)|^2 + \int_{-r}^{0} |x(s)|^2 d[\eta(t, s)] \tag{3.8}$$

and

$$2\langle x(t), f(t, E_t, x_{t-})\rangle + Tr[g^T(t, E_t, x_{t-})g(t, E_t, x_{t-})] \\ + \int_{|y|<c} Tr[b^T(t, E_t, x_{t-}, y)b(t, E_t, x_{t-}, y)]\nu(dy) \leq 0. \tag{3.9}$$

Then, the solution of (1.1) is h-stable in mean square sense if there exists $\beta > 0$ such that for any $t \in \mathbb{R}_+$,

$$\gamma(t) + \int_{-r}^{0} h^{\beta}(-s)d[\eta(t, s)] \leq -\beta. \tag{3.10}$$

Proof. Let us consider the following two cases:

Case 1. If $J \in (0, 1]$. Fix $K > 1$ and let $\xi \in D([-r, 0]; \mathbb{R}^d)$, such that $\mathbb{E}\|\xi\|_D^2 > 0$. We denote $x(t) := x(t, \xi)$ for simplicity, $t \geq -r$, where $x(t, \xi)$ is the solution of (1.1). Let

$$X(t) := \mathbb{E}|x(t)|^2, \quad t \geq 0$$

and

$$Z(t) := K\mathbb{E}\|\xi\|_D^2 h^{-\beta}(t), \quad t \geq 0.$$

For convenient, we define $h(t) = h(0) = 1$ for $t \in [-r, 0]$. Then, we deduce that from $K > 1$, $\mathbb{E}\|\xi\|_D^2 > 0$ that $X(t) < Z(t)$ for $t \in [-r, 0]$. We will show

$$X(t) \leq Z(t), \quad \forall t \geq 0. \tag{3.11}$$

Assume on the contrary that there exists $t_1 > 0$ such that $X(t_1) > Z(t_1)$. Let

$$t_* := \inf\{t > 0 : Y(t) > Z(t)\}.$$

By continuity of $X(t)$ and $Z(t)$,

$$X(t) \leq Z(t), \ t \in [0, t_*], \quad X(t_*) = Z(t_*), \tag{3.12}$$

and

$$\mathbb{E}|x(t_m)|^2 > K\mathbb{E}\|\xi\|_D^2 h^{-\beta}(t_m),$$

for some $t_m \in (t_*, t_* + \frac{1}{m})$, $m \in \mathbb{N}$.

Choosing $0 < \delta < \beta$ and applying the Itô formula to the function

$$V(t, x) = h^{\delta}(t)|x(t)|^2$$

with the solution $x(t)$ of (1.1), we have

$$\begin{aligned} h^{\delta}(t)|x(t)|^2 =& |\xi(0)|^2 + \int_0^t \delta h^{\delta}(s)\frac{h'(s)}{h(s)}|x(s)|^2 ds + 2\int_0^t h^{\delta}(s)\langle x(s), u(s, E_s, x_{s-})\rangle ds \\ &+ 2\int_0^t h^{\delta}(s)\langle x(s), f(s, E_s, x_{s-})\rangle dE_s + 2\int_0^t h^{\delta}(s)\langle x(s), u(s, E_s, x_{s-})\rangle dB_{E_s} \\ &+ 2\int_0^t \int_{|y|<c} h^{\delta}(s)\langle x(s), b(s, E_s, x_{s-}, y)\rangle \tilde{N}(dE_s, dy) \\ &+ \int_0^t h^{\delta}(s)Tr[g^T(s, E_s, x_{s-})g(s, E_s, x_{s-})]dE_s \\ &+ \int_0^t \int_{|y|<c} Tr[b^T(s, E_s, x_{s-}, y)b(s, E_s, x_{s-}, y)]\nu(dy)dE_s. \end{aligned}$$

By the standard property of the Itô's integral, we have

$$\mathbb{E}\Big(\int_0^t h^\delta(s)\langle x(s), g(s, E_s, x_{s-})\rangle dB_{E_s}\Big) = 0$$

and

$$\mathbb{E}\Big(\int_0^t \int_{|y|<c} h^\delta(s)\langle x(s), b(s, E_s, x_{s-}, y)\rangle \tilde{N}(dE_s, dy)\Big) = 0.$$

From (3.8) and (3.9), and the Fubini's theorem, it follows that

$$\begin{aligned} h^\delta(t)\mathbb{E}|x(t)|^2 \leq & |\xi(0)|^2 + \int_0^t h^\delta(s)\mathbb{E}|x(s)|^2(J\delta + \gamma(s))ds \\ & + \int_0^t h^\delta(s)\Big(\int_{-r}^0 \mathbb{E}|x(s+\theta)|^2 d[\eta(s,\theta)]\Big)ds. \end{aligned} \tag{3.13}$$

Let $K_1 := K\mathbb{E}\|\xi\|^2$. Since $\eta(s,\theta)$ is non-decreasing in θ on $[-r, 0]$, we obtain that from (3.11)

$$\int_{-r}^0 \mathbb{E}|x(s+\theta)|^2 d[\eta(s,\theta)] \leq K_1 \int_{-r}^0 h^{-\beta}(s+\theta)d[\eta(s,\theta)]$$

for any $s \leq t_*$.

If $s + \theta \leq 0$, then $s \leq -\theta$. Since h is increasing in $\mathbb{R}_+$, we have

$$h^{-\beta}(s)h^\beta(-\theta) \geq 1 = h^{-\beta}(s+\theta).$$

If $s + \theta \geq 0$, by (iii) of the Definition 2.1, we also have

$$h^{-\beta}(s+\theta) \leq h^{-\beta}(s)h^\beta(-\theta).$$

So, we get for each one $t_* \geq s$

$$\int_{-\tau}^0 \mathbb{E}|y(s+\theta)|^2 d[\eta(s,\theta)] \leq K_1 \int_{-\tau}^0 h^{-\beta}(s)h^\beta(-\theta)d[\eta(s,\theta)].$$

Then, combining (3.10) and (3.13) we have for all $s \leq t_*$

$$\begin{aligned} h^\delta(t)\mathbb{E}|y(t)|^2 \leq & |\xi(0)|^2 + K_1 \int_0^t h^{\delta-\beta}(s)(\delta + \gamma(s))ds \\ & + \int_0^t h^{\delta-\beta}(s)\Big(\int_{-\tau}^0 h^\beta(-\theta)d[\eta(s,\theta)]\Big)ds \\ \leq & |\xi(0)|^2 + K_1 \int_0^t h^{\delta-\beta}(s)(\delta - \beta)ds. \end{aligned}$$

Noticing that $J \in (0, 1]$, we have $-h(s) \leq h'(s)$. Since $\delta - \beta < 0$, it follows that

$$h^\delta(t)\mathbb{E}|x(t)|^2 \leq \mathbb{E}|\xi(0)|^2 + K_1 \int_0^t (\delta - \beta)h^{\delta-\beta-1}(s)h'(s)ds.$$

By using the fact that $K > 1$ we have

$$\begin{aligned} h^{\delta}(t_*)\mathbb{E}|x(t_*)|^2 \leq & |\xi(0)|^2 + K_1[h^{\delta-\beta}(t_*) - 1] \\ = & (|\xi(0)|^2 - K\|\xi\|_D^2) + K\mathbb{E}\|\xi\|_D^2 h^{\delta-\beta}(t_*) \\ < & K\mathbb{E}\|\xi\|_D^2 h^{\delta-\beta}(t_*). \end{aligned}$$

Thus,

$$\mathbb{E}|x(t_*)|^2 < K\mathbb{E}\|\xi\|_D^2 h^{-\beta}(t_*),$$

which conflicts with (3.12), Therefore,

$$\mathbb{E}|x(t)|^2 \leq K\mathbb{E}\|\xi\|_D^2 h^{-\beta}(t), \text{for all } t \geq 0.$$

So, the solution of (1.1) is h-stable in mean square sense.

Case 2. If $J \in (1, +\infty)$. Choose $0 < \delta < \frac{\beta}{J}$. By simple calculation we know that

$$0 < \frac{J\delta - \beta}{J(\delta - \beta)} < 1.$$

Fix

$$K > \frac{J(\delta - \beta)}{J\delta - \beta} > 1$$

and let

$$\xi \in D([-r, 0]; \mathbb{R}^d)$$

such that $\mathbb{E}\|\xi\|_D^2 > 0$. Similarly, We also shall show for all $t \geq 0$

$$X(t) \leq Z(t). \tag{3.14}$$

Assume on the contrary that there exists $t_1 > 0$ such that $X(t_1) > Z(t_1)$. Let

$$t_* := \inf\{t > 0 : X(t) > Z(t)\}.$$

By continuity of $X(t)$ and $Z(t)$,

$$X(t) \leq Z(t), \ t \in [0, t_*], \ \ X(t_*) = Z(t_*), \tag{3.15}$$

and

$$\mathbb{E}|x(t_m)|^2 > K\mathbb{E}\|\xi\|_D^2 h^{-\beta}(t_m),$$

for some $t_m \in (t_*, t_* + \frac{1}{m})$, $m \in \mathbb{N}$.

Applying the Itô formula to the function

$$V(t, x) = h^{\delta}(t)|x(t)|^2$$

with the solution $x(t)$ of (1.1), we have for all $s \leq t_*$

$$\begin{aligned} h^{\delta}(t)\mathbb{E}|x(t)|^2 \leq & \mathbb{E}|\xi(0)|^2 + K_1 \int_0^t h^{\delta-\beta}(s)(J\delta + \gamma(s))ds \\ & + \int_0^t h^{\delta-\beta}(s)\Big(\int_{-r}^0 h^{\beta}(-\theta)d[\eta(s,\theta)]\Big)ds \\ \leq & \mathbb{E}|\xi(0)|^2 + K_1 \int_0^t h^{\delta-\beta}(s)(J\delta - \beta)ds. \end{aligned}$$

Noticing that $J \in (1, +\infty)$, we have $-h(s) \le \frac{1}{J}h'(s)$. Since $J\delta - \beta < 0$, it follows that

$$\begin{aligned} h^{\delta}(t)\mathbb{E}|x(t)|^2 &\le \mathbb{E}|\xi(0)|^2 + K_1 \int_0^t \frac{J\delta - \beta}{J} h^{\delta-\beta-1}(s)h'(s)ds \\ &= \mathbb{E}|\xi(0)|^2 + K_1 \frac{J\delta - \beta}{J(\delta - \beta)}[h^{\delta-\beta}(t_*) - 1]. \end{aligned}$$

Noticing that

$$0 < \frac{J\delta - \beta}{J(\delta - \beta)} < 1$$

and using the fact that

$$K > \frac{J(\delta - \beta)}{J\delta - \beta},$$

we have

$$\begin{aligned} h^{\delta}(t_*)\mathbb{E}|x(t_*)|^2 &\le \mathbb{E}|\xi(0)|^2 + K_1 h^{\delta-\beta}(t_*) - K\frac{J\delta - \beta}{J(\delta - \beta)}\mathbb{E}\|\xi\|_D^2 \\ &= (\mathbb{E}|\xi(0)|^2 - K\frac{J\delta - \beta}{J(\delta - \beta)}\mathbb{E}\|\xi\|_D^2) + K\mathbb{E}\|\xi\|_D^2 h^{\delta-\beta}(t_*) \\ &< K\mathbb{E}\|\xi\|_D^2 h^{\delta-\beta}(t_*). \end{aligned}$$

Thus,

$$\mathbb{E}|x(t_*)|^2 < K\mathbb{E}\|\xi\|_D^2 h^{-\beta}(t_*),$$

which conflicts with (3.15), Therefore,

$$\mathbb{E}|x(t)|^2 \le K\mathbb{E}\|\xi\|_D^2 h^{-\beta}(t), \text{ for all } t \ge 0.$$

So, the solution of (1.1) is h-stable in mean square sense. The proof is complete. □

Corollary 3.2. Let

$$\Upsilon(\cdot,\cdot) : \mathbb{R}_+ \times [-r, 0] \to \mathbb{R}_+,$$
$$\gamma_i(\cdot), h_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}, \ i = 1, 2, \cdots, n$$

with

$$0 \le h_1(t) \le h_2(t) \le \cdots \le h_n(t) \le r, \quad t \in \mathbb{R}_+$$

be locally bounded Borel measurable functions. If for any $t \in \mathbb{R}_+$, $x \in D([-r, 0]; \mathbb{R}^d)$

$$2\langle x(t), u(t, E_t, x_{t-})\rangle \le \sum_{i=1}^{n} \gamma_i(t)|\varphi(-h_i(t))|^2 + \int_{-r}^{0} \Upsilon(t, s)|\varphi(s)|^2 ds \tag{3.16}$$

and

$$\begin{aligned} &2\langle x(t), f(t, E_t, x_{t-})\rangle + Tr[g^T(t, E_t, x_{t-})g(t, E_t, x_{t-})] \\ &+ \int_{|y|<c} Tr[b^T(t, E_t, x_{t-}, y)b(t, E_t, x_{t-}, y)]\nu(dy) \le 0. \end{aligned} \tag{3.17}$$

Then, the solution of (1.1) is h-stable in mean square sense if there exists $\beta > 0$ such that for any $t \in \mathbb{R}_+$,

$$\sum_{i=1}^{n} h^{\beta}(h_i(t))\gamma_i(t) + \int_{-r}^{0} h^{\beta}(-s)\Upsilon(t, s)ds \le -\beta. \tag{3.18}$$

Proof. For $s \in [-r, 0]$ and $t \geq 0$, we define functions $u_i(t, s)$ and $\eta(t, s)$ as following:

$$u_i(t,s) := \begin{cases} 0, & \text{if } s \in [-r, -h_i(t)], \\ \gamma_i(t), & \text{if } s \in (-h_i(t), 0], \end{cases}$$

and

$$\eta(t,s) := \sum_{i=1}^{n} u_i(t,s) + \int_{-r}^{s} \Upsilon(t,r)dr.$$

By the properties of the Riemann-Stieltjes integrals, one has for $t \in \mathbb{R}_+$

$$\int_{-r}^{0} \phi(s)d\Big[\int_{-r}^{s} \Upsilon(t,u)du\Big] = \int_{-r}^{0} \phi(s)\Upsilon(t,s)ds,$$

for any $\phi(\cdot) \in D([-r, 0]; \mathbb{R})$. Then for any $t \in \mathbb{R}_+$, $\phi(\cdot) \in D([-r, 0]; \mathbb{R})$,

$$\int_{-r}^{0} \phi(s)d[\eta(t,s)] = \sum_{i=1}^{n} \gamma_i(t)\phi(-h_i(t)) + \int_{-r}^{0} \phi(s)\Upsilon(t,s)ds,$$

Therefore, (3.16) ensures that implies (3.6) holds and (3.18) implies that (3.10) holds. According to the Theorem 3.1, we can immediately derive the results of the Corollary 3.2. The proof is complete. □

Corollary 3.3. Let $\eta(\cdot)$: $[-r, 0] \to \mathbb{R}_+$ is a non-decreasing function. If there exists a constant $\gamma \in \mathbb{R}$ such that for any $t \in \mathbb{R}_+$, $x \in D([-r, 0]; \mathbb{R}^d)$,

$$2\langle x(t), u(t, E_t, x_{t-})\rangle \leq \gamma|\phi(0)|^2 + \int_{-\tau}^{0} |\phi(s)|^2 d[\eta(s)] \tag{3.19}$$

and

$$\begin{aligned} &2\langle x(t), f(t, E_t, x_{t-})\rangle + Tr[g^T(t, E_t, x_{t-})g(t, E_t, x_{t-})] \\ &+ \int_{|y|<c} Tr[b^T(t, E_t, x_{t-}, y)b(t, E_t, x_{t-}, y)]\nu(dy) \leq 0. \end{aligned} \tag{3.20}$$

Then, the solution of (1.1) is h-stable in mean square sense if

$$\gamma + \eta(0) - \eta(-r) < 0. \tag{3.21}$$

Proof. According to (3.21) and continuity of $h(t)$, we know that for some sufficiently small $\beta > 0$,

$$\gamma + h^{\beta}(r)[\eta(0) - \eta(-r)] < -\beta.$$

Since $\eta(\cdot)$ and $h(\cdot)$ are increasing, we obtain that

$$\gamma + \int_{-r}^{0} h^{\beta}(-s)d[\eta(s)] \leq \gamma + h^{\beta}(r)[\eta(0) - \eta(-r)] < -\beta,$$

which implies that (3.10) holds for some sufficiently small $\beta > 0$. The proof is complete. □

Combining the Corollary 3.2 with the Corollary 3.3, we can immediately derive the following Corollary 3.4.

Corollary 3.4. Let $h_i(\cdot)$: $\mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \cdots, n$ with $0 \le h_1(t) \le h_2(t) \le \cdots \le h_n(t) \le r$, $t \in \mathbb{R}_+$ be locally bounded Borel measurable functions. Assume that there exist some constants γ_i, $i = 1, 2, \cdots, n$ and the Borel measurable function θ: $[-r, 0] \to \mathbb{R}_+$, such that for any $t \in \mathbb{R}_+$, $x \in D([-r, 0]; \mathbb{R})$

$$2\langle x(t), u(t, E_t, x_{t-})\rangle \le \sum_{i=1}^{n} \gamma_i |\varphi(-h_i(t))|^2 + \int_{-r}^{0} \theta(s)|\varphi(s)|^2 ds \tag{3.22}$$

and

$$\begin{aligned} &2\langle x(t), f(t, E_t, x_{t-})\rangle + Tr[g^T(t, E_t, x_{t-})g(t, E_t, x_{t-})] \\ &+ \int_{|y|<c} Tr[b^T(t, E_t, x_{t-}, y)b(t, E_t, x_{t-}, y)]\nu(dy) \le 0. \end{aligned} \tag{3.23}$$

Then, the solution of (1.1) is h-stable in mean square sense if

$$\sum_{i=0}^{n} \gamma_i + \int_{-r}^{0} \theta(s) ds < 0. \tag{3.24}$$

Remark 3.3. We remark that the conditions (3.8) and (3.9) in the Theorem 3.2 are general of some existing conditions. As far as we known, the conditions (3.8) and (3.9) have not been used to study the h-stability in mean square of stochastic systems in the reported literature even deterministic differential equations. Our results are novel and very advantage to applications in "mixed" delay time-changed SDEs, which include the point delay, variable delay and distributed delay.

Remark 3.4. The conditions of the Theorems 3.1 and 3.2 indicate that the drift term "dt" seems to play the dominating role in judging the h-stability of the time changed system while "dE_t" and "dB_{E_t}" seem to be less important.

Now, we intend to present an example to explain the statement of the Remark 3.4. For simplicity, considering the following two time-changed SDEs

$$dy(t) = -y(t)dt + cy(t)dE_t + dy(t)dB_{E_t} \tag{3.25}$$

and

$$dy(t) = y(t)dt + cy(t)dE_t + dy(t)dB_{E_t}. \tag{3.26}$$

By the Corollary 3.4 we conclude that the time change Eq (3.25) is h-stable if $2c + d^2 \le 0$, while we can not conclude that the time change Eq (3.26) is the h-stable no matter what c and d are.

4. Some examples

In this section, we will list a few examples to show the superiority of our outcome.

Example 4.1. For simplicity, we consider the following functional stochastic differential equation with time-changed Brownian motion

$$dx(t) = (f_0(t, x(t)) + f_1(t, x_t))dt + u(t, E_t)x(t)dE_t + g(t, E_t)x(t)dB_{E_t} \tag{4.1}$$

with

$$y_0(\cdot) = \xi \in C([-r, 0]; \mathbb{R}^d).$$

Assume that there exists a continuous function $\gamma(t)$: $[-r,0] \to \mathbb{R}_+$ such that for any $t \geq 0$ and $x \in \mathbb{R}^d$

$$x^T f_0(t,x) \leq \alpha |x|^2, \tag{4.2}$$

$$|f_1(t,\phi)| \leq \int_{-r}^{0} \gamma(s)|\phi| ds, \quad t \geq 0,\ \phi \in C([-r,0];\mathbb{R}^d), \tag{4.3}$$

and for any $t_1, t_2 > 0$

$$2u(t_1,t_2) + g^2(t_1,t_2) \leq 0. \tag{4.4}$$

By using the Razumikhin-type theorem in [27], we can immediately conclude that the null solution of (4.1) is exponentially stable in mean square sense if

$$\alpha + \sqrt{r}\Big(\int_{-r}^{0} (\gamma(s))^2 ds\Big)^{1/2} < 0. \tag{4.5}$$

Noticing that (4.2) and (4.3) imply that

$$\phi^T(0) f(t,\phi) \leq \Big(\alpha + \frac{1}{2}\int_{-r}^{0}\gamma(s)ds\Big)|\phi(0)|^2 + \frac{1}{2}\int_{-r}^{0}\gamma(s)|\phi(s)|^2 ds.$$

By the Corollary 3.4, the null solution of (4.1) is h-stable in mean square if (4.4) holds and

$$\alpha + \int_{-r}^{0}\gamma(s)ds < 0. \tag{4.6}$$

Notice that

$$\int_{-r}^{0}\gamma(s)ds \leq \sqrt{r}\Big(\int_{-r}^{0}(\gamma(s))^2 ds\Big)^{1/2}$$

owing to the Hölder's inequality. So, (4.6) is more progressive than (4.5).

Example 4.2. To reveal further the effectiveness of our results, we consider the following scalar linear stochastic differential equation with variable delay

$$\begin{aligned} dx(t) =& (-c(t)x(t) + d(t)x(t-h_1(t)))dt + G(t,E_t)x(t)dE_t + H(t,E_t)x(t)dB_{E_t} \\ &+ \int_{|y|<c} F(t,E_t,y)x(t)\tilde{N}(dE_t,dy), \end{aligned} \tag{4.7}$$

where $c(t), d(t), h_1(t)$: $\mathbb{R}_+ \to \mathbb{R}$ are continuous functions and $h_1(t) \leq r$ for some $r > 0$.

Let

$$f(t,\phi) := -c(t)\phi(0) + d(t)\phi(-h_1(t))$$

for $t \in \mathbb{R}_+$, $\phi \in D([-r,0];\mathbb{R})$. Then, for all $t \in \mathbb{R}_+$, $\phi \in D([-r,0];\mathbb{R})$ we have

$$\begin{aligned} 2\phi(0)f(t,\phi) =& -2c(t)|\phi(0)|^2 + 2d(t)\phi(0)\phi(-h_1(t)) \\ \leq& -2c(t)|\phi(0)|^2 + |d(t)|(\phi^2(0) + \phi^2(-h_1(t))). \end{aligned} \tag{4.8}$$

Then, by view of the Corollary 3.2 we can deduce that if there exists a positive constant ζ such that for any $t \in \mathbb{R}_+$,

$$-2c(t) + |d(t)| + h^{\zeta}(h_1(t))|d(t)| \leq -\zeta, \tag{4.9}$$

and for any $t_1, t_2 > 0$

$$2G(t_1,t_2) + H^2(t_1,t_2) + \int_{|y|<c} F^2(t_1,t_2,y)\nu(dy) \leq 0. \tag{4.10}$$

Then, the null solution of (4.7) is h-stable in mean square sense.

On the other hand, by the continuity,

$$-c(t) + |d(t)| \leq 0 \tag{4.11}$$

means that (4.9) holds with some sufficiently small $\zeta > 0$. Therefore, we can deduce that the null solution of (4.7) is h-stable in mean square sense if (4.10) and (4.11) holds.

Example 4.3. Now, we will consider the following stochastic equation with distributed delay:

$$\begin{aligned} dx(t) =& \Big(-c(t)x(t) + \int_{-r}^{0} x(t+s)d[\eta(s)]\Big)dt - G(t,E_t)x(t)dE_t + H(t,E_t)x(t)dB_{E_t} \\ &+ \int_{|y|<c} F(t,E_t,y)x(t)\tilde{N}(dE_t,dy), \end{aligned} \tag{4.12}$$

for $t \geq 0$, where $\eta(t)$ is a bounded variation function on $[-r,0]$ and $c(t)$ is a continuous function.

Denote

$$\gamma(t) := c(t) - Var_{[-r,0]}\eta(\cdot), \quad t \geq 0.$$

Using the Razumikhin-type theorem in [27], we declare that the null solution of (4.12) is asymptotically stable in mean square if for any $t_1, t_2 > 0$,

$$-2G(t_1,t_2) + H^2(t_1,t_2) + \int_{|y|<c} F^2(t_1,t_2,y)\nu(dy) \leq 0 \tag{4.13}$$

and

$$\gamma := \inf_{t\geq 0} \gamma(t) > 0. \tag{4.14}$$

In fact, we can deduce (4.13) and (4.14) ensure that the null solution of (4.12) is h-stable in mean square. Let

$$f(t,\varphi) := -c(t)\varphi(0) + \int_{-r}^{0} \phi(s)d[\eta(s)],$$

for $t \geq 0$, $\phi \in C([-r,0];\mathbb{R})$. Define

$$V(s) := Var_{[-r,s]}\eta(\cdot), \quad s \in [-r,0].$$

Then $V(s)$ is increasing on $[-r,0]$. We obtain by the properties of the Riemann-Stieltjes integral

$$\Big| \int_{-r}^{0} \phi(0)\phi(s)d[\eta(s)] \Big| \leq \int_{-r}^{0} |\phi(0)\phi(s)|d[V(s)].$$

Thus,

$$\begin{aligned} \phi(0)f(t,\phi) &\leq -c(t)\phi^2(0) + \int_{-r}^{0} |\phi(0)\phi(s)|d[V(s)] \\ &\leq \Big(-c(t) + \frac{1}{2}\int_{-r}^{0} d[V(s)]\Big)\phi^2(0) + \frac{1}{2}\int_{-r}^{0} \phi^2(s)d[V(s)]. \end{aligned}$$

By the Theorem 3.2, the null solution of (4.12) is h-stable in mean square if (4.13) holds and there exists $\beta > 0$ such that for all $t \geq 0$

$$-c(t) + \frac{1}{2}\int_{-r}^{0} d[V(s)] + \frac{1}{2}\int_{-r}^{0} h^{\beta}(-s)d[V(s)] \leq -\beta. \tag{4.15}$$

It follows from (4.14) that for all $t \geq 0$

$$-c(t) + V(0) \leq -\gamma.$$

Setting $\beta \in (0, \frac{\gamma}{2})$ sufficiently small, we can immediately know that

$$\frac{1}{2}(h^{\beta}(r) - 1)V(0) < \frac{\gamma}{2},$$

which means that for any $t \geq 0$

$$-c(t) + \frac{1}{2}V(0) + \frac{1}{2}h^{\beta}(r)V(0) \leq -\frac{\gamma}{2} \leq -\beta.$$

Since $V(\cdot)$ is increasing, it follows that

$$\int_{-r}^{0} h^{\beta}(-s)d[V(s)] \leq h^{\beta}(r)V(0).$$

Therefore, we can obtain for any $t \geq 0$

$$-c(t) + \frac{1}{2}\int_{-r}^{0} d[V(s)] + \frac{1}{2}\int_{-r}^{0} h^{\beta}(-s)d[V(s)] \leq -c(t) + \frac{1}{2}V(0) + \frac{1}{2}h^{\beta}(\tau)V(0) \leq -\beta.$$

Let

$$c(t) = 1, G(t, E_t) = E_t^2,$$

$$H(t, E_t) = \sqrt{1 + E_t}$$

and

$$F(t, E_t, y) = -y^2.$$

After verification, they satisfy the corresponding assumed conditions.

Since the numerical solution can be regarded as the true solution when the step size is sufficiently small, such as $h = 2^{-16}$. Figure 1 is one the solution paths of Eq (4.12).

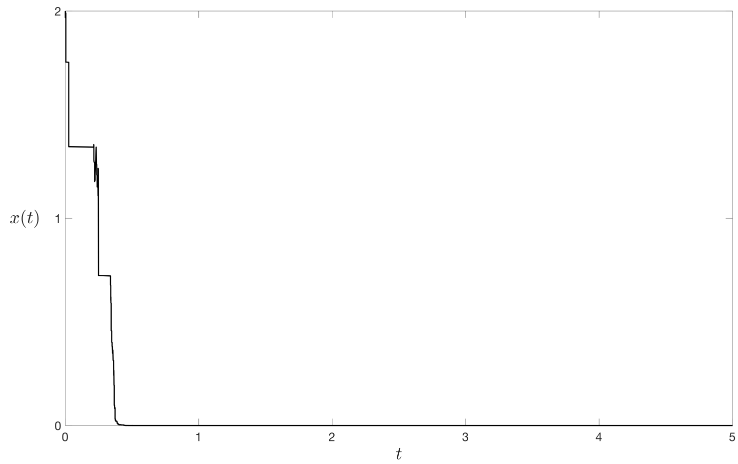

Figure 1. The solution path of $x(t)$ for (4.12).

From Definition 2.2 we know, the solution of $x(t, \xi)$ is said to be h-stable p-th moment sense, if for any initial value ξ, there exists some positive constant δ such that

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}\,|x(t,\xi)|}{\log h(t)} \leq -\delta. \tag{4.16}$$

From Figure 2 we see there exists a postive constant δ satisfying (4.16) when $h(t) = e^t$, then the solution for (4.12) is exponentially stable in p-th moment sense.

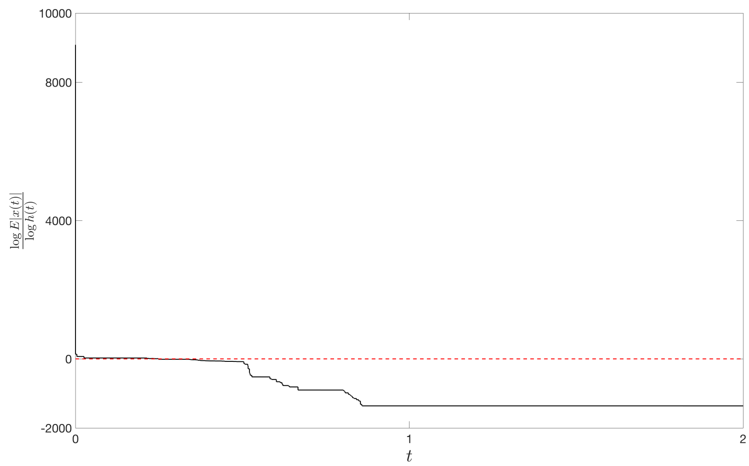

Figure 2. The solution of (4.12) is exponential stable when $h(t) = e^t$.

From Figure 3 we see there exists a postive constant δ satisfying (4.16) when $h(t) = 1 + t$, then the solution for (4.12) is polynomial stable in p-th moment sense.

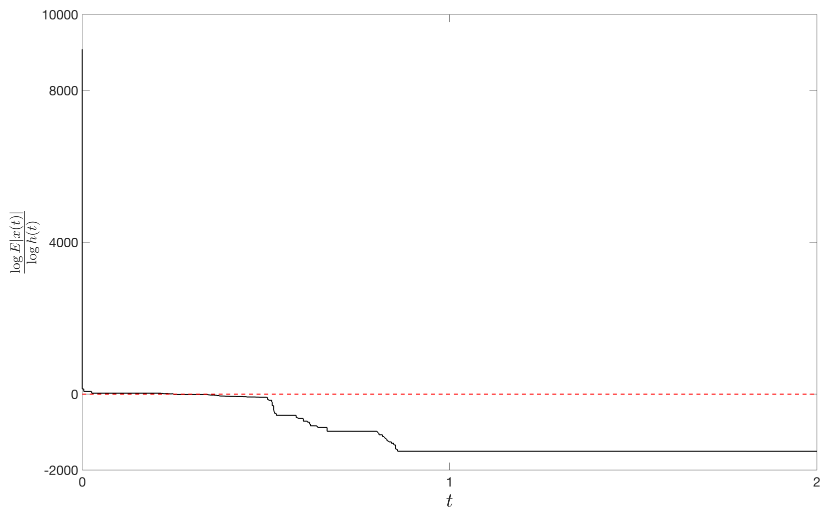

Figure 3. The solution of (4.12) is polynomial stable when $h(t) = 1 + t$.

From Figure 4 we see there exists a postive constant δ satisfying (4.16) when $h(t) = ln(e + t)$, then the solution for (4.12) is logarithmical stable in p-th moment sense.

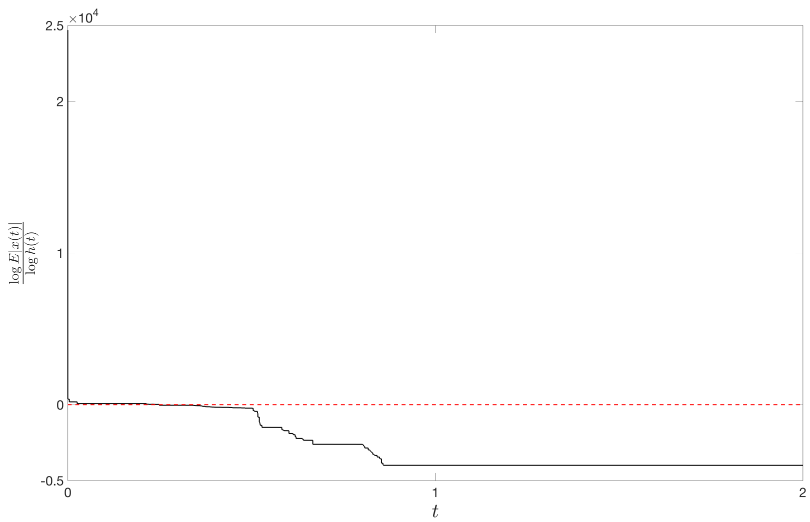

Figure 4. The solution of (4.12) is logarithmical stable when $h(t) = ln(e + t)$.

From Figure 5 we see there exists a postive constant δ satisfying (4.16) when $h(t) = (1 + t)e^t$, then the solution for (4.12) is exponential stable in p-th moment sense.

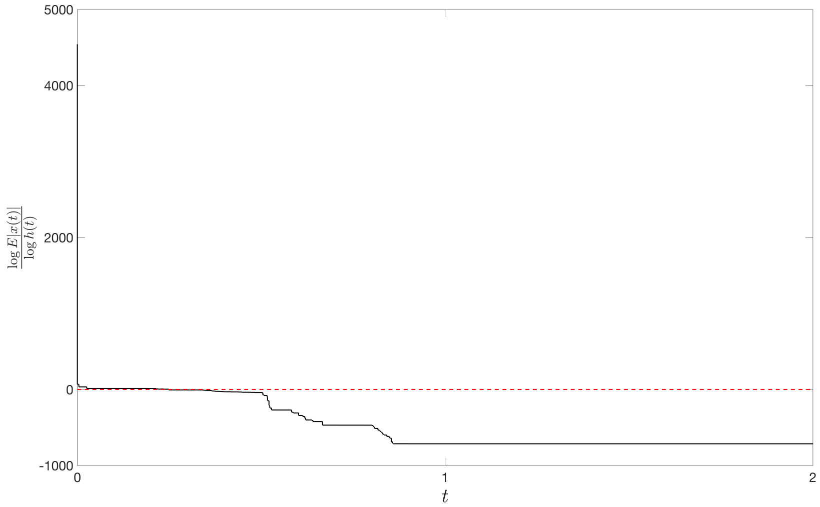

Figure 5. The solution of (4.12) is exponentially stable when $h(t) = (1 + t)e^t$.

5. Conlusions

In this work, we have used Lyapunov's technique to ensure that the solution of functional SDEs driven by time-changed Lévy process is h-stable in p-th moment sense. The stability conditions captured by using the Lyapunov's function method are often implicit and not easy to be examined. The advantages of this paper is that we used a comparison principle and a proof by contradiction to explore some new explicit conditions ensuring that the solution of functional SDEs driven by time-changed Lévy process is the h-stability in the mean square under some generalized assumptions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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