AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 22758–22765.
DOI: 10.3934/math.20231159
Received: 15 May 2023
Revised: 16 June 2023
Accepted: 24 June 2023
Published: 17 July 2023

Research article

Generalized differential identities on prime rings and algebras

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Abstract: The goal of this study is to bring out the following conclusion. Let R be a noncommutative prime ring with $2(m+n)!$ torsion freeness and let m and n be fixed, non-negative integers and d, g be Jordan derivations on R. If $x^{m+n}d(x) + x^m g(x)x^n \in Z(R)$ or $d(x)x^{m+n} + x^m g(x)x^n \in Z(R)$ or $x^n d(x)x^m + x^m g(x)x^n \in Z(R)$ then $d = g = 0$ follows for every $x \in R$.

Keywords: prime ring; derivation; generalized derivation; differential identities; Banach algebra
Mathematics Subject Classification: 16W25, 16N60, 16N80

1. Introduction

The present paper is particularly intended for algebraists whose work focuses on mappings of rings (algebras) with special qualities such as Lie and Jordan derivations, extended derivations, automorphisms, and linear preservers. The study is specifically devoted to ring theorists interested in polynomial (algebraic or differential) identities and generalizations of such identities. These identities have proven to be useful in solving several numerical problems in other branches of mathematics including operator theory, functional analysis and lattice theory. We point out that the theory of identities allows for some analogies with the theory of algebraic functions, at least at the level of basic definitions. The idea behind the theory of differential identities was inspired by Kharchenko, Herstein and Lanski. Kharchenko gave the basic idea to intrigue the differential identities of a ring as a prominence of usual identities for endomorphisms which satisfy some conditions in a ring.

The ring R will considered as an associative throughout having centre $Z(R)$. A ring R is n-torsion free for $n > 1$ is an integer, if $nw = 0$ implies $w = 0$ for every w in R. A prime ring R posses the property if $cRd = \{0\}$ gives that $c = 0$ or $d = 0$ and is said to be semiprime, if $cRc = \{0\}$ implicit that $c = 0$. A derivation d is a mapping on R that is additive and fulfilling the condition $d(wz) = d(w)z + wd(z)$ for each w, z in R. Map d will be noted as Jordan derivation if the last condition replaced by the condition

$d(w^2) = d(w)w + wd(w)$ for each w in R. Herstein [7] establish a result by choosing the identity $d(u)d(w) - d(w)d(u) = 0$, where d represents a derivation and p, w are noted as symmetric elements can not be true in the setting of prime ring with characteristic not 2 excepting that the ring satisfies S_4 , the fourth degree standard identity. In [6], the author mentioned that in a prime ring R having $char(R) \neq 2$, any Jordan derivation will behave as ordinary derivation. Cusack [2] derive a more fine version of the previous result by proving the following: Consider a ring R to be 2-torsion free and which posses a commutator that is not a zero divisor. Then, all Jordan derivation on R act as a derivation simply.

One more discussion need to put here by Herstein Hyper-centre theorem which states: If a ring R without nil ideals and a in R satisfies $d_a(x^n) = 0$ for every x in R then $d_a = 0$. Here, $n = n(x)$ is an integer greater than and equal to 1. Thus, d always be zero for such conditions. A simplification to this fact can be found with the conclusion of Felzenswalb [3].

When we talked about identities, there is an expression figure out in our mind expressing for elements of R or may be the elements for any subset of R, which involves maps on R. The general idea behind the study of identities is to construct the structural model of the maps involved when the ring structure can not describe. More precisely the commutative structure of ring. In this paper our conclusion includes the differential identities involved with Jordan derivation with prime ring and algebras.

A prime ring R and C is noted as its extended centroid and $\mathcal{U}$ its Martindale ring of quotients. Then, every element of $\mathcal{U}$ can be considered as equivalence classes of left R-module homomorphisms from ideals of R, to R. One can think that which structure $\mathcal{U}$ can have, we try to explain it by an easy example as:

Example 1.1. *Suppose K denotes a vector space of infinite-dimension over a field C. The notation $Hom_c(K, K)$ will be used for the set of all row finite matrices over C, with respect to a fixed well-ordered basis of K. Let S be the subring of $Hom(K, K)$ consisting of those matrices containing only a finite number of nonzero entries. Then, $\mathcal{U} = Hom_c(K, K)$ and $\mathcal{V}$ is the subring of $\mathcal{U}$ of all column finite matrices.*

To build out our main results we required the following lemma:

Lemma 1.1. *[19, Lemma] Let m be a fixed positive integer and R be a $m!$-torsion free ring having center $Z(R)$. Suppose that $y_1, y_2, ..., y_m \in R$ satisfy $\sum_{i=1}^{m} \lambda^i y_i \in Z(R)$ for $\lambda = 1, 2, ..., m$. Then, $y_i \in Z(R)$ for every i.*

2. Results on prime ring

Theorem 2.1. *Let the integers m, n be fixed and non negative, R be a non commutative prime ring possessing $2(m + n)!$ torsion freeness, d, g be Jordan derivations on R. If $x^{m+n}d(x) + x^m g(x)x^n \in Z(R)$ for every x in R, then $d = g = 0$.*

Proof. Given Jordan's derivations of R, namely d and g, by Herstien's result in [6], derivations of R are also d and g. From Theorem 2 of [10], R and $\mathcal{U}$ follows similar differential identities. This indicates a possibility

$$x^{m+n}d(x) + x^m g(x)x^n \in C \text{ for every } x \text{ in } \mathcal{U}. \tag{2.1}$$

Putting $x + \lambda y$ for x in (2.1), we find

$$(x+\lambda y)^{m+n}d(x+\lambda y)+(x+\lambda y)^{m}g(x+\lambda y)(x+\lambda y)^{n}\in C,$$

for every $x, y \in \mathcal{U}$. On expanding, we will get

$$\begin{aligned}&\Big[x^{m+n}+\tbinom{m+n}{1}x^{m+n-1}\lambda y+...+\lambda^{m+n}y^{m+n}\Big](d(x)+\lambda d(y))\\&+\Big[x^{m}+\tbinom{m}{1}x^{m-1}\lambda y+...+\lambda^{m}y^{m}\Big](g(x)+\lambda g(y))\Big[x^{n}+\tbinom{n}{1}x^{n-1}\lambda y+...+\lambda^{n}y^{n}\Big]\in C,\end{aligned} \tag{2.2}$$

for every $x, y \in \mathcal{U}$. An application of Lemma 1.1 implicit that

$$\begin{aligned}&P_1(x,y)=\tbinom{m+n}{1}x^{m+n-1}yd(x)+x^{m+n}d(y)+x^{m}g(y)x^{n}+\tbinom{n}{1}x^{m}g(x)\\&+\tbinom{n}{1}x^{m}g(x)x^{n-1}y+\tbinom{m}{1}x^{m-1}yg(x)x^{n}\in C,\end{aligned} \tag{2.3}$$

for every y, x in $\mathcal{U}$, where the word "$P_i(x, y)$" stands for the entire collection of terms involving the i factors of y. Since $\mathcal{U}$ possessing identity, so one can replace x by e in (2.3) to find that $\binom{m+n}{1}yd(e) + d(y) + g(y) + \binom{n}{1}g(e) + \binom{n}{1}g(e)y + \binom{m}{1}yg(e) \in C$. Since g and d are derivations, $d(e) = g(e) = 0$. Hence we get $d(y) + g(y) \in C$ for every y in $\mathcal{U}$. Using Posner's Theorem [13], we have $d(y) + g(y) = 0$ for every y in $\mathcal{U}$. Next, to find $P_2(x, y)$

$$\begin{aligned}P_2(x,y) \;=\;& \tbinom{m+n}{1}x^{m+n-1}yd(y)+\tbinom{m+n}{2}x^{m+n-2}y^{2}d(x)\\&+\tbinom{n}{2}x^{m}g(x)x^{n-2}y^{2}+\tbinom{n}{1}x^{m}g(y)x^{n-1}y\\&+\tbinom{n}{1}\tbinom{m}{1}x^{m-1}yg(x)x^{n-1}y+\tbinom{m}{1}x^{m-1}yg(y)x^{n}\in C.\end{aligned} \tag{2.4}$$

Again, replacing x by e in the above equation and using the fact that $d(e) = g(e) = 0$, we get $\binom{m+n}{1}yd(y)+ \binom{n}{1}g(y)y + \binom{m}{1}yg(y) \in C$. This implies that $(m + n)yd(y) + ng(y)y + myg(y) \in C$. Replace $d(y) = -g(y)$ to get $n[g(y), y] \in C$. Which implies that $[[g(y), y], r] = 0$ for each y, r in $\mathcal{U}$. Hence, $[g(y), y] \in C$ for every y in $\mathcal{U}$. Then, by Posner's Theorem $g = 0$. Now, if $g = 0$, then $d = 0$.

Theorem 2.2. *Let the integers m, n be non negative and fixed, R be a non commutative prime ring having $2(m + n)!$ torsion freeness, d, g be Jordan derivations on R. If $d(x)x^{m+n} + x^m g(x)x^n \in Z(R)$ for every x in R, then $d = g = 0$.*

Proof. Due to [6], d and g will be derivations of R. Using [10], we get

$$d(x)x^{m+n}+x^{m}g(x)x^{n}\in C \text{ for all } x\in\mathcal{U}. \tag{2.5}$$

Replacing $x + \lambda y$ for x in (2.5), we acquire

$$\begin{aligned}&(d(x)+\lambda d(y))\Big[x^{m+n}+\tbinom{m+n}{1}x^{m+n-1}\lambda y+...+\lambda^{m+n}y^{m+n}\Big]\\&+\Big[x^{m}+\tbinom{m}{1}x^{m-1}\lambda y+...+\lambda^{m}y^{m}\Big](g(x)+\lambda g(y))\Big[x^{n}+\tbinom{n}{1}x^{n-1}\lambda y+...+\lambda^{n}y^{n}\Big]\in C\end{aligned} \tag{2.6}$$

for all $x, y \in \mathcal{U}$. Applying Lemma 1.1, we obtain

$$\begin{aligned}&P_1(x,y)=\tbinom{m+n}{1}d(x)x^{m+n-1}y+d(y)x^{m+n}+x^{m}g(y)x^{n}+\tbinom{n}{1}x^{m}g(x)\\&+\tbinom{n}{1}x^{m}g(x)x^{n-1}y+\tbinom{m}{1}x^{m-1}yg(x)x^{n}\in C\end{aligned} \tag{2.7}$$

for each x, y in $\mathcal{U}$. Replace x by e in (2.7) and use the fact that $d(e) = g(e) = 0$ to find that $d(y)+g(y) \in C$ for every y in $\mathcal{U}$. Using Posner's Theorem [13], we have $d(y) + g(y) = 0$ for every y in $\mathcal{U}$. Next, to find $P_2(x, y)$

$$\begin{aligned} P_2(x, y) \;\; = \;\; & \tbinom{m+n}{1} d(y)x^{m+n-1}y + \tbinom{m+n}{2} d(x)x^{m+n-2}y^2 \\ & + \tbinom{n}{2} x^m g(x)x^{n-2}y^2 + \tbinom{n}{1} x^m g(y)x^{n-1}y \\ & + \tbinom{n}{1}\tbinom{m}{1} x^{m-1}yg(x)x^{n-1}y + \tbinom{m}{1} x^{m-1}yg(y)x^n \in C. \end{aligned} \tag{2.8}$$

Again, replacing x by e in the above equation and using the fact that $d(e) = g(e) = 0$, we get

$$(m + n)yd(y) + ng(y)y + myg(y) \in C.$$

Replacing $g(y) = -d(y)$, we get $m[d(y), y] \in C$. Which implies that $[[d(y), y], r] = 0$ for all $y, r \in \mathcal{U}$. Hence, $[d(y), y] \in C$ for every y in $\mathcal{U}$. Then, by Posner's Theorem we get the required conclusion.

Theorem 2.3. *Let the integers m, n be fixed and non negative, R be a non commutative prime ring possessing $2(m + n)!$ torsion freeness, d, g be Jordan derivations on R. If $x^n d(x)x^m + x^m g(x)x^n \in Z(R)$ for every x in R then $d = g = 0$.*

Proof. Here, d and g will be derivations on R due to Herstien Theorem in [6]. From Theorem 2 of [10], R and $\mathcal{U}$ follows similar differential identities. This indicates a possibility

$$x^n d(x)x^m + x^m g(x)x^n \in C \text{ for every } x \text{ in } \mathcal{U}. \tag{2.9}$$

Put $x + \lambda y$ for x in (2.9), to perceive

$$(x + \lambda y)^n d(x + \lambda y)(x + \lambda y)^m + (x + \lambda y)^m g(x + \lambda y)(x + \lambda y)^n \in C$$

for all $x, y \in \mathcal{U}$. On expanding, we will get

$$\begin{aligned} & \Big[x^n + \tbinom{n}{1} x^{n-1}\lambda y + ... + \lambda^n y^n\Big](d(x) + \lambda d(y))\Big[x^m + \tbinom{m}{1} x^{m-1}\lambda y + ... + \lambda^m y^m\Big] \\ & + \Big[x^m + \tbinom{m}{1} x^{m-1}\lambda y + ... + \lambda^m y^m\Big](g(x) + \lambda g(y))\Big[x^n + \tbinom{n}{1} x^{n-1}\lambda y + ... + \lambda^n y^n\Big] \in C \end{aligned} \tag{2.10}$$

for every x, y in $\mathcal{U}$. Using Lemma 1.1, we obtain

$$\begin{aligned} & P_1(x, y) = \tbinom{m}{1} x^n d(x)x^{m-1}y + x^n d(y)x^m + \tbinom{n}{1} x^{n-1}yd(x)x^m + x^m g(y)x^n \\ & + \tbinom{n}{1} x^m g(x)x^{n-1}y + \tbinom{m}{1} x^{m-1}yg(x)x^n \in C \end{aligned} \tag{2.11}$$

for every y, x in $\mathcal{U}$, where the word "$P_i(x, y)$" signifies the same as first result. Since $\mathcal{U}$ has identity element, one can replace x by e in (2.11) to find that

$$\binom{m+n}{1} d(e)y + d(y) + g(y) + \binom{n}{1} g(e) + \binom{n}{1} g(e)y + \binom{m}{1} yg(e) \in C.$$

Since g and d are derivations, $d(e) = g(e) = 0$. Hence we get $d(y) + g(y) \in C$ for all $y \in \mathcal{U}$. Using Posner's Theorem [13], we have $d(y) + g(y) = 0$ for every y in $\mathcal{U}$. Next, to find $P_2(x, y)$

$$\begin{aligned} P_2(x, y) \;\; = \;\; & \tbinom{m}{2} x^n d(x)x^{m-2}y^2 + \tbinom{m}{1} x^n d(y)x^{m-1}y \\ & + \tbinom{n}{1}\tbinom{m}{1} x^{n-1}yd(x)x^{m-1}y + \tbinom{n}{1} x^{n-1}yd(y)x^m \\ & + \tbinom{n}{2} x^{n-2}y^2 d(x)x^m + \tbinom{n}{2} x^m g(x)x^{n-2}y^2 \\ & + \tbinom{n}{1} x^m g(y)x^{n-1}y + \tbinom{n}{1}\tbinom{m}{1} x^{m-1}yg(x)x^{n-1}y \\ & + \tbinom{m}{1} x^{m-1}yg(y)x^n + \tbinom{m}{2} x^{m-2}y^2 g(x)x^n \in C. \end{aligned} \tag{2.12}$$

Again, replacing x by e in the above equation and using the fact that $d(e) = g(e) = 0$, we get

$$md(y)y + nyd(y) + ng(y)y + myg(y) \in C.$$

Replacing $g(y) = -d(y)$, we get $m - n[d(y), y] \in C$. Which implies that $[[d(y), y], r] = 0$ for all $y, r \in \mathcal{U}$. Hence, $[d(y), y] \in C$ for every y in $\mathcal{U}$. Then, by Posner's Theorem $d = 0$. Now, if $d = 0$, then $g = 0$.

Theorem 2.4. *Let the fixed integers m, n be non negative, R be a non commutative prime ring possessing $2(m + n)!$ torsion freeness, δ be a Jordan derivation on R. If $x^m\delta(x)x^n \in Z(R)$ for every x in R then $\delta = 0$.*

Proof. Here δ will be a derivation on R due to Herstien Theorem in [6]. Form Theorem 2 of [10], R and $\mathcal{U}$ follows equivalent differential identities. This indicates a possibility

$$x^m\delta(x)x^n \in C \text{ for every } x \text{ inside } \mathcal{U}. \tag{2.13}$$

Putting now $x + \lambda y$ for x in (2.13), we perceive

$$(x + \lambda y)^m\delta(x + \lambda y)(x + \lambda y)^n \in C$$

for every x, y in $\mathcal{U}$. On expanding, we will get

$$\left[x^m + \tbinom{m}{1}x^{m-1}\lambda y + ... + \lambda^m y^m\right](\delta(x) + \lambda\delta(y))\left[x^n + \tbinom{n}{1}x^{n-1}\lambda y + ... + \lambda^n y^n\right] \in C \tag{2.14}$$

for every x, y inside $\mathcal{U}$. Using Lemma 1.1, we obtain

$$\begin{aligned} &P_1(x, y) = x^m\delta(y)x^n \\ &+\tbinom{n}{1}x^m\delta(x)x^{n-1}y + \tbinom{m}{1}x^{m-1}y\delta(x)x^n \in C \end{aligned} \tag{2.15}$$

for every y, x in $\mathcal{U}$ where the word "$P_i(x, y)$" stands for the entire collection of terms involving the i factors of y. Since $\mathcal{U}$ having an identity element, one can switch out x for e in (2.15) to find that

$$\delta(y) + \binom{n}{1}\delta(e) + \binom{n}{1}\delta(e)y + \binom{m}{1}y\delta(e) \in C.$$

Since g and d are derivations, $\delta(e) = 0$. Hence we get $\delta(y) \in C$ for each y in $\mathcal{U}$. Using Posner's theorem [13], we conclude that $\delta(y) = 0$ for each y in $\mathcal{U}$.

3. Results on Banach algebra

Through the inspirational work of Sakai- Wermer-Singer-Hochschild, we got the idea to do this work. Let us start with the theorem of Singer-Wermer which is the first and influential result on Banach algebra that states that on a commutative Banach algebra, all continuous derivations map into its Jacobson radical [16]. Thomas [17] renew the idea of the last result which says that Singer-Wermer results will be true in the absence of continuity. This is widely known as the Singer-Wermer conjecture. Many generalizations can be found in the literature for similar kinds of research. Mathieu and Runde [12] obtained the finer version of the Singer-Wermer conjecture by establishing that on a Banach algebra, each centralizing derivation maps into its Jacobson radical. Recently work in [4, 5] is an outstanding contribution in the related algebraic structures. Our intention is to put out a conception of the above discussed results. In certain cases, we have a quick start as follows:

Theorem 3.1. *Let B be a noncommutative Banach algebra, d and g are derivations on B such that $d(b)b^{m+n} + b^m g(b)b^n \in Z_B$ for all $b \in B$ for some fixed integer $m, n > 1$ then $d(B) \subseteq rad(B)$ and $g(B) \subseteq rad(B)$.*

Proof. Consider K be a primitive ideal in B. Zorn's Lemma enable us to take a minimal ideal M of B contained in K. Making use of Lemma in [12], we obtain $d(M) \subseteq M$ and $g(M) \subseteq M$. If M is closed then it is reasonable to extend d and g to Banach algebra derivations $\frac{B}{M}$ given by

$$\hat{d}(B) = d(B) + M \text{ and } \hat{g}(B) = g(B) + M \text{ for all } \hat{b} \in \frac{B}{M} \text{ and } b \in B. \tag{3.1}$$

If $\frac{B}{M}$ is commutative then both $\hat{d}(\frac{B}{M})$ and $\hat{g}(\frac{B}{M})$ are included in the Jacobson radical of $\frac{B}{M}$ by following [17]. Next, think about the case when $\frac{B}{M}$ is non-commutative. For all $\hat{b}, \hat{c} \in \frac{B}{M}$ we have

$$[\hat{d}(\hat{b})\hat{b}^{m+n} + \hat{b}^m \hat{g}(\hat{b})\hat{b}^n, \hat{c}] = \hat{0}.$$

Since $\frac{B}{M}$ is prime and applying Theorem 2.1, we find that $\hat{d} = \hat{0}$ and $\hat{g} = \hat{0}$ on $\frac{B}{M}$. Therefore, we have $d(B) \subseteq K$ and $g(B) \subseteq K$. If P is not closed, the separating space of linear operator d, say $S(d)$ contained in K by Cusack [2]. This implies that $S(Q_{\hat{K}}d) = \hat{Q}_{\hat{K}}S(d) = 0$ where $Q_{\hat{K}}d$ is continuous on B. Hence, $Q_{\hat{K}}d(\hat{K}) = 0$ on $\frac{B}{K}$ and $d(\hat{K}) \subseteq K$. Thus, on Banach algebra $\frac{B}{\hat{K}}$ we have an induced derivation d given as $\hat{d}(\hat{b}) = d(b) + \hat{K}$ for all $b \in B$ and $\hat{b} \in \frac{B}{\hat{K}}$. Now, define a mapping $\hat{\eta}d^n Q_{\hat{K}} : B \to \frac{B}{\hat{K}} \to \frac{B}{M}$ such that $\hat{\eta}d^n Q_{\hat{K}}(b) = Q_M d^n(b)$ for all $b \in B$ and $n \in N$. In this way η is canonical inclusion map from $\frac{B}{\hat{K}}$ on $\frac{B}{M}$ and $\hat{d}$ is continuous on $\frac{B}{\hat{K}}$, we claim and $\|Q_K d^n\| \leq \|\hat{d}\|^n$ for all natural numbers n by [15]. By using lemma from [18], $d(K) \subseteq K$. Repeating the same arguments for g so that also we can obtain $g(K) \subseteq K$. This yields that d and g induces the derivation on $\frac{B}{K}$ given as

$$\hat{d}(\hat{b}) = d(b) + K \quad \text{and} \quad \hat{g}(\hat{b}) = g(b) + K$$

for all $\hat{b} \in \frac{B}{K}$, $b \in B$. Arguing in a similar manner as in (3.1) and using the primeness of $\frac{B}{K}$, we conclude that $g(B) \subseteq K$ for all primitive ideals K. This implies that $d(B) \subseteq rad(B)$ and $g(B) \subseteq rad(B)$, as desired.

Corollary 3.1. *Let B be a noncommutative semi-simple Banach algebra, d and g are derivations on B such that $d(b)b^{m+n} + b^m g(b)b^n \in Z_B$ for all $b \in B$ for some fixed integer $m, n > 1$ then $d = 0$ and $g = 0$.*

Theorem 3.2. *Let B be a noncommutative Banach algebra, d and g be two continuous Jordan derivations on B such that $d(b)b^{m+n} + b^m g(b)b^n \in radB$ for all $b \in B$ and for some fixed integers $m, n > 1$ then $d(B) \subseteq rad(B)$ and $g(B) \subseteq rad(B)$.*

Proof. Let K be a primitive ideal of B. We say that $d(K) \subseteq K$ and $g(K) \subseteq K$ as d and g are continuous by [14]. d and g can be extend to the Jordan derivations on $\frac{B}{K}$ such that

$$\hat{d}(\hat{b}) = d(b) + K \quad \text{and} \quad \hat{g}(\hat{b}) = g(b) + K$$

for all $\hat{b} \in \frac{B}{K}$, $b \in B$. But if K is a primitive ideal, then Banach algebra $\frac{B}{K}$ is semi-simple and prime. Therefore, $\hat{d}$ and $\hat{g}$ both are continuous derivation following [8]. An applications of Singer-Wermer theorem ensure us that there does not exits any nonzero derivations on a Banach algebra (commutative and semi-simple). Hence we get $\hat{g} = 0$ and $\hat{d} = 0$ when $\frac{B}{K}$ is commutative. Now we consider the non-commutative case of $\frac{B}{K}$. Since $\hat{d}(\hat{b})\hat{b}^{m+n} + \hat{b}^m \hat{g}(\hat{b})\hat{b}^n \in radB$ for all $b \in B$ and $\hat{b} \in \frac{B}{K}$. Making use of Theorem 2.2, we find $\hat{d} = 0$ and $\hat{g} = 0$. Hence in both cases we obtain $\hat{d} = 0$ and $\hat{g} = 0$. As K is arbitrary, we have $d(B) \subseteq rad(B)$ and $g(B) \subseteq rad(B)$. This completes our investigation and finished proof.

Corollary 3.2. *Let a semi-simple Banach algebra B be non-commutative, g and d be two continuous Jordan derivations on B. If $d(b)b^{m+n} + b^m g(b)b^n \in radB$ for all $b \in B$ and for $m, n > 1$ some fixed integers then $g = 0$ and $d = 0$.*

4. Conclusions

This article explores some of results on the structure of Jordan derivations satisfying particular generalized differential identities on a prime ring. In third section, we extend our results on Banach algebra and obtain the image of Jordan derivations contained in the radical. Future research on these differential identities in the context of additional algebraic structures, such as Von Neumann algebra, Lie algebra, etc., would be fascinating using the tools of algebra of linear operators (transformations).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

All authors are grateful for the referees suggestions to make the manuscript more presentable.

The authors of the paper extend their sincere gratitude to the Deanship of Scientific Research at Islamic University of Madinah for the support provided to the Post-Publishing Program 2.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. S. Ali, N. A. K. Ali, A. M. Ansari, On ∗-differential identities in prime rings with involution, *Hacet. J. Math. Stat.*, **49** (2020), 708–715. https://doi.org/10.15672/hujms.588726
2. J. Cusack, Jordan derivations on rings, *P. Am. Math. Soc.*, **53** (1975), 321–324.
3. B. Felzenswalb, Derivations in prime rings, *P. Am. Math. Soc.*, **84** (1982), 16–20. https://doi.org/10.1090/S0002-9939-1982-0633268-6
4. B. L. M. Ferreira, W. Feng, Mixed ∗-Jordan-type derivations on ∗-algebras, *J. Algebra Appl.*, **22** (2023), 2350100. https://doi.org/10.1142/S0219498823501001
5. B. L. M. Ferreira, H. Guzzo, R. N. Ferreira, F. Wei, Jordan derivations of alternative rings, *Commun. Algebra*, **48** (2020), 717–723. https://doi.org/10.1080/00927872.2019.1659285
6. I. N. Herstein, Jordan derivations of prime rings, *P. Am. Math. Soc.*, **8** (1957), 1104–1110.

7. I. N. Herstein, A theorem on derivations of prime rings with involution, *Can. J. Math.*, **34** (1982), 356–369. https://doi.org/10.4153/CJM-1982-023-x

8. B. E. Johnson, A. M. Sinclair, Continuity of derivations and a problem of Kaplansky, *Am. J. Math.*, **90** (1968), 1067–1073. https://doi.org/10.2307/2373290

9. C. Lanski, Differential identities in prime rings with involution, *T. Am. Math. Soc.*, **291** (1985), 765–787. https://doi.org/10.2307/2000109

10. T. K. Lee, Semiprime rings with differential identities, *Bull. Inst. Math. Acad.*, **20** (1992), 27–38. Available from: `http://140.112.114.62/handle/246246/121932.`

11. M. Mathieu, G. J. Murphy, Derivations mapping into the radical, *Arch. Math.*, **57** (1991), 469–474. https://doi.org/10.1007/BF01246745

12. M. Mathieu, V. Runde, Derivations mapping into the radical, II, *B. Lond. Math. Soc.*, **24** (1992), 485–487. https://doi.org/10.1112/blms/24.5.485

13. E. C. Posner, Derivations in prime rings, *P. Am. Math. Soc.*, **8** (1957), 1093–1100.

14. A. M. Sinclair, Jordan homomorphisms and derivations on semisimple Banach algebra, *P. Am. Math. Soc.*, **24** (1970), 209–214. https://doi.org/10.1090/S0002-9939-1970-0250069-3

15. A. M. Sinclair, *Automatic continuity of linear operators*, Cambridge University Press, 1976. https://doi.org/10.1017/CBO9780511662355

16. I. M. Singer, J. Werner, Derivations on commutative normed algebras, *Math. Ann.*, **129** (1955), 2–6.

17. M. P. Thomas, The image of a derivation is contained in the radical, *Ann. Math.*, **128** (1988), 435–460. https://doi.org/10.2307/1971432

18. M. P. Thomas, Primitive ideals and derivations on noncommutative Banach algebras, *Pac. J. Math.*, **159** (1993), 139–152.

19. F. Wei, Z. Xiao, Generalized derivations on (semi-) Prime rings and non commutative Banach algebras, *Rend. Semin. Mat. U. Pad.*, **122** (2009), 171–189. https://doi.org/10.4171/RSMUP/122-11