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Research article

Quasi self-dual codes over non-unital rings from three-class association schemes

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Abstract: Let E and I denote the two non-unital rings of order 4 in the notation of (Fine, 93) defined by generators and relations as $E = \langle a, b \mid 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$ and $I = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0\rangle$. Recently, Alahmadi et al classified quasi self-dual (QSD) codes over the rings E and I for lengths up to 12 and 6, respectively. The codes had minimum distance at most 2 in the case of I, and 4 in the case of E. In this paper, we present two methods for constructing linear codes over these two rings using the adjacency matrices of three-class association schemes. We show that under certain conditions the constructions yield QSD or Type IV codes. Many codes with minimum distance exceeding 4 are presented. The form of the generator matrices of the codes with these constructions prompted some new results on free codes over E and I.

Keywords: rings; self-orthogonal codes; free codes; association schemes; adjacency matrices
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1. Introduction

One of the many techniques for constructing self-dual codes over rings is to build their generator matrices from combinatorial matrices, that is to say, matrices related to graphs and designs. For instance, quadratic double circulant matrices related to the Paley graph [9, 13], and more generally, adjacency matrices of two-class association schemes [11] and three-class association schemes [6] were used to construct self-dual codes over fields and unital rings. The notion that plays the role of self-dual codes over non-unital rings of order four is that of quasi self-dual (QSD) codes, which are self-orthogonal codes of length n and size 2^n. The introduction of this new notion in [1, 2] was motivated by the fact that the usual relation between the size of a linear code and the size of its dual does not hold in general for linear codes over non-unital rings. In that context, Type IV codes were defined as QSD codes with all Hamming weights even. The non-unital rings E and I of order four in the list of Fine [12] have received some short length classification of QSD codes [1–4]. The codes in these classifications had minimum distance at most 2 in the case of I, and 4 in the case of E. This motivates us to look for general constructions in higher lengths to search for codes with better minimum distance.

Similar constructions to those in [11] were used to construct QSD codes over E in [17] and I in [5]. In this work, we describe two constructions (namely pure and bordered) for linear codes over the rings E and I in which we employ the adjacency matrices of symmetric or non-symmetric three-class association schemes. These constructions were first described for self-dual binary codes and codes over $\mathbb{Z}_k$ in [6]. The form of the generator matrices of the linear codes with these two constructions motivated some new results on free linear codes over E and I. These observations imply that all QSD codes over E formed by using either construction are equivalent as additive $\mathbb{F}_4$-codes to QSD codes over I. Consequently, we focus on studying QSD and Type IV codes over I and we give the conditions that guarantee such structures. Many codes meeting these conditions, and with minimum distance higher than 4, are presented. All computations are done via the additive codes package in Magma [7] using the connection between $\mathbb{F}_4$ and the non-unital rings as additive groups of order 4. Note that, since their inception, additive quaternary codes have some implications for quantum computing [8].

The paper is structured in the following way. Section 2 collects preliminary definitions and notations about rings, linear codes and association schemes with three classes. Section 3 studies free linear codes over the non-unital rings E and I. Section 4 describes the two constructions used to generate linear codes from the adjacency matrices of three-class association schemes and investigates the conditions required to obtain QSD codes as well as sufficient Type IV conditions. Section 5 concludes the paper.

2. Definitions and notations

2.1. Rings, modules and linear codes

The ring I defined by $I = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0\rangle$ consists of the four elements $\{0, a, b, c\}$ where $c = a + b$. It is a non-unital commutative ring with characteristic two. The multiplication table for I is given in Table 1.

Table 1. Multiplication table for the ring I.

$\times$	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

The ring E defined by $E = \langle a, b \mid 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$ consists of the four elements $\{0, a, b, c\}$ where $c = a + b$. It is a non-unital non-commutative ring with characteristic two. The multiplication table for E is given in Table 2.

Table 2. Multiplication table for the ring E.

$\times$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	b	b	0
c	0	c	c	0

Let $\mathcal{R}$ be either E or I. Throughout this paper, if the statement does not depend on which ring we are using, we shall denote the ring by $\mathcal{R}$.

For a positive integer n, $\mathcal{R}^n$ is an $\mathcal{R}$-module whose elements are n-tuples over $\mathcal{R}$. We will use the term *vectors* for these n-tuples. The (Hamming) **weight** $\mathrm{wt}(\mathbf{x})$ of $\mathbf{x} \in \mathcal{R}^n$ is the number of nonzero coordinates in $\mathbf{x}$.

A **linear code of length** n **over** I is an I-submodule of I^n whereas a **linear code of length** n **over** E is a left E-submodule of E^n. If C is a linear code of length n over $\mathcal{R}$ with a $k \times n$ **generator matrix** G, then

- for $\mathcal{R} = I$, $C = \left\{ \sum_{i=1}^{k} (\alpha_i \, \mathbf{g}_i + \beta_i \, a\mathbf{g}_i) \,\middle|\, \alpha_i, \beta_i \in \mathbb{F}_2 \right\}$,
- for $\mathcal{R} = E$, $C = \left\{ \sum_{i=1}^{k} (\alpha_i \, \mathbf{g}_i + \beta_i \, a\mathbf{g}_i + \gamma_i \, b\mathbf{g}_i) \,\middle|\, \alpha_i, \beta_i, \gamma_i \in \mathbb{F}_2 \right\}$,

where $\mathbf{g}_i$ is the ith row of G for each $1 \le i \le k$. A linear code with an **additive generator matrix** M is the $\mathbb{F}_2$-span of the rows of M.

With every linear code C over $\mathcal{R}$, we attach an additive code $\phi_{\mathcal{R}}(C)$ over $\mathbb{F}_4 = \mathbb{F}_2[\omega]$ such that

- $\phi_I(C)$ is defined by the alphabet substitution $0 \to 0,\ a \to \omega,\ b \to 1,\ c \to \omega^2$,
- $\phi_E(C)$ is defined by the alphabet substitution $0 \to 0,\ a \to \omega,\ b \to \omega^2,\ c \to 1$.

There are two binary linear codes of length n associated canonically with every linear code C of length n over $\mathcal{R}$, namely the **residue code** $\mathrm{res}(C)$ and the **torsion code** $\mathrm{tor}(C)$.
For $\mathcal{R} = I$,

$$\mathrm{res}(C) = \{\alpha(\mathbf{y}) \mid \mathbf{y} \in C\}$$

where $\alpha : I \to \mathbb{F}_2$ is the map defined by $\alpha(0) = \alpha(b) = 0$ and $\alpha(a) = \alpha(c) = 1$, extended componentwise from C to $\mathbb{F}_2^n$, and

$$\mathrm{tor}(C) = \{\mathbf{x} \in \mathbb{F}_2^n \mid b\mathbf{x} \in C\}.$$

For $\mathcal{R} = E$,

$$\mathrm{res}(C) = \{\alpha(\mathbf{y}) \mid \mathbf{y} \in C\}$$

where $\alpha : E \to \mathbb{F}_2$ is the map defined by $\alpha(0) = \alpha(c) = 0$ and $\alpha(a) = \alpha(b) = 1$, extended componentwise from C to $\mathbb{F}_2^n$, and

$$\mathrm{tor}(C) = \{\mathbf{x} \in \mathbb{F}_2^n \mid c\mathbf{x} \in C\}.$$

The two binary codes satisfy the inclusion $\mathrm{res}(C) \subseteq \mathrm{tor}(C)$ and their sizes are related to the size of C by $|C| = |\mathrm{res}(C)||\mathrm{tor}(C)|$ as shown in [1, 2]. Throughout the paper, we let $k_1 = \dim(\mathrm{res}(C))$ and

$k_2 = \dim(\mathrm{tor}(C)) - k_1$. The linear code C is said to be of **type** (k_1, k_2). We say that the linear code C is **free** if and only if $k_2 = 0$.

Two linear codes over $\mathcal{R}$ are **permutation equivalent** if there is a permutation of coordinates that maps one to the other.

An **inner product** on $\mathcal{R}^n$ is defined by $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$ where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathcal{R}^n$. A linear code C over $\mathcal{R}$ is **self-orthogonal** if for any $\mathbf{x}, \mathbf{y} \in C$, $\mathbf{x} \cdot \mathbf{y} = 0$. A linear code of length n is **quasi self-dual (QSD)** if it is self-orthogonal and of size 2^n. A QSD code with all weights even is called a **Type IV** code. A **quasi Type IV** code over I is a QSD code with an even torsion code.

The following matrix notations will be used consistently throughout this paper: I_k denotes the identity matrix of size k, O denotes the zero matrix of appropriate dimensions, and $\langle M_1, ..., M_n \rangle_{\mathbb{F}_2}$ denotes the $\mathbb{F}_2$-span of the rows of the matrices M_i with entries not necessarily from $\mathbb{F}_2$.

2.2. Three-class association schemes

For any positive integer m, an association scheme with m classes is a set together with $m+1$ relations defined on it satisfying certain conditions. Adopting Delsarte's [10] notations and conditions for $m = 3$, we have the following definition:

Definition 1. Let X be a set of size $n \geq 2$. Let $R = \{R_0, R_1, R_2, R_3\}$ be a family of four relations R_i on X. The pair (X, R) is called a **three-class association scheme** if the following conditions are satisfied:

(1) The set R is a partition of $X \times X$ and $R_0 = \{(x, x) \mid x \in X\}$.

(2) For $i \in \{0, 1, 2, 3\}$, $R_i^{-1} = \{(y, x) \mid (x, y) \in R_i\} = R_j$ for some $j \in \{0, 1, 2, 3\}$.

(3) For any triple of integers $i, j, k \in \{0, 1, 2, 3\}$, there exists a number $p_{ij}^k = p_{ji}^k$ such that for all $(x, y) \in R_k$,
$$p_{ij}^k = |\{z \in X \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}|.$$

We describe the relations R_i by their adjacency matrices $A_0 = I_n, A_1, A_2$ and $A_3 = J - I_n - A_1 - A_2$ where I_n is the identity matrix of size n and J is the all-one matrix of size n.

The values p_{ij}^k are called **intersection numbers**. It follows from the above definition that for any $i, j \in \{0, 1, 2, 3\}$,
$$A_i A_j = A_j A_i = \sum_{k=0}^{3} p_{ij}^k A_k. \tag{2.1}$$

For $0 \leq i \leq 3$, we have
$$A_i J = J A_i = \kappa_i J \tag{2.2}$$
where κ_i represents the number of ones per row (or column) in A_i. The matrix A_i is called a **valency** κ_i **matrix of degree** n.

It was shown in [14] that each p_{ij}^k is a non-negative integer that can be computed using the trace of the adjacency matrices as follows
$$p_{ij}^k = \frac{tr(A_k^T A_i A_j)}{n\kappa_k} \tag{2.3}$$
where κ_k is the valency of A_k.

If $R_i^{-1} = R_i$ for all i, then the association scheme is said to be **symmetric**, otherwise it is **non-symmetric**. If the association scheme is symmetric, then $A_i = A_i^T$ for all i and A_i is called a **non-directed matrix**. By counting the number of 1's in a non-directed valency κ matrix in two ways, the following lemma is obtained.

Lemma 1. [16] *There do not exist non-directed valency κ matrices of degree n if n and κ are odd.*

3. Results on linear codes over non-unital rings

Using the standard form of the generator matrices of linear codes over $\mathcal{R}$, we establish some new results on free codes as well as QSD codes.

3.1. Linear codes over I

We state the following result from [1] without proof.

Theorem 1. [1] *Assume C is a linear code over I of length n and type (k_1, k_2). Then, up to a permutation of columns, a generator matrix G of C is of the form*

$$G = \begin{pmatrix} aI_{k_1} & aX & Y \\ O & bI_{k_2} & bZ \end{pmatrix}$$

where I_{k_i} is the identity matrix of size k_i, X and Z are matrices with entries from $\mathbb{F}_2$, Y is a matrix with entries from I, and O is the $k_2 \times k_1$ zero matrix.

We will need the following structural theorem.

Theorem 2. *Let k be a positive integer and let C be a linear code over I. Then C is a free code of type $(k, 0)$ if and only if C is permutation equivalent to a code with generator matrix (aI_k, Y) where I_k is the identity matrix of size k and Y is a matrix with entries from I.*

Proof. It is immediate from Theorem 1 that every free code of type $(k, 0)$ over I is permutation equivalent to a code with generator matrix (aI_k, Y) where Y is a matrix with entries from I.

For the converse, let $G = (aI_k, Y)$ be a generator matrix for C where I_k and Y are as given in the hypothesis. Observe that

$$G' = \begin{pmatrix} G \\ aG \end{pmatrix} = \begin{pmatrix} aI_k & Y \\ bI_k & aY \end{pmatrix}$$

is an additive generator matrix for C where the rows are $\mathbb{F}_2$- linearly independent. Hence, $|C| = 2^{2k}$. Since $|C| = |\operatorname{res}(C)||\operatorname{tor}(C)|$, it follows that $2k = 2k_1 + k_2$. Since $\langle \alpha(G) \rangle_{\mathbb{F}_2} = \operatorname{res}(C)$ and $\alpha(G)$ is a binary matrix with rank k, it follows that $k = k_1$ and thus $k_2 = 0$. Hence, C is a free code of type $(k, 0)$. □

For $\mathbf{x}, \mathbf{y} \in I^n$, following [5], we denote by $\mathbf{x} \cap_{\neq} \mathbf{y}$ the vector in I^n which has a nonzero element $x + y \in I$ precisely in those positions where $\mathbf{x}$ has a nonzero x and $\mathbf{y}$ has a different nonzero y, and 0 elsewhere.

Lemma 2. [5] *Let m be an integer with $m \geq 2$. If $\mathbf{x}_i \in I^n$ for each $1 \leq i \leq m$, then*

$$\operatorname{wt}\Big(\sum_{i=1}^{m} \mathbf{x}_i\Big) \equiv \sum_{i=1}^{m} \operatorname{wt}(\mathbf{x}_i) + \sum_{j=1}^{m-1} \sum_{i=1}^{j} \operatorname{wt}\Big(\mathbf{x}_i \cap_{\neq} \mathbf{x}_{j+1}\Big) \pmod 2 .$$

Let C be a linear code over I. For $\mathbf{x}, \mathbf{y} \in C$ and $i, j \in I$, denote by $n_i(\mathbf{x})$ the number of coordinates of $\mathbf{x}$ that are i and denote by $n_{i,j}(\mathbf{x}, \mathbf{y})$ the number of coordinates satisfying simultaneously that i is a component in $\mathbf{x}$ and j is a component in $\mathbf{y}$. These notations are useful in gathering some facts about the weights of vectors related to the codewords of self-orthogonal codes over I as the following lemma shows.

Lemma 3. *If C is a self-orthogonal code over I, then the following hold for $\mathbf{x}, \mathbf{y} \in C$:*

(i) $\mathrm{wt}(\mathbf{x}) \equiv n_b(\mathbf{x}) \pmod 2$,

(ii) $\mathrm{wt}(a\mathbf{x}) \equiv 0 \pmod 2$,

(iii) $\mathrm{wt}(\mathbf{x} \cap_{\neq} a\mathbf{y}) \equiv 0 \pmod 2$.

Proof. The self-orthogonality of C implies that

$$n_a(\mathbf{x}) + n_c(\mathbf{x}) \equiv 0 \pmod 2 \tag{3.1}$$

and

$$n_{a,a}(\mathbf{x}, \mathbf{y}) + n_{a,c}(\mathbf{x}, \mathbf{y}) + n_{c,a}(\mathbf{x}, \mathbf{y}) + n_{c,c}(\mathbf{x}, \mathbf{y}) \equiv 0 \pmod 2. \tag{3.2}$$

Using the definition of $n_i(\mathbf{x})$, $n_{i,j}(\mathbf{x}, \mathbf{y})$ and $\mathbf{x} \cap_{\neq} \mathbf{y}$, we have the following:

(i) $\mathrm{wt}(\mathbf{x}) = n_a(\mathbf{x}) + n_b(\mathbf{x}) + n_c(\mathbf{x})$,

(ii) $\mathrm{wt}(a\mathbf{x}) = n_b(a\mathbf{x}) = n_a(\mathbf{x}) + n_c(\mathbf{x})$,

(iii) $\mathrm{wt}(\mathbf{x} \cap_{\neq} a\mathbf{y}) = n_{a,b}(\mathbf{x}, a\mathbf{y}) + n_{c,b}(\mathbf{x}, a\mathbf{y}) = n_{a,a}(\mathbf{x}, \mathbf{y}) + n_{a,c}(\mathbf{x}, \mathbf{y}) + n_{c,a}(\mathbf{x}, \mathbf{y}) + n_{c,c}(\mathbf{x}, \mathbf{y})$.

Substituting Eqs (3.1) and (3.2) into the above equations, we obtain the desired result. □

The next theorem gives a way to tell when a QSD code over I is Type IV.

Theorem 3. *Let C be a QSD code of length n over I and has a generator matrix G each of whose rows has an even weight. Then C is Type IV if and only if each distinct pair of rows $\mathbf{g}_i$ and $\mathbf{g}_j$ of G satisfies that $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j)$ is even.*

Proof. If C is a Type IV code, then every codeword in C has an even weight. In particular, $\mathrm{wt}(\mathbf{g}_i + \mathbf{g}_j)$ is even. Since $\mathrm{wt}(\mathbf{g}_i)$ and $\mathrm{wt}(\mathbf{g}_j)$ are even, by Lemma 2, $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j)$ is even.

For the converse, assume that $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j)$ is even. We need to prove that every codeword $\mathbf{c}$ in C has an even weight. By definition, $\mathbf{c}$ has one of the following forms:

$$\mathbf{g}_i\,,\ a\mathbf{g}_i\,,\ \sum \mathbf{g}_i\,,\ \sum a\mathbf{g}_i\,,\ \text{or}\ \sum (\mathbf{g}_i + a\mathbf{g}_j).$$

We will use Lemma 2 to prove that $\mathrm{wt}(\mathbf{c})$ is even. In particular, we will show that

$$\mathrm{wt}(\mathbf{g}_i), \mathrm{wt}(a\mathbf{g}_i), \mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j), \mathrm{wt}(a\mathbf{g}_i \cap_{\neq} a\mathbf{g}_j),\ \text{and}\ \mathrm{wt}(\mathbf{g}_i \cap_{\neq} a\mathbf{g}_j)$$

are even for all $1 \leq i, j \leq k$ where k is the number of rows of G. From the hypotheses, $\mathrm{wt}(\mathbf{g}_i)$ and $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j)$ are even. By definition, $a\mathbf{g}_i \cap_{\neq} a\mathbf{g}_j = \mathbf{0}$ and so $\mathrm{wt}(a\mathbf{g}_i \cap_{\neq} a\mathbf{g}_j) = 0$ proving that $\mathrm{wt}(a\mathbf{g}_i \cap_{\neq} a\mathbf{g}_j)$ is even. Since C is self-orthogonal, by Lemma 3, $\mathrm{wt}(a\mathbf{g}_i)$ and $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} a\mathbf{g}_j)$ are even. Hence, $\mathbf{c}$ has an even weight and thus C is Type IV. □

The following result is basic and useful.

Theorem 4. [5] *If C is a self-orthogonal code over I, then* $\mathrm{res}(C)$ *is even.*

Corollary 1. *Every free QSD code over I is quasi Type IV.*

Proof. Let C be a free QSD code. Then $\mathrm{res}(C) = \mathrm{tor}(C)$. Since C is self-orthogonal, by Theorem 4, $\mathrm{tor}(C)$ is even. Hence, C is quasi Type IV. □

3.2. Linear codes over E

We state the following result from [2] without proof.

Theorem 5. [2] *Assume C is a linear code over E of length n and type (k_1, k_2). Then, a generator matrix G of C is of the form*

$$G = \begin{pmatrix} aI_{k_1} & X & Y \\ O & cI_{k_2} & cZ \end{pmatrix}$$

where I_{k_i} is the identity matrix of size k_i, X and Y are matrices with entries from E, Z is a binary matrix, and O is the $k_2 \times k_1$ zero matrix.

The analogue of Theorem 2 does not hold in general for codes over E.

Example 1. The linear code C over E of length 2 and generator matrix $G = (a \;\; c)$ is not free as the residue and torsion codes of C have generator matrices $(1 \;\;\; 0)$ and I_2, respectively.

Theorem 6. *Let k be a positive integer and let C be a linear code over E. Then C is a free code of type $(k, 0)$ if and only if C is permutation equivalent to a code with generator matrix (aI_k, aM) where I_k is the identity matrix of size k and M is a binary matrix.*

Proof. Let C be a free code of type $(k, 0)$. Then $|C| = 2^{2k}$. By Theorem 5, we can write the generator matrix of C as $G = (aI_k, Y)$ where Y is a matrix with entries from E. As every element in E can be written in a c-adic decomposition form [2, Section 2.2], we can write $Y = aY_1 + cY_2$ where Y_1 and Y_2 are binary matrices. By linearity of C, we have $\langle aG, bG\rangle_{\mathbb{F}_2} \subseteq C$. Observe that

$$\begin{pmatrix} aG \\ bG \end{pmatrix} = \begin{pmatrix} aI_k & aY_1 \\ bI_k & bY_1 \end{pmatrix}$$

is a matrix with $2k$ linearly independent rows over $\mathbb{F}_2$. Hence, $|\langle aG, bG\rangle_{\mathbb{F}_2}| = 2^{2k} = |C|$ and thus $\langle aG, bG\rangle_{\mathbb{F}_2} = C$. We claim that Y_2 is a zero matrix. Suppose to the contrary that Y_2 is a nonzero matrix. Then, there exists a row $\mathbf{x}$ in G such that the first k components consist of exactly one a and $k - 1$ zeros, whereas the last $n - k$ components contain at least one c, where n is the length of C. Without loss of generality, let $\mathbf{x} = (a, 0, \ldots, 0, c, x_{k+2}, \ldots, x_n)$ where $x_i \in E$ for $k + 2 \leq i \leq n$. Since $\mathbf{x} \in C = \langle aG, bG\rangle_{\mathbb{F}_2}$, the first k components imply that $\mathbf{x}$ is a row in aG which is a contradiction as the entries of aG are from $\{0, a\}$. Thus Y_2 is a zero matrix, as claimed. Hence, $G = (aI_k, aY_1)$ where Y_1 is a binary matrix.

Conversely, if $G = (aI_k, aM)$ is a generator matrix for C where I_k and M are as given in the hypotheses, then $G = aG$ and therefore,

$$\begin{pmatrix} G \\ bG \end{pmatrix} = \begin{pmatrix} aI_k & aM \\ bI_k & bM \end{pmatrix}$$

is an additive generator matrix for C where the rows are $\mathbb{F}_2$- linearly independent. Hence, $|C| = 2^{2k}$. Since $|C| = |\operatorname{res}(C)||\operatorname{tor}(C)|$, it follows that $2k = 2k_1 + k_2$. Since $\langle\alpha(G)\rangle_{\mathbb{F}_2} = \operatorname{res}(C)$ and $\alpha(G)$ is a binary matrix with rank k, it follows that $k = k_1$ and thus $k_2 = 0$. Hence, C is a free code of type $(k, 0)$. □

Corollary 2. *Let C be a self-orthogonal code over E with generator matrix $G = (aI_n, Y)$ where Y is a square matrix of size n with entries from E. Then C is QSD if and only if C is a free code of type* $(n, 0)$.

Proof. Assume that C is a QSD code with generator matrix $G = (aI_n, Y)$ where Y is a square matrix of size n with entries from E. Then $|C| = 2^{2n}$. Using the same argument as in the proof of Theorem 6, it follows that $Y = aM$ where M is a binary square matrix of size n and hence C is a free code of type $(n, 0)$. The converse is immediate since $|C| = |\operatorname{res}(C)||\operatorname{tor}(C)| = 2^{2n}$. □

The residue and the torsion codes of QSD codes over E admit some useful properties as presented in the next theorem.

Theorem 7. [2] *If C is a QSD code over E, then*

(1) $\operatorname{res}(C) \subseteq \operatorname{res}(C)^\perp$,

(2) $\operatorname{tor}(C) = \operatorname{res}(C)^\perp$.

Furthermore, a QSD code C is Type IV if and only if $\operatorname{res}(C)$ *contains the all-one codeword.*

Corollary 3. *Every free QSD code over E is Type IV.*

Proof. Let C be a free QSD code. Then $\operatorname{res}(C) = \operatorname{tor}(C) = \operatorname{res}(C)^\perp$. Hence, $\operatorname{res}(C)$ is a binary self-dual code and so it contains the all-one codeword. By Theorem 7, C is Type IV. □

4. Linear codes constructed from three-class association schemes

Let A_0, A_1, A_2 and A_3 be the adjacency matrices of the relations of a three-class association scheme on a set of size n. Let $Q_{\mathcal{R}}(r, s, t, u) = rA_0 + sA_1 + tA_2 + uA_3$ where $r, s, t, u \in \mathcal{R}$. As $A_0 = I_n$ and $A_3 = J - I_n - A_1 - A_2$,

$$Q_{\mathcal{R}}(r, s, t, u) = (r - u)I_n + uJ + (s - u)A_1 + (t - u)A_2. \tag{4.1}$$

We describe two constructions using $Q_{\mathcal{R}}(r, s, t, u)$:

- The **pure** construction is the $n \times 2n$ matrix

$$P_{\mathcal{R}}(r, s, t, u) = (aI_n, Q_{\mathcal{R}}(r, s, t, u)).$$

  The linear code over $\mathcal{R}$ of length $2n$ and generator matrix $P_{\mathcal{R}}(r, s, t, u)$ is denoted by $C_{P_{\mathcal{R}}}(r, s, t, u)$.

- The **bordered** construction is the $(n + 1) \times (2n + 2)$ matrix

$$B_{\mathcal{R}}(r, s, t, u) = \left(\begin{array}{c|c|c|c} a & 0\cdots 0 & 0 & a\cdots a \\ \hline 0 & & a & \\ \vdots & aI_n & \vdots & Q_{\mathcal{R}}(r, s, t, u) \\ 0 & & a & \end{array}\right).$$

The linear code over $\mathcal{R}$ of length $2n+2$ and generator matrix $B_{\mathcal{R}}(r,s,t,u)$ is denoted by $C_{B_{\mathcal{R}}}(r,s,t,u)$.

Remark 1. Let $a \in \mathcal{R}$. Observe the following:

(1) The pure and the bordered constructions over $\mathcal{R}$ can be written as $G = (aI_m, Y)$ where Y is a square matrix of size m with entries from $\mathcal{R}$.

(2) If C is a self-orthogonal code over E with generator matrix $G = (aI_m, Y)$ where Y is a square matrix of size m, then by Theorem 6 and Corollary 2, C is QSD if and only if $Y = aS$ for some binary square matrix S of size m.

(3) Let M be a binary matrix of any size. If $G_{\mathcal{R}} = aM$ is a generator matrix for a linear code $C_{\mathcal{R}}$ over $\mathcal{R}$, then

$$G'_{\mathcal{R}} = \begin{pmatrix} aM \\ bM \end{pmatrix}$$

is an additive generator matrix for $C_{\mathcal{R}}$. By the definitions of the maps ϕ_E and ϕ_I, the additive codes $\phi_E(C_E)$ and $\phi_I(C_I)$ over $\mathbb{F}_4$ satisfy that

$$\phi_E(C_E) = \langle \omega M, \omega^2 M\rangle_{\mathbb{F}_2} = \langle \omega M, M\rangle_{\mathbb{F}_2} = \phi_I(C_I).$$

Hence, there is a one-to-one correspondence between the linear codes C_E and C_I and so they have the same size and weight distribution. Note that $C_{\mathcal{R}}$ is self-orthogonal if and only if $G_{\mathcal{R}}G_{\mathcal{R}}^T = O$ if and only if $a^2MM^T = O$. Since $a^2 \neq 0$, $C_{\mathcal{R}}$ is self-orthogonal if and only if $MM^T = O$ and so the self-orthogonality as well as the quasi-self duality and the Type IV property are preserved under this correspondence.

From the above three observations, it follows that if C is a QSD code over I from either the pure or the bordered construction with $r,s,t,u \in \{0,a\}$, then C can be regarded as a QSD code over E. Conversely, if C is a QSD code over E from either the pure or the bordered construction, then C can be regarded as a QSD code over I. Note that this is not generally true for self-orthogonal codes which are not QSD. Also this remark implies that the ring I may produce more free QSD codes and hence possibly more free Type IV codes than the ring E would.

Based on Remark 1 and since we are focusing on QSD and Type IV codes, it suffices to investigate the conditions which guarantee the quasi-self duality and the Type IV property of linear codes over the ring I. Indeed, in the case $r,s,t,u \in \{0,a\}$, such conditions would hold for QSD and Type IV codes over E. Hence, from this point forward, we only study linear codes over I.

4.1. QSD codes from non-symmetric three-class association schemes

Let (X,R) be a non-symmetric three-class association scheme. We can order the relations such that $R_2 = R_1^T$ and R_3 is a symmetric relation. The association scheme is uniquely determined by R_1. If we denote the adjacency matrix for R_1 by A, then the adjacency matrices for R_0, R_2 and R_3 are I_n, A^T and $J - I_n - A - A^T$, respectively. In this case, Eq (4.1) over I can be written as

$$Q_I(r,s,t,u) = (r-u)I_n + uJ + (s-u)A + (t-u)A^T. \tag{4.2}$$

To study self-orthogonality of the codes $C_{P_I}(r,s,t,u)$ and $C_{B_I}(r,s,t,u)$, we need to calculate $Q_I(r,s,t,u)Q_I(r,s,t,u)^T$. To do this, we make substantial use of the following lemma.

Lemma 4. [6] *If (X, R) is a non-symmetric three-class association scheme, then the following equations hold:*

$$AJ = JA = \kappa J, \tag{4.3}$$
$$AA^T = A^T A = \kappa I_n + \lambda(A + A^T) + \mu(J - I_n - A - A^T), \tag{4.4}$$
$$A^2 = \alpha A + \beta A^T + \gamma(J - I_n - A - A^T), \tag{4.5}$$

where $\kappa = p^0_{12} = p^0_{21}, \lambda = p^1_{12} = p^1_{21} = p^2_{12} = p^2_{21}, \mu = p^3_{12} = p^3_{21}, \alpha = p^1_{11}, \beta = p^2_{11}$, and $\gamma = p^3_{11}$. Moreover, $\alpha = \lambda$ and κ is the number of ones at each row and at each column of A.

Lemma 5. *Keep the notations in Lemma 4 and let $Q_I = Q_I(r, s, t, u)$. Then,*

$$Q_I Q_I^T = (r^2 + u^2 + (s^2 + t^2)(\kappa + \mu))I_n + (nu^2 + (s^2 + t^2)\mu)J$$
$$+ ((r + u)(t + s) + (s^2 + t^2)(\lambda + \mu) + (s + u)(t + u)(\beta + \lambda))(A + A^T).$$

Proof. By Eq (4.2) and the properties of I, we have

$$\begin{aligned}
Q_I Q_I^T &= ((r + u)I_n + uJ + (s + u)A + (t + u)A^T)((r + u)I_n + uJ + (s + u)A + (t + u)A^T)^T \\
&= ((r + u)I_n + uJ + (s + u)A + (t + u)A^T)((r + u)I_n + uJ + (s + u)A^T + (t + u)A) \\
&= (r^2 + u^2)I_n + (r + u)uJ + (r + u)(s + u)A^T + (r + u)(t + u)A + u(r + u)J + u^2J^2 + u(s + u)JA^T \\
&\quad + u(t + u)JA + (s + u)(r + u)A + (s + u)uAJ + (s^2 + u^2)AA^T + (s + u)(t + u)A^2 \\
&\quad + (t + u)(r + u)A^T + (t + u)uA^T J + (t + u)(s + u)(A^T)^2 + (t^2 + u^2)A^T A.
\end{aligned}$$

Applying Eq (4.3) gives

$$\begin{aligned}
Q_I Q_I^T &= (r^2 + u^2)I_n + (r + u)(s + u)A^T + (r + u)(t + u)A + nu^2J + u(s + u)\kappa J + u(t + u)\kappa J \\
&\quad + (s + u)(r + u)A + (s + u)u\kappa J + (s^2 + u^2)AA^T + (s + u)(t + u)A^2 + (t + u)(r + u)A^T \\
&\quad + (t + u)u\kappa J + (t + u)(s + u)(A^2)^T + (t^2 + u^2)A^T A \\
&= (r^2 + u^2)I_n + nu^2J + (r + u)(t + s)(A + A^T) + (s^2 + t^2)AA^T + (s + u)(t + u)(A^2 + (A^2)^T).
\end{aligned}$$

From Eqs (4.4) and (4.5), we obtain

$$\begin{aligned}
Q_I Q_I^T &= (r^2 + u^2)I_n + nu^2J + (r + u)(t + s)(A + A^T) + (s^2 + t^2)(\kappa I_n + \lambda(A + A^T) + \mu(J - I_n - A - A^T)) \\
&\quad + (s + u)(t + u)(\alpha A + \beta A^T + \alpha A^T + \beta A).
\end{aligned}$$

Since $\alpha = \lambda$ and I has characteristic two, it follows that

$$\begin{aligned}
Q_I Q_I^T &= (r^2 + u^2 + (s^2 + t^2)(\kappa + \mu))I_n + (nu^2 + (s^2 + t^2)\mu)J + ((r + u)(t + s) + (s^2 + t^2)(\lambda + \mu) \\
&\quad + (s + u)(t + u)(\beta + \lambda))(A + A^T).
\end{aligned}$$

□

In the next two theorems, all congruences are given in $\mathbb{F}_2$.

Theorem 8. *The code $C_{P_I}(r,s,t,u)$ constructed from a non-symmetric three-class association scheme is QSD if and only if the parameters are as in Table 3.*

Table 3. Conditions for linear codes over I formed from non-symmetric three-class association schemes by the pure construction to be QSD.

r	s	t	u	$C_{P_I}(r,s,t,u)$ is QSD
$\{0,b\}$	$\{0,b\}$	$\{0,b\}$	$\{0,b\}$	never
$\{0,b\}$	$\{0,b\}$	$\{0,b\}$	$\{a,c\}$	$n \equiv 0$ and $\beta \equiv \lambda$
$\{0,b\}$	$\{0,b\}$	$\{a,c\}$	$\{0,b\}$	$\kappa+1 \equiv \lambda \equiv \mu \equiv 0$
$\{0,b\}$	$\{0,b\}$	$\{a,c\}$	$\{a,c\}$	$\kappa \equiv \lambda+1 \equiv \mu \equiv n$
$\{0,b\}$	$\{a,c\}$	$\{0,b\}$	$\{0,b\}$	$\kappa+1 \equiv \lambda \equiv \mu \equiv 0$
$\{0,b\}$	$\{a,c\}$	$\{0,b\}$	$\{a,c\}$	$\kappa \equiv \lambda+1 \equiv \mu \equiv n$
$\{0,b\}$	$\{a,c\}$	$\{a,c\}$	$\{0,b\}$	never
$\{0,b\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$n \equiv 0$
$\{a,c\}$	$\{0,b\}$	$\{0,b\}$	$\{0,b\}$	always
$\{a,c\}$	$\{0,b\}$	$\{0,b\}$	$\{a,c\}$	never
$\{a,c\}$	$\{0,b\}$	$\{a,c\}$	$\{0,b\}$	$\kappa \equiv \lambda+1 \equiv \mu \equiv 0$
$\{a,c\}$	$\{0,b\}$	$\{a,c\}$	$\{a,c\}$	$\kappa+1 \equiv \lambda \equiv \mu \equiv n$
$\{a,c\}$	$\{a,c\}$	$\{0,b\}$	$\{0,b\}$	$\kappa \equiv \lambda+1 \equiv \mu \equiv 0$
$\{a,c\}$	$\{a,c\}$	$\{0,b\}$	$\{a,c\}$	$\kappa+1 \equiv \lambda \equiv \mu \equiv n$
$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{0,b\}$	$\beta \equiv \lambda$
$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	never

Proof. By Theorem 2, $C_{P_I}(r,s,t,u)$ is a free code of type $(n,0)$ and thus it has 2^{2n} codewords. Hence, to prove that $C_{P_I}(r,s,t,u)$ is QSD, we only need to show that it is self-orthogonal.

The code $C_{P_I}(r,s,t,u)$ is self-orthogonal if and only if $P_I(r,s,t,u)P_I(r,s,t,u)^T = O$ which occurs if and only if $Q_I(r,s,t,u)Q_I(r,s,t,u)^T = bI_n$. Then, by Lemma 5, the code $C_{P_I}(r,s,t,u)$ is self-orthogonal if and only if the following are satisfied:

$$r^2 + u^2 + (s^2+t^2)(\kappa+\mu) = b, \tag{4.6}$$
$$nu^2 + (s^2+t^2)\mu = 0, \tag{4.7}$$
$$(r+u)(s+t) + (s^2+t^2)(\lambda+\mu) + (s+u)(t+u)(\beta+\lambda) = 0. \tag{4.8}$$

To determine when the codes are self-orthogonal, we need to find the solutions that satisfy Eqs (4.6)–(4.8). We consider four cases depending on the values of the scalars s and t:

Case 1: $s,t \in \{0,b\}$.

Then $s^2+t^2 = 0$ and $s+t \in \{0,b\}$. Equations (4.6)–(4.8) then reduce to

$$\begin{aligned} r^2 + u^2 &= b,\\ nu^2 &= 0,\\ u^2(\beta+\lambda) &= 0. \end{aligned}$$

If $u \in \{0,b\}$, then $r \in \{a,c\}$. If $u \in \{a,c\}$, then $r \in \{0,b\}$, n is even, and $\beta \equiv \lambda \pmod 2$.

Case 2: $s, t \in \{a, c\}$.
Then $s^2 + t^2 = 0$ and $s + t \in \{0, b\}$. Equations (4.6)–(4.8) then reduce to

$$\begin{aligned} r^2 + u^2 &= b, \\ nu^2 &= 0, \\ (s + u)(t + u)(\beta + \lambda) &= 0. \end{aligned}$$

If $u \in \{0, b\}$, then $r \in \{a, c\}$ and $\beta \equiv \lambda \pmod 2$. If $u \in \{a, c\}$, then $r \in \{0, b\}$ and n is even.
Case 3: $s \in \{0, b\}$ and $t \in \{a, c\}$.
Then $s^2 + t^2 = b$. Equations (4.6)–(4.8) then reduce to

$$\begin{aligned} r^2 + u^2 + b(\kappa + \mu) &= b, \\ nu^2 + b\mu &= 0, \\ (r + u)t + b(\lambda + \mu) + u(t + u)(\beta + \lambda) &= 0. \end{aligned}$$

If $r, u \in \{0, b\}$, then $\kappa + 1 \equiv \lambda \equiv \mu \equiv 0 \pmod 2$. If $r, u \in \{a, c\}$, then $\kappa + 1 \equiv \lambda \equiv \mu \equiv n \pmod 2$. If $r \in \{0, b\}$ and $u \in \{a, c\}$, then $\lambda + 1 \equiv \kappa \equiv \mu \equiv n \pmod 2$. If $u \in \{0, b\}$ and $r \in \{a, c\}$, then $\lambda + 1 \equiv \kappa \equiv \mu \equiv 0 \pmod 2$.
Case 4: $t \in \{0, b\}$ and $s \in \{a, c\}$.
By interchanging t and s in Case 3, we get the same restrictions on r and u as well as the parameters of the association scheme. □

Theorem 9. *The code $C_{B_I}(r, s, t, u)$ constructed from a non-symmetric three-class association scheme is QSD if and only if the parameters are as in Table 4.*

Table 4. Conditions for linear codes over I formed from non-symmetric three-class association schemes by the bordered construction to be QSD.

r	s	t	u	$C_{B_I}(r, s, t, u)$ is QSD
$\{0, b\}$	$\{0, b\}$	$\{0, b\}$	$\{0, b\}$	never
$\{0, b\}$	$\{0, b\}$	$\{0, b\}$	$\{a, c\}$	$n \equiv 1$ and $\beta \equiv \lambda$
$\{0, b\}$	$\{0, b\}$	$\{a, c\}$	$\{0, b\}$	$\kappa + 1 \equiv \lambda \equiv \mu \equiv n \equiv 1$
$\{0, b\}$	$\{0, b\}$	$\{a, c\}$	$\{a, c\}$	$\kappa \equiv \lambda + 1 \equiv \mu \equiv n + 1 \equiv 0$
$\{0, b\}$	$\{a, c\}$	$\{0, b\}$	$\{0, b\}$	$\kappa + 1 \equiv \lambda \equiv \mu \equiv n \equiv 1$
$\{0, b\}$	$\{a, c\}$	$\{0, b\}$	$\{a, c\}$	$\kappa \equiv \lambda + 1 \equiv \mu \equiv n + 1 \equiv 0$
$\{0, b\}$	$\{a, c\}$	$\{a, c\}$	$\{0, b\}$	never
$\{0, b\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$n \equiv 1$
$\{a, c\}$	$\{0, b\}$	$\{0, b\}$	$\{0, b\}$	never
$\{a, c\}$	$\{0, b\}$	$\{0, b\}$	$\{a, c\}$	never
$\{a, c\}$	$\{0, b\}$	$\{a, c\}$	$\{0, b\}$	$\kappa \equiv \lambda + 1 \equiv \mu \equiv n \equiv 1$
$\{a, c\}$	$\{0, b\}$	$\{a, c\}$	$\{a, c\}$	$\kappa + 1 \equiv \lambda \equiv \mu \equiv n + 1 \equiv 0$
$\{a, c\}$	$\{a, c\}$	$\{0, b\}$	$\{0, b\}$	$\kappa \equiv \lambda + 1 \equiv \mu \equiv n \equiv 1$
$\{a, c\}$	$\{a, c\}$	$\{0, b\}$	$\{a, c\}$	$\kappa + 1 \equiv \lambda \equiv \mu \equiv n + 1 \equiv 0$
$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{0, b\}$	never
$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	never

Proof. By Theorem 2, $C_{B_I}(r, s, t, u)$ is a free code of type $(n + 1, 0)$ and thus it has 2^{2n+2} codewords. Hence, to prove that $C_{B_I}(r, s, t, u)$ is QSD, we only need to show that it is self-orthogonal.

The code $C_{B_I}(r, s, t, u)$ is self-orthogonal if and only if $B_I(r, s, t, u)B_I(r, s, t, u)^T = O$ which occurs if and only if the following are satisfied:

$$\begin{aligned}(n+1)b &= 0,\\ a(r + s\kappa + t\kappa + u(n - 2\kappa - 1)) &= 0,\\ Q_I(r, s, t, u)Q_I(r, s, t, u)^T &= b(I_n + J).\end{aligned}$$

Note that the first equation is the inner product of the top row with itself. The second equation is the inner product of the top row with any other row. The third equation ensures that the remaining rows are orthogonal to each other.

The first equation requires n to be odd. The remaining two equations then satisfy

$$\begin{aligned}a(r + s\kappa + t\kappa) &= 0,\\ Q_I(r, s, t, u)Q_I(r, s, t, u)^T &= b(I_n + J).\end{aligned}$$

Then, by Lemma 5, the code $C_{B_I}(r, s, t, u)$ is self-orthogonal if and only if n is odd and the following are satisfied:

$$a(r + s\kappa + t\kappa) = 0, \tag{4.9}$$
$$r^2 + u^2 + (s^2 + t^2)(\kappa + \mu) = b, \tag{4.10}$$
$$u^2 + (s^2 + t^2)\mu = b, \tag{4.11}$$
$$(r + u)(s + t) + (s^2 + t^2)(\lambda + \mu) + (s + u)(t + u)(\beta + \lambda) = 0. \tag{4.12}$$

To determine when the codes are self-orthogonal, we need to find the solutions that satisfy Eqs (4.9)–(4.12). We consider four cases depending on the values of the scalars s and t:

Case 1: $s, t \in \{0, b\}$.

Then $s^2 + t^2 = 0$ and $s + t \in \{0, b\}$. Equations (4.9)–(4.12) then reduce to

$$\begin{aligned}ar &= 0,\\ r^2 + u^2 &= b,\\ u^2 &= b,\\ u^2(\beta + \lambda) &= 0.\end{aligned}$$

Hence, we must have $r \in \{0, b\}$, $u \in \{a, c\}$, and $\beta \equiv \lambda \pmod 2$.

Case 2: $s, t \in \{a, c\}$.

Then $s^2 + t^2 = 0$ and $s + t \in \{0, b\}$. Equations (4.9)–(4.12) then reduce to

$$\begin{aligned}ar &= 0,\\ r^2 + u^2 &= b,\\ u^2 &= b,\\ (s + u)(t + u)(\beta + \lambda) &= 0.\end{aligned}$$

Hence, we must have $r \in \{0, b\}$ and $u \in \{a, c\}$.
Case 3: $s \in \{0, b\}$ and $t \in \{a, c\}$.
Then $s^2 + t^2 = b$. Equations (4.9)–(4.12) then reduce to

$$\begin{aligned} ar + b\kappa &= 0, \\ r^2 + u^2 + b(\kappa + \mu) &= b, \\ u^2 + b\mu &= b, \\ (r + u)t + b(\lambda + \mu) + u(t + u)(\beta + \lambda) &= 0. \end{aligned}$$

If $r, u \in \{0, b\}$, then $\lambda + 1 \equiv \mu + 1 \equiv \kappa \equiv 0 \pmod 2$. If $r, u \in \{a, c\}$, then $\kappa + 1 \equiv \lambda \equiv \mu \equiv 0 \pmod 2$. If $r \in \{0, b\}$ and $u \in \{a, c\}$, then $\lambda + 1 \equiv \mu \equiv \kappa \equiv 0 \pmod 2$. If $u \in \{0, b\}$ and $r \in \{a, c\}$, then $\lambda + 1 \equiv \mu \equiv \kappa \equiv 1 \pmod 2$.
Case 4: $t \in \{0, b\}$ and $s \in \{a, c\}$.
By interchanging t and s in Case 3, we get the same restrictions on r and u as well as the parameters of the association scheme. □

Theorem 10. *Every QSD code constructed from a non-symmetric three-class association scheme by either the pure or the bordered construction is quasi Type IV.*

Proof. Let C denote a QSD code constructed from a non-symmetric three-class association scheme by either the pure or the bordered construction. Then C is a free code by Theorem 2 and thus C is quasi Type IV by Corollary 1. □

Example 2. In Table 5, we present examples of (N, d) QSD codes over I of length N and minimum distance d satisfying the conditions in Theorems 8 and 9 constructed from non-symmetric three-class association schemes. The corresponding adjacency matrices with length n and parameters $(\kappa, \beta, \lambda, \mu)$ as defined in Lemma 4 of such schemes can be found in [15].

Table 5. QSD codes over I constructed from non-symmetric three-class association schemes.

Construction	$(n, \kappa, \beta, \lambda, \mu)$	Code	(N, d)	Type IV
Pure	$(6, 1, 1, 0, 0)$	$C_{P_I}(a, 0, a, a)$	$(12, 4)$	yes
	$(9, 1, 1, 0, 0)$	$C_{P_I}(a, b, b, 0)$	$(18, 2)$	yes
	$(12, 1, 1, 0, 0)$	$C_{P_I}(b, a, a, a)$	$(24, 4)$	no
	$(14, 3, 2, 1, 0)$	$C_{P_I}(0, a, a, a)$	$(28, 4)$	yes
	$(15, 1, 1, 0, 0)$	$C_{P_I}(0, a, 0, b)$	$(30, 2)$	yes
	$(18, 1, 1, 0, 0)$	$C_{P_I}(0, a, c, c)$	$(36, 4)$	no
	$(21, 3, 2, 1, 0)$	$C_{P_I}(a, 0, 0, b)$	$(42, 2)$	yes
	$(22, 5, 3, 2, 0)$	$C_{P_I}(c, 0, c, a)$	$(44, 8)$	yes
	$(28, 3, 2, 1, 0)$	$C_{P_I}(0, a, a, a)$	$(56, 4)$	yes
	$(33, 5, 3, 2, 0)$	$C_{P_I}(0, 0, a, 0)$	$(66, 6)$	yes
	$(38, 9, 5, 4, 0)$	$C_{P_I}(c, c, 0, c)$	$(76, 8)$	yes
Bordered	$(9, 1, 1, 0, 0)$	$C_{B_I}(0, c, c, a)$	$(20, 4)$	yes
	$(15, 1, 1, 0, 0)$	$C_{B_I}(0, a, a, a)$	$(32, 4)$	yes
	$(33, 5, 3, 2, 0)$	$C_{B_I}(c, 0, c, a)$	$(68, 8)$	yes

Lemma 6. *Let C be a QSD code of length $2n$ over I with generator matrix $G = (xI_n, yM)$ where $x, y \in \{a, c\}$ and M is a nonzero binary square matrix of size n. Then C is Type IV.*

Proof. Since C is self-orthogonal, by Lemma 3, $\mathrm{wt}(\mathbf{g}_i) \equiv n_b(\mathbf{g}_i) \pmod 2$ for each row $\mathbf{g}_i$ of G. As $x, y \in \{a, c\}$, $n_b(\mathbf{g}_i) = 0$ and thus $\mathrm{wt}(\mathbf{g}_i)$ is even. Since each column of G has entries from either $\{0, a\}$ or $\{0, c\}$, $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j) = 0$ and hence this weight is also even. By Theorem 3, C is Type IV. □

The conditions in the next two theorems are sufficient but not necessary as Table 5 shows.

Theorem 11. *A QSD $C_{P_I}(r, s, t, u)$ constructed from a non-symmetric three-class association scheme is Type IV if one of the following holds*

(i) $r, s, t, u \in \{0, a\}$,

(ii) $r, s, t, u \in \{0, c\}$,

(iii) $r = 0$, $s, t \in \{a, b, c\}$, $u \in \{a, c\}$, *and* $s = t \neq u$,

(iv) $r \in \{a, c\}$, $s = t \neq r$, *and* $u = 0$,

(v) $r \in \{a, c\}$, $s = t \neq r$, $u = b$, *and n is odd.*

Proof. If either (i) or (ii) are satisfied, then $C_{P_I}(r, s, t, u)$ is Type IV by Lemma 6.
To prove (iii)–(v), we will use Theorem 3. So we need to show that $\mathrm{wt}(\mathbf{g}_i)$ and $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j)$ are even where $\mathbf{g}_i$ and $\mathbf{g}_j$ for $1 \leq i, j \leq n$ are distinct rows of the generator matrix $P_I(r, s, t, u)$. By Lemma 3, $\mathrm{wt}(\mathbf{g}_i) \equiv n_b(\mathbf{g}_i) \pmod 2$. Since

$$n_b(\mathbf{g}_i) = \begin{cases} 0 & \text{in cases (iii) and (iv) with } s = t \neq b, \\ 2\kappa & \text{in cases (iii) and (iv) with } s = t = b, \\ n - 2\kappa - 1 & \text{in case (v) with } s = t \neq b, \\ n - 1 & \text{in case (v) with } s = t = b, \end{cases}$$

and n is odd in (v), $\mathrm{wt}(\mathbf{g}_i)$ is even.
Next, we will prove that $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j)$ is even. Let $\mathbf{q}_i$ be the ith row of the matrix $Q_I(r, s, t, u)$. Since $P_I = (aI_n, Q_I(r, s, t, u))$,

$$\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j) = \mathrm{wt}(\mathbf{q}_i \cap_{\neq} \mathbf{q}_j).$$

Since $s = t$, we can view the rows $\mathbf{q}_i$ and $\mathbf{q}_j$ of $Q_I(r, s, t, u)$ having the following entries:

$$\begin{array}{ccccccc} \mathbf{q}_i: & r & q_{ij} & s & s & u & u \\ \mathbf{q}_j: & q_{ji} & r & s & u & s & u \\ & \mathcal{A} & \mathcal{B} & \mathcal{C} & \mathcal{D} & \mathcal{E} & \mathcal{F} \end{array}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ are the numbers of coordinates where $\mathbf{q}_i$ and $\mathbf{q}_j$ satisfy the entries in the corresponding column. The jth entry in $\mathbf{q}_i$ and the ith entry in $\mathbf{q}_j$ are denoted by q_{ij} and q_{ji}, respectively. Note that $\mathcal{A} = \mathcal{B} = 1$ and $q_{ij}, q_{ji} \in \{s, u\}$.
Since $Q_I(r, s, t, u) = rI_n + s(A + A^T) + u(J - I - A - A^T)$, $Q_I(r, s, t, u)$ is symmetric and so $q_{ij} = q_{ji}$. Observe that in case (iii),

$$\mathrm{wt}(\mathbf{q}_i \cap_{\neq} \mathbf{q}_j) = \mathcal{D} + \mathcal{E},$$

in case (iv),

$$\mathrm{wt}(\mathbf{q}_i \cap_{\neq} \mathbf{q}_j) = \begin{cases} 0 & \text{if } q_{ij} = 0, \\ 2 & \text{if } q_{ij} \neq 0, \end{cases}$$

and in case (v),

$$\mathrm{wt}(\mathbf{q}_i \cap_{\neq} \mathbf{q}_j) = \begin{cases} \mathcal{D} + \mathcal{E} + 2 & \text{if } s = t \in \{a, c\}, \\ 0 & \text{if } s = t \in \{0, b\} \text{ and } q_{ij} = 0, \\ 2 & \text{if } s = t \in \{0, b\} \text{ and } q_{ij} = b. \end{cases}$$

Since $n_s(\mathbf{q}_i) = n_s(\mathbf{q}_j)$ along with the symmetry of $Q_I(r, s, t, u)$, it follows that $C + \mathcal{D} = C + \mathcal{E}$. Hence, $\mathcal{D} = \mathcal{E}$ and therefore $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j) = \mathrm{wt}(\mathbf{q}_i \cap_{\neq} \mathbf{q}_j)$ is even. Hence, $C_{P_I}(r, s, t, u)$ is Type IV. This proves (iii)–(v). □

Theorem 12. *A QSD $C_{B_I}(r, s, t, u)$ constructed from a non-symmetric three-class association scheme is Type IV if one of the following holds*

(i) $r, s, t, u \in \{0, a\}$,

(ii) $r, s, t, u \in \{0, c\}$,

(iii) $r = 0$ *and* $s, t, u \in \{a, c\}$ *such that* $s = t \neq u$.

Proof. If (i) is satisfied, then $C_{B_I}(r, s, t, u)$ is Type IV by Lemma 6.
To prove (ii) and (iii), we will use Theorem 3. So we need to show that $\mathrm{wt}(\mathbf{g}_i)$ and $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j)$ are even where $\mathbf{g}_i$ and $\mathbf{g}_j$ for $1 \leq i, j \leq n + 1$ are distinct rows of the generator matrix $B_I(r, s, t, u)$. By Lemma 3, $\mathrm{wt}(\mathbf{g}_i) \equiv n_b(\mathbf{g}_i) \pmod 2$. Since none of the entries of $B_I(r, s, t, u)$ equals b in cases (ii) and (iii), $n_b(\mathbf{g}_i) = 0$ and thus $\mathrm{wt}(\mathbf{g}_i)$ is even.
Now we will prove that $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j)$ is even.
The self-orthogonality of $C_{B_I}(r, s, t, u)$ implies that

$$n_{a,a}(\mathbf{g}_i, \mathbf{g}_j) + n_{a,c}(\mathbf{g}_i, \mathbf{g}_j) + n_{c,a}(\mathbf{g}_i, \mathbf{g}_j) + n_{c,c}(\mathbf{g}_i, \mathbf{g}_j) \equiv 0 \pmod 2. \tag{4.13}$$

In case (ii), since $r, s, t, u \in \{0, c\}$, Eq (4.13) implies that $n_{a,c}(\mathbf{g}_1, \mathbf{g}_j) \equiv 0 \pmod 2$ for $j > 1$. Observe that $\mathrm{wt}(\mathbf{g}_1 \cap_{\neq} \mathbf{g}_j) = n_{a,c}(\mathbf{g}_1, \mathbf{g}_j)$ and $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j) = 0$ for $i, j > 1$. Hence, $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j)$ is even for all i and j. Therefore, $C_{B_I}(r, s, t, u)$ is Type IV. This proves (ii).
In case (iii), observe that

$$\mathrm{wt}(\mathbf{g}_1 \cap_{\neq} \mathbf{g}_j) = n_{a,c}(\mathbf{g}_1, \mathbf{g}_j) = n_c(\mathbf{g}_j) = \begin{cases} 2\kappa & \text{if } s = t = c \text{ and } u = a, \\ n - 2\kappa - 1 & \text{if } s = t = a \text{ and } u = c. \end{cases}$$

Since n is odd in the bordered construction, $\mathrm{wt}(\mathbf{g}_1 \cap_{\neq} \mathbf{g}_j)$ is even. Now let $\mathbf{q}_i$ be the ith row of the matrix $Q_I(r, s, t, u)$. Then for $2 \leq i, j \leq n + 1$,

$$\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j) = \mathrm{wt}(\mathbf{q}_{i-1} \cap_{\neq} \mathbf{q}_{j-1}).$$

We can view the rows $\mathbf{q}_{i-1}$ and $\mathbf{q}_{j-1}$ of $Q_I(r, s, t, u)$ having the following entries:

$$\begin{array}{rcccccc} \mathbf{q}_{i-1}: & 0 & q_{j-1} & a & a & c & c \\ \mathbf{q}_{j-1}: & q_{i-1} & 0 & a & c & a & c \\ & \mathcal{A} & \mathcal{B} & C & \mathcal{D} & \mathcal{E} & \mathcal{F} \end{array}$$

where $\mathcal{A}, \mathcal{B}, C, \mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ are the numbers of coordinates where $\mathbf{q}_{i-1}$ and $\mathbf{q}_{j-1}$ satisfy the entries in the corresponding column. The entries q_{j-1} and q_{i-1} denote the components in the $j-1$ and $i-1$ positions of $\mathbf{q}_{i-1}$ and $\mathbf{q}_{j-1}$, respectively. Note that $\mathcal{A} = \mathcal{B} = 1$ and $q_{i-1}, q_{j-1} \in \{a, c\}$. Since $Q_I(r,s,t,u) = s(A + A^T) + u(J - I - A - A^T)$, $Q_I(r,s,t,u)$ is symmetric and so $q_{i-1} = q_{j-1}$. Since $n_a(\mathbf{q}_{i-1}) = n_a(\mathbf{q}_{j-1})$ along with the symmetry of $Q_I(r,s,t,u)$, it follows that $C + \mathcal{D} = C + \mathcal{E}$. Hence, $\mathcal{D} = \mathcal{E}$ and so $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j) = \mathrm{wt}(\mathbf{q}_{i-1} \cap_{\neq} \mathbf{q}_{j-1}) = \mathcal{D} + \mathcal{E}$ is even. Therefore, $C_{B_I}(r,s,t,u)$ is Type IV. This proves (iii). □

The next example shows that employing the same adjacency matrices of a three-class association scheme but interchanging the scalars within $\{0, b\}$ or $\{a, c\}$ leaves the quasi-self duality of the code unchanged but not the Type IV property or the weight distribution.

Example 3. Let (X, R) be the non-symmetric three-class association scheme of size 6 and relation matrix No. 4 of [15]. The corresponding adjacency matrices are as follows, $A_0 = I_6$,

$$A_1 = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0 \end{pmatrix}, \ A_3 = \begin{pmatrix} 0&0&0&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1\\ 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 1&1&1&0&0&0 \end{pmatrix}.$$

Recall that $Q_I(r,s,t,u) = rA_0 + sA_1 + tA_2 + uA_3$. Then

$$P_I(r,s,t,u) = (aI_6, Q_I(r,s,t,u)) = \left(\begin{array}{cccccc|cccccc} a&0&0&0&0&0&r&s&t&u&u&u\\ 0&a&0&0&0&0&t&r&s&u&u&u\\ 0&0&a&0&0&0&s&t&r&u&u&u\\ 0&0&0&a&0&0&u&u&u&r&s&t\\ 0&0&0&0&a&0&u&u&u&t&r&s\\ 0&0&0&0&0&a&u&u&u&s&t&r \end{array}\right)$$

is a generator matrix for $C_{P_I}(r,s,t,u)$. Using Eq (2.3), we calculate the parameters $(n, \kappa, \beta, \lambda, \mu) = (6,1,1,0,0)$ of (X, R). Hence, $\kappa + 1 \equiv \lambda \equiv \mu \equiv n \equiv 0 \pmod 2$. Consistent with Theorem 8, the following codes are QSD but not necessarily Type IV:

- $C_{P_I}(a,0,a,a)$ is a $(12,4)$ Type IV code with weight distribution $[<0,1>, <4,45>, <6,216>, <8,1755>, <10,1800>, <12,279>]$.
- $C_{P_I}(a,b,a,a)$ is a $(12,4)$ QSD but not Type IV code with weight distribution $[<0,1>, <4,45>, <6,152>, <7,384>, <8,795>, <9,1280>, <10,840>, <11,384>, <12,215>]$.
- $C_{P_I}(a,b,c,c)$ is a $(12,4)$ QSD but not Type IV code with weight distribution $[<0,1>, <4,15>, <5,24>, <6,216>, <7,312>, <8,975>, <9,840>, <10,1176>, <11,360>, <12,177>]$.
- $C_{P_I}(c,0,a,c)$ is a $(12,4)$ QSD but not Type IV code with weight distribution $[<0,1>, <4,15>, <5,24>, <6,200>, <7,408>, <8,735>, <9,1160>, <10,936>, <11,456>, <12,161>]$.

4.2. QSD codes from symmetric three-class association schemes

Let (X, R) be a symmetric three-class association scheme. Then all adjacency matrices are symmetric. In this case, Eq (4.1) over I can be written as

$$Q_I(r, s, t, u) = (r - u)I_n + uJ + (s - u)A_1 + (t - u)A_2. \tag{4.14}$$

To study self-orthogonality of the codes $C_{P_I}(r, s, t, u)$ and $C_{B_I}(r, s, t, u)$, we need to calculate $Q_I(r, s, t, u)Q_I(r, s, t, u)^T$. To do this, we make substantial use of the following lemma.

Lemma 7. *If (X, R) is a symmetric three-class association scheme, then the following equations hold:*

$$A_iJ = JA_i = \kappa_i J,$$
$$A_i^2 = A_iA_i^T = \kappa_i I_n + \alpha_i A_1 + \beta_i A_2 + \gamma_i(J - I_n - A_1 - A_2),$$

where $\kappa_i = p_{ii}^0, \alpha_i = p_{ii}^1, \beta_i = p_{ii}^2$, and $\gamma_i = p_{ii}^3$.

Proof. The equations follow immediately from Eqs (2.1) and (2.2). □

Lemma 8. *Keep the notations in Lemma 7 and let $Q_I = Q_I(r, s, t, u)$. Then,*

$$\begin{aligned}Q_IQ_I^T = &(r^2 + u^2 + (s^2 + u^2)(\kappa_1 + \gamma_1) + (t^2 + u^2)(\kappa_2 + \gamma_2))I_n + (nu^2 + (s^2 + u^2)\gamma_1 + (t^2 + u^2)\gamma_2)J \\ &+ ((s^2 + u^2)(\alpha_1 + \gamma_1) + (t^2 + u^2)(\alpha_2 + \gamma_2))A_1 + ((s^2 + u^2)(\beta_1 + \gamma_1) + (t^2 + u^2)(\beta_2 + \gamma_2))A_2.\end{aligned}$$

Proof. By Eq (4.14), Lemma 7 and the properties of I, we have

$$\begin{aligned}Q_IQ_I^T &= ((r + u)I_n + uJ + (s + u)A_1 + (t + u)A_2)((r + u)I_n + uJ + (s + u)A_1 + (t + u)A_2)^T \\ &= ((r + u)I_n + uJ + (s + u)A_1 + (t + u)A_2)((r + u)I_n + uJ + (s + u)A_1 + (t + u)A_2) \\ &= (r + u)^2 I_n + (r + u)uJ + (r + u)(s + u)A_1 + (r + u)(t + u)A_2 + u(r + u)J + u^2J^2 \\ &\quad + u(s + u)JA_1 + u(t + u)JA_2 + (s + u)(r + u)A_1 + (s + u)uA_1J + (s + u)^2A_1^2 \\ &\quad + (s + u)(t + u)A_1A_2 + (t + u)(r + u)A_2 + (t + u)uA_2J + (t + u)(s + u)A_2A_1 + (t + u)^2A_2^2 \\ &= (r^2 + u^2)I_n + u^2J^2 + (s^2 + u^2)A_1^2 + (t^2 + u^2)A_2^2 \\ &= (r^2 + u^2)I_n + nu^2J + (s^2 + u^2)(\kappa_1 I_n + \alpha_1A_1 + \beta_1A_2 + \gamma_1(J - I_n - A_1 - A_2)) \\ &\quad + (t^2 + u^2)(\kappa_2 I_n + \alpha_2A_1 + \beta_2A_2 + \gamma_2(J - I_n - A_1 - A_2)) \\ &= (r^2 + u^2 + (s^2 + u^2)(\kappa_1 + \gamma_1) + (t^2 + u^2)(\kappa_2 + \gamma_2))I_n + (nu^2 + (s^2 + u^2)\gamma_1 + (t^2 + u^2)\gamma_2)J \\ &\quad + ((s^2 + u^2)(\alpha_1 + \gamma_1) + (t^2 + u^2)(\alpha_2 + \gamma_2))A_1 + ((s^2 + u^2)(\beta_1 + \gamma_1) + (t^2 + u^2)(\beta_2 + \gamma_2))A_2.\end{aligned}$$

□

In the next two theorems, all congruences are given in $\mathbb{F}_2$.

Theorem 13. *The code $C_{P_I}(r, s, t, u)$ constructed from a symmetric three-class association scheme is QSD if and only if the parameters are as in Table 6.*

Table 6. Conditions for linear codes over I formed from symmetric three-class association schemes by the pure construction to be QSD.

r	s	t	u	$C_{P_I}(r,s,t,u)$ is QSD
$\{0,b\}$	$\{0,b\}$	$\{0,b\}$	$\{0,b\}$	never
$\{0,b\}$	$\{0,b\}$	$\{0,b\}$	$\{a,c\}$	$\alpha_1+\alpha_2 \equiv \beta_1+\beta_2 \equiv \gamma_1+\gamma_2 \equiv \kappa_1+\kappa_2 \equiv n$
$\{0,b\}$	$\{0,b\}$	$\{a,c\}$	$\{0,b\}$	$\alpha_2 \equiv \beta_2 \equiv \gamma_2 \equiv \kappa_2+1 \equiv 0$
$\{0,b\}$	$\{0,b\}$	$\{a,c\}$	$\{a,c\}$	$\alpha_1 \equiv \beta_1 \equiv \gamma_1 \equiv \kappa_1 \equiv n$
$\{0,b\}$	$\{a,c\}$	$\{0,b\}$	$\{0,b\}$	$\alpha_1 \equiv \beta_1 \equiv \gamma_1 \equiv \kappa_1+1 \equiv 0$
$\{0,b\}$	$\{a,c\}$	$\{0,b\}$	$\{a,c\}$	$\alpha_2 \equiv \beta_2 \equiv \gamma_2 \equiv \kappa_2 \equiv n$
$\{0,b\}$	$\{a,c\}$	$\{a,c\}$	$\{0,b\}$	$\alpha_1+\alpha_2 \equiv \beta_1+\beta_2 \equiv \gamma_1+\gamma_2 \equiv \kappa_1+\kappa_2+1 \equiv 0$
$\{0,b\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$n \equiv 0$
$\{a,c\}$	$\{0,b\}$	$\{0,b\}$	$\{0,b\}$	always
$\{a,c\}$	$\{0,b\}$	$\{0,b\}$	$\{a,c\}$	$\alpha_1+\alpha_2 \equiv \beta_1+\beta_2 \equiv \gamma_1+\gamma_2 \equiv \kappa_1+\kappa_2+1 \equiv n$
$\{a,c\}$	$\{0,b\}$	$\{a,c\}$	$\{0,b\}$	$\alpha_2 \equiv \beta_2 \equiv \gamma_2 \equiv \kappa_2 \equiv 0$
$\{a,c\}$	$\{0,b\}$	$\{a,c\}$	$\{a,c\}$	$\alpha_1 \equiv \beta_1 \equiv \gamma_1 \equiv \kappa_1+1 \equiv n$
$\{a,c\}$	$\{a,c\}$	$\{0,b\}$	$\{0,b\}$	$\alpha_1 \equiv \beta_1 \equiv \gamma_1 \equiv \kappa_1 \equiv 0$
$\{a,c\}$	$\{a,c\}$	$\{0,b\}$	$\{a,c\}$	$\alpha_2 \equiv \beta_2 \equiv \gamma_2 \equiv \kappa_2+1 \equiv n$
$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{0,b\}$	$\alpha_1+\alpha_2 \equiv \beta_1+\beta_2 \equiv \gamma_1+\gamma_2 \equiv \kappa_1+\kappa_2 \equiv 0$
$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	never

Proof. By Theorem 2, $C_{P_I}(r,s,t,u)$ is a free code of type $(n,0)$ and thus it has 2^{2n} codewords. Hence, to prove that $C_{P_I}(r,s,t,u)$ is QSD, we only need to show that it is self-orthogonal.

The code $C_{P_I}(r,s,t,u)$ is self-orthogonal if and only if $P_I(r,s,t,u)P_I(r,s,t,u)^T = O$ which occurs if and only if $Q_I(r,s,t,u)Q_I(r,s,t,u)^T = bI_n$. Then, by Lemma 8, the code $C_{P_I}(r,s,t,u)$ is self-orthogonal if and only if the following are satisfied:

$$r^2+u^2+(s^2+u^2)(\kappa_1+\gamma_1)+(t^2+u^2)(\kappa_2+\gamma_2) = b, \tag{4.15}$$

$$nu^2+(s^2+u^2)\gamma_1+(t^2+u^2)\gamma_2 = 0, \tag{4.16}$$

$$(s^2+u^2)(\alpha_1+\gamma_1)+(t^2+u^2)(\alpha_2+\gamma_2) = 0, \tag{4.17}$$

$$(s^2+u^2)(\beta_1+\gamma_1)+(t^2+u^2)(\beta_2+\gamma_2) = 0. \tag{4.18}$$

To determine when the codes are self-orthogonal, we need to find the solutions that satisfy Eqs (4.15)–(4.18). We consider four cases depending on the values of the scalars s and u:

Case 1: $s, u \in \{0,b\}$.

Then $s^2+u^2 = 0$ and Eqs (4.15)–(4.18) reduce to

$$\begin{aligned} r^2+t^2(\kappa_2+\gamma_2) &= b,\\ t^2\gamma_2 &= 0,\\ t^2(\alpha_2+\gamma_2) &= 0,\\ t^2(\beta_2+\gamma_2) &= 0. \end{aligned}$$

If $t \in \{0,b\}$, then $r \in \{a,c\}$.

If $t \in \{a,c\}$, then either $r \in \{0,b\}$ and $\alpha_2 \equiv \beta_2 \equiv \gamma_2 \equiv \kappa_2+1 \equiv 0 \pmod 2$, or $r \in \{a,c\}$ and $\alpha_2 \equiv \beta_2 \equiv \gamma_2 \equiv \kappa_2 \equiv 0 \pmod 2$.

Case 2: $s, u \in \{a, c\}$.
Then $s^2 + u^2 = 0$ and Eqs (4.15)–(4.18) reduce to

$$\begin{aligned} r^2 + b + (t^2 + b)(\kappa_2 + \gamma_2) &= b, \\ nb + (t^2 + b)\gamma_2 &= 0, \\ (t^2 + b)(\alpha_2 + \gamma_2) &= 0, \\ (t^2 + b)(\beta_2 + \gamma_2) &= 0. \end{aligned}$$

If $t \in \{a, c\}$, then $r \in \{0, b\}$ and n is even.
If $t \in \{0, b\}$, then either $r \in \{0, b\}$ and $\alpha_2 \equiv \beta_2 \equiv \gamma_2 \equiv \kappa_2 \equiv n \pmod 2$, or $r \in \{a, c\}$ and $\alpha_2 \equiv \beta_2 \equiv \gamma_2 \equiv \kappa_2 + 1 \equiv n \pmod 2$.

Case 3: $s \in \{0, b\}$ and $u \in \{a, c\}$.
Then $s^2 + u^2 = b$ and Eqs (4.15)–(4.18) reduce to

$$\begin{aligned} r^2 + b(1 + \kappa_1 + \gamma_1) + (t^2 + b)(\kappa_2 + \gamma_2) &= b, \\ b(n + \gamma_1) + (t^2 + b)\gamma_2 &= 0, \\ b(\alpha_1 + \gamma_1) + (t^2 + b)(\alpha_2 + \gamma_2) &= 0, \\ b(\beta_1 + \gamma_1) + (t^2 + b)(\beta_2 + \gamma_2) &= 0. \end{aligned}$$

If $r, t \in \{0, b\}$, then $\kappa_1 + \kappa_2 \equiv \alpha_1 + \alpha_2 \equiv \beta_1 + \beta_2 \equiv \gamma_1 + \gamma_2 \equiv n \pmod 2$.
If $r, t \in \{a, c\}$, then $\kappa_1 + 1 \equiv \alpha_1 \equiv \beta_1 \equiv \gamma_1 \equiv n \pmod 2$.
If $r \in \{0, b\}$ and $t \in \{a, c\}$, then $\kappa_1 \equiv \alpha_1 \equiv \beta_1 \equiv \gamma_1 \equiv n \pmod 2$.
If $r \in \{a, c\}$ and $t \in \{0, b\}$, then $\kappa_1 + \kappa_2 + 1 \equiv \alpha_1 + \alpha_2 \equiv \beta_1 + \beta_2 \equiv \gamma_1 + \gamma_2 \equiv n \pmod 2$.

Case 4: $u \in \{0, b\}$ and $s \in \{a, c\}$.
Then $s^2 + u^2 = b$ and Eqs (4.15)–(4.18) reduce to

$$\begin{aligned} r^2 + b(\kappa_1 + \gamma_1) + t^2(\kappa_2 + \gamma_2) &= b, \\ b\gamma_1 + t^2\gamma_2 &= 0, \\ b(\alpha_1 + \gamma_1) + t^2(\alpha_2 + \gamma_2) &= 0, \\ b(\beta_1 + \gamma_1) + t^2(\beta_2 + \gamma_2) &= 0. \end{aligned}$$

If $r, t \in \{0, b\}$, then $\kappa_1 + 1 \equiv \alpha_1 \equiv \beta_1 \equiv \gamma_1 \equiv 0 \pmod 2$.
If $r, t \in \{a, c\}$, then $\kappa_1 + \kappa_2 \equiv \alpha_1 + \alpha_2 \equiv \beta_1 + \beta_2 \equiv \gamma_1 + \gamma_2 \equiv 0 \pmod 2$.
If $r \in \{0, b\}$ and $t \in \{a, c\}$, then $\kappa_1 + \kappa_2 + 1 \equiv \alpha_1 + \alpha_2 \equiv \beta_1 + \beta_2 \equiv \gamma_1 + \gamma_2 \equiv 0 \pmod 2$.
If $r \in \{a, c\}$ and $t \in \{0, b\}$, then $\kappa_1 \equiv \alpha_1 \equiv \beta_1 \equiv \gamma_1 \equiv 0 \pmod 2$. □

Theorem 14. *The code $C_{B_I}(r, s, t, u)$ constructed from a symmetric three-class association scheme is QSD if and only if the parameters are as in Table 7.*

Table 7. Conditions for linear codes over I formed from symmetric three-class association schemes by the bordered construction to be QSD.

r	s	t	u	$C_{B_I}(r,s,t,u)$ is QSD
$\{0,b\}$	$\{0,b\}$	$\{0,b\}$	$\{0,b\}$	never
$\{0,b\}$	$\{0,b\}$	$\{0,b\}$	$\{a,c\}$	$\alpha_1+\alpha_2 \equiv \beta_1+\beta_2 \equiv \gamma_1+\gamma_2 \equiv n+1 \equiv 0$
$\{0,b\}$	$\{0,b\}$	$\{a,c\}$	$\{0,b\}$	$\alpha_2 \equiv \beta_2 \equiv \gamma_2 \equiv n \equiv 1$
$\{0,b\}$	$\{0,b\}$	$\{a,c\}$	$\{a,c\}$	$\alpha_1 \equiv \beta_1 \equiv \gamma_1 \equiv n+1 \equiv 0$
$\{0,b\}$	$\{a,c\}$	$\{0,b\}$	$\{0,b\}$	$\alpha_1 \equiv \beta_1 \equiv \gamma_1 \equiv n \equiv 1$
$\{0,b\}$	$\{a,c\}$	$\{0,b\}$	$\{a,c\}$	$\alpha_2 \equiv \beta_2 \equiv \gamma_2 \equiv n+1 \equiv 0$
$\{0,b\}$	$\{a,c\}$	$\{a,c\}$	$\{0,b\}$	$\alpha_1+\alpha_2 \equiv \beta_1+\beta_2 \equiv \gamma_1+\gamma_2 \equiv n \equiv 1$
$\{0,b\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$n \equiv 1$
$\{a,c\}$	I	I	I	never

Proof. By Theorem 2, $C_{B_I}(r,s,t,u)$ is a free code of type $(n+1,0)$ and thus it has 2^{2n+2} codewords. Hence, to prove that $C_{B_I}(r,s,t,u)$ is QSD, we only need to show that it is self-orthogonal.

The code $C_{B_I}(r,s,t,u)$ is self-orthogonal if and only if $B_I(r,s,t,u)B_I(r,s,t,u)^T = O$ which occurs if and only if the following are satisfied:

$$\begin{aligned}(n+1)b &= 0,\\ a(r+s\kappa_1+t\kappa_2+u(n-\kappa_1-\kappa_2-1)) &= 0,\\ Q_I(r,s,t,u)Q_I(r,s,t,u)^T &= b(I_n+J).\end{aligned}$$

Note that the first equation is the inner product of the top row with itself. The second equation is the inner product of the top row with any other row. The third equation ensures that the remaining rows are orthogonal to each other.
The first equation requires n to be odd. Therefore, by Lemma 1, κ_1 and κ_2 are even. The other two equations then satisfy

$$\begin{aligned}ar &= 0,\\ Q_I(r,s,t,u)Q_I(r,s,t,u)^T &= b(I_n+J).\end{aligned}$$

Then, by Lemma 8, the code $C_{B_I}(r,s,t,u)$ is self-orthogonal if and only if n is odd, κ_1 and κ_2 are even, $r \in \{0,b\}$, and the following are satisfied:

$$u^2+(s^2+u^2)\gamma_1+(t^2+u^2)\gamma_2 = b, \tag{4.19}$$
$$(s^2+u^2)(\alpha_1+\gamma_1)+(t^2+u^2)(\alpha_2+\gamma_2) = 0, \tag{4.20}$$
$$(s^2+u^2)(\beta_1+\gamma_1)+(t^2+u^2)(\beta_2+\gamma_2) = 0. \tag{4.21}$$

To determine when the codes are self-orthogonal, we need to find the solutions that satisfy Eqs (4.19)–(4.21). We consider four cases depending on the values of the scalars s and u:

Case 1: $s, u \in \{0, b\}$.
Then $s^2 + u^2 = 0$ and Eqs (4.19)–(4.21) reduce to

$$\begin{aligned} t^2\gamma_2 &= b, \\ t^2(\alpha_2 + \gamma_2) &= 0, \\ t^2(\beta_2 + \gamma_2) &= 0. \end{aligned}$$

The equations are satisfied when $t \in \{a, c\}$ and $\alpha_2 \equiv \beta_2 \equiv \gamma_2 \equiv 1 \pmod 2$.
Case 2: $s, u \in \{a, c\}$.
Then $s^2 + u^2 = 0$ and Eqs (4.19)–(4.21) reduce to

$$\begin{aligned} b + (t^2 + b)\gamma_2 &= b, \\ (t^2 + b)(\alpha_2 + \gamma_2) &= 0, \\ (t^2 + b)(\beta_2 + \gamma_2) &= 0. \end{aligned}$$

The equations are satisfied when either $t \in \{a, c\}$, or $t \in \{0, b\}$ together with $\gamma_2 \equiv \alpha_2 \equiv \beta_2 \equiv 0 \pmod 2$.
Case 3: $s \in \{0, b\}$ and $u \in \{a, c\}$.
Then $s^2 + u^2 = b$ and Eqs (4.19)–(4.21) reduce to

$$\begin{aligned} b(1 + \gamma_1) + (t^2 + b)\gamma_2 &= b, \\ b(\alpha_1 + \gamma_1) + (t^2 + b)(\alpha_2 + \gamma_2) &= 0, \\ b(\beta_1 + \gamma_1) + (t^2 + b)(\beta_2 + \gamma_2) &= 0. \end{aligned}$$

If $t \in \{0, b\}$, then $\gamma_1 + \gamma_2 \equiv \alpha_1 + \alpha_2 \equiv \beta_1 + \beta_2 \equiv 0 \pmod 2$. If $t \in \{a, c\}$, then $\gamma_1 \equiv \alpha_1 \equiv \beta_1 \equiv 0 \pmod 2$.
Case 4: $s \in \{a, c\}$ and $u \in \{0, b\}$.
Then $s^2 + u^2 = b$ and Eqs (4.19)–(4.21) reduce to

$$\begin{aligned} b\gamma_1 + t^2\gamma_2 &= b, \\ b(\alpha_1 + \gamma_1) + t^2(\alpha_2 + \gamma_2) &= 0, \\ b(\beta_1 + \gamma_1) + t^2(\beta_2 + \gamma_2) &= 0. \end{aligned}$$

If $t \in \{0, b\}$, then $\gamma_1 \equiv \alpha_1 \equiv \beta_1 \equiv 1 \pmod 2$. If $t \in \{a, c\}$, then $\gamma_1 + \gamma_2 \equiv \alpha_1 + \alpha_2 \equiv \beta_1 + \beta_2 \equiv 1 \pmod 2$. □

Theorem 15. *Every QSD code constructed from a symmetric three-class association scheme by either the pure or the bordered construction is quasi Type IV.*

Proof. Let C denote a QSD code constructed from a symmetric three-class association scheme by either the pure or the bordered construction. Then C is a free code by Theorem 2 and thus C is quasi Type IV by Corollary 1. □

The following remark will assist us in studying the conditions on QSD codes constructed from symmetric association schemes to be Type IV.

Remark 2. Let (X, R) be a symmetric three-class association scheme. From the third condition in Definition 1, for any $i, j, k \in \{0, 1, 2, 3\}$, $p_{ij}^k = p_{ji}^k$ is a constant number which does not depend on the choice of x and y that satisfy $(x, y) \in R_k$. By the symmetry of each R_i, we have

$$
\begin{aligned}
|\{z\in X \mid (x,z)\in R_i \text{ and } (y,z)\in R_j\}| &= |\{z\in X \mid (x,z)\in R_i \text{ and } (z,y)\in R_j\}|\\
&= p_{ij}^k\\
&= p_{ji}^k\\
&= |\{z\in X \mid (x,z)\in R_j \text{ and } (z,y)\in R_i\}|\\
&= |\{z\in X \mid (x,z)\in R_j \text{ and } (y,z)\in R_i\}|.
\end{aligned}
$$

Hence, for $\ell, m \in I$, the number of occurrences of $(\ell\ m)^T$ is the same as the number of occurrences of $(m\ \ell)^T$ in the columns of any two distinct rows of $Q_I(r,s,t,u)$, as defined in the symmetric case. That is, any two rows $\mathbf{x}$ and $\mathbf{y}$ in $Q_I(r,s,t,u)$ satisfy that $n_{\ell,m}(\mathbf{x},\mathbf{y}) = n_{m,\ell}(\mathbf{x},\mathbf{y})$.

Lemma 9. *Let C be a QSD code constructed from a symmetric three-class association scheme by either the pure or the bordered construction. If every row of the generator matrix of C has an even weight, then C is Type IV.*

Proof. By Theorem 3, it suffices to prove that $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j)$ is even for every distinct pair of rows $\mathbf{g}_i$ and $\mathbf{g}_j$ of the generator matrices of $C_{P_I}(r,s,t,u)$ and $C_{B_I}(r,s,t,u)$. Let $\mathbf{q}_i$ be the ith row of the matrix $Q_I(r,s,t,u)$.
In the pure construction, we have $P_I = (aI_n, Q_I(r,s,t,u))$. Hence, $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j) = \mathrm{wt}(\mathbf{q}_i \cap_{\neq} \mathbf{q}_j)$. Let $n_{\ell,m} = n_{\ell,m}(\mathbf{q}_i,\mathbf{q}_j)$. Then, by Remark 2, $n_{\ell,m} = n_{m,\ell}$ and so

$$\mathrm{wt}(\mathbf{q}_i \cap_{\neq} \mathbf{q}_j) = n_{a,b} + n_{a,c} + n_{b,a} + n_{b,c} + n_{c,a} + n_{c,b} = 2(n_{a,b} + n_{a,c} + n_{b,c}).$$

Therefore $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j) = \mathrm{wt}(\mathbf{q}_i \cap_{\neq} \mathbf{q}_j)$ is even. By Theorem 3, $C_{P_I}(r,s,t,u)$ is Type IV.
In the bordered construction, we have

$$B_I(r,s,t,u) = \left(\begin{array}{c|c|c|c} a & 0\cdots 0 & 0 & a\cdots a \\ \hline 0 & & a & \\ \vdots & aI_n & \vdots & Q_I(r,s,t,u) \\ 0 & & a & \end{array}\right).$$

Hence, for $j > 1$,

$$\mathrm{wt}(\mathbf{g}_1 \cap_{\neq} \mathbf{g}_j) = n_b(\mathbf{g}_j) + n_c(\mathbf{g}_j)$$

and for $2 \le i, j \le n+1$,

$$\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j) = \mathrm{wt}(\mathbf{q}_{i-1} \cap_{\neq} \mathbf{q}_{j-1}).$$

By Theorem 14, n is odd. By Lemma 1, κ_1, κ_2, and $n - \kappa_1 - \kappa_2 - 1$ are even. Observe that for $j > 1$

$$\mathrm{wt}(\mathbf{g}_j) = 2 + \mathrm{wt}(r) + \mathrm{wt}(s)\kappa_1 + \mathrm{wt}(t)\kappa_2 + \mathrm{wt}(u)(n - \kappa_1 - \kappa_2 - 1)$$

which is an even number, by assumption. So we must have $r = 0$. Hence, $n_b(\mathbf{g}_j)$ and $n_c(\mathbf{g}_j)$ are equal to any sum of the even values $\{0, \kappa_1, \kappa_2, n - \kappa_1 - \kappa_2 - 1\}$. Hence, $\mathrm{wt}(\mathbf{g}_1 \cap_{\neq} \mathbf{g}_j)$ is even. By the same argument as in the pure construction, $\mathrm{wt}(\mathbf{g}_i \cap_{\neq} \mathbf{g}_j) = \mathrm{wt}(\mathbf{q}_{i-1} \cap_{\neq} \mathbf{q}_{j-1})$ is even. By Theorem 3, $C_{B_I}(r,s,t,u)$ is Type IV. □

Theorem 16. *A QSD $C_{P_I}(r,s,t,u)$ constructed from a symmetric three-class association scheme is Type IV if and only if either $r,s,t,u \in \{0,a,c\}$ or at least one scalar is b and the parameters are as in Table 8, where all congruences are given in $\mathbb{F}_2$ and "$\neq$" means the scalar is different from b.*

Table 8. Conditions for QSD codes over I formed from symmetric three-class association schemes by the pure construction to be Type IV.

r	s	t	u	$C_{P_I}(r,s,t,u)$ is Type IV
$\neq$	$\neq$	$\neq$	b	$n \equiv \kappa_1 + \kappa_2 + 1$
$\neq$	$\neq$	b	$\neq$	$\kappa_2 \equiv 0$
$\neq$	$\neq$	b	b	$n \equiv \kappa_1 + 1$
$\neq$	b	$\neq$	$\neq$	$\kappa_1 \equiv 0$
$\neq$	b	$\neq$	b	$n \equiv \kappa_2 + 1$
$\neq$	b	b	$\neq$	$\kappa_1 \equiv \kappa_2$
$\neq$	b	b	b	$n \equiv 1$
b	$\neq$	$\neq$	$\neq$	never
b	$\neq$	$\neq$	b	$n \equiv \kappa_1 + \kappa_2$
b	$\neq$	b	$\neq$	$\kappa_2 \equiv 1$
b	$\neq$	b	b	$n \equiv \kappa_1$
b	b	$\neq$	$\neq$	$\kappa_1 \equiv 1$
b	b	$\neq$	b	$n \equiv \kappa_2$
b	b	b	$\neq$	$\kappa_1 + \kappa_2 \equiv 1$

Proof. Since $C_{P_I}(r,s,t,u)$ is self-orthogonal, by Lemma 3, $\mathrm{wt}(\mathbf{g}_i) \equiv n_b(\mathbf{g}_i) \pmod 2$ for every row $\mathbf{g}_i$ of the generator matrix $P_I(r,s,t,u)$. The quasi self-duality of $C_{P_I}(r,s,t,u)$, by Theorem 13, requires that at least one scalar is different from b. In Table 9, we compute the congruence of the weights of every row $\mathbf{g}_i$ in all possible cases.

Table 9. The weights of the rows of $P_I(r,s,t,u)$.

r	s	t	u	$\mathrm{wt}(\mathbf{g}_i) \pmod 2$
$\neq$	$\neq$	$\neq$	$\neq$	0
$\neq$	$\neq$	$\neq$	b	$n + \kappa_1 + \kappa_2 + 1$
$\neq$	$\neq$	b	$\neq$	κ_2
$\neq$	$\neq$	b	b	$n + \kappa_1 + 1$
$\neq$	b	$\neq$	$\neq$	κ_1
$\neq$	b	$\neq$	b	$n + \kappa_2 + 1$
$\neq$	b	b	$\neq$	$\kappa_1 + \kappa_2$
$\neq$	b	b	b	$n + 1$
b	$\neq$	$\neq$	$\neq$	1
b	$\neq$	$\neq$	b	$n + \kappa_1 + \kappa_2$
b	$\neq$	b	$\neq$	$\kappa_2 + 1$
b	$\neq$	b	b	$n + \kappa_1$
b	b	$\neq$	$\neq$	$\kappa_1 + 1$
b	b	$\neq$	b	$n + \kappa_2$
b	b	b	$\neq$	$\kappa_1 + \kappa_2 + 1$

From Table 9, it follows that $\mathrm{wt}(\mathbf{g}_i)$ is even if and only if either $r,s,t,u \in \{0,a,c\}$ or at least one scalar is b and the parameters are as in Table 8. By Lemma 9, $C_{P_I}(r,s,t,u)$ is Type IV. The converse follows immediately. □

Theorem 17. *A QSD $C_{B_I}(r,s,t,u)$ constructed from a symmetric three-class association scheme is Type IV if and only if $r = 0$.*

Proof. Since $C_{B_I}(r,s,t,u)$ is QSD, by Theorem 14, we know that $r \in \{0,b\}$.
Assume that $r = 0$. We will prove that $C_{B_I}(r,s,t,u)$ is Type IV. Let $\mathbf{g}_i$ be any arbitrary row of the generator matrix $B_I(0,s,t,u)$ for $1 \le i \le n+1$. Observe that $\mathrm{wt}(\mathbf{g}_1) = n+1$ and $\mathrm{wt}(\mathbf{g}_i) = 2 + \mathrm{wt}(s)\kappa_1 + \mathrm{wt}(t)\kappa_2 + \mathrm{wt}(u)(n-\kappa_1-\kappa_2-1)$ for $i > 1$. Since n is odd in the bordered construction, by Lemma 1, κ_1 and κ_2 are even. Hence, $\mathrm{wt}(\mathbf{g}_i)$ is even for all i. By Lemma 9, $C_{B_I}(0,s,t,u)$ is Type IV.
Now suppose that $r = b$. Then for $i > 1$, $\mathrm{wt}(\mathbf{g}_i) = 3 + \mathrm{wt}(s)\kappa_1 + \mathrm{wt}(t)\kappa_2 + \mathrm{wt}(u)(n-\kappa_1-\kappa_2-1)$. Since n is odd in the bordered construction, by Lemma 1, κ_1 and κ_2 are even and thus $\mathrm{wt}(\mathbf{g}_i)$ is odd. Hence, $C_{B_I}(b,s,t,u)$ is not Type IV. □

Example 4. In Table 10, we present examples of (N,d) QSD codes over I of length N and minimum distance d satisfying the conditions in Theorems 13, 14, 16 and 17 constructed from symmetric three-class association schemes. The corresponding adjacency matrices with length n and parameters as defined in Lemma 7 of such schemes can be found in [15].

Table 10. QSD codes over I constructed from symmetric three-class association schemes.

Construction	(n,κ_1,κ_2)	Code	(N,d)	Type IV
Pure	$(6,1,2)$	$C_{P_I}(a,0,a,a)$	$(12,4)$	yes
	$(7,2,2)$	$C_{P_I}(a,0,0,0)$	$(14,2)$	yes
	$(9,2,2)$	$C_{P_I}(a,b,0,b)$	$(18,2)$	yes
	$(10,2,2)$	$C_{P_I}(b,a,a,a)$	$(20,4)$	no
	$(12,1,5)$	$C_{P_I}(b,0,a,b)$	$(24,4)$	yes
	$(14,3,4)$	$C_{P_I}(a,0,a,0)$	$(28,6)$	yes
	$(16,3,3)$	$C_{P_I}(0,a,b,0)$	$(32,4)$	no
	$(18,2,4)$	$C_{P_I}(c,c,0,c)$	$(36,4)$	yes
	$(26,4,9)$	$C_{P_I}(a,a,0,a)$	$(52,8)$	yes
	$(28,3,12)$	$C_{P_I}(b,a,b,a)$	$(56,8)$	no
	$(32,6,10)$	$C_{P_I}(a,b,a,0)$	$(64,8)$	yes
Bordered	$(7,2,2)$	$C_{B_I}(0,a,a,a)$	$(16,4)$	yes
	$(9,2,2)$	$C_{B_I}(0,a,c,a)$	$(20,4)$	yes
	$(15,2,4)$	$C_{B_I}(0,c,a,b)$	$(32,8)$	yes
	$(19,6,6)$	$C_{B_I}(b,a,a,a)$	$(40,4)$	no
	$(21,4,8)$	$C_{B_I}(0,a,b,c)$	$(44,8)$	yes
	$(27,2,8)$	$C_{B_I}(0,c,a,0)$	$(56,8)$	yes
	$(33,2,10)$	$C_{B_I}(0,c,c,c)$	$(68,4)$	yes

5. Conclusions

In this work, we have described two methods for constructing linear codes over two non-unital rings, denoted E and I, using the adjacency matrices of three-class association schemes. We proved that the two constructions, under some restrictions, yield QSD codes and Type IV codes. Drawing on the rich literature on association schemes with small vertices in [15], many QSD codes and Type IV codes with minimum distance exceeding 4 were constructed. New results related to free codes over the two rings were given. Based on these results and the forms of the two described constructions, we remark that all QSD codes over E from the two constructions can be regarded as QSD codes over I. Consequently, our investigation focused on codes over I.

One possible direction for future research is to use m-class association schemes with $m \geq 4$ to construct QSD codes. Another direction is to consider the construction techniques presented in this paper over different non-unital rings that have been studied in the literature.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Patrick Solé is the Guest Editor of special issue "Mathematical Coding Theory and its Applications" for AIMS Mathematics. Prof. Patrick Solé was not involved in the editorial review and the decision to publish this article.

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