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Research article

On a conjecture for the difference equation $x_{n+1} = 1 + p\frac{x_{n-m}}{x_n^2}$

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Abstract: In [24], E. Tasdemir, et al. proved that the positive equilibrium of the nonlinear discrete equation $x_{n+1} = 1 + p\frac{x_{n-m}}{x_n^2}$ is globally asymptotically stable for $p \in (0, \frac{1}{2})$, locally asymptotically stable for $p \in (\frac{1}{2}, \frac{3}{4})$ and it was conjectured that for any p in the open interval $(\frac{1}{2}, \frac{3}{4})$ the equilibrium is globally asymptotically stable. In this paper, we prove that this conjecture is true for the closed interval $[\frac{1}{2}, \frac{3}{4}]$. In addition, it is shown that for $p \in (\frac{3}{4}, 1)$ the behaviour of the solutions depend on the delay m. Indeed, here we show that in case $m = 1$, there is an unstable equilibrium and an asymptotically stable 2-periodic solution. But, in case $m = 2$, there is an asymptotically stable equilibrium. These results are obtained by using linearisation, a method lying on the well known Perron's stability theorem ([17], p. 18). Finally, a conjecture is posed about the behaviour of the solutions for $m > 2$ and $p \in (\frac{3}{4}, 1)$.

Keywords: difference equations; asymptotic stability; equilibrium; periodic solutions
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1. Introduction

In their book [16], Kulenović and Ladas initiated a systematic study of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2, \cdots$$

for nonnegative real numbers $\alpha, \beta, \gamma, A, B, C$ such that $B + C > 0$ and $\alpha + \beta + \gamma > 0$, and for nonnegative or positive initial conditions x_{-1}, x_0. The periodicity of the solutions of this equation was discussed by Grove and Ladas in [10]. By setting $\alpha = A = C = 0$, we get

$$x_{n+1} = \frac{\beta}{B} + \frac{\gamma}{B}\frac{x_{n-1}}{x_n}, \tag{1.1}$$

an equation studied in several works, for instance, in Amleh et al. [3], Camouzis and Devault [5], Wan-Sheng He et al. [12], which is a special case of

$$x_{n+1} = p + \frac{x_{n-k}}{x_n}.$$

The behaviour of the solutions of this equation as well as of the more general equation

$$x_{n+1} = \alpha + \frac{x_{n-m}^s}{x_n^r}$$

was studied in a great number of papers especially by Stević (see, e.g. [18–22] and the references therein), as well as by Berenhaut and Stević [4] and El-Owaidy [7] and it differs completely from the behaviour of equation

$$y_{n+1} = A + \frac{y_n^p}{y_{n-k}^r}$$

studied e.g. by Stević [23] in the general case for p, r, but for $k = 1$ and by Abu-Saris and Devault in [1] when $p = r = 1$ and k is any positive integer. A more general version of Eq (1.1) is

$$x_{n+1} = \alpha + \frac{x_{n-k}}{f(x_n, \cdots, x_{n-k+1})},$$

investigated in [14, 15]. A basic condition in this situation is that the denominator f does not vanish at $(0, 0, \cdots, 0)$ and so it includes the specific case

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$$

investigated by e.g., El-Metwally et al. [8], Fotiades and Papaschinopoulos [9]. On the other hand, in [11] Hamza and Morsy studied the discrete equation

$$x_{n+1} = A + \frac{x_{n-1}}{x_n^k},$$

where $A > 0$ and $k \in \mathbb{N}$. See, also, Yalcinkaya [25].

One of the results of the present work is that for all p in the closed interval $[\frac{1}{2}, \frac{3}{4}]$, the unique positive equilibrium $\bar{y} = \frac{1}{2}(1 + \sqrt{1 + 4p})$ of the discrete equation

$$E_m : \quad x_{n+1} = 1 + p\frac{x_{n-m}}{x_n^2} \tag{1.2}$$

is globally asymptotically stable, for all values $m = 1, 2, \cdots$. Thus, we give a positive answer to a conjecture posed in a recent paper by Tasdemir, Göcen and Soykan, see [24]. In that work, it is shown that, if $m \geq 1$, then for $0 < p < \frac{3}{4}$, the equilibrium point is locally asymptotically stable, while, if $0 < p < \frac{1}{2}$ the equilibrium is globally asymptotically stable. Also, by using numerical simulations, it was conjectured that for p in the semi-closed interval $[\frac{1}{2}, \frac{3}{4})$ the equilibrium $\bar{y}$ is globally asymptotically stable. Here, we show that generally this conjecture is true and we have global stability for all p in the closed interval $[\frac{1}{2}, \frac{3}{4}]$. Notice that for $p \in (0, 1)$ all solutions of Eq (1.2) with positive initial values are bounded uniformly with initial values in bounded sets and stay greater than 1.

The results about stability for $p \in (0, \frac{3}{4})$ are independent of the delay m. This is not true for the values of p in the interval $(\frac{3}{4}, 1)$. In this paper, we present two more results when p belongs to this interval and $m = 1$, or $m = 2$. In the first case, we show that there exist a locally asymptotically stable 2-periodic solution and an unstable equilibrium point. An explicit example is also presented. For $m = 2$, we show that there is a unique equilibrium which is locally asymptotically stable. These results are shown by using the method by linearisation. Finally, we suggest that for the general value of the delay m the behaviour of the solutions do not change, and, if m is an odd positive integer, then they are as in case $m = 1$, while if m is even, they are as in case $m = 2$.

2. The case $\frac{1}{2} \leq p \leq \frac{3}{4}$

As we said previously, for the values of p in the interval $(0, \frac{1}{2})$ the global asymptotic stability of the equilibrium is proved in [24]. Now, first, we assume that $p \in [\frac{1}{2}, \frac{3}{4}]$. In this section we show the following result:

Theorem 2.1. *The equilibrium point $\bar{y} = \frac{1}{2}(1 + \sqrt{1 + 4p})$ of Eq (1.2) is unique and it is globally asymptotically stable if $\frac{1}{2} \leq p \leq \frac{3}{4}$.*

Proof. Let (x_n) be a solution with positive initial values. Then, we have $x_n > 1$, for all n and moreover

$$x_{n+1} \leq 1 + px_{n-m} \leq 1 + p + p^2 x_{n-2m-1} \leq 1 + p + p^2 + \cdots + p^k x_{n-km-(k-1)} \leq 1 + p + p^2 + \cdots + p^k B,$$

where $B := \max\{x_i : \ i = -m, -m+1, \cdots, 0\}$ and k is the integer part of the number $1 + \frac{n}{m+1}$. This integer is such that

$$1 + \frac{n}{1+m} \geq k > \frac{n}{1+m}.$$

Thus we obtain $-m \leq n - km - (k-1) \leq 0$. Hence, we have

$$x_n \leq 1 + p + p^2 + \cdots + B = \frac{1}{1-p} + B.$$

These facts guarantee that any solution with positive initial values is bounded uniformly with initial values in bounded sets and all its terms stay greater than 1.

Now consider any solution with positive initial values. According to [13], there are full limiting sequences S_n and I_n satisfying Eq (1.2) for all $n \in \mathbb{Z}$ and such that $S := S_0 = \limsup x_n$ and $I := I_0 = \liminf x_n$. This implies that

$$S = 1 + p\frac{S_{n-m}}{S_n^2} \leq 1 + p\frac{S}{I^2} \tag{2.1}$$

and

$$I = 1 + p\frac{I_{n-m}}{I_n^2} \geq 1 + p\frac{I}{S^2}. \tag{2.2}$$

Notice that $p < 1 \leq I \leq S$. From (2.1) and (2.2) we obtain

$$S + p\frac{I}{S} \leq SI \leq I + p\frac{S}{I}.$$

Thus

$$S - I \leq p\frac{S^2 - I^2}{SI},$$

which implies the following cases:

1) $I = S$, which case proves the result, and

2) $I < S$ and

$$1 \leq p(\frac{1}{I} + \frac{1}{S}) < 2p\frac{1}{I}.$$

It is clear that if $p = \frac{1}{2}$, then $I < 1$, which is impossible. Thus, the second case occurs only when $p \in (\frac{1}{2}, \frac{3}{4}]$. So, consider the second case and we shall arrive to a contradiction. Then, we have $1 < I < 2p =: a_1$.

Assume that $p < \frac{3}{4}$. From (2.2) we have

$$a_1(1 - \frac{p}{S^2}) = 2p(1 - \frac{p}{S^2}) > I(1 - \frac{p}{S^2}) \geq 1,$$

and so $S > \sqrt{\frac{pa_1}{a_1 - 1}} =: b_1$. It is easy to see that $b_1 > 1$. From (2.2) we get

$$b_1(1 - \frac{p}{I^2}) < S(1 - \frac{p}{I^2}) \leq 1$$

and therefore

$$I < \sqrt{\frac{pb_1}{b_1 - 1}} =: a_2.$$

Now, if $a_2 \leq 1$, it must be true that $I = 1$. Then, from (2.2) we get $1 \geq 1 + p\frac{1}{S^2}$, which is a contradiction. Thus, $a_2 > 1$ and, since $p > \frac{1}{2}$, we can easily see, that the inequality $a_1 > a_2$ holds. Again, from (2.1), we obtain

$$S > \sqrt{\frac{pa_2}{a_2 - 1}} =: b_2.$$

It is obvious that $b_1 < b_2$. We continue in this way and obtain two sequences (a_n) and (b_n) such that $I < a_n$, $b_n < S$ and

$$a_{n+1} = \sqrt{\frac{pb_n}{b_n - 1}}, \ \ b_n = \sqrt{\frac{pa_n}{a_n - 1}},$$

for all indices n.

Since $p < \frac{3}{4}$ we have $b_1 > a_1$. Hence, $b_1 > a_2$ and so $b_2 > a_2$. It follows that $b_n > a_n$, for all $n = 1, 2, \cdots$ and moreover the sequence (a_n) is decreasing and (b_n) is increasing. Since these sequences are bounded, they converge to some positive reals L_I and L_S respectively, such that $L_I < L_S$. Therefore, we have

$$L_I = \sqrt{\frac{pL_S}{L_S - 1}} \ \text{ and } \ L_S = \sqrt{\frac{pL_I}{L_I - 1}},$$

from which the relations

$$L_I = 1 + p\frac{L_I}{L_S^2}, \ \ L_S = 1 + p\frac{L_S}{L_I^2} \tag{2.3}$$

follow. From these relations we obtain

$$L_S L_I^2 + pL_S^2 = L_S^2 L_I^2 = L_S^2 L_I + pL_I^2.$$

So

$$L_I L_S = p(L_I + L_S).$$

Also, we have

$$L_S^2 L_I - L_S L_I^2 = L_S^2 - L_I^2 - p(L_S - L_I).$$

Thus

$$L_S L_I = L_S + L_I - p.$$

Hence,

$$L_I + L_S = \frac{p}{1-p} \text{ and } L_I L_S = \frac{p^2}{1-p}.$$

Therefore the quantities L_I and L_S are the roots of the quadratic equation

$$z^2 - \frac{p}{1-p} z + \frac{p^2}{1-p} = 0. \tag{2.4}$$

If $p < \frac{3}{4}$, this equation does not have real roots, which implies that in this case Eq (2.4) does not exist. So we must have $I = S$.

If $p = \frac{3}{4}$, then, this equation has equal roots. Thus, $L_I = L_S$, which is impossible. The proof of the theorem is complete.

3. The case $\frac{3}{4} < p < 1$

In this section we shall discuss the asymptotic behaviour of the solutions of Eq (1.2) when $\frac{3}{4} < p < 1$. For our purpose we need to rewrite the equation in a system form. By the use of this form we can extract some results about the existence of periodic solutions of the equation.

Let (x_n) be a solution with positive initial values. To simplify the writings we consider the function

$$\phi(u, v) := 1 + p\frac{u}{v^2}, \quad u, v > 0.$$

In the $(m + 1)$-vector space, define the sequence of vectors

$$y(j)_n := x_{(m+1)n+j-1}, \quad j = 1, 2, \cdots, m + 1, \quad n = 1, 2, \cdots$$

and observe that it satisfies the system of equations

$$\begin{aligned}
y(1)_{n+1} &= \phi(y(1)_n, y(m+1)_n), && (3.1)\\
y(2)_{n+1} &= \phi(y(2)_n, y(1)_{n+1}) = \phi(y(2)_n, \phi(y(1)_n, y(m+1)_n)), && (3.2)\\
y(3)_{n+1} &= \phi(y(3)_n, y(2)_{n+1}) = \phi(y(3)_n, \phi(y(2)_n, \phi(y(1)_n, y(m+1)_n))), && (3.3)\\
&\vdots\\
y(m+1)_{n+1} &= \phi(y(m+1)_n, \phi(y(m)_n, \cdots \phi(y(1)_n, y(m+1)_n))). && (3.4)
\end{aligned}$$

This system can be written in the simple vectorial form

$$Y_{n+1} = H(Y_n), \tag{3.5}$$

where Y_n is the $(m+1)$-vector $(y(1)_n, y(2)_n, \cdots, y(m+1)_n)^T$ and H is the vector valued function with coordinates the right parts of system (3.1)–(3.4).

From here we can show that Eq (1.2) has a $(m+1)$-periodic solution. To do that it is enough to prove that Eq (3.5) admits at least one constant solution. In the $(m+1)$-dimensional space $\mathbb{R}^n$ define the fixed point problem

$$X = F(X), \quad X \in I := [0,1]^{m+1}$$

where $X = (r_1, r_2, \cdots r_{m+1})^T$,

$$F(X) := (\theta(r_{m+1}), \theta(r_1), \cdots, \theta(r_m)$$

and

$$\theta(r) := 1 - pr^2.$$

It is clear that the continuous function F maps the compact connected set I into the set $[1-p,1]^{m+1} \subseteq I$, and therefore, due to the well known Brouwer's fixed point theorem (see, e.g. [2], p. 63), it has a (not necessarily unique) fixed point $Q := (q_1, q_2, \cdots q_{m+1})^T$, which, obviously, belongs to the set $[1-p,1)^{m+1}$. The vector equation $Q = F(Q)$ can be written in the form

$$q_j = \theta(q_{j-1}), \quad j = 1, 2, \cdots, m+1, \tag{3.6}$$

with $q_0 = q_{m+1}$. Solving this system with respect to any of the coordinates of Q, we see that each of them satisfies the equation

$$S(t) := \theta^{(m+1)}(t) = t. \tag{3.7}$$

We set $C := (c_1, c_2, \cdots, c_{m+1})$, where

$$c_j := 1/q_j, \quad j = 1, 2, \cdots, m+1$$

and observe that its coordinates are greater than 1 and they satisfy the algebraic system

$$c_j = 1 + p\frac{c_j}{c_{j-1}^2}, \quad j = 1, 2, \cdots, m+1,$$

where we have set $c_0 = c_{m+1}$. It is clear that the sequence $c_1, c_2, \cdots, c_{m+1}, c_1, c_2, \cdots$ is a $(m+1)$-periodic solution of the original equation.

The vector C is not necessarily unique and its coordinates might be equal. The latter means that the number $m+1$ is not necessarily the least period of C. Notice that, as we shall see later, for $m = 1$, C is a vector of the form $(a,a)^T$, namely, an equilibrium of the equation, or a 2-periodic solution $a, b, a, b, \cdots$ with $a \neq b$. But, for $m = 2$, C is a vector of the form $(a,a,a)^T$, with $a > 1$ and it is unique.

To proceed to our discussion we need to refer to the following result, which is implied from the classical result due to Perron ([17], p. 18):

Theorem 3.1. *([6], p. 311) If T is an n-dimensional differentiable function with fixed point X and J is the Jacobian matrix of T evaluated at X, then X is a locally stable fixed point if all eigenvalues of J have absolute value less than 1. If at least one of these absolute values is strictly greater than 1, the fixed point is unstable.*

In the sequel we shall discuss the cases $m = 0, m = 1, m = 2$ and, finally, we shall give some remarks for the general case.

3.1. The case E_0

It is obvious that in case $m = 0$, Eq (1.2) becomes $x_{n+1} = 1 + \frac{p}{x_n}$, which has the equilibrium $c = \frac{1}{2}(1 + \sqrt{1+4p})$ as a global attractor.

3.2. The case E_1

Here we discuss the behaviour of the solutions of the discrete equation (1.2) in case $m = 1$ and $p \in (\frac{3}{4}, 1)$ and we prove the following result:

Theorem 3.2. *Assume that* $m = 1$. *Then, the system of equations*

$$\alpha = 1 + p\frac{\alpha}{\beta^2}, \quad \beta = 1 + p\frac{\beta}{\alpha^2}, \tag{3.8}$$

has solutions the pair of numbers

$$\alpha = \frac{1}{2}\frac{p}{1-p}(1 + \sqrt{4p-3}), \ \beta = \frac{1}{2}\frac{p}{1-p}(1 - \sqrt{4p-3})$$

and the constant number

$$\gamma = \frac{1}{2}(1 + \sqrt{1+4p}).$$

The pair (α,β) *produces the sequence* $\alpha,\beta,\alpha,\beta,\cdots$, *which is a two-periodic solution of Eq* (1.2) *and it is asymptotically stable. The fixed point* γ *is an unstable point.*

Proof. In this case the constant solution of Eq (3.5) is the vector $(c(1), c(2))^T$, where $c(1) = \alpha$ and $c(2) = \beta$. Then, these numbers satisfy relations (2.3) and so they solve Eq (2.4). Thus, they are equal to the suggested values $\alpha = \frac{1}{2}\frac{p}{1-p}(1 + \sqrt{4p-3})$ and $\beta = \frac{1}{2}\frac{p}{1-p}(1 - \sqrt{4p-3})$. If $\alpha = \beta$, then we have the equation $\alpha = 1 + p\frac{1}{\alpha}$, whose the positive solution is equal to c.

Let (x_n) be a solution of Eq (1.2) with positive initial values. We set $y(1)_n := x_{2n}$ and $y(2)_n := x_{2n+1}$. These sequences satisfy the system

$$y(1)_{n+1} = \phi(y(1)_n, y(2)_n), \ \ y(2)_{n+1} = \phi(y(2)_n, \phi(y(1)_n, y(2)_n)).$$

Obviously this system has the equilibrium (α,β), whose the coordinates are the two roots of Eq (2.4), as well as the number k. Since $p > \frac{3}{4}$ we can easily see that $1 < \beta < \alpha$ and

$$\frac{p}{2(1-p)} < \alpha < \frac{p}{1-p}.$$

Notice that the numbers a, b, which are greater than 1 and satisfy the polynomial equation

$$x^4 - x^3 - p(x^2 - p)^2 = 0.$$

So, by setting $x := \frac{1}{t}$, we see that $\frac{1}{a}$ and $\frac{1}{b}$ satisfy the Eq (3.7) for $m = 1$, namely

$$\theta^2(t) = \theta(\theta(t)) = 0, \tag{3.9}$$

where $t \in (0, 1)$.

Next we set $u_n := y(1)_n$ and $v_n := y(2)_n$ and let z_n stand for the vector $(u_n, v_n)^T$. The sequences $(u_n), (v_n)$ satisfy the system of relations

$$u_{n+1} = 1 + p\frac{u_n}{(v_n)^2}, \quad v_{n+1} = 1 + p\frac{(v_n)^5}{[(v_n)^2 + p(u_n)]^2},$$

which can be written in the form

$$z_{n+1} = \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \phi(u_n, v_n) \\ \phi(v_n, \phi(u_n, v_n)) \end{pmatrix} =: K(z_n).$$

where $f(u, v) := \phi(u, v)$ and

$$g(u, v) := \phi(v, \phi(u, v))$$

are rational functions defined on the open square interval $(0, +\infty)^2$ and such that $f(\alpha,\beta) = \alpha$ and $g(\alpha,\beta) = \beta$. Let A be the Jacobian matrix of the operator K:

$$A := \begin{pmatrix} f_u(\alpha,\beta) & f_v(\alpha,\beta) \\ g_u(\alpha,\beta) & g_v(\alpha,\beta) \end{pmatrix} = \begin{pmatrix} \frac{p}{\beta^2} & \frac{-2(\alpha-1)}{\beta} \\ \frac{-2p^2}{\beta\alpha^3} & \frac{p(5\alpha-4)}{\alpha^3} \end{pmatrix}.$$

We shall show that the matrix A is stable, or equivalently, its spectral radius is less than 1. To prove it, first, we observe that

$$tr(A) = \frac{\alpha^3 - \alpha^2 + p(5\alpha - 4)}{\alpha^3}.$$

Since the function

$$\gamma(\alpha) := \alpha^3 - \alpha^2 + p(5\alpha - 4)$$

satisfies $\gamma(1) = p > 0$ and

$$\gamma'(\alpha) = 3\alpha^2 - 2\alpha + 5p > 0,$$

for all α, (notice that its discriminant is negative), it follows that the quantity $tr(A)$ is positive.

Also, the determinant $|A|$ of the matrix A is equal to

$$|A| = \frac{p^2}{\alpha^2\beta^2},$$

which is positive.

Next, we claim that the characteristic values are real numbers. Indeed, the discriminant of the characteristic equation

$$\lambda^2 - tr(A)\lambda + |A| = 0$$

is equal to

$$tr^2(A) - 4|A|.$$

The fact that this quantity is nonnegative is equivalent to the inequality

$$[(\alpha^2(\alpha - 1) + p(5\alpha - 4)]^2 - 4p\alpha^3(\alpha - 1) \geq 0.$$

Since $\alpha > 1$, the left side of the previous inequality is greater than or equal to $[(\alpha^2(\alpha - 1) - p(5\alpha - 4)]^2$, which is nonnegative and so the claim is proved.

It remains to show that the matrix A is stable. First, we observe that $tr(A) \leq 2$. Indeed, we have

$$2 - tr(A) = 2 - \frac{a^3 - a^2 + p(5a-4)}{a^3} = \frac{a^3 + a^2 - p(5a-4)}{a^3}$$

$$\geq \frac{a^2 + a - 5a + 4}{a^3} = \frac{a^2 - 4a + 4}{a^3} = \frac{(a-2)^2}{a^3} \geq 0.$$

Therefore, the fact that the greater root of the characteristic equation is strictly less than 1 is equivalent to the inequality

$$|A| - tr(A) + 1 > 0,$$

or, equivalently, to the inequality

$$\alpha^2 - 4p\alpha + 3p > 0. \tag{3.10}$$

Replacing a with its value, this inequality is equivalent to

$$6p^2 - \frac{31}{4}p + 3 + \frac{p}{2}(4p-3)^{3/2} > 0.$$

Obviously, this is true since the left side can be written as

$$6(p - \frac{31}{48})^2 + \frac{191}{384} + \frac{p}{2}(4p-3)^{3/2},$$

which is positive for p in the open interval $(\frac{3}{4}, 1)$. Therefore, relation (3.10) is true.

Now, we can apply Theorem 3.1 and the proof is complete as the equilibrium (α, β) is concerned.

Next, we shall check what is going on with the equilibrium γ. In this case, we see that all steps of the previous proof work equally well with γ in the place of α and β, except relation (3.10), which, we shall show, is not satisfied.

Indeed, to see that the inequality

$$[\frac{1 + \sqrt{1+4p}}{2}]^2 - 4p\frac{1 + \sqrt{1+4p}}{2} + 3p > 0$$

is not true, we observe that it is equivalent to

$$1 + 4p > (4p-1)\sqrt{1+4p},$$

or $\sqrt{1+4p} > 4p - 1$, or $p < \frac{3}{4}$, which is not true. This completes the proof of the theorem.

An application

Consider the case $p = 0.8$ and $m = 1$. Then we have $\alpha \approx 2.894$ and $\beta \approx 1.106$ approximately. For the initial values the points $x_0 = x_1 = 1$, the corresponding solution is as in the following matrix:

$x_0 = 1$	$x_1 = 1$	$x_2 = 1.8$	$x_3 = 1.247$	$x_4 = 1.926$	$x_5 = 1.269$
$x_6 = 1.957$	$x_7 = 1.265$	$x_8 = 1.978$	$x_9 = 1.258$	$x_{10} = 1.999$	$x_{11} = 1.251$
$x_{12} = 2.029$	$x_{13} = 1.236$	$x_{14} = 2.062$	$x_{15} = 1.232$	$x_{16} = 2.086$	$x_{17} = 1.226$
$x_{18} = 2.110$	$x_{19} = 1.220$	$x_{20} = 2.134$	$x_{21} = 1.214$	$x_{22} = 2.158$	$x_{23} = 1.206$
$x_{24} = 2.187$	$x_{25} = 1.201$	$x_{26} = 2.213$	$x_{27} = 1.196$	$x_{28} = 2.237$	$x_{29} = 1.191$
$x_{30} = 2.261$	$x_{31} = 1.186$	$x_{32} = 2.285$	$x_{33} = 1.181$	$x_{34} = 2.310$	$x_{35} = 1.177$
$x_{36} = 2.334$	$x_{37} = 1.172$	$x_{38} = 2.359$	$x_{39} = 1.168$	$x_{40} = 2.383$	$x_{41} = 1.164$
$x_{42} = 2.407$	$x_{43} = 1.160$	$x_{44} = 2.431$	$x_{45} = 1.157$	$x_{46} = 2.452$	$x_{47} = 1.153$
$x_{48} = 2.475$	$x_{49} = 1.150$	$x_{50} = 2.497$	$x_{51} = 1.147$	$x_{52} = 2.518$	$x_{53} = 1.144$
$x_{54} = 2.539$	$x_{55} = 1.141$	$x_{56} = 2.560$	$x_{57} = 1.139$	$x_{58} = 2.578$	$x_{59} = 1.137$
$x_{60} = 2.595$	$x_{61} = 1.135$	$x_{62} = 2.611$	$x_{63} = 1.133$	$x_{64} = 2.627$	$x_{65} = 1.131$
$x_{66} = 2.642$	$x_{67} = 1.129$	$x_{68} = 2.658$	$x_{69} = 1.127$	$x_{70} = 2.674$	$x_{71} = 1.126$
$x_{72} = 2.687$	$x_{73} = 1.124$	$x_{74} = 2.701$	$x_{75} = 1.123$	$x_{76} = 2.713$	$x_{77} = 1.122$
$x_{78} = 2.724$	$x_{79} = 1.120$	$x_{80} = 2.737$	$x_{81} = 1.119$	$x_{82} = 2.748$	$x_{83} = 1.118$
$x_{84} = 2.758$	$x_{85} = 1.117$	$x_{86} = 2.768$	$x_{87} = 1.116$	$x_{88} = 2.777$	$x_{89} = 1.115$
$x_{90} = 2.786$	$x_{91} = 1.114$	$x_{92} = 2.795$	$x_{93} = 1.114$	$x_{94} = 2.801$	$x_{95} = 1.113$
$x_{96} = 2.808$	$x_{97} = 1.112$	$x_{98} = 2.816$	$x_{99} = 1.112$	$x_{100} = 2.821$	$x_{101} = 1.111$
$x_{102} = 2.828$	$x_{103} = 1.111$	$x_{104} = 2.832$	$x_{105} = 1.110$	$x_{106} = 2.838$	$x_{107} = 1.110$
$x_{108} = 2.842$	$x_{109} = 1.109$	$x_{110} = 2.848$	$x_{111} = 1.109$	$x_{112} = 2.852$	$x_{113} = 1.109$
$x_{114} = 2.855$	$x_{115} = 1.108$	$x_{116} = 2.860$	$x_{117} = 1.108$	$x_{118} = 2.863$	$x_{119} = 1.108$
$x_{120} = 2.865$	$x_{121} = 1.107$	$x_{122} = 2.870$	$x_{123} = 1.107$	$x_{124} = 2.873$	$x_{125} = 1.107$
$x_{126} = 2.875$	$x_{127} = 1.107$	$x_{128} = 2.876$	$x_{129} = 1.107$	$x_{130} = 2.877$	$x_{131} = 1.106$
$x_{132} = 2.881$	$x_{133} = 1.106$	$x_{134} = 2.884$	$x_{135} = 1.106$	$x_{136} = 2.886$	$x_{137} = 1.106$
$x_{138} = 2.887$	$x_{139} = 1.106$	$x_{140} = 2.888$	$x_{141} = 1.106$	$x_{142} = 2.889$	$x_{143} = 1.106$

Note: The subsequence (x_{2n}) approaches the value $\alpha \approx 2.894$ and the subsequence (x_{2n+1}) approaches the value $\beta \approx 1.106$.

3.3. The case E_2

In this subsection we discuss the behaviour of the solutions of the discrete equation (1.2) where $m = 2$ and $p \in (\frac{3}{4}, 1)$.

In this case Eq (3.5) is formulated by using three variables u_n, v_n, w_n and it takes the form

$$u_{n+1} = \phi(u_n, w_n) \;\; v_{n+1} = \phi(v_n, \phi(u_n, w_n)), \;\; w_{n+1} = \phi(w_n, \phi(v_n, \phi(u_n, w_n))), \tag{3.11}$$

where $\phi(u, v) := 1 + p\frac{u}{v^2}$. A fixed point of this system is a triple a, b, c in the interval $(1, +\infty)$ satisfying the system

$$a = 1 + p\frac{a}{c^2}, \;\; b = 1 + p\frac{b}{a^2}, \;\; c = 1 + p\frac{c}{b^2}.$$

Since $a, b, c > 1$, we can easily see that all numbers a, b, c are smaller than $\frac{1}{1-p}$. Therefore we have

$$\frac{1}{1 - p(1 - p)^2} < a, b, c < \frac{1}{1 - p}. \tag{3.12}$$

Expressing b in terms of a and c in terms of b, substitute in the first equation and obtain the algebraic equation

$$Q(a) := a^8 - a^7 - p[a^4 - p(a^2 - p)^2]^2 = 0.$$

By symmetry, it follows that this equation is, also, satisfied by the numbers b and c. We shall show that this algebraic equation admits a unique root in the interval $(1, +\infty)$. To this end we put $t := \frac{1}{a} \in (0, 1)$ and see that t satisfies Eq (3.7) for $m = 2$, namely the equation

$$\theta^3(t) = \theta(\theta^2(t)) = t. \tag{3.13}$$

Now we observe that the algebraic equation

$$P(t) := p(1 - pt^2)^2 - 1 + \sqrt{\frac{1 - t}{p}} = 0,$$

has a unique root, because the function P vanishes at unique point in the interval $(0, 1)$. Indeed, it holds that

$$P(0) = p - 1 + \frac{1}{\sqrt{p}} > \frac{3}{4} - 1 + 1 > 0, \;\; P(1) = p(1 - p)^2 - 1 < 0,$$

and P is strictly decreasing on the interval $(0, 1)$, since,

$$P'(t) = -4p^2 t(1 - pt^2) - \frac{1}{2\sqrt{p}\sqrt{1 - t}} < 0.$$

This is because $t^2 < 1 < 1/p$.

Therefore the function P admits a unique real root in the interval $[\frac{3}{4}, 1]$. This means that the three numbers a, b, c are equal, obviously, to $\frac{1}{2}(1 + \sqrt{1 + 4p})$.

Now, we set

$$f(u, v, w) := \phi(u, w), \;\; g(u, v, w) := \phi(v, \phi(u, w)), \;\; h(u, v, w) := \phi(w, \phi(v, \phi(u, w)))$$

and let $F := (f, g, h)^T$. It is clear that it holds $F(a, a, a) = (a, a, a)^T$. To proceed, we form the Jacobian matrix A of the vector valued function F at the fixed point $(a, a, a)^T$ and obtain

$$A = \begin{bmatrix} \frac{a-1}{a} & 0 & \frac{-2(a-1)}{a} \\ -\frac{2(a-1)^2}{a^2} & \frac{a-1}{a} & \frac{4(a-1)^2}{a^2} \\ \frac{4(a-1)^3}{a^3} & \frac{-2(a-1)^2}{a^2} & \frac{a-1}{a} - \frac{8(a-1)^3}{a^3} \end{bmatrix}.$$

Next, we check the applicability of Theorem 3.1. This means that we have to show that the spectral radius of the matrix A is less than 1.

Indeed, the characteristic equation of A is

$$\lambda^3 + (8\zeta^2 - 3\zeta)\lambda^2 + 3\zeta^2\lambda - 8\zeta^5 - \zeta^3 = 0, \tag{3.14}$$

where ζ denotes the fraction $\frac{a-1}{a}$, whose maximum interval of existence is equal to $[\frac{1}{3}, \frac{3-\sqrt{5}}{2}] \subset (0, 1)$.

Setting $\lambda = x + iy$ we split the previous equation into the real part and imaginary part and obtain the pair of equations

$$x^3 - 3xy^2 + (8\zeta^2 - 3\zeta)(x^2 - y^2) + 3\zeta^2 x - 8\zeta^5 - \zeta^3 = 0, \tag{3.15}$$

$$y(3x^2 - y^2 + 2(8\zeta^2 - 3\zeta)x + 3\zeta^2) = 0. \tag{3.16}$$

If $y = 0$ we have a real eigenvalue λ_1, which is the real root of the first equation when $y = 0$. Thenit satisfies

$$\lambda_1^3 + (8\zeta^2 - 3\zeta)\lambda_1^2 + 3\zeta^2\lambda_1 - 8\zeta^5 - \zeta^3 = 0. \tag{3.17}$$

Assume that $|\lambda_1| \geq 1$. Since $\frac{3}{8} \in [\frac{1}{3}, \frac{3-\sqrt{5}}{2}]$, the interval $[\frac{1}{3}, \frac{3-\sqrt{5}}{2}]$ is divided into two subintervals:

$$[\frac{1}{3}, \frac{3-\sqrt{5}}{2}] = [\frac{1}{3}, \frac{3}{8}] \cup (\frac{3}{8}, \frac{3-\sqrt{5}}{2}].$$

Let $\zeta \in [\frac{1}{3}, \frac{3}{8}]$. Then, we have $3 > 8\zeta$. Therefore from (3.17), we obtain

$$1 \leq \frac{1}{|\lambda_1|}(3\zeta - 8\zeta^2) + 3\zeta^2\frac{1}{|\lambda_1|^2} + (8\zeta^5 + \zeta^3)\frac{1}{|\lambda_1|^3} \leq 3\zeta - 5\zeta^2 + 8\zeta^5 + \zeta^3$$
$$\leq 3\frac{3}{8} - 5(\frac{1}{3})^2 + 8(\frac{3}{8})^5 + (\frac{3}{8})^3 = \frac{25123}{36864} < 1,$$

a contradiction.

Let $\zeta \in [\frac{3}{8}, \frac{3-\sqrt{5}}{2}]$. Then $3 \leq 8\zeta$ and, if $|\lambda_1| \geq 1$, then

$$1 \leq \frac{1}{|\lambda_1|}(8\zeta^2 - 3\zeta) + 3\zeta^2\frac{1}{|\lambda_1|^2} + (8\zeta^5 + \zeta^3)\frac{1}{|\lambda_1|^3} \leq 11\zeta^2 - 3\zeta + 8\zeta^5 + \zeta^3$$
$$\leq 11(\frac{3-\sqrt{5}}{2})^2 - 3(\frac{3}{8}) + 8(\frac{3-\sqrt{5}}{2})^5 + (\frac{3-\sqrt{5}}{2})^3 = 538 + \frac{3}{8} - (240 + \frac{1}{2})\sqrt{5} < 1,$$

a contradiction. Therefore the real eigenvalue λ_1 of the matrix A has absolute value strictly less than 1.

Next, assume that $y \neq 0$. Then, from (3.16) we have

$$y^2 = 3x^2 + 2(8\zeta^2 - 3\zeta)x + 3\zeta^2. \tag{3.18}$$

Substituting y^2 into (3.15) we obtain the equation

$$B(x) := x^3 + (8\zeta^2 - 3\zeta)x^2 + \zeta^2(16\zeta^2 - 12\zeta + 3)x + \zeta^5 + 3\zeta^4 - \zeta^3 = 0.$$

Here, we observe that $B(0) = \zeta^3(\zeta^2 + 3\zeta - 1) > 0$, because

$$1 > \zeta \geq \frac{1}{3} > \frac{\sqrt{13} - 3}{2}.$$

Also, we have

$$B(-0.2) = \zeta^5 - 0.2\zeta^4 - 0.6\zeta^3 - 0.28\zeta^2 - 0.12\zeta - 0.008.$$

Assuming that $B(-0.2) > 0$, we must have

$$1 > \frac{0.2}{\zeta} + \frac{0.6}{\zeta^2} + \frac{0.28}{\zeta^3} + \frac{0.12}{\zeta^4} + \frac{0.008}{\zeta^5} > \frac{0.2}{\zeta} + \frac{0.6}{\zeta} \geq \frac{0.8}{\frac{3-\sqrt{5}}{2}} = 1.2 + 0.4\sqrt{5} > 1,$$

a contradiction. Thus the function B admits a real root $\hat{x}$ in the interval $(-0.2, 1)$. We claim that such a number $\hat{x}$ with this property is unique. Indeed, we observe that the derivative

$$B'(x) = 3x^2(8\zeta^2 - 3\zeta)x + \zeta^2(16\zeta^2 - 12\zeta + 3)$$

is positive, since its discriminant

$$16\zeta^4 - 16\zeta^3 = 16\zeta^3(\zeta - 1)$$

is negative, for $\zeta \in (0, 1)$. Hence, B is strictly increasing and so our claim is proved.

Next assume that λ_2, λ_3 are the two complex roots of (3.14). Obviously, these numbers are conjugate so they have the same absolute value $(\hat{x}^2 + \hat{y}^2)^{1/2}$, where $\hat{y}$ is any value of the variable y given in (3.18) with $x = \hat{x}$. Now we observe that

$$\hat{x}^2 + \hat{y}^2 = 4\hat{x}^2 + 2(8\zeta^2 - 3\zeta)\hat{x} + 3\zeta^2 \leq 4\hat{x}^2 + 2|8\zeta^2 - 3\zeta||\hat{x}| + 3\zeta^2 \leq 0.16 + 0.4|8\zeta^2 - 3\zeta| + 3\zeta^2.$$

If $\zeta \in (\frac{1}{3}, \frac{3}{8}]$, then

$$\hat{x}^2 + \hat{y}^2 \leq 0.16 + 1.2\zeta - 0.2\zeta^2 < 0.16 + 1.2 \times \frac{3}{8} = 0.61 < 1.$$

Also, if $\zeta > \frac{3}{8}$, then

$$\hat{x}^2 + \hat{y}^2 \leq 0.16 + 6.2\zeta^2 - 1.2\zeta = 6.2(\zeta - \frac{3}{31})^2 - \frac{9}{155} + 0.16$$
$$\leq 6.2(\frac{3 - \sqrt{5}}{2} - \frac{3}{31})^2 - \frac{9}{155} + 0.16 \approx 0.606 < 1.$$

Summarizing all previous results, we see that the three roots of the characteristic equation (3.14) have absolute values strictly less than 1. Hence, the matrix A is stable. After all these derivations, we apply Theorem 3.1 and conclude the following result:

Theorem 3.3. *If $p \in (\frac{3}{4}, 1)$ and $m = 2$ the discrete equation (1.2) admits a unique equilibrium $\bar{y} = \frac{1}{2}(1 + \sqrt{1 + 4p})$ which is asymptotically stable.*

3.4. The case E_m, for $m > 2$.

As we have shown in section 2, in the general case, a constant solution $Q := (q_1, q_2, \cdots, q_{m+1})$ satisfies Eq (3.7), where t is the inverse of any of the coordinates of Q. If $m = 1$ this equation becomes (3.9) and if $m = 2$, it becomes (3.13). In order to obtain the solutions of (3.7), we observe that a solution of equation $\theta(t) = t$ solves Eq (3.7), too, which in turn says that the point $a = (1 + \sqrt{1 + 4p})/2$ is a solution, for all $m = 0, 1, 2, 3, \cdots$ Also, if m is odd, then any solution of (3.9), is a solution of (3.7). Indeed, by using a graphing calculator we can see that if m is even, there is only one positive root of (3.9) and this is a. However, if m is odd, then there are three positive roots of it, as in case $m = 1$. We close this work with the following conjecture:

Conjecture: *For $p \in (\frac{3}{4}, 1)$ the solutions of E_m have a behaviour similar to E_1, for m odd, and similar to E_2, for m even.*

4. Discussion

We are interested in the asymptotic behaviour of equation E_m, when p is a real number in the interval $(0, 1)$. The results are coming to push further the study presented in [24], when the case $p \in (0, 1/2)$ is discussed. The existence of a globally asymptotically stable equilibrium is shown for the case $p \in [\frac{1}{2}, \frac{3}{4}]$ and any m. We give a partial answer to the problem when $p \in (\frac{3}{4}, 1)$ and $m = 1, 2$, but some graphing settings push us to believe that for p in this interval the behaviour of the solutions is exactly like the two cases, we have examined. See Figure 1.

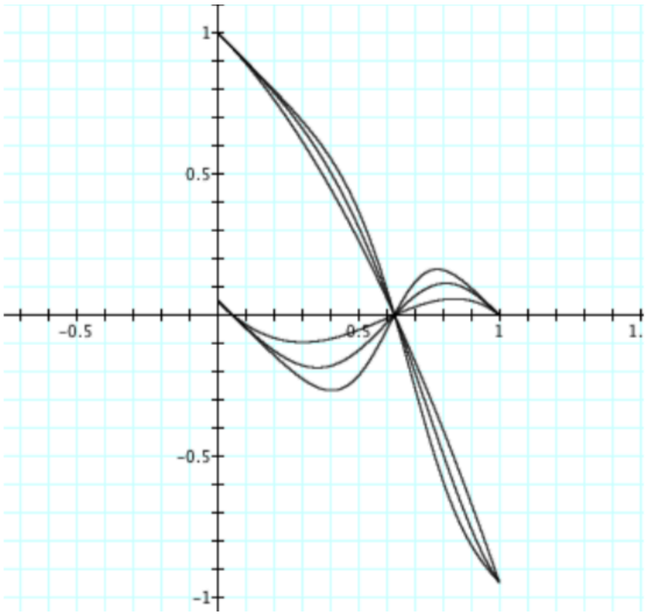

Figure 1. *For* $p \in (\frac{3}{4}, 1)$ *Eq* (3.7) *admits three roots in the interval* $(0, 1)$ *if m is odd and it admits only one root if m is even.*

5. Conclusions

It is proved that for $p \in [\frac{1}{2}, \frac{3}{4}]$ the equilibrium $\bar{y}$ of Eq (1.2) is globally asymptotically stable for solutions with positive initial values. For $p \in (\frac{3}{4}, 1)$ there is no unified behaviour for the stability of solutions, but it depends on the value of the delay m. If $m = 1$, Eq (1.2) admits an unstable equilibrium plus a locally stable 2-periodic solution. However, if $m = 2$, then there is a unique (positive) asymptotically stable equilibrium point.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he does not have competing interests.

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