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Research article

Sums of the higher divisor function of diagonal homogeneous forms in short intervals

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Abstract: Let $d_k(n)$ denote the k-th divisor function. In this paper, we give the asymptotic formula of the sum

$$\sum_{\substack{x-y\leq n_i^r\leq x+y\\ i=1,2,\ldots,l}} d_k(n_1^r+n_2^r+\ldots+n_l^r),$$

where $n_1, n_2, \ldots, n_l \in \mathbb{Z}^+$, $k \geq 2$, $r \geq 3$ and $l > 2^{r-1}$ are integers.

Keywords: circle method; divisor problem; short intervals
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1. Introduction

For any integer $k \geq 2$, let

$$d_k(n) = \sum_{\substack{m_1m_2\ldots m_k=n\\ m_1,m_2,\ldots,m_k\in\mathbb{Z}^+}} 1$$

define the k-th divisor function and $d(n) = d_2(n)$.

Recently, the sums of divisor function about the quadratic form $n_1^2 + n_2^2 + \ldots + n_l^2$ with $l \geq 3$ has drawn researchers' attention. For $k = 2$ and $l = 3$, in 2000, Calderón and de Velasco [1] gave the following asymptotic formula,

$$\sum_{1\leq n_1,n_2,n_3\leq x^{\frac{1}{2}}} d(n_1^2+n_2^2+n_3^2) = \frac{4\zeta(3)}{5\zeta(4)}x^{\frac{3}{2}}\log x + O(x^{\frac{3}{2}}).$$

Later, Guo and Zhai [4] used the classical circle method to solve the case of $k = 2$ and $l = 3$. Furthermore, the error term was eventually upgraded by Zhao [14] to $x\log^7 x$. For $k = 2$ and $l \geq 3$, Zhang [15] proved that

$$\sum_{1\le n_1,\cdots,n_l\le x^{\frac{1}{2}}} d(n_1^2+\ldots+n_l^2)=c_1x^{\frac{l}{2}}\log x+c_2x^{\frac{l}{2}}+O(x^{\frac{l+1}{4}}\log^{l+4}x+x^{\frac{l-2}{2}}\log x),$$

where c_1 and c_2 are constants. Moreover, Lü and Mu [10] considered the nonhomogeneous case, the leading term of

$$\sum_{\substack{1\le n_1,n_2\le x^{\frac{1}{2}}\\ 1\le n_3\le x^{\frac{1}{k}}}} d(n_1^2+n_2^2+n_3^k)$$

is $x^{1+\frac{1}{k}}(1+\log x)$.

In 2016, Sun and Zhang [12] began to take up the higher divisor function. For $k=3$ and $l=3$, they proved that

$$\sum_{1\le n_1,n_2,n_3\le x^{\frac{1}{2}}} d_3(n_1^2+n_2^2+n_3^2)=c_3x^{\frac{3}{2}}\log^2 x+c_4x^{\frac{3}{2}}\log x+c_5x^{\frac{3}{2}}+O(x^{\frac{3}{2}-\frac{1}{8}+\varepsilon}),$$

where c_3, c_4 and c_5 are constants. Later, Hu and Yang [8] considered the cases of $l=4$. For $k\ge 4$ and $l\ge 3$, Hu and Lü [5] investigated that the main term of the sum

$$\sum_{1\le n_1,n_2,\ldots,n_l\le x^{\frac{1}{2}}} d_k(n_1^2+n_2^2+\ldots+n_l^2)$$

is $x^{\frac{l}{2}}\log^{k-1}x$.

Recently, the above results were summarized by Zhou and Ding [17], for $r\ge 2$, $k\ge 2$ and $l>2^{r-1}$,

$$\sum_{1\le n_1,n_2,\ldots,n_l\le x^{\frac{1}{r}}} d_k(n_1^r+n_2^r+\ldots+n_l^r)\sim c_6x^{\frac{l}{r}}\log^{k-1}x,$$

where c_6 is a constant.

Inspired by the above research involving the divisor function, Hu and Yao [7] considered the sums of divisor of the ternary quadratic form in short intervals. It is stated that, for $\theta=\frac{1}{2}+2\varepsilon$ and $y=x^{\theta}$, there holds

$$\sum_{x-y<m_1,m_2,m_3\le x+y} d(m_1^2+m_2^2+m_3^2)=c_7L_1(x,y)+c_8L_2(x,y)+O(y^{3-\varepsilon}),$$

where c_7 and c_8 are constants, and $L_j(x,y)$ $(j=1,2)$ satisfies $L_1(x,y)\asymp y^3\log y$, $L_2(x,y)\asymp y^3$.

Later, Hu and Liu [6] explored the case of a quaternary quadratic form in short intervals. Moreover, Zhang and Li [16] studied the nonhomogenous case in short intervals. They proved that, $y=x^{1-\delta_k+4\varepsilon}$ with $\delta_3=\frac{2}{15}$, $\delta_k=\frac{1}{k(2^{k-2}+1)}$ for $4\le k\le 7$ and $\delta_k=\frac{1}{k(k^2-k+1)}$ for $k\ge 8$, there holds

$$\sum_{\substack{x-y<m_3^k\le x+y\\ x-y<m_i^2\le x+y\\ i=1,2}} d(m_1^2+m_2^2+m_3^k)=c_9\mathfrak{L}_1(x,y)+c_{10}\mathfrak{L}_2(x,y)+O(y^3x^{-2+\frac{1}{k}-\varepsilon}),$$

where c_9 and c_{10} are constants, and $\mathfrak{L}_j(x,y)$ $(j = 1,2)$ satisfies $\mathfrak{L}_1(x,y) \asymp y^3 x^{-2+\frac{1}{k}} \log x$, $\mathfrak{L}_2(x,y) \asymp y^3 x^{-2+\frac{1}{k}}$.

In this paper, we will extend the result of Zhou and Ding [17] in short intervals. We want to consider the asymptotic formula of the sum

$$S_k(x,y) = \sum_{\substack{x-y\le n_i^r\le x+y\\ i=1,2,\ldots,l}} d_k(n_1^r + n_2^r + \ldots + n_l^r),$$

where $y = x^\theta$ with $0 < \theta \le 1$.

To give our result, let

$$S_r(q,a) = \sum_{h=1}^{q} e\left(\frac{ah^r}{q}\right). \tag{1.1}$$

For $0 \le j \le k-1$, we define

$$A_j(q) = \sum_{b=1}^{q} e\left(-\frac{ab}{q}\right) c_{j+1}(b,q), \tag{1.2}$$

where $a > 0$ is a positive integer. We learn from [3, p.372] that for $(a,q) = 1$, $A_j(q)$ is independent of a. The coefficients $c_j(b,q)$ can be written as

$$\sum_{b_1 b_2 \equiv b (mod\ q)} f(b_1)$$

for some function f.

Meanwhile, the number of terms in $c_j(b,q)$ depends only on k. More accurately, the coefficients $c_j(b,q)$ are given explicitly in [2, (2.13)]. Also, from [3, (4.8)] we can get

$$A_j(q) \ll_k q^{-1}. \tag{1.3}$$

Our result is as follows.

Theorem 1.1. *Let $k \ge 2$, $r \ge 3$ and $l > 2^{r-1}$ be integers. We have*

$$S_k(x,y) = \sum_{q=1}^{\infty} \sum_{\substack{1\le a\le q\\ (a,q)=1}} \frac{S_r^l(q,a)}{q^l} \sum_{j=0}^{k-1} A_j(q) \frac{1}{j!} \mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon}),$$

where

$$\theta = \begin{cases} \dfrac{(1-\frac{1}{r})(\frac{r}{2}+\frac{l-2^{r-1}}{k\cdot 2^{r-1}}-\varepsilon)}{\frac{r}{2}+\frac{l-2^{r-1}}{k\cdot 2^{r-1}}-\frac{1}{2}-\varepsilon} & \text{if } 3 \le r \le 7,\\[2ex] \dfrac{(1-\frac{1}{r})(\frac{r}{2}+\frac{l-2r(r-1)}{k\cdot 2r(r-1)}-\varepsilon)}{\frac{r}{2}+\frac{l-2r(r-1)}{k\cdot 2r(r-1)}-\frac{1}{2}-\varepsilon} & \text{if } r \ge 8, \end{cases}$$

and $S_r(q,a)$, $A_j(q)$ are defined in (1.1) and (1.2), respectively. Moreover, $\mathfrak{S}(x,y)$ is defined as

$$\mathfrak{S}(x,y) = \frac{1}{r^l} \sum_{\substack{x-y\le n_i\le x+y\\ i=1,2\ldots l}} \frac{1}{(n_1\ldots n_i)^{1-\frac{1}{r}}} \sum_{\substack{l(x-y)\le n\le l(x+y)\\ n_1+n_2+\ldots+n_l=n}} \log^j n \tag{1.4}$$

satisfying $\mathfrak{S}(x,y) \asymp y^l x^{-l+\frac{l}{r}} \log^j x$.

For this problem, unlike other results, we are interested in how to find a smaller suitable value of θ to make sure that $S_k(x,y)$ has an asymptotic formula. We will establish Theorem 1.1 by the circle method and employ the estimate of the sum of the divisors over the arithmetic progression and some estimates of exponential sum in short intervals.

Notation: Throughout this paper, x always denotes a sufficiently large positive integer and $y = x^\theta$ with $0 < \theta \leq 1$, we take $N_1 = x - y$ and $N_2 = x + y$, $f(x) \ll g(x)$ means that $f(x) = O(g(x))$, $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$. As usual, $\|\alpha\|$ denotes the distance from α to the nearest integer, $e(x) = e^{2\pi i x}$, ε always denotes an arbitrary small positive constant, which may not be the same at different occurrences. For $\lambda \in \mathbb{R}$, we set

$$T_r\left(N_1^{\frac{1}{r}}, N_2^{\frac{1}{r}}\right) = \frac{1}{r} \sum_{N_1 \leq n \leq N_2} n^{\frac{1}{r}-1} e(n\lambda),$$

$$T^*(lN_1, lN_2) = \sum_{lN_1 \leq n \leq lN_2} (\log^j n) e(-n\lambda).$$

We define

$$f_r(\alpha) = \sum_{N_1^{\frac{1}{r}} \leq n \leq N_2^{\frac{1}{r}}} e(\alpha n^r), \ g(\alpha) = \sum_{lN_1 \leq n \leq lN_2} d_k(n) e(-n\alpha). \tag{1.5}$$

2. Lemmas

We list the following lemmas that will be used in subsequent sections.

Lemma 2.1. *For any real numbers α and $\tau \geq 1$, and integers a and q satisfying $(a,q) = 1, 1 \leq q \leq \tau$, α can be written as*

$$\alpha = \frac{a}{q} + \lambda, |\lambda| \leq \frac{1}{q\tau}.$$

Proof. See Pan and Pan [11, Lemma 5.19]. □

Lemma 2.2. *For any $a, q \in \mathbb{Z}$ with $1 \leq a \leq q$, $(a,q) = 1$.*

$$S_r(q,a) = \sum_{h=1}^{q} e\left(\frac{ah^r}{q}\right) \ll q^{1-\frac{1}{r}}.$$

Proof. See Vaughan [13, Theorem 4.2]. □

Lemma 2.3. *Let $r \geq 3$ and N_1, N_2 be defined as above. Then we get*

$$T_r\left(N_1^{\frac{1}{r}}, N_2^{\frac{1}{r}}\right) \ll \min\left(yx^{-1+\frac{1}{r}}, \frac{1}{x^{1-\frac{1}{r}}\|\lambda\|}\right), \tag{2.1}$$

$$T^*(lN_1, lN_2) \ll \log^j(lN_2) \min(y, \frac{1}{\|\lambda\|}), \tag{2.2}$$

$$T^*(lN_1, lN_2) = \int_{lN_1}^{lN_2} e(-\lambda u)(\log^j u) du + O\Big(\log^r(lN_2)(1 + y|\lambda|)\Big). \tag{2.3}$$

Proof. For (2.1), (2.2) and (2.3), see [16, Lemmas 2.5 and 2.9]. □

Lemma 2.4. *For $r \geq 3$, We have*

$$f_r(\alpha) = \frac{S_r(q,a)}{q} T_r\left(N_1^{\frac{1}{r}}, N_2^{\frac{1}{r}}\right) + O\left(q^{1-\frac{1}{r}}(1+|\lambda|y)\right).$$

Proof. See [16, Lemma 2.10]. □

Lemma 2.5. *Let $g(\alpha)$ be defined in* (1.5). *Then there holds*

$$\int_0^1 |g(\alpha)|^2 d\alpha \ll y \log^{k^2-1} x.$$

Proof. By the definition of $g(\alpha)$, we have

$$\begin{aligned}\int_0^1 |g(\alpha)|^2 d\alpha &= \sum_{lN_1 \leq n_1 \leq lN_2} \sum_{lN_1 \leq n_2 \leq lN_2} d_k(n_1) d_k(n_2) \int_0^1 e((n_1 - n_2)\alpha) d\alpha \\ &= \sum_{n \leq lN_2} d_k^2(n) - \sum_{n \leq lN_1} d_k^2(n).\end{aligned}$$

It follows from [9, Theorem 1] that,

$$\sum_{n \leq x} d_k^2(n) = x\left(\log^{k^2-1} x + \log^{k^2-2} x + \ldots + \log x\right) + O\left(\exp\left(e^{c_{11}k^2}\right) x^{1-c_{12}k^{-\frac{4}{3}}+\varepsilon}\right),$$

where $k = o(\sqrt{\log \log x})$ and $c_{11}, c_{12} > 0$ are constants. Then we have

$$\begin{aligned}&\int_0^1 |g(\alpha)|^2 d\alpha \\ =& lN_2 \log^{k^2-1}(lN_2) - lN_1 \log^{k^2-1}(lN_1) + \ldots + lN_2 \log(lN_2) - lN_1 \log(lN_1) \\ &+ O\left(\exp\left(e^{c_{11}k^2}\right) N_2^{1-c_{12}k^{-\frac{4}{3}}+\varepsilon}\right).\end{aligned}$$

Noting the fact that $\log(1+u) \leq u$ for $u \geq 0$, we can deduce that

$$\begin{aligned}&lN_2 \log^{k^2-1}(lN_2) - lN_1 \log^{k^2-1}(lN_1) \\ &= (lN_2 - lN_1) \log^{k^2-1}(lN_2) + lN_1 \left(\log^{k^2-1}(lN_2) - \log^{k^2-1}(lN_1)\right) \\ &\ll y \log^{k^2-1} x + N_1 \log\left(\frac{N_2}{N_1}\right) \log^{k^2-2} x \\ &\ll y \log^{k^2-1} x + N_1 \log\left(1 + \frac{N_2 - N_1}{N_1}\right) \log^{k^2-2} x \\ &\ll y \log^{k^2-1} x.\end{aligned}$$

Therefore, by similar arguments, it's not hard to derive that

$$\int_0^1 |g(\alpha)|^2 d\alpha \ll y \log^{k^2-1} x.$$

□

Lemma 2.6. *For $r \geq 2$ and $f_r(\alpha)$ defined in* (1.5)*, we have*

$$\int_0^1 |f_r(\alpha)|^{2^r} d\alpha \ll (yx^{-1+\frac{1}{r}})^{2^r-r+\varepsilon}.$$

Proof. The proof is similar to that of [13, Lemma 2.5]. We notice that the key of proof of [13, Lemma 2.5] depends on the summation interval. It is showed that the only difference between these two proofs is the length of interval and easily to get $N_2^{\frac{1}{r}} - N_1^{\frac{1}{r}} = (x+y)^{\frac{1}{r}} - (x-y)^{\frac{1}{r}} \asymp yx^{-1+\frac{1}{r}}$. This completes the proof of Lemma 2.6. □

Lemma 2.7. *Suppose that $(a, q) = 1$, $q \leq Q \leq y^{\frac{1}{k}}$. We have*

$$g(\alpha) = \sum_{j=0}^{k-1} A_j(q)I_j(\lambda) + O(Q^{k+\varepsilon} + y^{\eta+\varepsilon}Q),$$

where $\eta = \frac{k-1}{k+1}$ for $2 \leq k \leq 3$, $\eta = \frac{k-1}{k+2}$ for $k \geq 4$, and $A_j(q)$ is defined as in (1.2).

Proof. When $2 \leq k \leq 3$, by the definition of (1.5), we write

$$\begin{aligned} g(\alpha) &= \sum_{lN_1 \leq n \leq lN_2} d_k(n)e(-\alpha n) \\ &= \sum_{b=1}^{q} e\left(\frac{ab}{q}\right) \sum_{\substack{lN_1 \leq n \leq lN_2 \\ n \equiv b(\mod q)}} d_k(n)e(-n\lambda) \\ &= \sum_{b=1}^{q} e\left(\frac{ab}{q}\right) \int_{lN_1}^{lN_2} e(-\lambda u)d\left(D_k(u; b, q)\right), \end{aligned}$$

where

$$D_k(u; b, q) = \sum_{\substack{1 \leq n \leq u \\ n \equiv b(\mod q)}} d_k(n).$$

According to the divisor problem for arithmetic progressions, we utilize it to isolate the main term and estimate the remainder. For the sake of simplicity, we denote

$$D_k(u; b, q) = M_k(u; b, q) + \triangle_k(u; b, q).$$

We learn from [2, Theorem 1] that main term has the form

$$M_k(u; b, q) = \sum_{j=0}^{k-1} c_{j+1}(b, q)L_j(u),$$

where $L_j(u)$ is an antiderivative of $\log^j u/j!$. Thus, the contribution of main term to $g(\alpha)$ can be written as

$$g(\alpha) = \sum_{j=0}^{k-1} A_j(q)I_j(\lambda),$$

where $A_j(q)$ is defined as in (1.2), and $I_j(\lambda)$ is defined as in (2.4). After integrating by parts, one has

$$I_j(\lambda) = \int_{lN_1}^{lN_2} e(-\lambda u)\frac{\log^j u}{j!}du \ll y^{\varepsilon}\min(y, \lambda^{-1}). \tag{2.4}$$

It follows from [3, (4.4)] that the contribution from error term is bounded by

$$O(Q^{k+\varepsilon} + y^{\eta+\varepsilon}Q),$$

when $k \geq 4$, the proof is similar, so we omit it.

□

3. Proof of Theorem 1.1

In order to apply the circle method, we set

$$x^{\varepsilon} \ll Q < \tau,\ Q\tau \asymp yx^{1-\frac{1}{r}},\ Q = y^{\frac{1}{k}}x^{\frac{1}{kr}-\frac{1}{k}}. \tag{3.1}$$

By Lemma 2.1, each $\alpha \in I := \left[\frac{1}{\tau}, 1+\frac{1}{\tau}\right)$ can be written as

$$\alpha = \frac{a}{q} + \lambda, |\lambda| \leq \frac{1}{q\tau}$$

for two positive integers a, q with $1 \leq a \leq q \leq \tau$ and $(a, q) = 1$.

We define the major arcs $\mathfrak{M}$ and $\mathfrak{m}$ as follows:

$$\mathfrak{M} = \bigcup_{q\leq Q}\ \bigcup_{\substack{1\leq a\leq q\\(a,q)=1}} \mathfrak{M}(q,a), \mathfrak{m} = I\setminus\mathfrak{M},$$

where $\mathfrak{M}(q,a) = \left\{\alpha : \alpha = \frac{a}{q} + \lambda, |\lambda| \leq \frac{1}{q\tau}\right\}$.

It is obvious that the major arcs $\mathfrak{M}$ are actually disjoint when $1 < Q < \frac{\tau}{2}$. Noting that $f_r(\alpha)$ and $g(\alpha)$ are defined in (1.5), we obtain

$$\begin{aligned} S_k(x,y) &= \int_{\frac{1}{\tau}}^{1+\frac{1}{\tau}} f_r^l(\alpha)\, g(\alpha)\, d\alpha \\ &= \int_{\mathfrak{M}} f_r^l(\alpha)\, g(\alpha)\, d\alpha + \int_{\mathfrak{m}} f_r^l(\alpha)\, g(\alpha)\, d\alpha. \end{aligned}$$

First, we deal with the integrals over the minor arcs.

Lemma 3.1. *Suppose $\alpha \in \mathfrak{m}$ with $Q < q \leq \tau$.*

- For $3 \leq r \leq 7$, we have

$$\int_{\mathfrak{m}} f_r^l(\alpha)\, g(\alpha)\, d\alpha \ll y^{l-\frac{r}{2}+\frac{1}{2}}x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon}(Q^{\frac{2^{r-1}-l}{2^{r-1}}} + y^{\frac{2^{r-1}-l}{2^{r-1}}}x^{(\frac{1}{r}-1)(\frac{2^{r-1}-l}{2^{r-1}})}).$$

- For $r \geq 8$, we have

$$\int_{\mathfrak{m}} f_r^l(\alpha)\, g(\alpha)\, d\alpha \ll y^{l-\frac{r}{2}+\frac{1}{2}}x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon}(Q^{\frac{2r(r-1)-l}{2r(r-1)}} + y^{\frac{2r(r-1)-l}{2r(r-1)}}x^{(\frac{1}{r}-1)(\frac{2r(r-1)-l}{2r(r-1)})}).$$

Proof. The Cauchy's inequality gives us that

$$\int_{\mathfrak{m}} f_r^l(\alpha)\, g(\alpha)\, d\alpha \ll \sup_{\alpha\in\mathfrak{m}} |f_r(\alpha)|^{l-2^{r-1}} \left(\int_0^1 |f_r(\alpha)|^{2^r} d\alpha\right)^{\frac{1}{2}} \left(\int_0^1 g^2(\alpha) d\alpha\right)^{\frac{1}{2}}.$$

We now use the result of [16] and get

$$\begin{cases} \sup\limits_{\alpha\in\mathfrak{m}} f_r(\alpha) \ll y x^{-1+\frac{1}{r}+\varepsilon}(Q^{-1} + y^{-1}x^{1-\frac{1}{r}})^{\frac{1}{2^{r-1}}}, & \text{if } 3 \le \text{r} \le 7, \\ \sup\limits_{\alpha\in\mathfrak{m}} f_r(\alpha) \ll y x^{-1+\frac{1}{r}+\varepsilon}(Q^{-1} + y^{-1}x^{1-\frac{1}{r}})^{\frac{1}{2r(r-1)}}, & \text{if r} \ge 8. \end{cases} \tag{3.2}$$

From that Lemma 2.5, Lemma 2.6 and (3.2), we get result of Lemma 3.1. □

3.1. The major arcs

By the definition of major arcs, we clearly have

$$\begin{aligned} \int_{\mathfrak{M}} f_r^l(\alpha)\, g(\alpha)\, d\alpha &= \sum_{q\le Q} \sum_{\substack{1\le a\le q \\ (a,q)=1}} \int_{\mathfrak{M}(q,a)} f_r^l(\alpha) g(\alpha) d\alpha \\ &= \sum_{q\le Q} \sum_{\substack{1\le a\le q \\ (a,q)=1}} \int_{|\lambda|\le \frac{1}{q\tau}} f_r^l(\frac{a}{q}+\lambda) g(\frac{a}{q}+\lambda) d\lambda. \end{aligned} \tag{3.3}$$

By (2.1), Lemmas 2.2 and 2.4, we can conclude that

$$\begin{aligned} f_r^l(\alpha) =& \frac{S_r^l(q,a)}{q^l} T_r^l(N_1^{\frac{1}{r}}, N_2^{\frac{1}{r}}) \\ &+ O\left(q^{1-\frac{l}{r}} \min\left(y^{l-1}x^{(-1+\frac{1}{r})(l-1)}, \frac{1}{x^{(1-\frac{1}{r})(l-1)}\|\lambda\|^{l-1}}\right)(1+|\lambda|y)\right). \end{aligned}$$

This combined with Lemma 2.7 gives that

$$f_r^l(\alpha) g(\alpha) = \mathscr{U}(q,\lambda) + O\left(\sum\nolimits_1 + \sum\nolimits_2\right),$$

where

$$\mathscr{U}(q,\lambda) = \frac{S_r^l(q,a)}{q^l} T_r^l(N_1^{\frac{1}{r}}, N_2^{\frac{1}{r}}) \sum_{j=0}^{k-1} A_j(q) I_j(\lambda),$$

and

$$\sum\nolimits_1 = \left|f_r^l(\alpha)\right| (Q^{k+\varepsilon} + y^{\eta+\varepsilon} Q),$$

$$\sum\nolimits_2 = \left(q^{1-\frac{l}{r}} \min(y^{l-1}x^{(-1+\frac{1}{r})(l-1)}, \frac{1}{x^{(1-\frac{1}{r})(l-1)}\|\lambda\|^{l-1}})(1+|\lambda|y)\right) \left|\sum_{j=0}^{k-1} A_j(q) I_j(\lambda)\right|.$$

Taking these into (3.3), we obtain

$$\int_{\mathfrak{M}} f_r^l(\alpha)\, g(\alpha)\, d\alpha = \sum_{q\le Q} \sum_{\substack{1\le a\le q \\ (a,q)=1}} \int_{|\lambda|\le \frac{1}{q\tau}} \mathscr{U}(q,\lambda) d\lambda + \mathscr{U}_1 + \mathscr{U}_2,$$

where the error terms $\mathscr{U}_i (i = 1, 2)$ satisfy

$$\mathscr{U}_i \ll \sum_{q\le Q} \sum_{\substack{1\le a\le q\\(a,q)=1}} \int_{|\lambda|\le\frac{1}{q\tau}} \sum\nolimits_i d\lambda.$$

3.1.1. Estimate of $\mathscr{U}_1$

The estimate of $\mathscr{U}_1$ will be separated into two cases.

If $2^{r-1} < l < 2^r$, it can be obtained by Hölder's inequality and Lemma 2.6 that

$$\begin{aligned}
&\sum_{q\le Q} \sum_{\substack{1\le a\le q\\(a,q)=1}} \int_{|\lambda|\le\frac{1}{q\tau}} \sum\nolimits_1 d\lambda \\
&= (Q^{k+\varepsilon} + y^{\eta+\varepsilon}Q) \int_{\mathfrak{M}} |f_r(\alpha)|^l d\alpha \\
&\ll (Q^{k+\varepsilon} + y^{\eta+\varepsilon}Q)\left(\int_0^1 |f_r(\alpha)|^{2^r} d\alpha\right)^{\frac{l}{2^r}} \\
&\ll y^{(1-\frac{r}{2^r})l} x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon} Q^{k+\varepsilon} + y^{(1-\frac{r}{2^r})l+\eta} x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon} Q.
\end{aligned}$$

If $l \ge 2^r$, by Lemma 2.6, we have

$$\begin{aligned}
\sum_{q\le Q} \sum_{\substack{1\le a\le q\\(a,q)=1}} \int_{|\lambda|\le\frac{1}{q\tau}} \sum\nolimits_1 d\lambda &= (Q^{k+\varepsilon} + y^{\eta+\varepsilon}Q) \int_{\mathfrak{M}} |f_r(\alpha)|^l d\alpha \\
&\ll (Q^{k+\varepsilon} + y^{\eta+\varepsilon}Q) \sup_{\alpha\in I} |f_r(\alpha)|^{l-2^r} \left(\int_0^1 |f_r(\alpha)|^{2^r} d\alpha\right) \\
&\ll y^{l-r} x^{(l-r)(\frac{1}{r}-1)+\varepsilon} Q^{k+\varepsilon} + y^{l-r+\eta} x^{(l-r)(\frac{1}{r}-1)+\varepsilon} Q.
\end{aligned}$$

3.1.2. Estimate of $\mathscr{U}_2$

By (1.3) and (2.4), we have

$$\begin{aligned}
&\sum_{q\le Q} \sum_{\substack{1\le a\le q\\(a,q)=1}} \int_{|\lambda|\le\frac{1}{q\tau}} \sum\nolimits_2 d\lambda \\
&\ll y^{l-1} x^{(\frac{1}{r}-1)(l-1)} \sum_{q\le Q} q^{1-\frac{l}{r}} \sum_{\substack{1\le a\le q\\(a,q)=1}} \sum_{j=0}^{k-1} |A_j(q)| \int_{|\lambda|\le\frac{1}{q\tau}} |I_j(\lambda)|1 + \lambda y| d\lambda \\
&\ll y^l x^{(\frac{1}{r}-1)(l-1)} \tau^{-1} y^{\varepsilon} \sum_{q\le Q} q^{-\frac{l}{r}} \\
&\ll y^l x^{(\frac{1}{r}-1)(l-1)+\varepsilon} \tau^{-1} Q^{-\frac{l}{r}+1}.
\end{aligned}$$

3.1.3. Major arcs for $\mathscr{U}(q,\lambda)$

$$\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\int_{|\lambda|\le\frac{1}{q\tau}}\mathscr{U}(q,\lambda)d\lambda$$
$$=\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\frac{S_r^l(q,a)}{q^l}\sum_{j=0}^{k-1}A_j(q)\int_{|\lambda|\le\frac{1}{q\tau}}I_j(\lambda)T_r^l(N_1^{\frac{1}{r}},N_2^{\frac{1}{r}})d\lambda.$$

By means of (2.3) and (2.4), we have

$$I_j(\lambda)T_r^l(N_1^{\frac{1}{r}},N_2^{\frac{1}{r}})=\frac{1}{j!}T^*(lN_1,lN_2)\cdot T_r^l(N_1^{\frac{1}{r}},N_2^{\frac{1}{r}})+O\Big(|T_r(N_1^{\frac{1}{r}},N_2^{\frac{1}{r}})|^l\log^j(lN_2)(1+y|\lambda|)\Big).$$

First, according to (1.3), (2.1) and Lemma 2.2, the contribution of the O-term to the integrals over the major arcs is

$$\ll\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\frac{S_r^l(q,a)}{q^l}\sum_{j=0}^{k-1}A_j(q)\Big\{\int_0^{\frac{1}{y}}y^lx^{-l+\frac{l}{r}}\log^j(lN_2)(1+y|\lambda|)d\lambda$$
$$+\int_{\frac{1}{y}}^{\frac{1}{q\tau}}x^{-l+\frac{l}{r}}\lambda^{-l}\log^j(lN_2)(1+y|\lambda|)d\lambda\Big\}$$
$$\ll\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\frac{S_r^l(q,a)}{q^l}\sum_{j=0}^{k-1}A_j(q)x^{-l+\frac{l}{r}+\varepsilon}y^{l-1}\ll y^{l-1}x^{-l+\frac{l}{r}+\varepsilon}Q^{-\frac{l}{r}+1}. \tag{3.4}$$

Hence, combining (3.4) and integrating over the major arcs, we obtain

$$\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\int_{|\lambda|\le\frac{1}{q\tau}}\mathscr{U}(q,\lambda)d\lambda$$
$$=\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\frac{S_r^l(q,a)}{q^l}\sum_{j=0}^{k-1}A_j(q)\frac{1}{j!}\int_{|\lambda|\le\frac{1}{q\tau}}T^*(lN_1,lN_2)T_r^l(N_1^{\frac{1}{r}},N_2^{\frac{1}{r}})d\lambda$$
$$+O(y^{l-1}x^{-l+\frac{l}{r}+\varepsilon}Q^{-\frac{l}{r}+1}). \tag{3.5}$$

In the integral of above identity, we extend the interval to $[-\frac{1}{2},\frac{1}{2}]$. By (2.1) and (2.2)

$$\int_{\frac{1}{q\tau}}^{\frac{1}{2}}T^*(lN_1,lN_2)\cdot T_r^l(N_1^{\frac{1}{r}},N_r^{\frac{1}{r}})d\lambda\ll\int_{\frac{1}{q\tau}}^{\frac{1}{2}}x^{\frac{l}{r}-l}(\log x)\lambda^{-l-1}d\lambda$$
$$\ll x^{\frac{l}{r}-l}q^l\tau^l(\log x). \tag{3.6}$$

Noting the condition (1.3), (3.1) and Lemma 2.2, the contribution of the above upper bound to the integrals over the major arcs is

$$\ll \sum_{q\le Q} x^{\frac{l}{r}-l} q^{l-\frac{l}{r}} \tau^l (\log x) \ll x^{\frac{l}{r}-l}\tau^l Q^{l-\frac{l}{r}+1}\log x \ll y^l x^{\frac{l}{r}-l-\varepsilon}. \tag{3.7}$$

From (3.4)–(3.7), we have

$$\begin{aligned}
&\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\int_{|\lambda|\le\frac{1}{q\tau}} \mathscr{U}(q,\lambda)d\lambda\\
&=\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\frac{S_r^l(q,a)}{q^l}\sum_{j=0}^{k-1}A_j(q)\frac{1}{j!}\mathfrak{S}(x,y)\\
&+O(y^{l-1}x^{-l+\frac{l}{r}+\varepsilon}Q^{-\frac{l}{r}+1})\\
&+O(y^l x^{\frac{l}{r}-l-\varepsilon}),
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{S}(x,y)&=\int_{-\frac12}^{\frac12} T^*(lN_1,lN_2)\cdot T_r^l(N_1^{\frac1r},N_2^{\frac1r})d\lambda\\
&=\frac{1}{r^l}\sum_{\substack{N_1\le n_i\le N_2\\ i=1,2\ldots l}}\frac{1}{(n_1\ldots n_i)^{1-\frac1r}}\sum_{\substack{lN_1\le n\le lN_2\\ n_1+n_2+\ldots+n_l=n}}\log^j n\\
&\asymp \log^j x\sum_{\substack{N_1\le n_i\le N_2\\ i=1,2\cdots l}}\frac{1}{(n_1\ldots n_i)^{1-\frac1r}}\asymp y^l x^{-l+\frac{l}{r}}\log^j x.
\end{aligned}$$

Therefore, we have established the following identity

$$\begin{aligned}
&\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\int_{|\lambda|\le\frac{1}{q\tau}} \mathscr{U}(q,\lambda)d\lambda\\
&=\sum_{q=1}^{\infty}\sum_{\substack{1\le a\le q\\(a,q)=1}}\frac{S_r^l(q,a)}{q^l}\sum_{j=0}^{k-1}A_j(q)\frac{1}{j!}\mathfrak{S}(x,y)+O\Big(y^l x^{-l+\frac{l}{r}}(\log^j x)\sum_{q>Q}q^{-\frac{l}{r}}\Big)\\
&+O(y^{l-1}x^{-l+\frac{l}{r}+\varepsilon}Q^{-\frac{l}{r}+1})+O(y^l x^{-l+\frac{l}{r}-\varepsilon})\\
&=\sum_{q=1}^{\infty}\sum_{\substack{1\le a\le q\\(a,q)=1}}\frac{S_r^l(q,a)}{q^l}\sum_{j=0}^{k-1}A_j(q)\frac{1}{j!}\mathfrak{S}(x,y)+O(y^l x^{-l+\frac{l}{r}-\varepsilon}).
\end{aligned}\tag{3.8}$$

By §3.1.1, §3.1.2 and (3.8), we get the following lemma.

Lemma 3.2. *For $\alpha\in\mathfrak{M}$, let $k\ge 2$, $r\ge 3$, $l>2^{r-1}$ be integers.*

- For $2^{r-1} < l < 2^r$, we have

$$\int_{\mathfrak{M}} f_r^l(\alpha)\, g(\alpha)\, d\alpha = \sum_{q=1}^{\infty} \sum_{\substack{1\le a\le q\\(a,q)=1}} \frac{S_r^l(q,a)}{q^l} \sum_{j=0}^{k-1} A_j(q) \frac{1}{j!} \mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon}) +$$
$$+ O(y^{(1-\frac{r}{2^r})l} x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l} Q^{k+\varepsilon}) + O(y^{(1-\frac{r}{2^r})l+\eta} x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon} Q)$$
$$+ O(y^l x^{(\frac{1}{r}-1)(l-1)+\varepsilon} \tau^{-1} Q^{-\frac{l}{r}+1}).$$

- For $l \ge 2^r$, we have

$$\int_{\mathfrak{M}} f_r^l(\alpha)\, g(\alpha)\, d\alpha = \sum_{q=1}^{\infty} \sum_{\substack{1\le a\le q\\(a,q)=1}} \frac{S_r^l(q,a)}{q^l} \sum_{j=0}^{k-1} A_j(q) \frac{1}{j!} \mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon}) +$$
$$+ O(y^{l-r} x^{(l-r)(\frac{1}{r}-1)+\varepsilon} Q^{k+\varepsilon}) + O(y^{l-r+\eta} x^{(l-r)(\frac{1}{r}-1)+\varepsilon} Q)$$
$$+ O(y^l x^{(\frac{1}{r}-1)(l-1)+\varepsilon} \tau^{-1} Q^{-\frac{l}{r}+1}),$$

where $S_r(q,a)$, $A_j(q)$ and $\mathfrak{S}(x,y)$ are defined in (1.1), (1.2) and (1.4), respectively.

4. Completion of the proof

Finally, combining Lemma 3.1, Lemma 3.2 and noting that $\eta = \frac{k-1}{k+1}$ for $2 \le k \le 3$ and $\eta = \frac{k-1}{k+2}$ for $k \ge 4$ by condition of Lemma 2.7, we can get the following that.

Case 1. For $2^{r-1} < l < 2^r$, $2 \le k \le 3$, $3 \le r \le 7$,

$$S_k(x,y) = \sum_{q=1}^{\infty} \sum_{\substack{1\le a\le q\\(a,q)=1}} \frac{S_r^l(q,a)}{q^l} \sum_{j=0}^{k-1} A_j(q) \frac{1}{j!} \mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon})$$
$$+ O(y^{(1-\frac{r}{2^r})l} x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon} Q^{k+\varepsilon}) + O(y^{(1-\frac{r}{2^r})l+\frac{k-1}{k+1}} x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon} Q)$$
$$+ O(y^{l-\frac{r}{2}+\frac{1}{2}} x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon} (Q^{\frac{2^{r-1}-l}{2^{r-1}}} + y^{\frac{2^{r-1}-l}{2^{r-1}}} x^{(\frac{1}{r}-1)(\frac{2^{r-1}-l}{2^{r-1}})}))$$
$$+ O(y^l x^{(\frac{1}{r}-1)(l-1)+\varepsilon} \tau^{-1} Q^{-\frac{l}{r}+1}).$$

Case 2. For $2^{r-1} < l < 2^r$, $k \ge 4$, $3 \le r \le 7$,

$$S_k(x,y) = \sum_{q=1}^{\infty} \sum_{\substack{1\le a\le q\\(a,q)=1}} \frac{S_r^l(q,a)}{q^l} \sum_{j=0}^{k-1} A_j(q) \frac{1}{j!} \mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon})$$
$$+ O(y^{(1-\frac{r}{2^r})l} x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon} Q^{k+\varepsilon}) + O(y^{(1-\frac{r}{2^r})l+\frac{k-1}{k+2}} x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon} Q)$$
$$+ O(y^{l-\frac{r}{2}+\frac{1}{2}} x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon} (Q^{\frac{2^{r-1}-l}{2^{r-1}}} + y^{\frac{2^{r-1}-l}{2^{r-1}}} x^{(\frac{1}{r}-1)(\frac{2^{r-1}-l}{2^{r-1}})}))$$
$$+ O(y^l x^{(\frac{1}{r}-1)(l-1)+\varepsilon} \tau^{-1} Q^{-\frac{l}{r}+1}).$$

Case 3. For $l \ge 2^r$, $2 \le k \le 3$, $3 \le r \le 7$,

$$S_k(x,y) = \sum_{q=1}^{\infty} \sum_{\substack{1\le a\le q\\(a,q)=1}} \frac{S_r^l(q,a)}{q^l} \sum_{j=0}^{k-1} A_j(q) \frac{1}{j!} \mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon})$$

$$+ O(y^{l-r}x^{(l-r)(\frac{1}{r}-1)+\varepsilon}Q^{k+\varepsilon}) + O(y^{l-r+\frac{k-1}{k+1}}x^{(l-r)(\frac{1}{r}-1)+\varepsilon}Q)$$
$$+ O(y^{l-\frac{r}{2}+\frac{1}{2}}x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon}(Q^{\frac{2^{r-1}-l}{2^{r-1}}} + y^{\frac{2^{r-1}-l}{2^{r-1}}}x^{(\frac{1}{r}-1)(\frac{2^{r-1}-l}{2^{r-1}})}))$$
$$+ O(y^{l}x^{(\frac{1}{r}-1)(l-1)+\varepsilon}\tau^{-1}Q^{-\frac{l}{r}+1}).$$

Case 4. For $l \geq 2^r$, $k \geq 4$, $3 \leq r \leq 7$,

$$S_k(x,y) = \sum_{q=1}^{\infty}\sum_{\substack{1\leq a\leq q\\(a,q)=1}}\frac{S_r^l(q,a)}{q^l}\sum_{j=0}^{k-1}A_j(q)\frac{1}{j!}\mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon})+$$
$$+ O(y^{l-r}x^{(l-r)(\frac{1}{r}-1)+\varepsilon}Q^{k+\varepsilon}) + O(y^{l-r+\frac{k-1}{k+2}}x^{(l-r)(\frac{1}{r}-1)+\varepsilon}Q)$$
$$+ O(y^{l-\frac{r}{2}+\frac{1}{2}}x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon}(Q^{\frac{2^{r-1}-l}{2^{r-1}}} + y^{\frac{2^{r-1}-l}{2^{r-1}}}x^{(\frac{1}{r}-1)(\frac{2^{r-1}-l}{2^{r-1}})}))$$
$$+ O(y^{l}x^{(\frac{1}{r}-1)(l-1)+\varepsilon}\tau^{-1}Q^{-\frac{l}{r}+1}).$$

Case 5. For $2^{r-1} < l < 2^r$, $2 \leq k \geq 3$, $r \geq 8$,

$$S_k(x,y) = \sum_{q=1}^{\infty}\sum_{\substack{1\leq a\leq q\\(a,q)=1}}\frac{S_r^l(q,a)}{q^l}\sum_{j=0}^{k-1}A_j(q)\frac{1}{j!}\mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon})+$$
$$+ O(y^{(1-\frac{r}{2^r})l}x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon}Q^{k+\varepsilon}) + O(y^{(1-\frac{r}{2^r})l+\frac{k-1}{k+1}}x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon}Q)$$
$$+ O(y^{l-\frac{r}{2}+\frac{1}{2}}x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon}(Q^{\frac{2r(r-1)-l}{2r(r-1)}} + y^{\frac{2r(r-1)-l}{2r(r-1)}}x^{(\frac{1}{r}-1)(\frac{2r(r-1)-l}{2r(r-1)})}))$$
$$+ O(y^{l}x^{(\frac{1}{r}-1)(l-1)+\varepsilon}\tau^{-1}Q^{-\frac{l}{r}+1}).$$

Case 6. For $2^{r-1} < l < 2^r$, $k \geq 4$, $r \geq 8$,

$$S_k(x,y) = \sum_{q=1}^{\infty}\sum_{\substack{1\leq a\leq q\\(a,q)=1}}\frac{S_r^l(q,a)}{q^l}\sum_{j=0}^{k-1}A_j(q)\frac{1}{j!}\mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon})+$$
$$+ O(y^{(1-\frac{r}{2^r})l+}x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon}Q^{k+\varepsilon}) + O(y^{(1-\frac{r}{2^r})l+\frac{k-1}{k+2}}x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon}Q)$$
$$+ O(y^{l-\frac{r}{2}+\frac{1}{2}}x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon}(Q^{\frac{2r(r-1)-l}{2r(r-1)}} + y^{\frac{2r(r-1)-l}{2r(r-1)}}x^{(\frac{1}{r}-1)(\frac{2r(r-1)-l}{2r(r-1)})}))$$
$$+ O(y^{l}x^{(\frac{1}{r}-1)(l-1)+\varepsilon}\tau^{-1}Q^{-\frac{l}{r}+1}).$$

Case 7. For $l \geq 2^r$, $2 \leq k \leq 3$, $r \geq 8$,

$$S_k(x,y) = \sum_{q=1}^{\infty}\sum_{\substack{1\leq a\leq q\\(a,q)=1}}\frac{S_r^l(q,a)}{q^l}\sum_{j=0}^{k-1}A_j(q)\frac{1}{j!}\mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon})+$$
$$+ O(y^{l-r}x^{(l-r)(\frac{1}{r}-1)+\varepsilon}Q^{k+\varepsilon}) + O(y^{l-r+\frac{k-1}{k+1}}x^{(l-r)(\frac{1}{r}-1)+\varepsilon}Q)$$
$$+ O(y^{l-\frac{r}{2}+\frac{1}{2}}x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon}(Q^{\frac{2r(r-1)-l}{2r(r-1)}} + y^{\frac{2r(r-1)-l}{2r(r-1)}}x^{(\frac{1}{r}-1)(\frac{2r(r-1)-l}{2r(r-1)})}))$$
$$+ O(y^{l}x^{(\frac{1}{r}-1)(l-1)+\varepsilon}\tau^{-1}Q^{-\frac{l}{r}+1}).$$

Case 8. For $l \geq 2^r$, $k \geq 4$, $r \geq 8$,

$$S_k(x,y) = \sum_{q=1}^{\infty} \sum_{\substack{1\leq a\leq q \\ (a,q)=1}} \frac{S_r^l(q,a)}{q^l} \sum_{j=0}^{k-1} A_j(q)\frac{1}{j!}\mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon}) +$$
$$+ O(y^{l-r} x^{(l-r)(\frac{1}{r}-1)+\varepsilon} Q^{k+\varepsilon}) + O(y^{l-r+\frac{k-1}{k+2}} x^{(l-r)(\frac{1}{r}-1)+\varepsilon} Q)$$
$$+ O(y^{l-\frac{r}{2}+\frac{1}{2}} x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon}(Q^{\frac{2r(r-1)-l}{2r(r-1)}} + y^{\frac{2r(r-1)-l}{2r(r-1)}} x^{(\frac{1}{r}-1)(\frac{2r(r-1)-l}{2r(r-1)})}))$$
$$+ O(y^l x^{(\frac{1}{r}-1)(l-1)+\varepsilon} \tau^{-1} Q^{-\frac{l}{r}+1}).$$

For Cases 1–4, we take $y = x^{\theta}, \theta = \frac{(1-\frac{1}{r})(\frac{r}{2}+\frac{l-2^{r-1}}{k\cdot 2^{r-1}}-\varepsilon)}{\frac{r}{2}+\frac{l-2^{r-1}}{k\cdot 2^{r-1}}-\frac{1}{2}-\varepsilon}$ by condition of Theorem 1.1 and replace Q and τ in (3.1) by

$$x^{\varepsilon} \ll Q < \tau,\ Q\tau \asymp yx^{1-\frac{1}{r}},\ Q = y^{\frac{1}{k}} x^{\frac{1}{kr}-\frac{1}{k}}. \tag{4.1}$$

It is easy to see that

$$y^{l-\frac{r}{2}+\frac{1}{2}} x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon} Q^{\frac{2^{r-1}-l}{2^{r-1}}} > y^{l-\frac{r}{2}+\frac{1}{2}+\frac{2^{r-1}-l}{2^{r-1}}} x^{(\frac{1}{r}-1)(l-\frac{r}{2}+\frac{2^{r-1}-l}{2^{r-1}})+\varepsilon}.$$

In fact, we can check that the argument of Theorem 1.1 is valid if the parameters Q, τ and y satisfy the conditions

$$\begin{aligned}\max\{y^{(1-\frac{r}{2^r})l} x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon} Q^{k+\varepsilon}, y^{(1-\frac{r}{2^r})l+\frac{k-1}{k+1}} x^{(\frac{1}{r}-1)(1-\frac{r}{2^r})l+\varepsilon} Q\} \ll y^l x^{-l+\frac{l}{r}-\varepsilon},\\ \max\{y^{l-r} x^{(l-r)(\frac{1}{r}-1)+\varepsilon} Q^{k+\varepsilon}, y^{l-r+\frac{k-1}{k+1}} x^{(l-r)(\frac{1}{r}-1)+\varepsilon} Q\} \ll y^l x^{-l+\frac{l}{r}-\varepsilon},\end{aligned} \tag{4.2}$$

and

$$y^{l-\frac{r}{2}+\frac{1}{2}} x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon} Q^{\frac{2^{r-1}-l}{2^{r-1}}} \ll y^l x^{-l+\frac{l}{r}-\varepsilon}, y^l x^{(\frac{1}{r}-1)(l-1)+\varepsilon} \tau^{-1} Q^{-\frac{l}{r}+1} \ll y^l x^{-l+\frac{l}{r}-\varepsilon}.$$

Therefore,

$$S_k(x,y) = \sum_{q=1}^{\infty} \sum_{\substack{1\leq a\leq q \\ (a,q)=1}} \frac{S_r^l(q,a)}{q^l} \sum_{j=0}^{k-1} A_j(q)\frac{1}{j!}\mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon}).$$

For Cases 4–8, we take $y = x^{\theta}, \theta = \frac{(1-\frac{1}{r})(\frac{r}{2}+\frac{l-2r(r-1)}{k\cdot 2r(r-1)}-\varepsilon)}{\frac{r}{2}+\frac{l-2r(r-1)}{k\cdot 2r(r-1)}-\frac{1}{2}-\varepsilon}$ by condition of Theorem 1.1. According to (4.1), we find

$$y^{l-\frac{r}{2}+\frac{1}{2}} x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon} Q^{\frac{2r(r-1)-l}{2r(r-1)}} > y^{l-\frac{r}{2}+\frac{1}{2}+\frac{2r(r-1)-l}{2r(r-1)}} x^{(\frac{1}{r}-1)(l-\frac{r}{2}+\frac{2r(r-1)-l}{2r(r-1)})+\varepsilon}.$$

Actually, we can also deduce that Theorem 1.1 is available if the parameters Q, τ and y satisfying (4.2) and

$$y^{l-\frac{r}{2}+\frac{1}{2}} x^{(\frac{1}{r}-1)(l-\frac{r}{2})+\varepsilon} Q^{\frac{2r(r-1)-l}{2r(r-1)}} \ll y^l x^{-l+\frac{l}{r}-\varepsilon}, y^l x^{(\frac{1}{r}-1)(l-1)+\varepsilon} \tau^{-1} Q^{-\frac{l}{r}+1} \ll y^l x^{-l+\frac{l}{r}-\varepsilon}.$$

Similarly,

$$S_k(x,y) = \sum_{q=1}^{\infty} \sum_{\substack{1\leq a\leq q \\ (a,q)=1}} \frac{S_r^l(q,a)}{q^l} \sum_{j=0}^{k-1} A_j(q)\frac{1}{j!}\mathfrak{S}(x,y) + O(y^l x^{-l+\frac{l}{r}-\varepsilon}).$$

This completes the proof of Theorem 1.1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interests.

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