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Research article

Some properties and inequalities for generalized class of harmonical Godunova-Levin function via center radius order relation

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Abstract: There are many benefits derived from the speculation regarding convexity in the fields of applied and pure science. According to their definitions, convexity and integral inequality are linked concepts. The construction and refinement of classical inequalities for various classes of convex and nonconvex functions have been extensively studied. In convex theory, Godunova-Levin functions play an important role, because they make it easier to deduce inequalities when compared to convex functions. Based on Bhunia and Samanta's total order relation, harmonically cr-h-Godunova-Levin function is defined in this paper. Utilizing center order (CR) relationship, various types of inequalities can be introduced. (CR)-order relation enables us to derive some Hermite-Hadamard ($\mathcal{H}.\mathcal{H}$) inequality along with a Jensen-type inequality for harmonically h-Godunova-Levin interval-valued functions (GL-$\mathcal{IVFS}$). Many well-known and new convex functions are unified by this kind of convexity. For further verification of the accuracy of our findings, we provide some numerical examples.

Keywords: Jensen inequality; Hermite-Hadamard inequality; Godunova-Levin function; cr-order relation; interval-valued function; harmonic convexity
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1. Introduction

The interval analysis discipline addresses uncertainty using interval variables in contrast to variables in the form of points, the calculation results are reported as intervals, preventing mistakes that may lead to false conclusions. Despite its long history, Moore [1], used interval analysis for the first time in 1969 to analyze automatic error reports. This led to an improvement in calculation performance, which attracted many scholars' attention. Due to their ability to be expressed as uncertain variables, intervals are commonly used in uncertain problems, such as computer graphics [2], decision-making analysis [3], multi-objective optimization [4], and error analysis [5]. Consequently, interval analysis has produced numerous excellent results, and interested readers can consult. [6–8].

Meanwhile, numerous disciplines, including economics, control theory, and optimization, use convex analysis and many scholars have studied it, see [9–12]. Recently, generalized convexity of interval-valued functions ($\mathcal{IVFS}$) has received extensive research and has been utilized in a large number of fields and applications, see [13–16]. The (A, s)-convex and (A, s)-concave mappings describe the continuity of $\mathcal{IVFS}$, as described by Breckner in [17]. Numerous inequalities have recently been established for $\mathcal{IVFS}$. By applying the generalized Hukuhara derivative to $\mathcal{IVFS}$, Chalco-Cano et al. [18] derived some Ostrowski-type inclusions. Costa [19], established Opial type inequalities for the generalized Hukuhara differentiable $\mathcal{IVFS}$. In general, we can define a classical Hermite Hadamard inequality as follows:

$$\eta\left(\frac{t+u}{2}\right) \leq \frac{1}{u-t}\int_t^u \eta(v)dv \leq \frac{\eta(t)+\eta(u)}{2}. \tag{1.1}$$

Considering this inequality was the first geometrical interpretation of convex mappings in elementary mathematics, it has gained a lot of attention. The following are some variations and generalizations of this inequality, see [20–23]. Initially in 2007, Varoşanec [24] developed the notion of h-convex. Several authors have contributed to the development of inequalities based on $\mathcal{H}.\mathcal{H}$ using h-convex functions , see [25–28]. The harmonically h-convex functions introduced by Noor [29], are important generalizations of convex functions. Here are some recent results relating to harmonically h-convexity, see [30–35]. At present, these results are derived from inclusion relations and interval LU-order relationships, both of which have significant flaws because these are partial order relations. It can be demonstrated the validity of the claim by comparing examples from the literature with those derived from these old relations. In light of this, determining how to use a total order relation to investigate convexity and inequality is crucial. As an additional observation, the interval differences between endpoints are much closer in examples than in these old partial order relations. Because of this, the ability to analyze convexity and inequalities using a total order relation is essential. Therefore, we will focus our entire paper on Bhunia et al. [36], (CR)- order relation. Using cr-order, Rahman [37], studied nonlinear constrained optimization problems with cr-convex functions. Based on the notions of cr-order relation, Wei Liu and his co-authors developed a modified version of $\mathcal{H}.\mathcal{H}$ and Jensen-type inequalities for h-convex and harmonic h-convex functions by using center radius order relation, see [38, 39].

Theorem 1.1 (See [38])**.** *Let* $\eta : [t,u] \to R_I{}^+$. *Consider* $h : (0,1) \to R^+$ *and* $h\left(\frac{1}{2}\right) \neq 0$. *If* $\eta \in SHX(cr$-

$h, [t, u], R_I{}^+)$ *and* $\eta \in IR_{[t,u]}$, *then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\eta\left(\frac{2tu}{t+u}\right) \preceq_{cr} \frac{ut}{u-t}\int_t^u \frac{\eta(\nu)}{\nu^2}d\nu \preceq_{cr} [\eta(t)+\eta(u)]\int_0^1 h(x)dx. \tag{1.2}$$

In addition, a Jensen-type inequality was also proved with harmonic cr-h-convexity.

Theorem 1.2 (See [38]). *Let* $d_i \in R^+$, $z_i \in [t, u]$, $\eta : [t, u] \to R_I{}^+$. *If* h *is super multiplicative and non-negative function and* $\eta \in SHX(cr\text{-}h, [t, u], R_I{}^+)$. *Then the inequality become as:*

$$\eta\left(\frac{1}{\frac{1}{D_k}\sum_{i=1}^k d_i z_i}\right) \preceq_{cr} \sum_{i=1}^k h\left(\frac{d_i}{D_k}\right)\eta(z_i). \tag{1.3}$$

Using the h-GL function, Ohud Almutairi and Adem Kiliman have proven the following result in 2019, see [40].

Theorem 1.3. *Let* $\eta : [t, u] \to R$. *If* η *is h-Godunova-Levin function and* $h(\frac{1}{2}) \neq 0$. *Then*

$$\frac{h\left(\frac{1}{2}\right)}{2}\eta\left(\frac{t+u}{2}\right) \leq \frac{1}{u-t}\int_t^u \eta(\nu)d\nu \leq [\eta(t)+\eta(u)]\int_0^1 \frac{dx}{h(x)}. \tag{1.4}$$

This study is unique in that it introduces a notion of interval-valued harmonical h-Godunova-Levin functions that are related to a total order relation, called Center-Radius order, which is novel in the literature. By incorporating cr-interval-valued functions into inequalities, this article opens up a new avenue of research in inequalities. In contrast to classical interval-valued analysis, cr-order interval-valued analysis follows a different methodology. Based on the concept of center and radius, we calculate intervals as follows: $t_c = \frac{\underline{t}+\bar{t}}{2}$ and $t_r = \frac{\bar{t}-\underline{t}}{2}$, respectively, where $\bar{t}$ and $\underline{t}$ are endpoints of interval t.

Inspired by. [15, 34, 38, 39, 41], This study introduces a novel class of harmonically cr-h-GL functions based on cr-order. First, we derived some $\mathcal{H}.\mathcal{H}$ inequalities, then we developed the Jensen inequality using this new class. In addition, the study presents useful examples in support of its conclusions.

Lastly the paper is designed as follows: In section 2, preliminary information is provided. The key problems are described in section 3. There is a conclusion at the end of section 6.

2. Preliminaries

Some notions are used in this paper that aren't defined in this paper, see [38, 41]. The collection of intervals is denoted by R_I of R, while the collection of all positive intervals can be denoted by R_I^+. For $\nu \in R$, the scalar multiplication and addition are defined as

$$t + u = [\underline{t}, \bar{t}] + [\underline{u}, \bar{u}] = [\underline{t} + \underline{u}, \bar{t} + \bar{u}]$$

$$\nu t = \nu.[\underline{t}, \bar{t}] = \begin{cases} [\nu\underline{t}, \mu\bar{t}], \; if\; \nu > 0, \\ \{0\}, \; if\; \nu = 0, \\ [\nu\bar{t}, \nu\underline{t}], if\; \nu < 0, \end{cases}$$

respectively. Let $t = [\underline{t}, \bar{t}] \in R_I$, $t_c = \frac{\underline{t}+\bar{t}}{2}$ is called center of interval t and $t_r = \frac{\bar{t}-\underline{t}}{2}$ is said to be radius of interval t. In the case of interval t, this is the (CR) form

$$t = \left(\frac{\underline{t}+\bar{t}}{2}, \frac{\bar{t}-\underline{t}}{2}\right) = (t_c, t_r).$$

An order relation between radius and center can be defined as follows.

Definition 2.1. (See [25]). Consider $t = [\underline{t}, \bar{t}] = (t_c, t_r)$, $u = [\underline{u}, \overline{u}] = (u_c, u_r) \in R_I$, then centre-radius order (In short cr-order) relation is defined as

$$t \preceq_{cr} u \Leftrightarrow \begin{cases} t_c < u_c, t_c \neq u_c, \\ \\ t_c \leq u_c, t_c = u_c. \end{cases}$$

Further, we represented the concept of Riemann integrable (in short IR) in the context of $\mathcal{IVFS}$ [39].

Theorem 2.1 (See [39])**.** *Let $\varphi : [t, u] \to R_I$ be $\mathcal{IVF}$ given by $\eta(v) = [\underline{\eta}(v), \overline{\eta}(v)]$ for each $v \in [t, u]$ and $\underline{\eta}, \overline{\eta}$ are IR over interval $[t, u]$. In that case, we would call η is IR over interval $[t, u]$, and*

$$\int_t^u \eta(v)dv = \left[\int_t^u \underline{\eta}(v)dv, \int_t^u \overline{\eta}(v)dv\right].$$

All Riemann integrables (IR) $\mathcal{IVFS}$ over the interval should be assigned $IR_{[t,u]}$.

Theorem 2.2 (See [39])**.** *Let $\eta, \zeta : [t, u] \to R_I^+$ given by $\eta = [\underline{\eta}, \overline{\eta}]$, and $\zeta = [\underline{\zeta}, \overline{\zeta}]$. If $\eta, \zeta \in IR_{[t,u]}$, and $\eta(v) \preceq_{cr} \zeta(v)$ $\forall v \in [t, u]$, then*

$$\int_t^u \eta(v)dv \preceq_{cr} \int_t^u \zeta v dv.$$

See interval analysis notations for a more detailed explanation, see [38, 39].

Definition 2.2 (See [39])**.** Consider $h : [0, 1] \to R^+$. We say that $\eta : [t, u] \to R^+$ is known harmonically h-convex function, or that $\eta \in SHX(h, [t, u], R^+)$, if $\forall$ $t_1, u_1 \in [t, u]$ and $v \in [0, 1]$, we have

$$\eta\left(\frac{t_1 u_1}{v t_1 + (1-v)u_1}\right) \leq h(v)\eta(t_1) + h(1-v)\eta(u_1). \tag{2.1}$$

If in (2.1) $\leq$ replaced with $\geq$ it is called harmonically h-concave function or $\eta \in SHV(h, [t, u], R^+)$.

Definition 2.3. (See [27]). Consider $h : (0, 1) \to R^+$. We say that $\eta : [t, u] \to R^+$ is known as harmonically h-GL function, or that $\eta \in SGHX(h, [t, u], R^+)$, if $\forall$ $t_1, u_1 \in [t, u]$ and $v \in (0, 1)$, we have

$$\eta\left(\frac{t_1 u_1}{v t_1 + (1-v)u_1}\right) \leq \frac{\eta(t_1)}{h(v)} + \frac{\eta(u_1)}{h(1-v)}. \tag{2.2}$$

If in (2.2) $\leq$ replaced with $\geq$ it is called harmonically h-Godunova-Levin concave function or $\eta \in SGHV(h, [t, u], R^+)$.

Now let's look at the $\mathcal{IVF}$ concept with respect to cr-h-convexity.

Definition 2.4 (See [39]). Consider $h : [0,1] \to R^+$. We say that $\eta = [\underline{\eta}, \overline{\eta}] : [t,u] \to R_I^+$ is called harmonically cr-h-convex function, or that $\eta \in SHX(\text{cr-h}, [t,u], R_I^+)$, if $\forall\ t_1, u_1 \in [t,u]$ and $\nu \in [0,1]$, we have

$$\eta\left(\frac{t_1 u_1}{\nu t_1 + (1-\nu)u_1}\right) \preceq_{cr} h(\nu)\eta(t_1) + h(1-\nu)\eta(u_1). \tag{2.3}$$

If in (2.3) $\preceq_{cr}$ replaced with $\succeq_{cr}$ it is called harmonically cr-h-concave function or $\eta \in SHV(\text{cr-h}, [t,u], R_I^+)$.

Definition 2.5. (See [39]) Consider $h : (0,1) \to R^+$. We say that $\eta = [\underline{\eta}, \overline{\eta}] : [t,u] \to R_I^+$ is called harmonically cr-h-Godunova-Levin convex function, or that $\eta \in SGHX(\text{cr-h}, [t,u], R_I^+)$, if $\forall\ t_1, u_1 \in [t,u]$ and $\nu \in (0,1)$, we have

$$\eta\left(\frac{t_1 u_1}{\nu t_1 + (1-\nu)u_1}\right) \preceq_{cr} \frac{\eta(t_1)}{h(\nu)} + \frac{\eta(u_1)}{h(1-\nu)}. \tag{2.4}$$

If in (2.4) $\preceq_{cr}$ replaced with $\succeq_{cr}$ it is called harmonically cr-h-Godunova-Levin concave function or $\eta \in SGHV(\text{cr-h}, [t,u], R_I^+)$.

Remark 2.1.

(i) If $h(\nu) = 1$, in this case, Definition 2.5 becomes a harmonically cr-P-function [28].

(ii) If $h(\nu) = \frac{1}{h(\nu)}$, in this case, Definition 2.5 becomes a harmonically cr h-convex function [28].

(iii) If $h(\nu) = \nu$, in this case, Definition 2.5 becomes a harmonically cr-Godunova-Levin function [28].

(iv) If $h(\nu) = \frac{1}{\nu^s}$, in this case, Definition 2.5 becomes a harmonically cr-s-convex function [28].

(v) If $h(\nu) = \nu^s$, in this case, Definition 2.5 becomes a harmonically cr-s-GL function [28].

3. Main results

Proposition 3.1. *Define $h_1, h_2 : (0,1) \to R^+$ functions that are non-negative and*

$$\frac{1}{h_2} \leq \frac{1}{h_1}, \nu \in (0,1).$$

If $\eta \in SGHX(cr\text{-}h_2, [t,u], {R_I}^+)$, then $\eta \in SGHX(cr\text{-}h_1, [t,u], {R_I}^+)$.

Proof. Since $\eta \in SGHX(cr\text{-}h_2, [t,u], {R_I}^+)$, then for all $t_1, u_1 \in [t,u], \nu \in (0,1)$, we have

$$\eta\left(\frac{t_1 u_1}{\nu t_1 + (1-\nu)u_1}\right) \preceq_{cr} \frac{\eta(t_1)}{h_2(\nu)} + \frac{\eta(u_1)}{h_2(1-\nu)}$$
$$\preceq_{cr} \frac{\eta(t_1)}{h_1(\nu)} + \frac{\eta(u_1)}{h_1(1-\nu)}.$$

Hence, $\eta \in SGHX(cr\text{-}h_1, [t,u], {R_I}^+)$. □

Proposition 3.2. *Let $\eta : [t,u] \to R_I$ given by $[\underline{\eta}, \overline{\eta}] = (\eta_c, \eta_r)$. If η_c and η_r are harmonically h-GL over $[t,u]$, then η is known as harmonically cr-h-GL function over $[t,u]$.*

Proof. Since η_c and η_r are harmonically h-GL over $[t, u]$, then for each $\nu \in (0, 1)$ and for all $t_1, u_1 \in [t, u]$, we have

$$\eta_c\left(\frac{t_1u_1}{\nu t_1 + (1-\nu)u_1}\right) \preceq_{cr} \frac{\eta_c(t_1)}{h(\nu)} + \frac{\eta_c(u_1)}{h(1-\nu)},$$

and

$$\eta_r\left(\frac{t_1u_1}{\nu t_1 + (1-\nu)u_1}\right) \preceq_{cr} \frac{\eta_r(t_1)}{h(\nu)} + \frac{\eta_r(u_1)}{h(1-\nu)}.$$

Now, if

$$\eta_c\left(\frac{t_1u_1}{\nu t_1 + (1-\nu)u_1}\right) \neq \frac{\eta_c(t_1)}{h(\nu)} + \frac{\eta_c(u_1)}{h(1-\nu)},$$

then for each $\nu \in (0, 1)$ and for all $t_1, u_1 \in [t, u]$,

$$\eta_c\left(\frac{t_1u_1}{\nu t_1 + (1-\nu)u_1}\right) < \frac{\eta_c(t_1)}{h(\nu)} + \frac{\eta_c(u_1)}{h(1-\nu)}.$$

Accordingly,

$$\eta_c\left(\frac{t_1u_1}{\nu t_1 + (1-\nu)u_1}\right) \preceq_{cr} \frac{\eta_c(t_1)}{h(\nu)} + \frac{\eta_c(u_1)}{h(1-\nu)}.$$

Otherwise, for each $\nu \in (0, 1)$ and for all $t_1, u_1 \in [t, u]$,

$$\eta_r\left(\frac{t_1u_1}{\nu t_1 + (1-\nu)u_1}\right) \leq \frac{\eta_r(t_1)}{h(\nu)} + \frac{\eta_r(u_1)}{h(1-\nu)} \Rightarrow \eta\left(\frac{t_1u_1}{\nu t_1 + (1-\nu)u_1}\right) \preceq_{cr} \frac{\eta(t_1)}{h(\nu)} + \frac{\eta(u_1)}{h(1-\nu)}.$$

Based on all the above, and Definition 2.1, this can be expressed as follows:

$$\eta\left(\frac{t_1u_1}{\nu t_1 + (1-\nu)u_1}\right) \preceq_{cr} \frac{\eta(t_1)}{h(\nu)} + \frac{\eta(u_1)}{h(1-\nu)}$$

for each $\nu \in (0, 1)$ and for all $t_1, u_1 \in [t, u]$.

This completes the proof. □

4. Hermite-Hadamard type inequality

This section developed the $\mathcal{H}.\mathcal{H}$ inequalities for harmonically cr-h-GL functions.

Theorem 4.1. *Consider $h : (0, 1) \to R^+$ and $h(\frac{1}{2}) \neq 0$. Let $\eta : [t, u] \to {R_I}^+$, if $\eta \in SGHX($cr-$h, [t, u], {R_I}^+)$ and $\eta \in IR_{[t,u]}$, we have*

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{2} f\left(\frac{2tu}{t+u}\right) \preceq_{cr} \frac{tu}{u-t}\int_t^u \frac{\eta(\nu)}{\nu^2}d\nu \preceq_{cr} [\eta(t) + \eta(u)]\int_0^1 \frac{dx}{h(x)}. \tag{4.1}$$

Proof. Since $\eta \in SGHX(\text{cr-h}, [t, u], {R_I}^+)$, we have

$$h\left(\frac{1}{2}\right)\eta\left(\frac{2tu}{t+u}\right) \preceq_{cr} \eta\left(\frac{tu}{xt + (1-x)u}\right) + \eta\left((\frac{tu}{1-x)t + xu}\right).$$

On integration over $(0,1)$, we have

$$h\left(\frac{1}{2}\right)\eta\left(\frac{2tu}{t+u}\right) \preceq_{cr} \left[\int_0^1 \eta\left(\frac{tu}{xt+(1-x)u}\right)dx + \int_0^1 \eta\left(\frac{tu}{(1-x)t+xu}\right)dx\right]$$
$$= \left[\int_0^1 \underline{\eta}\left(\frac{tu}{xt+(1-x)u}\right)dx + \int_0^1 \underline{\eta}\left(\frac{tu}{(1-x)t+xu}\right)dx,\right.$$
$$\left.\int_0^1 \overline{\eta}\left(\frac{tu}{xt+(1-x)u}\right)dx + \int_0^1 \overline{\eta}\left(\frac{tu}{(1-x)t+xu}\right)dx\right]$$
$$= \left[\frac{2tu}{u-t}\int_t^u \frac{\underline{\eta}(\nu)}{\nu^2}d\nu, \frac{2tu}{u-t}\int_t^u \frac{\overline{\eta}(\nu)}{\nu^2}d\nu\right]$$
$$= \frac{2tu}{u-t}\int_t^u \frac{\eta(\nu)}{\nu^2}d\nu. \tag{4.2}$$

By Definition 2.5, we have

$$\eta\left(\frac{tu}{xt+(1-x)u}\right) \preceq_{cr} \frac{\eta(t)}{h(x)} + \frac{\eta(u)}{h(1-x)}.$$

On integration over (0,1), we have

$$\int_0^1 \eta\left(\frac{tu}{xt+(1-x)u}\right)dx \preceq_{cr} \eta(t)\int_0^1 \frac{dx}{h(x)} + \eta(u)\int_0^1 \frac{dx}{h(1-x)}.$$

Accordingly,

$$\frac{ut}{u-t}\int_t^u \frac{\eta(\nu)}{\nu^2}d\nu \preceq_{cr} [\eta(t)+\eta(u)]\int_0^1 \frac{dx}{h(x)}. \tag{4.3}$$

Adding (4.2) and (4.3), results are obtained as expected

$$\frac{h\left(\frac{1}{2}\right)}{2}\eta\left(\frac{2tu}{t+u}\right) \preceq_{cr} \frac{ut}{u-t}\int_t^u \frac{\eta(\nu)}{\nu^2}d\nu \preceq_{cr} [\eta(t)+\eta(u)]\int_0^1 \frac{dx}{h(x)}.$$

□

Remark 4.1.

(i) If $h(x) = 1$, in this case, Theorem 4.1 becomes result for harmonically cr- P-function:

$$\frac{1}{2}\eta\left(\frac{2tu}{t+u}\right) \preceq_{cr} \frac{ut}{u-t}\int_t^u \frac{\eta(\nu)}{\nu^2}d\nu \preceq_{cr} [\eta(t)+\eta(u)].$$

(ii) If $h(x) = \frac{1}{x}$, in this case, Theorem 4.1 becomes result for harmonically cr-convex function:

$$\eta\left(\frac{2tu}{t+u}\right) \preceq_{cr} \frac{ut}{u-t}\int_t^u \frac{\eta(\nu)}{\nu^2}d\nu \preceq_{cr} \frac{[\eta(t)+\eta(u)]}{2}.$$

(iii) If $h(x) = \frac{1}{(x)^s}$, in this case, Theorem 4.1 becomes result for harmonically cr-s-convex function:

$$2^{s-1}\eta\left(\frac{2tu}{t+u}\right) \preceq_{cr} \frac{ut}{u-t}\int_t^u \frac{\eta(\nu)}{\nu^2}d\nu \preceq_{cr} \frac{[\eta(t)+\eta(u)]}{s+1}.$$

Example 4.1. Let $[t, u] = [1, 2]$, $h(x) = \frac{1}{x}$, $\forall\ x \in\ (0, 1)$. $\eta : [t, u] \to R_I{}^+$ is defined as

$$\eta(\nu) = \left[\frac{-1}{\nu^4} + 2, \frac{1}{\nu^4} + 3\right],$$

where

$$\frac{h\left(\frac{1}{2}\right)}{2}\eta\left(\frac{2tu}{t+u}\right) = \eta\left(\frac{4}{3}\right) = \left[\frac{431}{256}, \frac{849}{256}\right],$$

$$\frac{ut}{u-t}\int_t^u \frac{\eta(\nu)}{\nu^2}d\nu = 2\left[\int_1^2 \left(\frac{2\nu^4-1}{\nu^6}\right)d\nu, \int_1^2 \left(\frac{3\nu^4+1}{\nu^6}\right)d\nu\right] = \left[\frac{258}{160}, \frac{542}{160}\right],$$

$$[\eta(t) + \eta(u)]\int_0^1 \frac{dx}{h(x)} = \left[\frac{47}{8}, \frac{113}{8}\right].$$

As a result,

$$\left[\frac{431}{256}, \frac{849}{256}\right] \preceq_{cr} \left[\frac{258}{160}, \frac{542}{160}\right] \preceq_{cr} \left[\frac{47}{8}, \frac{113}{8}\right].$$

Thus, proving the theorem above.

Theorem 4.2. *Consider $h : (0, 1) \to R^+$ and $h\left(\frac{1}{2}\right) \neq 0$. Let $\eta : [t, u] \to R_I{}^+$, if $\eta \in SGX(cr\text{-}h, [t, u], R_I{}^+)$ and $\eta \in\ IR_{[t,u]}$, we have*

$$\frac{\left[h(\frac{1}{2})\right]^2}{4}\eta\left(\frac{2tu}{t+u}\right) \preceq_{cr} \triangle_1 \preceq_{cr} \frac{1}{u-t}\int_t^u \frac{\eta(\nu)}{\nu^2}d\nu \preceq_{cr} \triangle_2$$

$$\preceq_{cr} \left\{[\eta(t) + \eta(u)]\left[\frac{1}{2} + \frac{1}{h(\frac{1}{2})}\right]\right\}\int_0^1 \frac{dx}{h(x)},$$

where

$$\triangle_1 = \frac{[h(\frac{1}{2})]}{4}\left[\eta\left(\frac{4tu}{3t+u}\right) + \eta\left(\frac{4tu}{3u+t}\right)\right],$$

$$\triangle_2 = \left[\eta\left(\frac{2tu}{t+u}\right) + \frac{\eta(t) + \eta(u)}{2})\right]\int_0^1 \frac{dx}{h(x)}.$$

Proof. Consider $\left[t, \frac{t+u}{2}\right]$, we have

$$\eta\left(\frac{4tu}{t+3u}\right) \preceq_{cr} \frac{\eta\left(\frac{t\frac{2tu}{t+u}}{xt+(1-x)\frac{2tu}{t+u}}\right)}{\left[h\left(\frac{1}{2}\right)\right]} + \frac{\eta\left(\frac{t\frac{2tu}{t+u}}{(1-x)t+x\frac{2tu}{t+u}}\right)}{\left[h\left(\frac{1}{2}\right)\right]}.$$

Integration over $(0, 1)$, we have

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\eta\left(\frac{4tu}{u+3t}\right) \preceq_{cr} \frac{ut}{u-t}\int_u^{\frac{2tu}{t+u}} \frac{\eta(\nu)}{\nu^2}d\nu. \tag{4.4}$$

Similarly for interval $\left[\frac{t+u}{2}, u\right]$, we have

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\eta\left(\frac{4tu}{t+3u}\right) \preceq_{cr} \frac{ut}{u-t}\int_{\frac{2tu}{t+u}}^{t} \frac{\eta(\nu)}{\nu^2}d\nu. \tag{4.5}$$

Adding inequalities (4.4) and (4.5), we get

$$\triangle_1 = \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left[\eta\left(\frac{4tu}{u+3t}\right)+\eta\left(\frac{4ut}{t+3u}\right)\right] \preceq_{cr} \frac{ut}{u-t}\int_{t}^{u} \frac{\eta(\nu)}{\nu^2}d\nu.$$

Now

$$\begin{aligned}
&\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\eta\left(\frac{2tu}{t+u}\right)\\
&= \frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\eta\left(\frac{1}{2}\left(\frac{4tu}{3u+t}\right)+\frac{1}{2}\left(\frac{4tu}{3t+u}\right)\right)\\
&\preceq_{cr} \frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\left[\frac{\eta\left(\frac{4tu}{u+3t}\right)}{h\left(\frac{1}{2}\right)}+\frac{\eta\left(\frac{4tu}{3u+t}\right)}{h\left(\frac{1}{2}\right)}\right]\\
&= \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left[\eta\left(\frac{4tu}{u+3t}\right)+\eta\left(\frac{4tu}{3u+t}\right)\right]\\
&= \triangle_1\\
&\preceq_{cr} \frac{ut}{u-t}\int_{u}^{t}\frac{\eta(\nu)}{\nu^2}d\nu\\
&\preceq_{cr} \frac{1}{2}\left[\eta(t)+\eta(u)+2\eta\left(\frac{2tu}{t+u}\right)\right]\int_0^1\frac{dx}{h(x)}\\
&= \triangle_2\\
&\preceq_{cr} \left[\frac{\eta(t)+\eta(u)}{2}+\frac{\eta(t)}{h\left(\frac{1}{2}\right)}+\frac{\eta(u)}{h\left(\frac{1}{2}\right)}\right]\int_0^1\frac{dx}{h(x)}\\
&\preceq_{cr} \left[\frac{\eta(t)+\eta(u)}{2}+\frac{1}{h\left(\frac{1}{2}\right)}\left[\eta(t)+\eta(u)\right]\right]\int_0^1\frac{dx}{h(x)}\\
&\preceq_{cr} \left\{\left[\eta(t)+\eta(u)\right]\left[\frac{1}{2}+\frac{1}{h\left(\frac{1}{2}\right)}\right]\right\}\int_0^1\frac{dx}{h(x)}.
\end{aligned}$$

□

Example 4.2. Thanks to example 4.1, we have

$$\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\eta\left(\frac{2tu}{t+u}\right) = \eta\left(\frac{4}{3}\right) = \left[\frac{431}{256}, \frac{849}{256}\right],$$

$$\triangle_1 = \frac{1}{2}\left[\eta\left(\frac{8}{5}\right) + \eta\left(\frac{8}{7}\right)\right] = \left[\frac{6679}{4096}, \frac{13801}{4096}\right],$$

$$\triangle_2 = \left[\frac{\eta(1)+\eta(2)}{2} + \eta\left(\frac{4}{3}\right)\right]\int_0^1 \frac{dx}{h(x)},$$

$$\triangle_2 = \left[\frac{1935}{512}, \frac{4465}{512}\right],$$

$$\left\{[\eta(t)+\eta(u)]\left[\frac{1}{2} + \frac{1}{h(\frac{1}{2})}\right]\right\}\int_0^1 \frac{dx}{h(x)} = \left[\frac{47}{8}, \frac{113}{8}\right].$$

Thus, we obtain

$$\left[\frac{431}{256}, \frac{849}{256}\right] \preceq_{cr} \left[\frac{6679}{4096}, \frac{13801}{4096}\right] \preceq_{cr} \left[\frac{258}{160}, \frac{542}{160}\right] \preceq_{cr} \left[\frac{1935}{512}, \frac{4465}{512}\right] \preceq_{cr} \left[\frac{47}{8}, \frac{113}{8}\right].$$

This proves the above theorem.

Theorem 4.3. *Let $\eta, \zeta : [t,u] \to {R_I}^+, h_1, h_2 : (0,1) \to R^+$ such that $h_1, h_2 \neq 0$. If $\eta \in SGHX(cr$ $-h_1, [t,u], {R_I}^+)$, $\zeta \in SGHX(cr\text{-}h_2, [t,u], {R_I}^+)$ and $\eta, \zeta \in IR_{[v,w]}$ then, we have*

$$\frac{ut}{u-t}\int_t^u \frac{\eta(v)\zeta(v)}{v^2}dv \preceq_{cr} M(t,u)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx + N(t,u)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx, \tag{4.6}$$

where

$$M(t,u) = \eta(t)\zeta(t) + \eta(u)\zeta(u), N(t,u) = \eta(t)\zeta(u) + \eta(u)\zeta(t).$$

Proof. Conider $\eta \in SGHX(cr\text{-}h_1, [t,u], {R_I}^+)$, $\zeta \in SGHX(cr\text{-}h_2, [t,u], {R_I}^+)$ then, we have

$$\eta\left(\frac{tu}{tx+(1-x)u}\right) \preceq_{cr} \frac{\eta(t)}{h_1(x)} + \frac{\eta(u)}{h_1(1-x)},$$

$$\zeta\left(\frac{tu}{tx+(1-x)u}\right) \preceq_{cr} \frac{\zeta(t)}{h_2(x)} + \frac{\zeta(u)}{h_2(1-x)}.$$

Then,

$$\eta\left(\frac{tu}{tx+(1-x)u}\right)\zeta\left(\frac{tu}{tx+(1-x)u}\right)$$

$$\preceq_{cr} \frac{\eta(t)\zeta(t)}{h_1(x)h_2(x)} + \frac{\eta(t)\zeta(u)}{h_1(x)h_2(1-x)} + \frac{\eta(u)\zeta(t)}{h_1(1-x)h_2(x)} + \frac{\eta(u)\zeta(u)}{h_1(1-x)h_2(1-x)}.$$

Integration over (0,1), we have

$$\int_0^1 \eta\left(\frac{tu}{tx+(1-x)u}\right)\zeta\left(\frac{tu}{tx+(1-x)u}\right)dx$$
$$= \left[\int_0^1 \underline{\eta}\left(\frac{tu}{tx+(1-x)u}\right)\underline{\zeta}\left(\frac{tu}{tx+(1-x)u}\right)dx,\right.$$

$$\int_0^1 \overline{\eta}\left(\frac{tu}{tx+(1-x)u}\right)\overline{\zeta}\left(\frac{tu}{tx+(1-x)u}\right)dx\Bigg]$$

$$= \left[\frac{ut}{u-t}\int_t^u \frac{\underline{\eta}(v)\underline{\zeta}(v)}{v^2}dv, \frac{ut}{u-t}\int_t^u \frac{\overline{\eta}(v)\overline{\zeta}(v)}{v^2}dv\right] = \frac{ut}{u-t}\int_t^u \frac{\eta(v)\zeta(v)}{v^2}dv$$

$$\leq_{cr} \int_0^1 \frac{[\eta(t)\zeta(t)+\eta(u)\zeta(u)]}{h_1(x)h_2(x)}dx + \int_0^1 \frac{[\eta(t)\zeta(u)+\eta(u)\zeta(t)]}{h_1(x)h_2(1-x)}dx.$$

It follows that

$$\frac{ut}{u-t}\int_t^u \frac{\eta(v)\zeta(v)}{v^2}dv$$

$$\leq_{cr} M(t,u)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx + N(t,u)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx.$$

Theorem is proved. □

Example 4.3. Let $[t,u]=[1,2]$, $h_1(x)=h_2(x)=\frac{1}{x}$ $\forall\ x\in\ (0,1)$. $\eta,\zeta:[t,u]\to {R_I}^+$ be defined as

$$\eta(v)=\left[\frac{-1}{v^4}+2, \frac{1}{v^4}+3\right], \zeta(v)=\left[\frac{-1}{v}+1, \frac{1}{v}+2\right].$$

Then,

$$\frac{ut}{u-t}\int_t^u \frac{\eta(v)\zeta(v)}{v^2}dv = \left[\frac{282}{640}, \frac{5986}{640}\right],$$

$$M(t,u)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx = M(1,2)\int_0^1 x^2 dx = \left[\frac{31}{96}, \frac{629}{96}\right],$$

$$N(t,u)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx = N(1,2)\int_0^1 (x-x^2)dx = \left[\frac{1}{12}, \frac{307}{96}\right].$$

It follows that

$$\left[\frac{282}{640}, \frac{5986}{640}\right] \leq_{cr} \left[\frac{31}{96}, \frac{629}{96}\right] + \left[\frac{1}{12}, \frac{307}{96}\right] = \left[\frac{13}{32}, \frac{39}{4}\right].$$

This proves the above theorem.

Theorem 4.4. *Let* $\eta,\zeta:[t,u]\to {R_I}^+, h_1,h_2:(0,1)\to R^+$ *such that* $h_1,h_2\neq 0$. *If* $\eta\in SGHX(cr\text{-}h_1,[t,u],{R_I}^+)$, $\zeta\in SGHX(cr\text{-}h_2,[t,u],{R_I}^+)$ *and* $\eta,\zeta\in\ IR_{[v,w]}$ *then, we have*

$$\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{2}\eta\left(\frac{2tu}{t+u}\right)\zeta\left(\frac{2tu}{t+u}\right)$$

$$\leq_{cr} \frac{ut}{u-t}\int_t^u \frac{\eta(v)\zeta(v)}{v^2}d\mu + M(t,u)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx + N(t,u)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx.$$

Proof. Since $\eta\in SGHX(cr\text{-}h_1,[t,u],{R_I}^+)$, $\zeta\in SGHX(cr\text{-}h_2,[t,u],{R_I}^+)$, we have

$$\eta\left(\frac{2tu}{t+u}\right) \leq_{cr} \frac{\eta\left(\frac{tu}{tx+(1-x)u}\right)}{h_1\left(\frac{1}{2}\right)} + \frac{\eta\left(\frac{tu}{t(1-x)+xu}\right)}{h_1\left(\frac{1}{2}\right)},$$

$$\zeta\left(\frac{2tu}{t+u}\right) \preceq_{cr} \frac{\zeta\left(\frac{tu}{tx+(1-x)u}\right)}{h_2\left(\frac{1}{2}\right)} + \frac{\zeta\left(\frac{tu}{t(1-x)+xu}\right)}{h_2\left(\frac{1}{2}\right)}.$$

Then,

$$\begin{aligned}
&\eta\left(\frac{2tu}{t+u}\right)\zeta\left(\frac{2tu}{t+u}\right)\\
&\preceq_{cr} \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\eta\left(\frac{tu}{tx+(1-x)u}\right)\zeta\left(\frac{tu}{tx+(1-x)u}\right)+\eta\left(\frac{tu}{t(1-x)+xu}\right)\zeta\left(\frac{tu}{t(1-x)+xu}\right)\right]\\
&+\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\eta\left(\frac{tu}{tx+(1-x)u}\right)\zeta\left(\frac{tu}{t(1-x)+xu}\right)+\eta\left(\frac{tu}{t(1-x)+xu}\right)\zeta\left(\frac{tu}{tx+(1-x)u}\right)\right]\\
&\preceq_{cr} \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\eta\left(\frac{tu}{tx+(1-x)u}\right)\zeta\left(\frac{tu}{tx+(1-x)u}\right)+\eta\left(\frac{tu}{t(1-x)+xu}\right)\zeta\left(\frac{tu}{t(1-x)+xu}\right)\right]\\
&+\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\left(\frac{\eta(t)}{h_1(x)}+\frac{\eta(u)}{h_1(1-x)}\right)\left(\frac{\zeta(u)}{h_2(1-x)}+\frac{\zeta(u)}{h_2(x)}\right)+\left(\frac{\eta(t)}{h_1(1-x)}+\frac{\eta(u)}{h_1(x)}\right)\left(\frac{\zeta(t)}{h_2(x)}+\frac{\zeta(u)}{h_2(1-x)}\right)\right]\\
&\preceq_{cr} \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\eta\left(\frac{tu}{tx+(1-x)u}\right)\zeta\left(\frac{tu}{tx+(1-x)u}\right)+\eta\left(\frac{tu}{t(1-x)+ux}\right)\zeta\left(\frac{tu}{t(1-x)+ux}\right)\right]\\
&+\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\left(\frac{1}{h_1(x)h_2(1-x)}+\frac{1}{h_1(1-x)h_2(x)}\right)M(t,u)+\left(\frac{1}{h_1(x)h_2(x)}+\frac{1}{h_1(1-x)h_2(1-x)}\right)N(t,u)\right].
\end{aligned}$$

Integration over $(0,1)$, we have

$$\begin{aligned}
&\int_0^1 \eta\left(\frac{2tu}{t+u}\right)\zeta\left(\frac{2tu}{t+u}\right)dx = \left[\int_0^1 \underline{\eta}\left(\frac{2tu}{t+u}\right)\underline{\zeta}\left(\frac{2tu}{t+u}\right)dx, \int_0^1 \overline{\eta}\left(\frac{2tu}{t+u}\right)\overline{\zeta}\left(\frac{2tu}{t+u}\right)dx\right]\\
&= \eta\left(\frac{2tu}{t+u}\right)\zeta\left(\frac{2tu}{t+u}\right)dx\\
&\preceq_{cr} \frac{2}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\left[\frac{ut}{u-t}\int_t^u \frac{\eta(\nu)\zeta(\nu)}{\nu^2}d\nu\right] + \frac{2}{h\left(\frac{1}{2}\right)h(\frac{1}{2})}\left[M(t,u)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx + N(t,u)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx\right].
\end{aligned}$$

Multiply both sides by $\frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{2}$ above equation, we get required result

$$\begin{aligned}
&\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{2}\eta\left(\frac{2tu}{t+u}\right)\zeta\left(\frac{2tu}{t+u}\right)\\
&\preceq_{cr} \frac{ut}{u-t}\int_t^u \frac{\eta(\nu)\zeta(\nu)}{\nu^2}d\mu + M(t,u)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx + N(t,u)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx.
\end{aligned}$$

□

Example 4.4. Let $[t,u]=[1,2]$, $h_1(x)=h_2(x)=\frac{1}{x}$, $\forall\ x \in\ (0,1)$. $\eta,\zeta : [t,u] \to {R_I}^+$ be defined as

$$\eta(\nu) = \left[\frac{-1}{\nu^4}+2, \frac{1}{\nu^4}+3\right], \zeta(\nu) = \left[\frac{-1}{\nu}+1, \frac{1}{\nu}+2\right].$$

Then,

$$\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{2}\eta\left(\frac{2tu}{t+u}\right)\zeta\left(\frac{2tu}{t+u}\right) = 2\eta\left(\frac{4}{3}\right)\zeta\left(\frac{4}{3}\right) = \left[\frac{431}{512}, \frac{9339}{512}\right],$$

$$\frac{ut}{u-t}\int_t^u \frac{\eta(v)\zeta(v)}{v^2}dv = \left[\frac{282}{640}, \frac{5986}{640}\right],$$
$$M(t,u)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx = M(1,2)\int_0^1 (x-x^2)dx = \left[\frac{31}{192}, \frac{629}{192}\right],$$
$$N(t,u)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx = N(1,2)\int_0^1 x^2dx = \left[\frac{1}{6}, \frac{307}{48}\right].$$

It follows that

$$\left[\frac{431}{512}, \frac{9339}{512}\right] \preceq_{cr} \left[\frac{282}{640}, \frac{5986}{640}\right] + \left[\frac{31}{192}, \frac{629}{192}\right] + \left[\frac{1}{6}, \frac{307}{48}\right] = \left[\frac{123}{160}, \frac{761}{40}\right].$$

This proves the above theorem.

5. Jensen type inequality

Theorem 5.1. *Let $d_i \in R^+$, $z_i \in [t,u]$. If h is non-negative and super multiplicative function or $\eta \in SGHX(cr\text{-}h, [t,u], R_I{}^+)$. Then the inequality become as :*

$$\eta\left(\frac{1}{\frac{1}{D_k}\sum_{i=1}^k d_i z_i}\right) \preceq_{cr} \sum_{i=1}^k \left[\frac{\eta(z_i)}{h\left(\frac{d_i}{D_k}\right)}\right], \tag{5.1}$$

where $D_k = \sum_{i=1}^k d_i$.

Proof. If $k = 2$, inequality (5.1) holds. Assume that inequality (5.1) also holds for $k-1$, then

$$\begin{aligned}
\eta\left(\frac{1}{\frac{1}{D_k}\sum_{i=1}^k d_i z_i}\right) &= \eta\left(\frac{1}{\frac{d_k}{D_k}z_k + \sum_{i=1}^{k-1}\frac{d_i}{D_k}z_i}\right)\\
&= \eta\left(\frac{1}{\frac{d_k}{D_k}z_k + \frac{D_{k-1}}{D_k}\sum_{i=1}^{k-1}\frac{d_i}{D_{k-1}}z_i}\right)\\
&\preceq_{cr} \frac{\eta(z_k)}{h\left(\frac{d_k}{D_k}\right)} + \frac{\eta\left(\sum_{i=1}^{k-1}\frac{d_i}{D_{k-1}}z_i\right)}{h\left(\frac{D_{k-1}}{D_k}\right)}\\
&\preceq_{cr} \frac{\eta(z_k)}{h\left(\frac{d_k}{D_k}\right)} + \sum_{i=1}^{k-1}\left[\frac{\eta(z_i)}{h\left(\frac{d_i}{D_{k-1}}\right)}\right]\frac{1}{h\left(\frac{D_{k-1}}{D_k}\right)}\\
&\preceq_{cr} \frac{\eta(z_k)}{h\left(\frac{d_k}{D_k}\right)} + \sum_{i=1}^{k-1}\left[\frac{\eta(z_i)}{h(\frac{d_i}{D_k})}\right]\\
&\preceq_{cr} \sum_{i=1}^{k}\left[\frac{\eta(z_i)}{h\left(\frac{d_i}{D_k}\right)}\right].
\end{aligned}$$

Therefore, the result can be proved by mathematical induction. □

Remark 5.1.

(i) If $h(x) = 1$*, in this case, Theorem 5.1 becomes result for harmonically cr- P-function:*

$$\eta\left(\frac{1}{\frac{1}{D_k}\sum_{i=1}^{k} d_i z_i}\right) \preceq_{cr} \sum_{i=1}^{k} \eta(z_i).$$

(ii) If $h(x) = \frac{1}{x}$*, in this case, Theorem 5.1 becomes result for harmonically cr-convex function:*

$$\eta\left(\frac{1}{\frac{1}{D_k}\sum_{i=1}^{k} d_i z_i}\right) \preceq_{cr} \sum_{i=1}^{k} \frac{d_i}{D_k}\eta(z_i).$$

(iii) If $h(x) = \frac{1}{(x)^s}$*, in this case, Theorem 5.1 becomes result for harmonically cr-s-convex function:*

$$\eta\left(\frac{1}{\frac{1}{D_k}\sum_{i=1}^{k} d_i z_i}\right) \preceq_{cr} \sum_{i=1}^{k} \left(\frac{d_i}{D_k}\right)^s \eta(z_i).$$

6. Conclusions

This study presents a harmonically cr-h-GL concept for $\mathcal{IVFS}$. Using this new concept, we study Jensen and $\mathcal{H}.\mathcal{H}$ inequalities for $\mathcal{IVFS}$. This study generalizes results developed by Wei Liu [38, 39] and Ohud Almutairi [34]. Several relevant examples are provided as further support for our basic conclusions. It might be interesting to determine equivalent inequalities for different types of convexity in the future. Under the influence of this concept, a new direction begins to emerge in convex optimization theory. Using the cr-order relation, we will study automatic error analysis with intervals and apply harmonically cr-h-GL functions to optimize problems. Using this concept, we aim to benefit and advance the research of other scientists in various scientific disciplines.

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Conflict of interest

The authors declare that there is no conflicts of interest in publishing this paper.

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