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Research article

New extension to fuzzy dynamic system and fuzzy fixed point results with an application

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Abstract: In this paper we introduce the notion of fuzzy dynamic system in b-metric-like space. By applying this, discuss some new refinements of the F-fuzzy Suzuki-type fixed point results for the fuzzy operators are presented. Also, establish the concept fuzzy dynamic system instead of the Piscard iterative sequence, which improves the existing results for such analysis as those presented here. Includes some tangible instances and an application are given to highlight the usability and validity of the theoretical results.

Keywords: fuzzy dynamic system; fuzzy mappings; fuzzy fixed points; b-metric-like space; Hukuhara differentiability problem
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1. Introduction

Dynamic process is a mightful formalistic apparatus for association with a large spectrum analysis of multistage decision making problems. Such problems appear and are congruent in essentially all human activities. Unfavourably, for explicit reasons, the analysis of fuzzy dynamic process is difficult. Fuzzy dynamic process are characteristic of all dynamic process where the variables associated are state and decision variables. Fuzzy dynamic iterative process is established as a process getting preprocessed inputs and having outputs that are furthermore defuzzified for realistic applications.

In the light of epistemic access, the term fuzzy sets appear as descriptions or perceptions of nonexistent underlying crisp values. As an example, it is noted that the temperature was high form but the numerical value is uncharted. This leads the way, to a number of classical problems which usually provide themselves to fuzzification fashions like Zadeh's generalization theorem [19].

In functional analysis, the field Banach fixed point theory originate as an imperative apparatus over the last some decades in non-linear sciences and engineering via behavioral science, economics, etc see ([4, 6–8, 11, 12, 14, 16, 20, 21, 23–26, 29, 31]). To be unequivocal, while codifying an experiment mathematically, many number of researchers to interrogate the solvability of a functional equation in terms of differential equations, integral equations, or fractional differential equations. Such as the existence and uniqueness of a solution are often achieved by finding fixed point of a particular contraction mapping, (see more [1, 3, 9, 10, 13, 15, 18, 30]). The three major structure in Banach fixed point theory are metric structure, topological structure, and discrete structure. These idea was extend by either generalized metric spaces into by modifying the structure of the contraction operators. However, Nadler [22] display the concept of Hausdorff metric discoursed the Banach fixed point theory for multi-valued mapping rather than single-valued mappings.

On the other hand, Alghamdi et al. [2] improved the idea of partial metric space to b-metric-like space. They produced interesting theorems of fixed point in the newly defined frame. Their concept was expedited by various researchers in many ways (see more [17, 27, 28]).

This article regards fuzzy dynamic process as fuzzy dynamic process on b-metric-like space, specifically the mapping of set-valued (extended) fuzzy intervals endowed with the b-metric-like. From that point of view, a natural topic is convergence theorems via fuzzy dynamic process in the class of b-metric-like space. Our view of convergence theorems in b-metric-like space, then, disposes of fuzzy dynamic process entirely. Instead, we just adopt the standard setting of fuzzy dynamic process in b-metric-like space which defines convergence theorems in generalized $\mathcal{F}$-contraction via expectations of fuzzy Suzuki Hardy Rogers type contraction operators. Subsequently, corollaries are originated from the main result. To explain the example in the main section, a table and diagram has been created that best illustrates the Fuzzy dynamic process to the readers. At the end, gives an application of our results in solving Hukuhara differentiability through the fuzzy initial valued problem and fuzzy functions. The pivotal role of Hukuhara differentiability in Fuzzy dynamic process is stated. At last, a summary of the article is described in the conclusion section.

2. Preliminaries

Formally, an fuzzy set is defined as [32]:

A fuzzy set on G is a mapping that assigns every value of G to some element in $[0, 1]$. The family of all such mappings is expressed as $F(G)$. For a fuzzy set A on G and $\mu \in G$, the value $A(\mu)$ is known as the membership grade of μ in A. The $\alpha-$level set of A expressed as $[A]_\alpha$ is given by

$$\begin{cases} [A]_\alpha = \{\mu : A(\mu) \geqq \alpha\}, \alpha \in (0, 1]; \\ [A]_0 = \overline{\{\mu : A(\mu) > 0\}}. \end{cases}$$

For a nonempty set G and an ms G', a mapping $T : G \to F(G')$ is a fuzzy mapping and is a fuzzy subset of $G \times G'$ having the membership function $T(g)(g')$. $T(g)(g')$ describes the membership grade of g' in $T(g)$, while $[T(g)]_\alpha$ states the $\alpha-$level set of $T(g)$, for more details see [5].

Definition 2.1. [5] A point $g \in G$ is called a fuzzy fixed point of a fuzzy mapping $T : G \to F(G)$ if there is $\alpha \in (0, 1]$ such that $g \in [T(g)]_\alpha$.

In the recent past, Wardowski [31] provided the term known as F-contraction and implemented on Banach fixed point theory. Which is the efficient generalization of Banach fixed point theory. Formally, an F-contraction is defined as follows [31]:

Definition 2.2. Let $\nabla_{\mathcal{F}}$ is the set of mappig $\mathcal{F} : R^+ \longrightarrow R$ satisfying $(\mathrm{F}_i) - (\mathrm{F}_{iii})$:
(F_i) $\mu_1 < \mu_2$ implies $\mathcal{F}(\mu_1) < \mathcal{F}(\mu_2)$ for all $\mu_1, \mu_2 \in (0, +\infty)$;
(F_{ii}) For every sequence $\{\mu_\sigma\}$ in R^+ such that

$$\lim_{\sigma \longrightarrow +\infty} \mu_\sigma = 0 \text{ if and only if } \lim_{\sigma \longrightarrow +\infty} \mathcal{F}(\mu_\sigma) = -\infty;$$

(F_{iii}) There exist $k \in (0, 1)$ such that $\lim_{\mu \to 0}()\mu^k \mathcal{F}(\mu) = 0$.

A mapping $T : G \to G$ is called an $\mathcal{F}$-contraction on a metric space (G, d), if there is $\tau \in \mathbb{R}^+ / \{0\}$ such that

$$d(T\mu_1, T\mu_2) > 0 \;\Rightarrow\; \tau + \mathcal{F}(T\mu_1, T\mu_2) \leqq \mathcal{F}(d(\mu_1, \mu_2)) \text{ for each } \mu_1, \mu_2 \in G.$$

After, we recall the following some basic idea of dynamic system:

Let $\xi : G \to C(G)$ be a mapping. A set

$$\check{D}(\xi, \mu_0) = \{(\mu_a)_{a \geq 0} : \mu_a \in \xi\mu_{a-1} \text{ for all} a \in N\}.$$

is called dynamic process $\check{D}(\xi, \mu_0)$ of μ with starting point μ_0. Where $\mu_0 \in G$ be arbitrary and fixed. In the light of $\check{D}(\xi, \mu_0)$, $(\mu_a)_{a \in N - \{0\}}$ onward has the form (μ_a) (see more [18]).

Further, the literature contains many generalizations of the idea of fixed point theory in metric spaces and its topological behavior. In particularly, Alghamdi et al. [2] designed the fashion of b-metric-like space as follows:

Definition 2.3. [2] Let G be a b-metric-like space with $G \neq \phi$ and $s \geq 1$. A function $d : G \times G \to \mathbb{R}^+ \cup \{0\}$ such that for every $\mu_1, \mu_2, \mu_3 \in G$, the following conditions (b_i), (b_{ii}) and (b_{iii}) hold true:
(b_i) the condition: $d(\mu_1, \mu_2) = 0$ implies $\mu_1 = \mu_2$;
(b_{ii}) the condition is hold true: $d(\mu_1, \mu_2) = d(\mu_2, \mu_1)$;
(b_{iii}) the condition is satisfied: $d(\mu_1, \mu_3) \leqq s[d(\mu_1, \mu_2) + d(\mu_2, \mu_3)]$.
The pair $(G\ , d)$ is known as a b-metric-like space.

Example 2.4. Define (G, d) with $s = 2$ by

$$d(0,0) = 0, d(1,1) = d(2,2) = d(0,2) = 2, d(0,1) = 4, d(1,2) = 1,$$

with

$$d(\mu_1, \mu_2) = d(\mu_2, \mu_1),$$

for all $\mu_1, \mu_2 \in G = \{0, 1, 2\}$. Then, (G, d) is a b-metric-like space. Clearly, it is neither a b-metric nor a metric-like space, see more detail in [2].

Remark 2.5. Owing to above definition (2.3), every partial metric is a b-metric-like space but converse may not hold true in general, see more [2]

Nadler [22], design the idea of Hausdorff metric and extended the Banach contraction theorem for multi-valued operators instead of single-valued operators. Hereinafter, we investigate the concept of Hausdorff b-metric-like as follows. Let (G,μ) be a b-metric-like space. For $\mu_1 \in G$ and $L_1 \subseteq G$, let $d_b(\mu_1, L_2) = \inf\{d(\mu_1,\mu_2) : \mu_2 \in L_2\}$. Define $\hat{H}_b : CB(G)\times CB(G) \to [0,+\infty)$ by

$$\hat{H}_b(L_1, L_2) = \max\left\{\sup_{\mu_1\in L_1} d_b(\mu_1, L_2), \sup_{\mu_2\in L_2} d_b(\mu_1, L_1)\right\},$$

for each $L_1, L_2 \in CB(G)$. Where $CB(G)$ denote the family of all non-empty closed and bounded-subsets of G and $CL(G)$ the family of all non-empty closed-subsets of G.

Definition 2.6. [5] Let $L_1, L_2 \in V(G)$, $\alpha \in (0,1]$. Then $d_\alpha(L_1, L_2) = inf_{g\in L_{1\alpha}, g'\in L_{2\alpha}} d(g, g')$,

$$H_\alpha(L_1, L_2) = \hat{H}_{b^l}(L_{1\alpha}, L_{2\alpha}),$$

where $\hat{H}_{b^l}$ is the Hausdorff distance.

Lemma 2.7. *Let L_1 and L_2 be nonempty proximal subsets of a b-MLS (G,d). If $g \in L_1$, then*

$$d(g, L_2) \leqq H(L_1, L_2).$$

Lemma 2.8. *Let (G,d) be a b-metric-like space. For all $L_1, L_2 \in CB(G)$ and for any $g \in L_1$ such that $d(g, L_2) = d(g, g')$, where $g' \in L_2$. Then, $\hat{H}_{b^l}(L_1, L_2) \geq d(g, g')$.*

In the following, the concept of fuzzy dynamic process as a generalization of dynamic process, and some elementary facts about these concepts are discussed.

3. Fuzzy dynamic process: $\check{D}([T\mu]_\alpha, \mu_0)$

In this section, first we deal with some new aspects of the fuzzy dynamic process as follows:

Definition 3.1. Let $T : G \to F(G)$ be a fuzzy mapping. If there is $\alpha \in (0,1]$, and let $\mu_0 \in G$ be arbitrary and fixed such that

$$\check{D}([T\mu]_\alpha, \mu_0) = \left\{(\mu_j)_{j\in\mathbb{N}\cup\{0\}} : \mu_j \in \left[T\mu_{j-1}\right]_\alpha,\ \forall\ j \in \mathbb{N}\right\}.$$

Every membership value of $\check{D}([T\mu]_\alpha, \mu_0)$ is called a fuzzy dynamic process of T starting point μ_0. The fuzzy dynamic process $(\mu_j)_{j\in\mathbb{N}\cup\{0\}}$ onward is written as $\left(\mu_j\right)$.

Example 3.2. Let $G = C([0,1])$ be a Banach space with norm $\|\mu\| = \sup_{r\in[0,1]} |\mu(r)|$ for $\mu \in G$. Let $T : G \to F(G)$ be a fuzzy mapping. If there is $\alpha \in (0,1]$ such that for every $\mu \in G$, $[T\mu]_\alpha$ is a set of the function

$$\delta \longmapsto k\int_0^\delta \mu(r)\,dr,\ k \in [0,1],$$

that is,

$$\check{D}([T\mu]_\alpha(\delta), \mu_0) = \{k\int_0^\delta \mu(r)\,dr : k \in [0,1]\},\ \mu \in G,$$

and let $\mu_0(\delta) = \delta$, $\delta \in [0, 1]$. Then the iterative sequence

$$\mu_j = \begin{cases} (\frac{1}{j!(j+1)!}\delta^{j+1}), & j \geq 0; \\ 0 \text{ elsewehere.} \end{cases}$$

is a fuzzy dynamic process of mapping T with starting point μ_0. The mapping $T : G \to F(\mathbb{R})$ is said to be $\check{D}([T\mu]_\alpha, \mu_0)$ fuzzy dynamic lower semi-continuous at $\mu \in G$, if for every fuzzy dynamic process $(\mu_j) \in D(T, \mu_0)$ and for every subsequence $(\mu_{j(i)})$ of (μ_j) convergent to μ

$$[T\mu]_\alpha \leq \lim \inf_{i \to +\infty} \left[T\mu_{j(i)}\right]_\alpha.$$

In this case, T is fuzzy dynamic lower semi-continuous $\check{D}([T\mu]_\alpha, \mu_0)$. If T is fuzzy dynamic lower semi-continuous $\check{D}([T\mu]_\alpha, \mu_0)$ at each $\mu \in G$, then T is known as lower semi-continuous. For every sequence $(\mu_j) \subset G$ and $\mu \in G$ such that $(\mu_j) \to \mu$, we have $[T\mu]_\alpha \leq \lim\inf_{i \to +\infty} [T\mu(j)]_\alpha$.

Example 3.3. Let $G = \mathbb{R}^+ \cup \{0\}$. Define $T : G \to F(G)$ by

$$T(\mu)(\mu') = \begin{cases} 1, & \text{if } 0 \leq \mu' \leq \frac{\mu}{4}; \\ \frac{1}{2}, & \text{if } \frac{\mu}{4} < \mu' \leq \frac{\mu}{3}; \\ \frac{1}{4}, & \text{if } \frac{\mu}{3} < \mu' \leq \frac{\mu}{2}; \\ 0, & \text{if } \frac{\mu}{2} < \mu' \leq 1. \end{cases}$$

all $\mu \in G$, there is $\alpha(\mu) = 1$ such that $[T\mu]_{\alpha(\mu)} = \left[0, \frac{\mu}{2}\right]$. Apply the following iterative procedure to generate a sequence $\{\mu_n\}$ of fuzzy sets is given by (see Table 1 and Figure 1)

$$\mu_i = \begin{cases} \mu_0 h^{i-1}, & \text{if } i \geq 2; \\ 0, & \text{elsewhere.} \end{cases}$$

Where $\mu_0 = 2$ is intial point and $h = \frac{1}{2}$.

Table 1. Fuzzy dynamic process.

$i \geq 2$	$\mu_i = \mu_0 g^{i-1}$	$\in$	$[T\mu]_{\alpha(\mu)} = \left[0, \frac{\mu}{2}\right]$
$\mu_{i=2}$	1	–	$[T\mu_1]_{\alpha(\mu_1)} = [0, 1]$
$\mu_{i=3}$	$\frac{1}{2}$	–	$[T\mu_2]_{\alpha(\mu_2)} = [0, \frac{1}{2}]$
$\mu_{i=4}$	$\frac{1}{4}$	–	$[T\mu_3]_{\alpha(\mu_3)} = [0, \frac{1}{4}]$
$\mu_{i=5}$	$\frac{1}{8}$	–	$[T\mu_3]_{\alpha(\mu_4)} = [0, \frac{1}{8}]$
$\mu_{i=6}$	$\frac{1}{16}$	–	$[T\mu_3]_{\alpha(\mu_5)} = [0, \frac{1}{16}]$
$\mu_{i=7}$	$\frac{1}{32}$	–	$[T\mu_3]_{\alpha(\mu_6)} = [0, \frac{1}{32}]$
$\mu_{i=8}$	$\frac{1}{64}$	–	$[T\mu_3]_{\alpha(\mu_7)} = [0, \frac{1}{64}]$
$\mu_{i=9}$	$\frac{1}{128}$	–	$[T\mu_3]_{\alpha(\mu_8)} = [0, \frac{1}{128}]$
$\mu_{i=10}$	$\frac{1}{256}$	–	$[T\mu_4]_{\alpha(\mu_9)} = [0, \frac{1}{256}]$

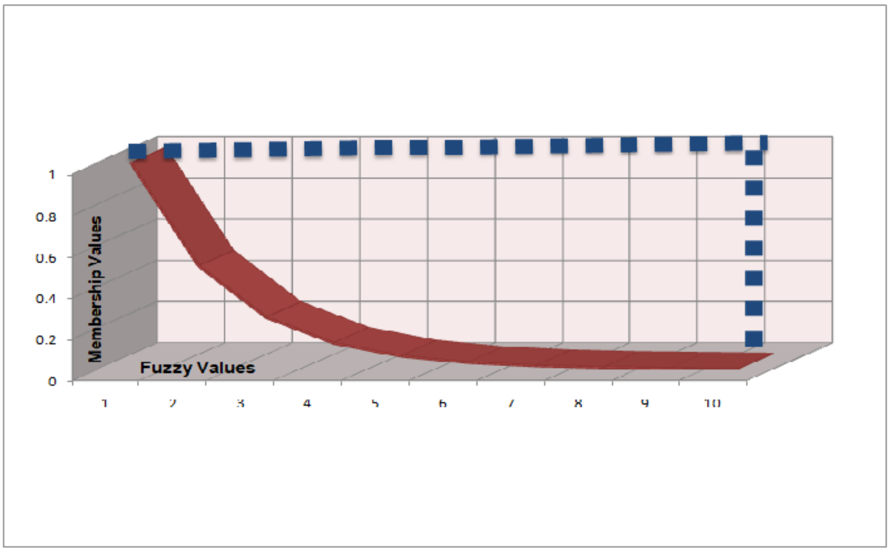

Figure 1. Fuzzy dynamic process: $\check{D}([T\mu]_{\alpha}, \mu_0)$.

We obtain,

$$\check{D}([T\mu]_{\alpha}, \mu_0) = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}\}$$

is a fuzzy dynamic process of T starting at point $\mu_0 = 2$.

Further, in the following we develop fuzzy fixed point theorems with respect to fuzzy dynamic process $\check{D}([T\mu]_{\alpha}, \mu_0)$ as follows.

4. Fuzzy fixed point theorems with respect to fuzzy dynamic process: $\check{D}([T\mu]_{\alpha}, \mu_0)$

Now, we start with the following main definition:

Definition 4.1. Let (G, d) be a b-metric-like space with $s \geq 1$. A mapping $T : G \to F(G)$ is called a F-fuzzy Suzuki-Hardy-Rogers (abbr., F-FSHR) type contraction with respect to $\check{D}([T\mu]_{\alpha}, \mu_0)$ and $\alpha : G \to (0, 1]$ such that $[T\mu_i]_{\alpha(i)}$ are nonempty closed subsets of G if for some $\mathcal{F} \in \nabla_F$ and $\tau : (0, +\infty) \to (0, +\infty)$ such that

$$\frac{1}{2s} d_b(\mu_{i-1}, [T\mu_{i-1}]_{\alpha(i-1)}) \leqq d(\mu_{i-1}, \mu_i),$$

we have

$$\tau(U(\mu_{i-1}, \mu_i)) + \mathcal{F}[\hat{H}_b([T\mu_i]_{\alpha(i)}, [T\mu_{i+1}]_{\alpha(i+1)})] \leqq \mathcal{F}(U(\mu_{i-1}, \mu_i)), \tag{4.1}$$

where

$$U(\mu_{i-1}, \mu_i) = e_1 [d(\mu_{i-1}, \mu_i)] + e_2[d_b(\mu_{i-1}, [T\mu_{i-1}]_{\alpha(i-1)})] + e_3[d_b(\mu_i, [T\mu_i]_{\alpha(i)})]$$

$$+\frac{e_4}{2s}[d_b(\mu_{i-1},[T\mu_i]_{\alpha(i)})]+\frac{e_5}{2s}[d_b(\mu_i,[T\mu_{i-1}]_{\alpha(i-1)})],$$

for all $\mu_i \in \check{D}([T\mu]_\alpha, \mu_0)$, $\hat{H}_b([T\mu_i]_{\alpha(i)}, [T\mu_{i+1}]_{\alpha(i+1)}) > 0$, where $e_1, e_2, e_3, e_4, e_5 \in [0,1]$ such that $e_1 + e_2 + e_3 + e_4 + e_5 = 1$ and $1 - e_3 - e_5 > 0$.

Remark 4.2. To continue with our results, the behavior of self distance in b-metric-like space is defined by

$$d(\mu_1,\mu_1) \le 2d(\mu_1,\mu_2).$$

Additionally, we assume that $\mu_i \in \check{D}([T\mu]_\alpha, \mu_0)$ satisfying fuzzy dynamic process for below condition:

$$d_b(\mu_i,[T\mu_i])_{\alpha(i)} > 0,\ d_b(\mu_{i-1},[T\mu_{i-1}]_{\alpha(i-1)}) > 0, \tag{4.2}$$

for all $i \in \mathbb{N}$. If for the investigated process that does not satisfy (4.2), there is some $i_0 \in \mathbb{N}$ such that

$$d_b(\mu_{i_0},[T\mu_{i_0}]_{\alpha(i_0)}) > 0,$$

and

$$d_b(\mu_{i_{0-1}},[T\mu_{i_{0-1}}]_{\alpha(i_{0-1})}) = 0,$$

then we get $\mu_{i_{0-1}} = \mu_{i_0} \in [T\mu_{i_{0-1}}]_{\alpha(i_{0-1})}$ which implies the existence of fuzzy fixed point. In the light of this consideration, fuzzy dynamic process satisfying (4.2) does not depreciate a generality of our analysis.

Now, we proceed to our main result:

Theorem 4.3. *Let (G,d,s) be a complete b-metric-like space. Let $T : G \to \mu_\alpha(G)$ be an F-FSHR type contraction with respect to μ_i. Assume that the following holds:*
(i) There is a fuzzy dynamic iterative process $\mu_i \in \check{D}([T\mu]_\alpha, \mu_0)$ such that for each $l \ge 0$ $\liminf_{k\to l^+} \tau(k) > 0$;
(ii) A mapping $G \ni \mu_i \longmapsto d_b(\mu_i,[T\mu_i]_{\alpha(i)})$ is fuzzy dynamic lower semi-continuous $\check{D}([T\mu]_\alpha, \mu_0)$;
(iii) If, in addition, $\mathcal{F}$ is super-additive, i.e., for $\mu_1,\mu_2,\xi_1,\xi_2 \in R^+$ we have

$$\mathcal{F}(\xi_1\mu_1 + \xi_2\mu_2) \le \xi_2 F(\mu_1) + \xi_2 F(\mu_2).$$

Then T has a fuzzy fixed point.

Proof. Choose an arbitrary point $\mu_0 \in G$. In veiw of fuzzy dynamic iterative process, we have

$$\check{D}([T\mu]_\alpha, \mu_0) = \{(\mu_i)_{i\in\mathbb{N}\cup\{0\}} : \mu_{i+1} = \mu_i \in [T\mu_{i-1}]_{\alpha(i-1)} \text{ for all } i \in \mathbb{N}\}.$$

In case that there is $i_0 \in \mathbb{N}$ such that $\mu_{i_0} = \mu_{i_{0+1}}$, then our proof of Theorem (4.3) go ahead as follows. If we let $\mu_i \ne \mu_{i+1}$ for all $i \in \mathbb{N}$, then we have

$$\frac{1}{2s}d_b(\mu_i,[T\mu_i]_{\alpha(i)}) \le d(\mu_i,\mu_{i+1}),\ \text{for all } i \in \mathbb{N}. \tag{4.3}$$

From (4.1) and in the light of Lemma (2.8), we have

$$\mathcal{F}(d(\mu_{i+1},\mu_{i+2}) \quad \le \quad \mathcal{F}[\hat{H}_b([T\mu_i]_{\alpha(i)}, [T\mu_{i+1}]_{\alpha(i+1)})] \tag{4.4}$$

$$\begin{aligned}
&\leq \mathcal{F}\Big[e_1 d(\mu_i,\mu_{i+1}) + e_2 d_b\left(\mu_i,[T\mu_i]_{\alpha(i)}\right) + e_3 d_b\left(\mu_{i+1},[T\mu_{i+1}]_{\alpha(i+1)}\right) \\
&\quad + \frac{e_4}{2s} d_b\left(\mu_i,[T\mu_{i+1}]_{\alpha(i+1)}\right) + \frac{e_5}{2s} d_b\left(\mu_{i+1},[T\mu_i]_{\alpha(i)}\right)\Big] \\
&\quad -\tau\Big[e_1 d(\mu_i,\mu_{i+1}) + e_2 d_b\left(\mu_i,[T\mu_i]_{\alpha(i)}\right) + e_3 d_b\left(\mu_{i+1},[T\mu_{i+1}]_{\alpha(i+1)}\right) \\
&\quad + \frac{e_4}{2s} d_b\left(\mu_i,[T\mu_{i+1}]_{\alpha(i+1)}\right) + \frac{e_5}{2s} d_b\left(\mu_{i+1},[T\mu_i]_{\alpha(i)}\right)\Big].
\end{aligned}$$

Now, we survey to the following inequality

$$d_b(\mu_{i+1},[T\mu_{i+1}]_{\alpha(i+1)}) < d_b\left(\mu_i,[T\mu_i]_{\alpha(i)}\right), \tag{4.5}$$

for all $i \in \mathbb{N}$. Suppose, on the contrary, there is $i_0 \in \mathbb{N}$ such that $d(\mu_{i_0+1},[T\mu_{i_0+1}]_{\alpha(i_0+1)}) \geq d(\mu_{i_0},[T\mu_{i_0}]_{\alpha(i_0)})$. By (4.4) and Lemma (2.8), we have

$$\begin{aligned}
\mathcal{F}\left[d_b(\mu_{i_0+1},[T\mu_{i_0+1}]_{\alpha(i_0+1)})\right] &= \mathcal{F}\left[d(\mu_{i_0+1},\mu_{i_0+2})\right] \\
&\leq \mathcal{F}\left[\hat{H}_b\left([T(\mu_{i_0})]_{\alpha(i_0)},[T(\mu_{i_0+1})]_{\alpha(i_0+1)}\right)\right] - \tau\left(U(\mu_{i_0},\mu_{i_0+1})\right) \\
&\leq \mathcal{F}\Big[e_1\left(d_b\left(\mu_{i_0},[T(\mu_{i_0})]_{\alpha(i_0)}\right)\right) + e_2\left(d_b\left(\mu_{i_0},[T(\mu_{i_0})]_{\alpha(i_0)}\right)\right) \\
&\quad + e_3\left(d_b\left(\mu_{i_0+1},[T(\mu_{i_0+1})]_{\alpha(i_0+1)}\right)\right) \\
&\quad + \frac{e_4}{2s}\left(d_b\left(\mu_{i_0},[T(\mu_{i_0+1})]_{\alpha(i_0+1)}\right)\right) \\
&\quad + \frac{e_5}{2s}\left(d_b\left(\mu_{i_0+1},[T(\mu_{i_0})]_{\alpha(i_0)}\right)\right)\Big] - \tau\left(U(\mu_{i_0},\mu_{i_0+1})\right) \\
&\leq \mathcal{F}\Big[e_1 d_b\left(\mu_{i_0},[T(\mu_{i_0})]_{\alpha(i_0)}\right) + e_2\left(d_b\left(\mu_{i_0},[T(\mu_{i_0})]_{\alpha(i_0)}\right)\right) \\
&\quad + e_3\left(d_b\left(\mu_{i_0+1},[T(\mu_{i_0+1})]_{\alpha(i_0+1)}\right)\right) \\
&\quad + \frac{se_4}{2s} d_b\left(\mu_{i_0},[T(\mu_{i_0})]_{\alpha(i_0)}\right) \\
&\quad + \frac{se_4}{2s} d_b\left([T(\mu_{i_0})]_{\alpha(i_0)},[T(\mu_{i_0+1})]_{\alpha(i_0+1)}\right) \\
&\quad + \frac{2se_5}{2s}\left(d_b(\mu_{i_0},[T(\mu_{i_0})]_{\alpha(i_0)}\right)\Big] \\
&\quad -\tau\left(U(\mu_{i_0},\mu_{i_0+1})\right).
\end{aligned} \tag{4.6}$$

Owing to the above hypothesis, this, in turn, yields:

$$\begin{aligned}
\mathcal{F}\left[d_b(\mu_{i_0+1},[T\mu_{i_0+1}]_{\alpha(i_0+1)})\right] &\leq \mathcal{F}\Big[e_1 d_b\left(\mu_{i_0},[T(\mu_{i_0})]_{\alpha(i_0)}\right) + e_2\left(d_b\left(\mu_{i_0},[T(\mu_{i_0})]_{\alpha(i_0)}\right)\right) \\
&\quad + e_3\left(d_b\left(\mu_{i_0+1},[T(\mu_{i_0+1})]_{\alpha(i_0+1)}\right)\right) \\
&\quad + e_4\left(d_b\left([T(\mu_{i_0})]_{\alpha(i_0)},[T(\mu_{i_0+1})]_{\alpha(i_0+1)}\right)\right) \\
&\quad + e_5\left(d_b\left(\mu_{i_0},[T(\mu_{i_0})]_{\alpha(i_0)}\right)\right)\Big] - \tau\left(U(\mu_{i_0},\mu_{i_0+1})\right).
\end{aligned}$$

Since $\mathcal{F}$ is super-additive, we can write

$$\mathcal{F}\left[d_b(\mu_{i_0+1},[T\mu_{i_0+1}]_{\alpha(i_0+1)})\right] \leq \frac{(e_1+e_2+e_5)}{(1-e_3-e_4)}\mathcal{F}\left[d_b(\mu_{i_0},[T\mu_{i_0}]_{\alpha(i_0)})\right] - \frac{\tau\left(U(\mu_{i_0-1},\mu_{i_0})\right)}{(1-e_3-e_4)}.$$

From this, By given condition $e_1 + e_2 + e_3 + e_4 + e_5 = 1$, we have

$$\mathcal{F}\left[d_b(\mu_{i_0+1}, [T\mu_{i_0+1}]_{\alpha(i_0+1)})\right] \leq \mathcal{F}\left[d_b(\mu_{i_0}, [T\mu_{i_0}]_{\alpha(i_0)})\right] - \frac{\tau\left(U(\mu_{i_0-1}, \mu_{i_0})\right)}{(1 - e_3 - e_4)}, \tag{4.7}$$

a contradiction. Hence (4.5) holds true. In the light of above hypothesis,Therefore $d_b\left(\mu_i, [T\mu_i]_{\alpha(i)}\right)$ is a decreasing sequence with respect to real number and it is bounded from below . Suppose that there is $\Psi \geq 0$ such that

$$\Psi = \lim_{i\to+\infty} d_b\left(\mu_i, [T\mu_i]_{\alpha(i)}\right) = \inf\left\{d_b\left(\mu_i, [T\mu_i]_{\alpha(i)}\right) : i \in \mathbb{N}\right\}. \tag{4.8}$$

We now to prove that $\Psi = 0$. Suppose, based on contrary that $\Psi > 0$. Then, for every $\varepsilon > 0$, there is a natural number j such that

$$d_b\left(\mu_j, \left[T\mu_j\right]_{\alpha(j)}\right) < \Psi + \varepsilon.$$

By $(\mathcal{F}_i)$,

$$\mathcal{F}\left[d_b\left(\mu_j, \left[T\mu_j\right]_{\alpha(j)}\right)\right] < \mathcal{F}(\Psi + \varepsilon). \tag{4.9}$$

Also, by applying (4.3), we have

$$\frac{1}{2s} d_b\left(\mu_j, \left[T\mu_j\right]_{\alpha(j)}\right) \leq d_b\left(\mu_j, \mu_{j+1}\right), \text{ for all } i \in \mathbb{N}.$$

Since F-FSHR type contraction with respect to $\check{D}(T, \mu_0)$, we have

$$\begin{aligned}
\mathcal{F}\left[d_b(\mu_{j+1}, \left[T\mu_{j+1}\right]_{\alpha(j+1)})\right] &= \mathcal{F}\left[d(\mu_{j+1}, \mu_{j+2})\right] \\
&\leq \mathcal{F}\left[\hat{H}_b\left(\left[T\left(\mu_j\right)\right]_{\alpha(j)}, \left[T\left(\mu_{j+1}\right)\right]_{\alpha(j+1)}\right)\right] - \tau\left(U(\mu_j, \mu_{j+1})\right) \\
&\leq \mathcal{F}\left[e_1\left(d_b\left(\mu_j, \left[T\left(\mu_j\right)\right]_{\alpha(j)}\right)\right) + e_2\left(d_b\left(\mu_j, \left[T\left(\mu_j\right)\right]_{\alpha(j)}\right)\right)\right. \\
&\quad + e_3\left(d_b\left(\mu_{j+1}, \left[T\left(\mu_{j+1}\right)\right]_{\alpha(j+1)}\right)\right) \\
&\quad + \frac{e_4}{2s}\left(d_b\left(\mu_j, \left[T\left(\mu_{j+1}\right)\right]_{\alpha(j+1)}\right)\right) \\
&\quad \left. + \frac{e_5}{2s}\left(d_b\left(\mu_{j+1}, \left[T\left(\mu_j\right)\right]_{\alpha(j)}\right)\right)\right] - \tau\left(U(\mu_j, \mu_{j+1})\right).
\end{aligned}$$

Due to the above hypothesis, this, in turn, yields:

$$\begin{aligned}
\mathcal{F}\left[d_b(\mu_{j+1}, \left[T\mu_{j+1}\right]_{\alpha(j+1)})\right] &\leq \mathcal{F}\left[e_1 d_b\left(\mu_j, \left[T\left(\mu_j\right)\right]_{\alpha(j)}\right) + e_2\left(d_b\left(\mu_j, \left[T\left(\mu_j\right)\right]_{\alpha(j)}\right)\right)\right. \\
&\quad + e_3\left(d_b\left(\mu_{j+1}, \left[T\left(\mu_{j+1}\right)\right]_{\alpha(j+1)}\right)\right) \\
&\quad + \frac{se_4}{2s} d_b\left(\mu_j, \left[T\left(\mu_j\right)\right]_{\alpha(j)}\right) \\
&\quad + \frac{se_4}{2s} d_b\left(\left[T\left(\mu_j\right)\right]_{\alpha(j)}, \left[T\left(\mu_{j+1}\right)\right]_{\alpha(j+1)}\right) \\
&\quad \left. + \frac{2se_5}{2s}\left(d_b(\mu_j, \left[T\left(\mu_j\right)\right]_{\alpha(j)}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
& -\tau\left(U(\mu_j,\mu_{j+1})\right) \\
\leq\ & \mathcal{F}\left[e_1 d_b\left(\mu_j,\left[T\left(\mu_j\right)\right]_{\alpha(j)}\right)+e_2\left(d_b\left(\mu_j,\left[T\left(\mu_j\right)\right]_{\alpha(j)}\right)\right)\right. \\
& +e_3\left(d_b\left(\mu_{j+1},\left[T\left(\mu_{j+1}\right)\right]_{\alpha(j+1)}\right)\right) \\
& +e_4\left(d_b\left(\mu_j,\left[T\left(\mu_j\right)\right]_{\alpha(j)}\right)\right) \\
& \left.+e_5\left(d_b\left(\mu_j,\left[T\left(\mu_j\right)\right]_{\alpha(j)}\right)\right)\right]-\tau\left(U(\mu_j,\mu_{j+1})\right).
\end{aligned}
$$

This implies

$$\mathcal{F}\left[d_b(\mu_{j+1},\left[T\mu_{j+1}\right]_{\alpha(j+1)})\right]\leq\mathcal{F}\left[d_b(\mu_j,\left[T\mu_j\right]_{\alpha(j)})\right]-\frac{\tau\left(U(\mu_j,\mu_{j+1})\right)}{1-e_3}.$$

Since

$$\frac{1}{2s}d_b\left(\mu_{j+1},\left[T\mu_{j+1}\right]_{\alpha(j+1)}\right)\leq d_b\left(\mu_{j+1},\mu_{j+2}\right),\ \text{for all } i\in\mathbb{N}.$$

By appealing to above observation, we obtain

$$\mathcal{F}\left[d_b(\mu_{j+2},\left[T\mu_{j+2}\right]_{\alpha(j+2)})\right]\leq\mathcal{F}\left[d_b(\mu_{j+1},\left[T\mu_{j+1}\right]_{\alpha(j)})\right]-\frac{\tau\left(U(\mu_{j+1},\mu_{j+2})\right)}{1-e_3}. \tag{4.10}$$

Continuing these fashion, we obtain

$$
\begin{aligned}
\mathcal{F}\left[d_b(\mu_{j+i},\left[T\mu_{j+i}\right]_{\alpha(j+i)})\right] &\leq \mathcal{F}\left[d_b\left(\mu_{j+(i-1)},\left[T\mu_{j+(i-1)}\right]_{\alpha(j+(i-1))}\right)\right]-\frac{\tau\left(U(\mu_{j+(i-1)},\mu_{j+i})\right)}{1-e_3} \\
&\leq \mathcal{F}\left[d_b\left(\mu_{j+(i-2)},\left[T\mu_{j+(i-2)}\right]_{\alpha(j+(i-2))}\right)\right]-\left\{\begin{array}{l}\frac{\tau\left(U(\mu_{j+(i-2)},\mu_{j+(i-1)})\right)}{1-e_3} \\ +\frac{\tau\left(U(\mu_{j+(i-1)},\mu_{j+i})\right)}{1-e_3}\end{array}\right. \\
&\ \ \vdots \\
&\leq \mathcal{F}\left[d_b\left(\mu_{j_0},\left[T\mu_{j_0}\right]_{\alpha(j_0)}\right)\right]-\frac{(n-j_0)\,\tau\left(U(\mu_{j_{0-1}},\mu_{j_0})\right)}{1-e_3} \\
&< \mathcal{F}(\Psi+\varepsilon)-\frac{(n-j_0)\,\tau\left(U(\mu_{j_{0-1}},\mu_{j_0})\right)}{1-e_3}.
\end{aligned} \tag{4.11}
$$

Upon setting $i\to+\infty$, we have

$$\lim_{i\to+\infty}\mathcal{F}\left[d_b\left(\mu_{j+i},\left[T\mu_{j+i}\right]_{\alpha(j+i)}\right)\right]=-\infty.$$

Also, in veiw of $(\mathcal{F}_{ii})$, we get

$$\lim_{i\to+\infty}\left[d_b(\mu_{j+i},\left[T\mu_{j+i}\right]_{\alpha(j+i)})\right]=0.$$

So, there is $i_1\in\mathbb{N}$ such that $d_b(\mu_{j+i},\left[T\mu_{j+i}\right]_{\alpha(j+i)})<\Psi$ for all $i>i_1$, which is a contradiction with repect to Ψ. Therefore, we have

$$\lim_{i\to+\infty}\left[d_b\left(\mu_i,\left[T\mu_i\right]_{\alpha(i)}\right)\right]=0. \tag{4.12}$$

Now, we show that

$$\lim_{i,m\to+\infty} d(\mu_i,\mu_m) = 0. \tag{4.13}$$

Let us assume on the contrary that, for every $\varepsilon > 0$ there are sequences $\gamma(i)$ and $\delta(i)$ in $\mathbb{N}$ such that

$$d(\mu_{\gamma(i)},\mu_{\delta(i)}) \geq \varepsilon,\, d_b\left(\mu_{\delta(i)-1},\left[T\mu_{\gamma(i)-1}\right]_{\alpha\left(\gamma(i)-1\right)}\right) < \varepsilon,\, \gamma(i) > \delta(i) > i, \tag{4.14}$$

for all $i \in \mathbb{N}$. So, we have

$$\begin{aligned} d(\mu_{\gamma(i)},\mu_{\delta(i)}) &\leq sd_b\left(\mu_{\gamma(i)-1},\left[T\mu_{\gamma(i)-1}\right]_{\alpha\left(\gamma(i)-1\right)}\right) + sd_b\left(\left[T\mu_{\gamma(i)-1}\right]_{\alpha\left(\gamma(i)-1\right)},\mu_{\delta(i)}\right) \\ &< sd_b\left(\mu_{\gamma(i)},\left[T\mu_{\gamma(i)-2}\right]_{\alpha\left(\gamma(i)-2\right)}\right) + s\varepsilon. \end{aligned} \tag{4.15}$$

By (4.12), $\exists\, i_2 \in \mathbb{N}$ such that

$$d_b\left(\mu_{\gamma(i)-1},\left[T\mu_{\gamma(i)-1}\right]_{\alpha\left(\gamma(i)-1\right)}\right) < \varepsilon,\; d_b\left(\mu_{\gamma(i)},\left[T\mu_{\gamma(i)}\right]_{\alpha\left(\gamma(i)\right)}\right) < \varepsilon,\; d_b\left(\mu_{\delta(i)},\left[T\mu_{\delta(i)}\right]_{\alpha\left(\delta(i)\right)}\right) < \varepsilon, \tag{4.16}$$

for all $i > i_2$, which together with (4.15) yields

$$d(\mu_{\gamma(i)},\mu_{\delta(i)}) < 2s\varepsilon) \text{ for all } i > i_2.$$

In view of $(\mathcal{F}_i)$, we can write

$$\mathcal{F}\left(d(\mu_{\gamma(i)},\mu_{\delta(i)})\right) < \mathcal{F}(2s\varepsilon) \text{ for all } i > i_2. \tag{4.17}$$

From (4.14) and (4.16), we write

$$\frac{1}{2s}d_b\left(\mu_{\gamma(i)},\left[T\mu_{\gamma(i)}\right]_{\alpha\left(\gamma(i)\right)}\right) < \frac{\varepsilon}{2s} < d(\mu_{\gamma(i)},\mu_{\delta(i)}) \text{ for all } i > i_2. \tag{4.18}$$

Applying the triangle inequality, we find that

$$\begin{aligned} \epsilon \leq d(\mu_{\gamma(i)},\mu_{\gamma(i)}) &\leq sd\left(\mu_{\gamma(i)},\mu_{\gamma(i)+1}\right) \\ &+ s^2d\left(\mu_{\gamma(i)+1},\mu_{\delta(i)+1}\right) + s^2d\left(\mu_{\delta(i)+1},\mu_{\delta(i)}\right). \end{aligned} \tag{4.19}$$

Next, if we setting to the limit $i \to +\infty$ in (4.19) and make use of (4.12), then,

$$\frac{\epsilon}{s^2} \leq \lim_{i\to+\infty} \inf d\left(\mu_{\gamma(i)+1},\mu_{\delta(i)+1}\right).$$

Also, there is $i_3 \in \mathbb{N}$ such that

$$d\left(\mu_{\gamma(i)+1},\mu_{\delta(i)+1}\right) > 0,$$

for all $i > i_3$, that is, $d\left(\mu_{\gamma(i)+1},\mu_{\delta(i)+1}\right) > 0 > 0$ for $i > i_3$. Further, from (4.1) and Lemma (2.8), we can write

$$\mathcal{F}\left[d\left(\mu_{\gamma(i)+1},\mu_{\delta(i)+1}\right)\right] \leq \mathcal{F}\left(\hat{H}_b\left(\left[T\mu_{\gamma(i)}\right]_{\alpha\left(\gamma(i)\right)},\left[T\mu_{\delta(i)}\right]_{\alpha\left(\delta(i)\right)}\right)\right) - \tau(U(\mu_{\gamma(i)},\mu_{\delta(i)})) \tag{4.20}$$

$$
\begin{aligned}
&\leq \mathcal{F}\Big[e_1(d(\mu_{\gamma(i)},\mu_{\delta(i)})) + e_2\Big(d_b\Big(\mu_{\gamma(i)},\big[T\mu_{\gamma_{(i)}}\big]_{\alpha(\gamma_{(i)})}\Big)\Big) \\
&\quad +e_3\Big(d_b\Big(\mu_{\delta(i)},\big[T\mu_{\delta_{(i)}}\big]_{\alpha(\delta_{(i)})}\Big)\Big) + e_4\Big(d_b\Big(\mu_{\gamma(i)},\big[T\mu_{\delta_{(i)}}\big]_{\alpha(\delta_{(i)})}\Big)\Big) \\
&\quad +e_5\Big(d_b\Big(\mu_{\delta(i)},\big[T\mu_{\gamma_{(i)}}\big]_{\alpha(\gamma_{(i)})}\Big)\Big)\Big] - \tau(U(\mu_{\gamma(i)},\mu_{\delta(i)})) \\
&\leq \mathcal{F}\Big[e_1(d(\mu_{\gamma(i)},\mu_{\delta(i)})) + e_2\Big(d_b\Big(\mu_{\gamma(i)},\big[T\mu_{\gamma_{(i)}}\big]_{\alpha(\gamma_{(i)})}\Big)\Big) \\
&\quad +e_3\Big(d_b\Big(\mu_{\delta(i)},\big[T\mu_{\delta_{(i)}}\big]_{\alpha(\delta_{(i)})}\Big)\Big) + se_4\left(d\left(\mu_{\gamma(i)},\mu_{\delta(i)}\right)\right) \\
&\quad +se_4\Big(d_b\Big(\mu_{\delta(i)},\big[T\mu_{\delta_{(i)}}\big]_{\alpha(\delta_{(i)})}\Big)\Big) + se_5\left(d\left(\mu_{\delta(i)},\mu_{\gamma(i)}\right)\right) \\
&\quad +se_5\Big(d_b\Big(\mu_{\gamma(i)},\big[T\mu_{\gamma_{(i)}}\big]_{\alpha(\gamma_{(i)})}\Big)\Big)\Big] - \tau\left(U(\mu_{\gamma(i)},\mu_{\delta(i)})\right),
\end{aligned}
$$

for all $i > \max\{i_1, i_2\}$. In view of (4.16)–(4.18), inequaility (4.20) yields

$$
\begin{aligned}
\mathcal{F}\left[d\left(\mu_{\gamma_{(i)+1}},\mu_{\delta_{(i)+1}}\right)\right] &\leq \mathcal{F}\Big(\hat{H}_b\Big(\big[T\mu_{\gamma_{(i)}}\big]_{\alpha(\gamma_{(i)})},\big[T\mu_{\delta_{(i)}}\big]_{\alpha(\delta_{(i)})}\Big)\Big) \\
&\leq \mathcal{F}\Big[e_1(2s\varepsilon)) + e_2\Big(d_b\Big(d(\mu_{\gamma(i)},\big[T\mu_{\gamma_{(i)}}\big]_{\alpha(\gamma_{(i)})}\Big)\Big) \\
&\quad +e_3\Big(d_b\Big(\mu_{\delta(i)},\big[T\mu_{\delta_{(i)}}\big]_{\alpha(\delta_{(i)})}\Big)\Big) \\
&\quad +\frac{e_4}{2}(s\varepsilon + s\varepsilon) + \frac{e_5}{2}(s\varepsilon + \varepsilon))] \\
&\quad -\tau\left(U(\mu_{\gamma(i)},\mu_{\delta(i)}))\right),
\end{aligned}
\tag{4.21}
$$

for all $i > \max\{i_1, i_2\}$. Taking the limit $i \to +\infty$ in (4.21), we get

$$
\lim_{i\to+\infty} \mathcal{F}\left[d\left(\mu_{\gamma_{(i)+1}},\mu_{\delta_{(i)+1}}\right)\right] = -\infty,
$$

which by vertue of $(\mathcal{F}_{ii})$, implies that $\lim_{i\to+\infty} d\left(\mu_{\gamma_{(i)+1}},\mu_{\delta_{(i)+1}}\right) = 0$. In the light of (4.19), we can write $\lim_{i\to+\infty} d(\mu_{\gamma_{(i)}},\mu_{\delta_{(i)}}) = 0$, which contradicts. Hence (4.13) holds true. Hence $\{\mu_i\}$ is a Cauchy sequence in G. Since G is a complete b-metric-like space, there is a point $c \in G$ such that

$$
d(c, c) = \lim_{i\to+\infty} d(\mu_i, c) = \lim_{i,j\to+\infty} d(\mu_i, \mu_j) = 0. \tag{4.22}
$$

Now, we show futher the following inequatlity

$$
\frac{1}{2s} d_b\left(\mu_i, [T\mu_i]_{\alpha(i)}\right) < d(\mu_i, c) \text{ or } \frac{1}{2s} d_b\left(\mu_{i+1}, [T\mu_{i+1}]_{\alpha(i+1)}\right) < d\left(\mu_{i+1}, c\right). \tag{4.23}
$$

Assume on the contrary that $\exists\, i_0 \in \mathbb{N}$ such that

$$
\frac{1}{2s} d_b(\mu_{i_0}, [T\mu_{i_0}]_{\alpha(i_0)}) \geq d(\mu_{i_0}, c),\ \frac{1}{2s} d_b\left(\mu_{i_{0+1}}, [T\mu_{i_0+1}]_{\alpha(i_0+1)}\right) \geq d\left(\mu_{i_0+1}, c\right). \tag{4.24}
$$

Then from (4.5) and (4.24), we have

$$
d_b\left(\mu_{i_0}, [T\mu_{i_0}]_{\alpha(i_0)}\right) \leq sd(\mu_{i_0}, c) + sd_b\left(c, [T\mu_{i_0}]_{\alpha(i_0)}\right)
$$

$$
\begin{aligned}
&\leq \frac{1}{2s} s d_b\left(\mu_{i_0}, [T\mu_{i_0}]_{\alpha(i_0)}\right) + \frac{1}{2s} s d_b\left(\mu_{i_{0+1}}, [T\mu_{i_{0+1}}]_{\alpha(i_{0+1})}\right) \\
&\leq \frac{1}{2} d_b\left(\mu_{i_0}, [T\mu_{i_0}]_{\alpha(i_0)}\right) + \frac{1}{2} d_b\left(\mu_{i_0}, [T\mu_{i_0}]_{\alpha(i_0)}\right) \\
&\leq d_b\left(\mu_{i_0}, [T\mu_{i_0}]_{\alpha(i_0)}\right),
\end{aligned}
$$

a contradiction. Thus (4.23) holds true. So,we can write

$$
\begin{aligned}
\mathcal{F}\left(d_b\left(\mu_{i+1}, [T(c)]_{\alpha(c)}\right)\right) &\leq \mathcal{F}\left[\hat{H}_b([T(\mu_i)]_{\alpha(i)}, [T(c)]_{\alpha(c)})\right] - \tau(U(\mu_i, c)) \\
&\leq \mathcal{F}\left[e_1(d(\mu_i, c)) + e_2 d_b\left(\mu_i, [T(\mu_i)]_{\alpha(i)}\right)\right. \\
&\quad + e_3 d_b\left(c, [T(c)]_{\alpha(c)}\right) + \frac{e_4}{2s} d_b\left(\mu_i, [T(c)]_{\alpha(c)}\right) \\
&\quad \left. + \frac{e_5}{2s} d_b\left(c, [T(\mu_i)]_{\alpha(i)}\right)\right] - \tau(U(\mu_i, c)),
\end{aligned} \tag{4.25}
$$

or

$$
\begin{aligned}
\mathcal{F}\left(d\left(\mu_{i+2}, [T(c)]_{\alpha(c)}\right)\right) &\leq \mathcal{F}\left[\hat{H}_b([T(\mu_{i+1})]_{\alpha(i+1)}, [T(c)]_{\alpha(c)})\right] - \tau(U(\mu_i, c)) \\
&\leq \mathcal{F}\left[e_1(d(\mu_{i+1}, c)) + e_2 d_b\left(\mu_{i+1}, [T(\mu_{i+1})]_{\alpha(i+1)}\right)\right. \\
&\quad + e_3 d_b\left(c, [T(c)]_{\alpha(c)}\right) + \frac{e_4}{2s}\left(\mu_{i+1}, [T(c)]_{\alpha(c)}\right) \\
&\quad \left. + \frac{e_5}{2s} d_b\left(c, [T(\mu_{i+1})]_{\alpha(i+1)}\right)\right] - \tau(U(\mu_{i+1}, c)).
\end{aligned} \tag{4.26}
$$

Now, let us now examine the following cases:
Case 1. Assume that (4.25) holds true. From (4.25), we have

$$
\begin{aligned}
\mathcal{F}\left(d_b\left(\mu_{i+1}, [T(c)]_{\alpha(c)}\right)\right) &\leq \mathcal{F}\left[\hat{H}_b([T(\mu_i)]_{\alpha(i)}, [T(c)]_{\alpha(c)})\right] - \tau(U(\mu_i, c)) \\
&\leq \mathcal{F}\left[e_1(d(\mu_i, c)) + e_2 d_b\left(\mu_i, [T(\mu_i)]_{\alpha(i)}\right)\right. \\
&\quad + e_3 d_b\left(c, [T(c)]_{\alpha(c)}\right) + \frac{e_4}{2} d_b(\mu_i, c) \\
&\quad + \frac{e_4}{2} d_b\left(c, [T(c)]_{\alpha(c)}\right) + \frac{e_5}{2} d(c, \mu_i) \\
&\quad \left. + \frac{e_5}{2} d_b\left(\mu_i, [T(\mu_i)]_{\alpha(i)}\right)\right] - \tau(U(\mu_i, c)).
\end{aligned} \tag{4.27}
$$

By (4.12) and (4.22), there is $i_4 \in \mathbb{N}$ such that for some $\varepsilon_1 > 0$

$$
d(c, \mu_i) < \varepsilon_1,\ d_b(\mu_i, [T(\mu_i)]_{\alpha(i)}) < \varepsilon_1,\ \text{for } i > i_4. \tag{4.28}
$$

From (4.27) and (4.28), we have

$$\begin{aligned}\mathcal{F}\left(d_b\left(\mu_{i+1},[T\,(c)]_{\alpha(c)}\right)\right) &\leq \mathcal{F}\left[\hat{H}_b([T(\mu_i)]_{\alpha(i)},[T\,(c)]_{\alpha(c)})\right]-\tau\,(U(\mu_i,c)), \\ &\leq \mathcal{F}\left[e_1(d(\mu_i,c))+e_2d_b\left(\mu_i,[T\,(\mu_i)]_{\alpha(i)}\right)\right. \\ &\quad +e_3d_b\left(c,[T\,(c)]_{\alpha(c)}\right)+\frac{e_4}{2}\,(\varepsilon_1) \\ &\quad +\frac{e_4}{2}d_b\left(c,[T\,(c)]_{\alpha(c)}\right)+e_5\,(\varepsilon_1) \\ &\quad -\tau\,(U(\mu_i,c)),\end{aligned} \tag{4.29}$$

for all $i > i_4$. Taking the limit as $i \to +\infty$ in (4.29), we find that $\lim_{i\to+\infty}\mathcal{F}\left(d_b\left(\mu_{i+1},[T\,(c)]_{\alpha(c)}\right)\right) = -\infty$. By means of $(\mathcal{F}_{ii})$, we have

$$\lim_{i\to+\infty} d_b(\mu_{i+1},[T\,(c)]_{\alpha(c)}) = 0.$$

On the other hand, we see that

$$d_b\left(c,[T\,(c)]_{\alpha(c)}\right) \leq d\,(c,\mu_{i+1})+d_b\left(\mu_{i+1},[T\,(c)]_{\alpha(c)}\right).$$

Further, in the light of above hypothesis with respect to $G \ni c \longmapsto d_b(c,[T\,(\mu_i)]_{\alpha(i)})$ is $\check{D}(T,\mu_0)$-fuzzy dynamic lower semi-continuous, we have

$$d_b(c,[T\,(c)]_{\alpha(c)}) \leq \lim_{n\to+\infty}\inf d_b(c,[T\,(\mu_i)]_{\alpha(i)})+0 = 0.$$

Also, the closedness of $[T\,(c)]_{\alpha(c)}$ implies that $c \in [T\,(c)]_{\alpha(c)}$ which means that c is a fuzzy fixed point of T.

Case 2. Assume that (4.26) holds true. From (4.26), we can write

$$\begin{aligned}\mathcal{F}\left(d_b\left(\mu_{i+2},[T\,(c)]_{\alpha(c)}\right)\right) &\leq \mathcal{F}\left[\hat{H}_b([T(\mu_{i+1})]_{\alpha(i+1)},[T\,(c)]_{\alpha(c)})\right]-\tau\,(U(\mu_{i+1},c)) \\ &\leq \mathcal{F}\left[e_1(d(\mu_{i+1},c))+e_2d_b\left(\mu_{i+1},[T\,(\mu_{i+1})]_{\alpha(i+1)}\right)\right. \\ &\quad +e_3d_b\left(c,[T\,(c)]_{\alpha(c)}\right)+\frac{e_4}{2}d\,(\mu_{i+1},c) \\ &\quad +\frac{e_4}{2}d_b\left(c,[T\,(c)]_{\alpha(c)}\right)+\frac{e_5}{2}d\,(c,\mu_{i+1}) \\ &\quad \left.+\frac{e_5}{2}d_b\left(\mu_{i+1},[T\,(\mu_{i+1})]_{\alpha(i+!)}\right)\right]-\tau\,(U(\mu_{i+1},c)).\end{aligned} \tag{4.30}$$

From (4.12) and (4.22), there is $i_5 \in \mathbb{N}$ such that for some $\varepsilon_2 > 0$

$$d(c,\mu_{i+1}) < \varepsilon_2,\ d_b(\mu_{i+1},[T\,(\mu_{i+1})]_{\alpha(i+1)}) < \varepsilon_2,\ \text{ for } i > i_5. \tag{4.31}$$

Now, from (4.30) and (4.31), we have

$$\begin{aligned}\mathcal{F}\left(d_b\left(\mu_{i+2},[T\,(c)]_{\alpha(c)}\right)\right) &\leq \mathcal{F}\left[\hat{H}_b([T(\mu_{i+1})]_{\alpha(i+1)},[T\,(c)]_{\alpha(c)})\right]-\tau\,(U(\mu_{i+1},c)) \\ &\leq \mathcal{F}\left[e_1(d(\mu_{i+1},c))+e_2d_b\left(\mu_{i+1},[T\,(\mu_{i+1})]_{\alpha(i+1)}\right)\right. \\ &\quad +e_3d_b\left(c,[T\,(c)]_{\alpha(c)}\right)+\frac{e_4}{2}\,(\varepsilon_1)\end{aligned} \tag{4.32}$$

$$+\frac{e_4}{2}d_b\left(c,[T\left(c\right)]_{\alpha(c)}\right)+e_5\left(\varepsilon_1\right)$$
$$-\tau\left(U(\mu_{i+1},c)\right).$$

for all $i > i_5$. Taking the limit as $i \to +\infty$ in (4.32), we see that $\lim_{i\to+\infty}\mathcal{F}\left(d_b\left(\mu_{i+2},[T\left(c\right)]_{\alpha(c)}\right)\right)=-\infty$. By means of $(\mathcal{F}_{ii})$, we have

$$\lim_{i\to+\infty}d_b(\mu_{i+2},[T\left(c\right)]_{\alpha(c)})=0.$$

Consequently,

$$d_b\left(c,[T\left(c\right)]_{\alpha(c)}\right)\leq d\left(c,\mu_{i+2}\right)+d_b\left(\mu_{i+2},[T\left(c\right)]_{\alpha(c)}\right).$$

Further, in view of above fashion with respect to $G \ni c \longmapsto d_b(c,[T\left(\mu_i\right)]_{\alpha(i)})$ is $\check{D}(T,\mu_0)$-fuzzy dynamic lower semi-continuous, we have

$$d_b(c,[T\left(c\right)]_{\alpha(c)})\leq\lim_{i\to+\infty}\inf d_b(c,[T\left(\mu_{i+1}\right)]_{\alpha(i+1)})+0=0.$$

Also, the closedness of $[T\left(c\right)]_{\alpha(c)}$, which implies that $c\in[T\left(c\right)]_{\alpha(c)}$. Hence, c is a fuzzy fixed point of T. □

Corollary 4.4. *Let (G,d) be a b-metric-like space with $s\geq1$. Assume that $T:G\to\mu(G)$ is a F-fuzzy Suzuki-Kannan (abbr., F-FSK) type contraction with respect to fuzzy dynamic system $\check{D}\left([T\mu]_\alpha,\mu_0\right)$ and $\alpha:G\to[0,1]$ such that $\left[T\left(\mu_i\right)\right]_{\alpha(i)}$ are nonempty closed subsets of G. Assume that for some $\mathcal{F}\in\nabla_F$ and $\tau:(0,+\infty)\to(0,+\infty)$ such that*

$$\frac{1}{2s}d_b\left(\mu_{i-1},[T\left(\mu_{i-1}\right)]_{\alpha(i-1)}\right)\leq d\left(\mu_{i-1},\mu_i\right),$$

we have

$$\tau(U(\mu_{i-1},\mu_i))+\mathcal{F}\left[\hat{H}_b\left(\left[T\left(\mu_i\right)\right]_{\alpha(i)},\left[T\left(\mu_{i+1}\right)\right]_{\alpha(i+1)}\right)\right]\leq\mathcal{F}(U(\mu_{i-1},\mu_i)),$$

where

$$U(\mu_{i-1},\mu_i)=e_2d_b\left(\mu_{i-1},[T\left(\mu_{i-1}\right)]_{\alpha(i-1)}\right)+e_3d_b\left(\mu_i,[T\left(\mu_i\right)]_{\alpha(i)}\right),$$

for all $\mu_i\in\check{D}\left([T\mu]_\alpha,\mu_0\right)$, $\hat{H}_b\left(\left[T\left(\mu_i\right)\right]_{\alpha(i)},\left[T\left(\mu_{i+1}\right)\right]_{\alpha(i+1)}\right)>0$, where $e_2,e_3\in[0,1]$ such that $e_1+e_2=1$. Assume that (i)–(iii) are satisfied. Then T has a fuzzy fixed point.

Corollary 4.5. *Let (G,d) be a b-metric-like space with $s\geq1$. Assume that $T:G\to\mu(G)$ is a F-fuzzy Suzuki-Chatterjea (abbr., F-FSC) type contraction with respect to fuzzy dynamic system $\check{D}\left([T\mu]_\alpha,\mu_0\right)$ and $\alpha:G\to[0,1]$ such that $\left[T\left(\mu_i\right)\right]_{\alpha(i)}$ are nonempty closed subsets of G. Assume that for some $\mathcal{F}\in\nabla_F$ and $\tau:(0,+\infty)\to(0,+\infty)$ such that*

$$\frac{1}{2s}d_b\left(\mu_{i-1},[T\left(\mu_{i-1}\right)]_{\alpha(i-1)}\right)\leq d\left(\mu_{i-1},\mu_i\right),$$

we have

$$\tau(U(\mu_{i-1},\mu_i))+\mathcal{F}\left[\hat{H}_b\left(\left[T\left(\mu_i\right)\right]_{\alpha(i)},\left[T\left(\mu_{i+1}\right)\right]_{\alpha(i+1)}\right)\right]\leq\mathcal{F}(U(\mu_{i-1},\mu_i)),$$

where

$$U(\mu_{i-1},\mu_i)=e_4d_b\left(\mu_{i-1},[T\left(\mu_i\right)]_{\alpha(i)}\right)+e_5d_b\left(\mu_i,[T\left(\mu_{i-1}\right)]_{\alpha(i-1)}\right),$$

for all $\mu_i\in\check{D}\left([T\mu]_\alpha,\mu_0\right)$, $\hat{H}_b\left(\left[T\left(\mu_i\right)\right]_{\alpha(i)},\left[T\left(\mu_{i+1}\right)\right]_{\alpha(i+1)}\right)>0$, where $e_4,e_5\in[0,\frac{1}{2})$. Assume that (i)–(iii) are satisfied. Then T has a fuzzy fixed point.

Corollary 4.6. *Let (G, d) be a b-metric-like space with $s \geq 1$. Assume that $T : G \to \mu(G)$ is a F-fuzzy Suzuki-Banach (abbr., F-FSB) type contraction with respect to fuzzy dynamic system $\check{D}\left([T\mu]_{\alpha}, \mu_0\right)$ and $\alpha : G \to [0, 1]$ such that $\left[T\left(\mu_i\right)\right]_{\alpha(i)}$ are nonempty closed subsets of G. Assume that for some $\mathcal{F} \in \nabla_F$ and $\tau : (0, +\infty) \to (0, +\infty)$ such that*

$$\frac{1}{2s} d_b\left(\mu_{i-1}, \left[T\left(\mu_{i-1}\right)\right]_{\alpha(i-1)}\right) \leq d\left(\mu_{i-1}, \mu_i\right),$$

we have

$$\tau(d(\mu_{i-1}, \mu_i)) + \mathcal{F}\left[\hat{H}_b\left(\left[T\left(\mu_i\right)\right]_{\alpha(i)}, \left[T\left(\mu_{i+1}\right)\right]_{\alpha(i+1)}\right)\right] \leq \mathcal{F}(e_1 d(\mu_{i-1}, \mu_i)),$$

for all $\mu_i \in \check{D}\left([T\mu]_{\alpha}, \mu_0\right)$, $\hat{H}_b\left(\left[T\left(\mu_i\right)\right]_{\alpha(i)}, \left[T\left(\mu_{i+1}\right)\right]_{\alpha(i+1)}\right) > 0$, where $e_1 \in [0, 1)$. Assume that (i) and (ii) are satisfied. Then T has a fuzzy fixed point.

Corollary 4.7. *Let (G, d) be a b-metric-like space with $s \geq 1$. Assume that $T : G \to \mu(G)$ is a F-fuzzy Banach (abbr., F-FB) type contraction with respect to fuzzy dynamic system $\check{D}\left([T\mu]_{\alpha}, \mu_0\right)$ and $\alpha : G \to [0, 1]$ such that $\left[T\left(\mu_i\right)\right]_{\alpha(i)}$ are nonempty closed subsets of G. Assume that for some $\mathcal{F} \in \nabla_F$ and $\tau : (0, +\infty) \to (0, +\infty)$ such that*

$$\tau(d(\mu_{i-1}, \mu_i)) + \mathcal{F}\left[\hat{H}_b\left(\left[T\left(\mu_i\right)\right]_{\alpha(i)}, \left[T\left(\mu_{i+1}\right)\right]_{\alpha(i+1)}\right)\right] \leq \mathcal{F}(e_1 d(\mu_{i-1}, \mu_i))$$

for all $\mu_i \in \check{D}\left([T\mu]_{\alpha}, \mu_0\right)$, $\hat{H}_b\left(\left[T\left(\mu_i\right)\right]_{\alpha(i)}, \left[T\left(\mu_{i+1}\right)\right]_{\alpha(i+1)}\right) > 0$, where $e_1 \in [0, 1)$. Assume that (i) and (ii) are satisfied. Then T has a fuzzy fixed point.

Example 4.8. Let $G = \mathbb{R}^+ \cup \{0\}$ and $d : G \times G \to \mathbb{R}^+ \cup \{0\}$ be a function defined by

$$d\left(\mu_1, \mu_2\right) = \left(\max\left\{\mu_1, \mu_2\right\}\right)^2.$$

Clearly, (d, G) is a complete b-metric-like space with $s = \frac{4}{3}$. Define a fuzzy mapping $T : G \to F(G)$ by

$$T\left(\mu\right)\left(\mu'\right) = \begin{cases} 1, \text{ if } 0 \leq \mu' \leq \frac{\mu}{4}; \\ \frac{1}{2}, \text{ if } \frac{\mu}{4} < \mu' \leq \frac{\mu}{3}; \\ \frac{1}{4}, \text{ if } \frac{\mu}{3} < \mu' \leq \frac{\mu}{2}; \\ 0, \text{ if } \frac{\mu}{2} < \mu' \leq 1. \end{cases}$$

Define $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ and $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ by $\mathcal{F}(\mu) = \ln(\mu)$ and

$$\tau(h) = \begin{cases} \ln(1), & \text{for } \mu = 0, 1; \\ \frac{1}{100}, & \text{for } \mu \in (1, +\infty). \end{cases}$$

For all $\mu \in \check{D}\left([T\mu]_{\alpha}, \mu_0\right)$, there is $\alpha\left(\mu\right) = 1$ such that $[T\mu]_{\alpha(\mu)} = \left[0, \frac{\mu}{2}\right]$. Then we have

$$\frac{1}{2s} d_b\left(\mu_i, \left[T\left(\mu_i\right)\right]_{\alpha(i)}\right) \leq d\left(\mu_i, \mu_{i+1}\right),$$

setting $e_2 = e_3 = e_4 = e_4 = 0$ and $e_1 = 1$, we obtain

$$\tau(d(\mu_i, \mu_{i+1})) + \mathcal{F}\left[\hat{H}_b\left(\left[T\left(\mu_i\right)\right]_{\alpha(i)}, \left[T\left(\mu_{i+1}\right)\right]_{\alpha(i+1)}\right)\right] \leq \mathcal{F}(\alpha d(\mu_i, \mu_{i+1})).$$

Hence all the required possible hypothesis of Corollary 4.6 are satisfied, Thus T has a fuzzy fixed point.

5. An application to Hukuhara fuzzy differentiability problem

Fuzzy differential equations and fuzzy integral equations have always been of key importance in dynamical programming and engineering problems. Therefore, various authors used different techniques for solving an fuzzy differential equations and fuzzy integral equations. Among those, Hukuhara differentiability for fuzzy valued function is the most celebrated problem. This section renders solution of a fuzzy differential equations. For this we explore Hukuhara differentiability for fuzzy functions and fuzzy initial valued problem in the setting of b-metric-like space.

Definition 5.1. A function $g : \mathbb{R} \to [0, 1]$ is called a fuzzy real number if

(i) g is normal, i.e., there is $\mu_0 \in \mathbb{R}$ in such a way that $g(\mu_0) = 1$;

(ii) ga is fuzzy convex, i.e., $g(\beta(\mu_1) + (1-\beta)\mu_2) \geq \min\{g(\mu_1), g(\mu_2))\}$, $0 \leq \beta \leq 1$, for all $\mu_1, \mu_2 \in \mathbb{R}$;

(iii) g is upper semi-continuous;

$(iiii)$ $[g]^0 = cl\{\mu \in R : g(\mu) > 0\}$ is compact.

Note that, for $\alpha \in (0, 1]$,

$$[g]^\alpha = cl\{\mu \in R : g(\mu) > \alpha\} = \left[g^\alpha_{s_1}, g^\alpha_{s_2}\right],$$

expresses $\alpha - cut$ of the fuzzy set g. For $g \in P^1$, where P^1 represents the family of fuzzy real numbers, one can write $[g]^\alpha \in C_c(\mathbb{R})\, forall\alpha \in [0, 1]$, where $C_c(R)$ denotes the set of all compact and convex subsets of $\mathbb{R}$. The supremum on P^1 endowed with the b-metric-like is defined by

$$d^*(g_1, g_2) = \sup_{\alpha\in[0,1]} \left[\left|g^\alpha_{1,s_1} - g^\alpha_{2,s_1}\right| + \left|g^\alpha_{1,s_2} - g^\alpha_{2,s_2}\right|\right]^2,$$

for all $g_1, g_2 \in P^1$, $g^\alpha_{1,s_1} - g^\alpha_{2,s_1} = diam([g])$. Consider the continuous fuzzy function defined on $[0, \Gamma]$, for $\Gamma > 0$ as $C\left([0, \Gamma], P^1\right)$ endowed with the complete b-metric-like with respect to b-metric-like as:

$$d(g_1, g_2) = \sup_{\mu\in[0,1]} \left[d^*(g_1, g_2)\right],$$

for all $g_1, g_2 \in C^1\left([0, \Gamma], P^1\right)$. Consider the fuzzy initial valued problem:

$$\begin{cases} g'(\mu) = f(\mu, g(\mu)),\ \mu \in I = [0, \Gamma]; \\ g(0) = 0, \end{cases} \tag{5.1}$$

where g' is the Hukuhara differentiability and f is the fuzzy function, i.e., $f : I \times P^1 \to P^1$ is continuous. Denote the set of all continuous fuzzy functions $f : I \to P^1$ which have continuous derivatives by $C^1\left(I, P^1\right)$. A family $\mu \in C^1\left(I, P^1\right)$ is a solution of fuzzy initial valued problem (5.1) if and only if

$$g(\mu) = g_0 \Theta_E (-1)^\mu_0 f(r, g(r))\, dr,\ \mu \in I = [0, \Gamma], \tag{5.2}$$

where (5.2) is called a fuzzy Volterra integral equation.

Theorem 5.2. *Let $f : I \times P^1 \to P^1$ be a continuous function such that*

$$g < g' \text{ implies } f(\mu, g(\mu)) < f(\mu, g'(\mu)),$$

for $g, g' \in P^1$, In addition, assume that $\tau : (0, +\infty) \to (0, +\infty)$ such that

$$\left[\left|f(\mu, g(\mu)) - f(\mu, g'(\mu))\right|\right]^2 \leq \tau e^{-\tau} \max_{\mu \in I} (d^*(g_1(\mu), g_2(\mu)) e^{-\tau\mu}),$$

where $g < g'$ forall$\mu \in I$ and $g, g' \in P^1$. Then the FIVP (5.1) has a fuzzy solution with respect to $C^1\left(I, P^1\right)$.

Proof. Let $\tau : (0, +\infty) \to (0, +\infty)$ and the family $C^1\left(I, P^1\right)$ endow with the b-metric-like as:

$$d_\tau(g, g') = \sup_{\mu \in [0,1]} [d^*(g(\mu), g'(\mu)) e^{-\tau\mu}],$$

for all $g, g' \in C^1\left(I, P^1\right)$. Let $S : G \to (0, 1]$. Due to (5.2) for $g \in G$, one can write

$$Y_g(\mu) = g_0 \Theta_E (-1)_0^\mu f(r, g(r))\, dr, \ \mu \in I.$$

Assume that $g < g$. Then we have

$$\begin{aligned} Y_g(\mu) &= g_0 \Theta_E (-1)_0^\mu f(r, g(r))\, dr \\ &< g_0 \Theta_E (-1)_0^\mu f(r, g'(r))\, dr \\ &= Y_{g'}(\mu). \end{aligned}$$

This implies $Y_g(\mu) \neq Y_{g'}(\mu)$. Assume a fuzzy mapping $T : G \to P^G$ is defined by

$$\begin{cases} \eta_{Tg}(t) = \begin{cases} Y(g), & t(\mu) = Y_g(\mu); \\ 0, & \text{otherwise.} \end{cases} \\ \eta_{Tg'}(t) = \begin{cases} Y(g'), & t(\mu) = Y_{g'}(\mu); \\ 0, & \text{otherwise.} \end{cases} \end{cases}$$

Owing to $\alpha(g) = S(g)$ and $\alpha(g') = S(g')$, we have

$$[Tg]_{\alpha(g)} = \{t \in G : Tg(\mu) \geq S(g)\} = Y_g(\mu),$$

and on the same fashion, we have

$$[Tg']_{\alpha(g')} = \{t \in G : Tg'(\mu) \geq S(g')\} = Y_{g'}(\mu).$$

Therefore,

$$\begin{aligned} \hat{H}_b\left([Tg]_{\alpha(g)}, [Tg']_{\alpha(g')}\right) &= \max \left\{ \begin{array}{l} \sup_{g \in [Tg]_{\alpha(g)}} \inf_{g' \in [Tg']_{\alpha(g')}} d(g, g'), \\ \sup_{g' \in [Tg']_{\alpha(g')}} \inf_{g \in [Tg]_{\alpha(g)}} d(g, g'), \end{array} \right\} \\ &\leq \max \left\{ \sup_{\mu \in I} \left[\left|Y_g(\mu)\right| + \left|Y_{g'}(\mu)\right|\right]^2 \right\} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\mu\in I}\left[\left|Y_g(\mu)\right| + \left|Y_{g'}(\mu)\right|\right]^2 \\
&= \sup_{\mu\in I}\left[\left|g_0\Theta_E(-1)_0^{\mu} f(r,g(r))\,dr\right| + \left|g_0\Theta_E(-1)_0^{\mu} f(r,g'(r))\,dr\right|\right]^2 \\
&= \sup_{\mu\in I}\left[g_0\Theta_E(-1)_0^{\mu}\left(\left|f(r,g(r))\,dr\right| + \left|f(r,g'(r))\,dr\right|\right)\right]^2.
\end{aligned}
$$

Then, in view of above hypothesis we have:,

$$
\begin{aligned}
\hat{H}_b\left([Tg]_{\alpha(g)},[Tg']_{\alpha(g')}\right) &\leq \sup_{\mu\in I}\left[\left|{}_0^{\mu}f(r,g(r))\right| + \left|{}_0^{\mu}f(r,g'(r))\right|dr\right]^2 \\
&\leq \sup_{\mu\in I}\left[\left|{}_0^{\mu}f(r,g(r))\right|^{\frac{1}{2}} + \left|{}_0^{\mu}f(r,g'(r))\right|^{\frac{1}{2}}dr\right]^2 \\
&\leq \sup_{\mu\in I}\left\{{}_0^{\mu}\tau e^{-\tau}\left|g(r)-g'(r)\right|e^{-\tau r}e^{\tau r}dr\right\} \\
&= \tau e^{-\tau}\frac{1}{\tau}d_\tau(g,g')e^{\tau r}.
\end{aligned}
$$

By appealing to the above fashion, we obtain

$$
\hat{H}_b\left([Tg]_{\alpha(g)},[Tg']_{\alpha(g')}\right)e^{-\tau r} \leq e^{-\tau}d_\tau(g,g'),
$$

or equivalently,

$$
\hat{H}_b\left([Tg]_{\alpha(g)},[Tg']_{\alpha(g')}\right) \leq e^{-\tau}d_\tau(g,g').
$$

Owing to logarithms, we have

$$
\ln\left(\hat{H}_b\left([Tg]_{\alpha(g)},[Tg']_{\alpha(g')}\right)\right) \leq \ln\left(e^{-\tau}d_\tau(g,g')\right),
$$

Owing to the above speculation, this, in turn, yields:

$$
\tau\left(d_\tau(g,g')\right) + \ln\left(\hat{H}_b\left([Tg]_{\alpha(g)},[Tg']_{\alpha(g')}\right)\right) \leq \ln\left(d_\tau(g,g')\right).
$$

Due to $\mathcal{F}$-contraction, with the setting $\mathcal{F}(\mu) = \ln\mu, forall\mu \in C^1\left(I,P^1\right)$, we have

$$
\tau\left(d_\tau(g,g')\right) + \mathcal{F}\left(\hat{H}_b\left([Tg]_{\alpha(g)},[Tg']_{\alpha(g')}\right)\right) \leq \mathcal{F}\left(d_\tau(g,g')\right).
$$

It follows that there is $c \in C^1\left(I,P^1\right)$ such that $c \in [Tc]_{\alpha(c)}$. Hence all the possible hypothesis of Corollary 4.7 are satisfied and consequently fuzzy initial valued problem (5.1) has a fuzzy solution $c \in C^1\left(I,P^1\right)$ in $C^1\left(I,P^1\right)$. □

6. Concluding conclusions and observations

The article regards with new approach of fuzzy dynamic process on b-metric-like space, specifically the mapping of set-valued (extended) fuzzy intervals endowed with the b-metric-like. After we just adopt the standard setting of fuzzy dynamic process in b-metric-like space which defines convergence theorems in generalized $\mathcal{F}$-contraction via expectations of fuzzy Suzuki-type contraction mappings.

Subsequently, corollaries are originated from the main result. To explain the example in the main section, a graphically interpretation has been created that best illustrates the fuzzy dynamic process to the readers. At the end, gives an application of our results in solving Hukuhara differentiability through the fuzzy initial valued problem and fuzzy functions. The pivotal role of Hukuhara differentiability in fuzzy dynamic process is stated. In future, this methodology can be inspected intuitionistic fuzzy and picture fuzzy sets the fuzzy dynamic process for a hybrid pair of mappings can be examined.

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Conflict of interest

The authors declare that they have no conflicts of interests.

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