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Research article

Existence, uniqueness and stability of solutions for generalized proportional fractional hybrid integro-differential equations with Dirichlet boundary conditions

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Abstract: In this work, the existence of solutions for nonlinear hybrid fractional integro-differential equations involving generalized proportional fractional (GPF) derivative of Caputo-Liouville-type and multi-term of GPF integrals of Reimann-Liouville type with Dirichlet boundary conditions is investigated. The analysis is accomplished with the aid of the Dhage's fixed point theorem with three operators and the lower regularized incomplete gamma function. Further, the uniqueness of solutions and their Ulam-Hyers-Rassias stability to a special case of the suggested hybrid problem are discussed. For the sake of corroborating the obtained results, an illustrative example is presented.

Keywords: incomplete gamma function; Caputo-Liouville proportional fractional derivative; hybrid fractional integro-differential equation; fixed point theorem; Ulam-Hyers stability
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1. Introduction

Fractional calculus has received a lot of attention in the last decades [1]. This type of calculus has become an important area of research due to its intensive development and diverse applications with several complex problems of everyday life have been modeled by differential equations and integro-differential equations of fractional order [2–5]. Such applications can be found in variety of fields like economics, physics, chemistry, astronomy, thermal acoustic engineering, biology, probability theory, control theory, viscous fluid dynamics, signal processing, electromagnetism, robotics, anomalous diffusion, potential theory and electrical statistics, etc. We refer the reader to the papers [6–8] and the references cited therein for more details.

Many authors have taken into consideration the numerical solutions of boundary value problems or initial value problems of ordinary/partial value problems (see [9, 10] and the references mentioned). However, studying the qualitative properties of differential equations has still had priority.

In mathematical analysis, boundary conditions are constraints on the solutions of ordinary or partial differential equations in a given domain. There is a large number of possible boundary conditions depending on the formulation of the number of variables involved and thus depending on the nature of the differential equation. Boundary value problems of differential equations of fractional order have been extensively studied. In many works, the authors presented some excellent results about the existence of solutions and utilized the fixed point theorems for this purpose; for example, see [11–16]. Concerning the fractional integro-differential equations, please refer to [17–21]. On the other hand, some researchers were interested in studying the stability of solutions to fractional differential equations. One of the most important types of stability that played a noticeable role in the development of fractional differential equations is the so-called Ulam-Hyers stability. Ulam-Hyers stability analysis was recognized as a simple method of investigation of the solutions of a differential equation. We refer the readers to [22–27] and the references contained therein.

Some basic theory for the hybrid differential (or hybrid integro-differential) equations was discussed in some articles; for instance, in [28], Dhage and Lakshmikantham discussed the existence and the uniqueness of solutions for the following classical Cauchy problem of hybrid differential equations

$$\begin{cases} \frac{d}{dt}\left(\frac{u(t)}{f(t,u(t))}\right) = g\left(t, u\left(t\right)\right), \quad t \in [0, T], \\ u(0) = u_0 \end{cases} \tag{1.1}$$

where $g \in C([0, T] \times \mathbb{R}, \mathbb{R})$ and $f \in C([0, T] \times \mathbb{R}, \mathbb{R}\backslash\{0\})$.

Herzallah and Baleanu [29] discussed the existence of solutions and some fundamental differential inequalities for the following Cauchy problems of hybrid differential equations involving Caputo-Liouville derivative.

$$\begin{cases} {}^{C}D^{\beta}_{0^+}\left(\frac{u(t)}{f(t,u(t))}\right) = g\left(t, u\left(t\right)\right), \\ u(0) = u_0 \end{cases} \tag{1.2}$$

and

$$\begin{cases} {}^{C}D^{\beta}_{0^+}\left(u\left(t\right) - h(t, u(t))\right) = g\left(t, u\left(t\right)\right), \\ u(0) = u_0, \end{cases} \tag{1.3}$$

where $t \in [0, T]$, $0 < \beta \leq 1$, $g \in C([0, T] \times \mathbb{R}, \mathbb{R})$ and $f, h \in C([0, T] \times \mathbb{R}, \mathbb{R}\backslash\{0\})$.

Bashir et al. [30] investigated the existence of solutions for a nonlocal boundary value problem of hybrid fractional integro-differential inclusions given by

$$\begin{cases} {}^{C}D^{\beta}_{0^+}\left(\frac{u(t)-\sum_{i=1}^{i=m} I^{\alpha_i}_{0^+} h_i(t,u(t))}{f(t,u(t))}\right) \in g\left(t, u\left(t\right)\right), \quad t \in [0, 1], \\ u(0) = \mu(u), \ u(1) = A, \end{cases} \tag{1.4}$$

where where ${}^{C}D^{\beta}_{0^+}$ denotes the Caputo-Liouville fractional derivative of order $\beta \in (1, 2]$, $I^{\alpha_i}_{0^+}$ denotes the Reimann-Liouville fractional integral of order α_i and $h_i \in C([0, T] \times \mathbb{R}, \mathbb{R})$, $i = 1, ..., m$.

In [31], Dhage proved the existence and approximations of the solutions for the following initial value problems of nonlinear hybrid fractional integro-differential equations

$$\begin{cases} {}^{c}D_{0^+}^{\beta}\left(\frac{u(t)-I_{0^+}^{\alpha}h(t,u(t))}{f(t,u(t))}\right) = g\left(t, u(t), \int_0^t k(s,u(s))ds\right), \quad t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{1.5}$$

where $0 < \beta \leq 1$, $\alpha > 0$ and $h, k : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$, $g : [0,T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are given functions.

Motivated by the above mentioned works, in this article, we handle the existence of solutions for the following hybrid proportional fractional integro-differential equations with Dirichlet boundary conditions

$$\begin{cases} {}_{p}^{C}\mathfrak{D}_{0^+}^{\beta,\sigma}\left(\frac{u(t)-\sum_{i=1}^{i=m} J_{0^+}^{\alpha_i,\sigma} h_i(t,u(t))}{f(t,u(t))}\right) = g(t, u(t)), \quad t \in [0,T], \\ u(0) = u_0, u(T) = u_T, \end{cases} \tag{1.6}$$

where $\sigma \in (0,1]$, $1 < \beta \leq 2$, $\alpha_i > 0, (i = 1, ..., m), m \in \mathbb{N}^*$, $u_0, u_T \in \mathbb{R}$ and ${}_{p}^{C}\mathfrak{D}_{0^+}^{\beta,\sigma}$ represents the Caputo-Liouville proportional fractional derivative of order β and $J_{0^+}^{\alpha_i,\sigma}$ denotes the Reimann-Liouville proportional fractional integral of order α_i, and $h_i : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}, g : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are given functions. We discuss existence of solutions by applying the Dhage's fixed point and benefiting from the incomplete Gamma function and its properties that were presented by Laadjal et al. [32]. On the top of this, we discuss the uniqueness of these solutions and their Ulam-Hyers-Rassias stability for the case $f(t, u(t)) = 1$ for all $t \in [0,T]$ of the hybrid problem (1.6).

We note that, we deduce the following main cases from problem (1.6).

Case 1. *If* $\sigma = 1$. *The problem (1.6) reduces to a hybrid integro-differential equations involving the usual Caputo-Liouville fractional derivative with the usual Reimann-Liouville fractional integral.*

Case 2. *If* $f(t, u(t)) = 1$ *and* $h_i(t, u(t)) = 0$, $(i = 1, ..., m)$ *for all* $(t, u(t)) \in [0,T] \times \mathbb{R}$, *we get the following Caputo-Liouville proportional fractional boundary value problem*

$$\begin{cases} {}_{p}^{C}\mathfrak{D}_{0^+}^{\beta,\sigma} u(t) = g(t, u(t)), t \in [0,T], \\ u(0) = u_0, u(T) = u_T, \end{cases} \tag{1.7}$$

Case 3. *If* $\sigma = 1, f(t, u(t)) = 1$, *and* $h_i(t, u(t)) = 0$, $(i = 1, ..., m)$ *for all* $(t, u(t)) \in [0,T] \times \mathbb{R}$, *we get the following Caputo-Liouville fractional boundary value problem*

$$\begin{cases} {}^{C}D_{0^+}^{\beta} u(t) = g(t, u(t)), t \in [0,T], \\ u(0) = u_0, u(T) = u_T. \end{cases} \tag{1.8}$$

The paper is organized as follows: Next section presents some definitions and some properties needed in next sections. Section 3 proves the existence of solutions for the given problem. Section 4 investigates the uniqueness result of solutions for the given problem (with $f(t, u(t)) = 1$). Section 5 discusses the Ulam-Hyers-Rassias stability results. Lastly, section 6 provides an example to clarify the results obtained.

2. Preliminaries

In this section, we present some definitions and properties associated with the fractional calculus [33, 34] and the incomplete gamma function [32] that are helpful for our discussion.

Definition 4 ([33]). *Let $\beta \geq 0$. The left fractional integral of Reimann-Liouville type of the function $\Theta \in L^1([0,T],\mathbb{R})$ is defined by $I^0_{a^+}\Theta(t) = \Theta(t)$ and*

$$I^{\beta}_{a^+}\Theta(t) = \frac{1}{\Gamma(\beta)}\int_a^t (t-\rho)^{\beta-1}\Theta(\rho)\,d\rho, \text{ for } \beta > 0, \tag{2.1}$$

where $t \in [a,b]$.

Definition 5 ([33]). *Let $\beta \geq 0$. The left fractional derivative of Caputo-Liouville type of the function $\Theta \in C^n([0,T],\mathbb{R})$ is defined by ${}^{C}D^0_{a^+}\Theta(t) = \Theta(t)$ and*

$$\begin{aligned} {}^{C}D^{\beta}_{a^+}\Theta(t) &= I^{n-\beta}_{a^+}(D^n\Theta)(t) \\ &= \frac{1}{\Gamma(n-\beta)}\int_a^t (t-\rho)^{n-\beta-1} D^n\Theta(\rho)\,d\rho \text{ for } \beta > 0, \end{aligned} \tag{2.2}$$

where $n-1 < \beta \leq n$, $n \in \mathbb{N}$.

Definition 6 ([34]). *Let $\sigma \in (0,1]$, and $\beta \geq 0$. The left generalized proportional fractional integral of Reimann-Liouville type of the function $\Theta \in L^1([0,T],\mathbb{R})$ is defined by $J^{0,\sigma}_{a^+}\Theta(t) = \Theta(t)$ and*

$$J^{\beta,\sigma}_{a^+}\Theta(t) = \frac{1}{\sigma^{\beta}\Gamma(\beta)}\int_a^t (t-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(t-\rho)}\Theta(\rho)\,d\rho, \text{ for } \beta > 0, \tag{2.3}$$

where $t \in [a,b]$.

Definition 7 ([34]). *Let $\sigma \in (0,1]$, and $\beta \geq 0$. The left generalized proportional fractional derivative of Caputo-Liouville type of the function $\Theta \in C^n([0,T],\mathbb{R})$ is defined by ${}^{C}_{p}\mathfrak{D}^{0,\sigma}_{a^+}\Theta(t) = \Theta(t)$ and*

$$\begin{aligned} {}^{C}_{p}\mathfrak{D}^{\beta,\sigma}_{a^+}\Theta(t) &= J^{n-\beta,\sigma}_{a^+}(D^{n,\sigma}\Theta)(t) \\ &= \frac{1}{\sigma^{\beta}\Gamma(n-\beta)}\int_a^t (t-\rho)^{n-\beta-1} e^{\frac{\sigma-1}{\sigma}(t-\rho)}(D^{n,\sigma}\Theta)(\rho)\,d\rho \text{ for } \beta > 0, \end{aligned} \tag{2.4}$$

where $n-1 < \beta \leq n$, $n \in \mathbb{N}$. and $(D^{1,\sigma}\Theta)(t) = (D^{\sigma}\Theta)(t) = (1-\sigma)\Theta(t) + \sigma\Theta'(t)$, and

$$(D^{n,\sigma}\Theta)(t) = \begin{cases} \Theta(t), & \text{for } n = 0, \\ (\underbrace{D^{\sigma}D^{\sigma}\cdots D^{\sigma}}_{n\text{-times}}\Theta)(t), & \text{for } n \geq 1. \end{cases} \tag{2.5}$$

Remark 8. *Note that, for $\sigma = 1$, Definitions 6 and 7 reduce to the usual definitions of Riemann-Liouville fractional integral and Caputo-Liouville fractional derivative, respetively.*

Proposition 9 ([34]). *Let $\sigma \in (0,1]$, $\alpha > 0$ and $\beta > 0$ with $n-1 < \beta \leq n$, and $\Theta \in L^1([0,T],\mathbb{R})$, we have the following properties.*

$$J^{\beta,\sigma}_{a^+}\left(J^{\alpha,\sigma}_{a^+}\Theta\right)(t) = J^{\alpha,\sigma}_{a^+}\left(J^{\beta,\sigma}_{a^+}\Theta\right)(t) = J^{\beta+\alpha,\sigma}_{a^+}\Theta(t); \tag{2.6}$$

$${}^{C}_{p}\mathfrak{D}^{\beta,\sigma}_{a^+}\left(J^{\beta,\sigma}_{a^+}\Theta\right)(t) = \Theta(t); \tag{2.7}$$

$$J^{\beta,\sigma}_{a^+}\left({}^{C}_{p}\mathfrak{D}^{\beta,\sigma}_{a^+}\Theta\right)(t) = \Theta(t) - \sum_{k=0}^{n-1} c_k\,(t-a)^k\, e^{\frac{\sigma-1}{\sigma}(t-a)},\ \textit{(here } \Theta \in C^n([0,T],\mathbb{R})\textit{)}, \tag{2.8}$$

where $c_k = \frac{(D^{k,\sigma}\Theta)(a)}{\sigma^k k!}$. *In particular, when* $1 < \beta \leq 2$ *and* $a = 0$, *we have*

$$J^{\beta,\sigma}_{0^+}\left({}^{C}_{p}\mathfrak{D}^{\beta,\sigma}_{0^+}\Theta\right)(t) = \Theta(t) - c_0 e^{\frac{\sigma-1}{\sigma}t} - c_1 t e^{\frac{\sigma-1}{\sigma}t}. \tag{2.9}$$

Definition 10 ([32, 35, 36]). *Let* $\beta \in (\Re(\beta) > 0)$.
The upper incomplete gamma function is defined by

$$\Gamma(\beta,t) = \int_t^{+\infty} x^{\beta-1}e^{-x}dx, \quad t \geq 0. \tag{2.10}$$

The lower incomplete gamma function is defined by

$$\gamma(\beta,t) = \int_0^t x^{\beta-1}e^{-x}dx, \quad t \geq 0. \tag{2.11}$$

The upper regularized incomplete gamma function is defined by

$$Q(\beta,t) = \frac{\Gamma(\beta,t)}{\Gamma(\beta)}. \tag{2.12}$$

The lower regularized incomplete gamma function is defined by

$$P(\beta,t) = 1 - Q(\beta,t) = \frac{\gamma(\beta,t)}{\Gamma(\beta)}. \tag{2.13}$$

Remark 11 ([32]). *Let* $\varsigma \in \mathbb{R}^+$ *and* $\beta \in \mathbb{C}$, *where* $(\Re(\beta) > 0)$, *It is clear that* $P(\beta,\varsigma(t-a))$ *is a non-decreasing function with respect to* $t \in [a,b]$. *Moreover*

$$P(\beta,\varsigma(t-a)) \in [0,1]\ \textit{for all}\ t \geq a; \tag{2.14}$$

$$\max_{t\in[a,b]} P(\beta,\varsigma(t-a)) = P(\beta,\varsigma(t-a))|_{t=b} = P(\beta,\varsigma(b-a)); \tag{2.15}$$

$$\min_{t\in[a,b]} P(\beta,\varsigma(t-a)) = P(\beta,\varsigma(t-a))|_{t=a} = 0. \tag{2.16}$$

Lemma 12 ([32, 37]). *Let* $\sigma \in (0,1]$, *and* $\omega > 0$. *Then*

$$\left(J^{\omega,\sigma}_{a^+}1\right)(t) = \begin{cases} \frac{P(\omega,\frac{1-\sigma}{\sigma}(t-a))}{(1-\sigma)^\omega}, & \textit{for } \sigma \in (0,1), \\ \left(I^{\omega}_{a^+}1\right)(t) = \frac{(t-a)^\omega}{\Gamma(\omega+1)}, & \textit{for } \sigma = 1, \end{cases} \tag{2.17}$$

where $t \in [a,b]$ *and the function* P *is defined by Eq (2.13). Moreover*

$$\lim_{\sigma\to 1}\frac{P(\omega,\frac{1-\sigma}{\sigma}(t-a))}{(1-\sigma)^\omega} = (I^{\omega}_{a^+}1)(t) = \frac{(t-a)^\omega}{\Gamma(\omega+1)} \tag{2.18}$$

and

$$\max_{t\in[a,b]}\left(\lim_{\sigma\to 1}\frac{P(\omega,\frac{1-\sigma}{\sigma}(t-a))}{(1-\sigma)^\omega}\right) = \frac{(b-a)^\omega}{\Gamma(\omega+1)}. \tag{2.19}$$

Lemma 13 ([32,37]). *Let $\sigma \in (0,1)$, $t_1, t_2 \in [a,b]$ $(t_1 \le t_2)$ and $\omega > 0$. Then*

$$\int_{t_1}^{t_2} (b-\rho)^{\omega-1}\, e^{\frac{\sigma-1}{\sigma}(b-\rho)} d\rho = \frac{\sigma^{\omega}\Gamma(\omega)}{(1-\sigma)^{\omega}} \left[P(\omega, \frac{1-\sigma}{\sigma}(b-t_1)) - P(\omega, \frac{1-\sigma}{\sigma}(b-t_2)) \right], \tag{2.20}$$

where the function P is defined by Eq (2.13).

Lemma 14 ([32,37]). *Let $\sigma \in (0,1]$, $\beta > 0$ and $a \le \rho \le t_1 < t_2 \le b$. Then*

$$\lim_{t_2 \to t_1} \int_a^{t_1} \left| (t_2-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(t_2-\rho)} - (t_1-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(t_1-\rho)} \right| d\rho = 0. \tag{2.21}$$

Theorem 15 (Dhage [38]). *Let $\mho$ be a non-empty, closed convex and bounded subset of a Banach algebra Σ and let $A, C : \Sigma \to \Sigma$ and $B : \mho \to \Sigma$ be three operators satisfying the following conditions.*
(1) A and C are Lipschitz with Lipschitz constants $\widetilde{A}$ and $\widetilde{C}$, respectively
(2) B is completely continuous
(3) $\widetilde{A}M_B + \widetilde{C} < 1$, where $M_B = \|B(\mho)\| = \sup\{\| Bx \| : x \in \mho\}$
(4) $AxBy + Cx = x \Rightarrow x \in \mho$ for all $y \in \mho$.
Then, the operator equation

$$AxBx + Cx = x \tag{2.22}$$

has a solution in $\mho$.

3. Existence results

In this section, we provide our essential findings concerning the problem defined by (1.6) and then derive the existence results for the special cases 1, 2 and 3.

Lemma 16. *Let $\sigma \in (0,1]$, $1 < \beta \le 2$, $\alpha_i > 0, (i = 1, ..., m)$. For $\Theta \in C([0,T], \mathbb{R})$, the solution of*

$$\begin{cases} {}^{C}_{p}\mathfrak{D}^{\beta,\sigma}_{0^+} \left(\frac{u(t) - \sum_{i=1}^{i=m} J^{\alpha_i,\sigma} h_i(t,u(t))}{f(t,u(t))} \right) = \Theta(t) \\ u(0) = u_0, u(T) = u_T, \end{cases} \tag{3.1}$$

is equivalent to the integral equation

$$\begin{aligned} u(t) &= f(t,u(t)) \Big[\frac{u_0}{f(0,u_0)} \left(1 - \frac{t}{T}\right) e^{\frac{\sigma-1}{\sigma}t} + \frac{u_T \frac{t}{T} e^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T) e^{\frac{\sigma-1}{\sigma}T}} \\ &- \frac{t e^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T) T e^{\frac{\sigma-1}{\sigma}T}} \sum_{i=1}^{i=m} \frac{1}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \int_0^T (T-\rho)^{\alpha_i-1}\, e^{\frac{\sigma-1}{\sigma}(T-\rho)} h_i(\rho, u(\rho)) d\rho \\ &- \frac{t e^{\frac{\sigma-1}{\sigma}t}}{T e^{\frac{\sigma-1}{\sigma}T}} \frac{1}{\sigma^{\beta}\Gamma(\beta)} \int_0^T (T-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(T-\rho)} \Theta(\rho)\, d\rho \\ &+ \frac{1}{\sigma^{\beta}\Gamma(\beta)} \int_0^t (t-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(t-\rho)} \Theta(\rho)\, d\rho \Big] \\ &+ \sum_{i=1}^{m} \frac{1}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \int_0^t (t-\rho)^{\alpha_i-1}\, e^{\frac{\sigma-1}{\sigma}(t-\rho)} h_i(\rho, u(\rho)) d\rho. \end{aligned} \tag{3.2}$$

Proof. Applying the operator $J_{0^+}^{\beta,\sigma}$ on both sides of the equation in Eq (3.1), we get

$$\frac{u(t) - \sum_{i=1}^{i=m} J_{0^+}^{\alpha_i,\sigma} h_i(t,u(t))}{f(t,u(t))} - c_0 e^{\frac{\sigma-1}{\sigma}t} - c_1 t e^{\frac{\sigma-1}{\sigma}t} = J_{0^+}^{\beta,\sigma}\Theta(t). \tag{3.3}$$

From the boundary conditions $u(0) = u_0$ and $u(T) = u_T$ we get

$$c_0 = \frac{u_0}{f(0,u_0)}, \tag{3.4}$$

and

$$c_1 = \frac{u_T - \sum_{i=1}^{i=m} J_{0^+}^{\alpha_i,\sigma} h_i(.,u(.))(T)}{f(T,u_T)Te^{\frac{\sigma-1}{\sigma}T}} - \frac{u_0}{f(0,u_0)T} - \frac{1}{Te^{\frac{\sigma-1}{\sigma}T}} J_{0^+}^{\beta,\sigma}\Theta(T). \tag{3.5}$$

Substituting the values of c_0 and c_1 in Eq (3.3), we obtain

$$\begin{aligned} u(t) &= f(t,u(t))\Big[\frac{u_0}{f(0,u_0)}\left(1-\frac{t}{T}\right)e^{\frac{\sigma-1}{\sigma}t} + \frac{u_T\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T)e^{\frac{\sigma-1}{\sigma}T}} \\ &\quad - \frac{te^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T)Te^{\frac{\sigma-1}{\sigma}T}} \sum_{i=1}^{i=m} J_{0^+}^{\alpha_i,\sigma} h_i(\cdot,u(\cdot))(T) \\ &\quad - \frac{te^{\frac{\sigma-1}{\sigma}t}}{Te^{\frac{\sigma-1}{\sigma}T}} J^{\beta,\sigma}\Theta(T) + J_{0^+}^{\beta,\sigma}\Theta(t)\Big] + \sum_{i=1}^{i=m} J_{0^+}^{\alpha_i,\sigma} h_i(t,u(t)). \end{aligned}$$

Inversely, it is obvious that if $u(t)$ satisfies Eq (3.2), then Eq (3.1) holds. The proof is complete. □

Now, consider the Banach algebra space defined by $\Sigma = C([0,T],\mathbb{R})$, with the norm $\|u\| = \sup_{0\le t\le T} |u(t)|$. Then, we define the multiplication in Σ by $(uv)(t) = u(t)v(t)$ for all $u, v \in \Sigma$ and the operator $\Phi : \Sigma \to \Sigma$ by

$$\begin{aligned} \Phi u(t) &= f(t,u(t))\Big[\frac{u_0}{f(0,u_0)}\left(1-\frac{t}{T}\right)e^{\frac{\sigma-1}{\sigma}t} + \frac{u_T\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T)e^{\frac{\sigma-1}{\sigma}T}} \\ &- \frac{\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T)e^{\frac{\sigma-1}{\sigma}T}} \sum_{i=1}^{i=m} \frac{\int_0^T (T-\rho)^{\alpha_i-1}\, e^{\frac{\sigma-1}{\sigma}(T-\rho)} h_i(\rho,u(\rho))d\rho}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \\ &- \frac{\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{\sigma^{\beta}\Gamma(\beta)\, e^{\frac{\sigma-1}{\sigma}T}} \int_0^T (T-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(T-\rho)} g(\rho,u(\rho))\,d\rho \\ &+ \frac{1}{\sigma^{\beta}\Gamma(\beta)} \int_0^t (t-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(t-\rho)} g(\rho,u(\rho))\,d\rho\Big] \\ &+ \sum_{i=1}^{i=m} \frac{1}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \int_0^t (t-\rho)^{\alpha_i-1}\, e^{\frac{\sigma-1}{\sigma}(t-\rho)} h_i(\rho,u(\rho))d\rho. \end{aligned} \tag{3.6}$$

Remark 17. *The problem described by Eq (1.6) has a solution $u \in \Sigma$ if only if u is fixed point of the operator Φ.*

To prove the existence, we need the following assumption:

(H_1) Assume that $g:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous function, there exists a continuous function $\widetilde{g}$ from $[0,T]$ to $\mathbb{R}_+$ and a continuous nondecreasing function ψ from $\mathbb{R}_+$ to $\mathbb{R}_+\backslash\{0\}$ such that $|g(t,u(t))|\leq \widetilde{g}(t)\psi(\|u\|)$, for all $t\in[0,T]$ with $\sup\limits_{0\leq t\leq T}\widetilde{g}(t)=\|\widetilde{g}\|$.

(H_2) Assume that $h_i:[0,T]\times\mathbb{R}\to\mathbb{R}$, $i=1,...,4$, are continuous functions, there exist $L_{h_i}>0$, $i=1,...,m$ such that $|h_i(t,\lambda_1)-h_i(t,\lambda_2)|\leq L_{h_i}|\lambda_1-\lambda_2|$, for all $(t,\lambda_1,\lambda_2)\in[0,T]\times\mathbb{R}\times\mathbb{R}$ and $|h_i(t,0)|\leq\theta_i(t)$, where $\theta_i(t)$ are positive and continuous functions on $[0,T]$ for $i=1,...,m$ with $\sup\limits_{0\leq t\leq T}\theta_i(t)=\|\theta_i\|$.

(H_3) Assume that there exists $L_f>0$ such that $|f(t,\lambda_1)-f(t,\lambda_2)|\leq L_f|\lambda_1-\lambda_2|$, for all $(t,\lambda_1,\lambda_2)\in[0,T]\times\mathbb{R}\times\mathbb{R}$, and $|f(t,0)|\leq\zeta(t)$, where $\zeta(t)$ is positive and continuous function on $[0,T]$ with $\sup\limits_{0\leq t\leq T}\zeta(t)=\|\zeta\|$.

Now, we present the following existence theorem for the problem described in Eq (1.6) using the Dhage's fixed point theorem.

Theorem 18. *Let $\sigma\in(0,1)$ and the hypotheses $(H_1),(H_2)$ and (H_3) hold. If there exists a real number $R>0$ such that*

$$R\geq\frac{\varrho\|\zeta\|+\sum_{i=1}^{i=m}\frac{\|\theta_i\|P(\alpha_i,\frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}}}{1-\varrho L_f-\sum_{i=1}^{i=m}\frac{L_{h_i}P(\alpha_i,\frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}}},\tag{3.7}$$

where P denotes the lower regularized incomplete gamma function with

$$\varrho L_f+\sum_{i=1}^{i=m}\frac{L_{h_i}P(\alpha_i,\frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}}<1,\tag{3.8}$$

and

$$\begin{aligned}\varrho &= \frac{|u_0|}{|f(0,u_0)|}+\frac{1}{|f(T,u_T)|\,e^{\frac{\sigma-1}{\sigma}T}}\Big[|u_T|+\sum_{i=1}^{i=m}\frac{(L_{h_i}R+\|\theta_i\|)\,P(\alpha_i,\frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}}\Big]\\ &\quad+\left(\frac{1}{e^{\frac{\sigma-1}{\sigma}T}}+1\right)\frac{\|\widetilde{g}\|\,\psi(R)P(\beta,\frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\beta}}.\end{aligned}\tag{3.9}$$

Then, the problem described in Eq (1.6) has at least one solution on $[0,T]$.

Proof. Consider $\mho=\{u\in\Sigma,\ \text{s.t. } \|u\|\leq R\}$ and let

$$\Phi u=AuBu+Cu,\tag{3.10}$$

where $A,C:\Sigma\to\Sigma$ and $B:\mho\to\Sigma$ are three operators defined by

$$Au\,(t)=f(t,u(t)),\tag{3.11}$$

$$Bu(t) = \frac{u_0}{f(0,u_0)}\left(1-\frac{t}{T}\right)e^{\frac{\sigma-1}{\sigma}t}+\frac{u_T\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T)e^{\frac{\sigma-1}{\sigma}T}}$$

$$
\begin{aligned}
&- \frac{\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T)e^{\frac{\sigma-1}{\sigma}T}}\sum_{i=1}^{i=m}\frac{\int_0^T (T-\rho)^{\alpha_i-1}\, e^{\frac{\sigma-1}{\sigma}(T-\rho)}h_i(\rho,u(\rho))d\rho}{\sigma^{\alpha_i}\Gamma(\alpha_i)}\\
&- \frac{\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{\sigma^{\beta}\Gamma(\beta)\, e^{\frac{\sigma-1}{\sigma}T}}\int_0^T (T-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(T-\rho)}g(\rho,u(\rho))\, d\rho\\
&+ \frac{1}{\sigma^{\beta}\Gamma(\beta)}\int_0^t (t-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(t-\rho)}g(\rho,u(\rho))\, d\rho \qquad (3.12)
\end{aligned}
$$

and

$$
Cu(t) = \sum_{i=1}^{i=m}\frac{1}{\sigma^{\alpha_i}\Gamma(\alpha_i)}\int_0^t (t-\rho)^{\alpha_i-1}\, e^{\frac{\sigma-1}{\sigma}(t-\rho)}h_i(\rho,u(\rho))d\rho. \qquad (3.13)
$$

Claim (1). A and C are Lipschitz with Lipschitz constants $\widetilde{A}$ and $\widetilde{C}$, respectively.
Let $u, v \in \Sigma$, for all $t \in [0,T]$ we have

$$
|Au(t) - Av(t)| = |f(t,u(t)) - f(t,v(t))| \leq L_f\, \|u - v\|,
$$

which yields

$$
\|Au - Av\| \leq \widetilde{A}\, \|u - v\|, \text{ where } \widetilde{A} = L_f.
$$

Next, we have

$$
\begin{aligned}
|Cu(t) - Cv(t)| &\leq \sum_{i=1}^{i=m}\frac{1}{\sigma^{\alpha_i}\Gamma(\alpha_i)}\int_0^t (t-\rho)^{\alpha_i-1}\, e^{\frac{\sigma-1}{\sigma}(t-\rho)}\, |h_i(\rho,u(\rho)) - h_i(\rho,v(\rho))|\, d\rho\\
&\leq \sum_{i=1}^{i=m}\frac{L_{h_i}}{\sigma^{\alpha_i}\Gamma(\alpha_i)}\int_0^t (t-\rho)^{\alpha_i-1}\, e^{\frac{\sigma-1}{\sigma}(t-\rho)}d\rho\, \|u - v\|\\
&= \sum_{i=1}^{i=m}\frac{L_{h_i}P(\alpha_i, \frac{1-\sigma}{\sigma}t)}{(1-\sigma)^{\alpha_i}}\, \|u - v\|\\
&\leq \sum_{i=1}^{i=m}\frac{L_{h_i}P(\alpha_i, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}}\, \|u - v\|,
\end{aligned}
$$

which yields

$$
\|Cu - Cv\| \leq \widetilde{C}\, \|u - v\|, \text{ where } \widetilde{C} = \sum_{i=1}^{i=m}\frac{L_{h_i}P(\alpha_i, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}}. \qquad (3.14)
$$

Hence A and C are Lipschitz with Lipschitz constants $\widetilde{A}$ and $\widetilde{C}$, respectively.
Claim (2). B is completely continuous.
Step 1. We will show that the operator $B : \mho \to \Sigma$ is uniformly bounded.
For any $u \in \mho$, we have

$$
|Bu(t)| = \frac{|u_0|}{|f(0,u_0)|}\left(1 - \frac{t}{T}\right)e^{\frac{\sigma-1}{\sigma}t} + \frac{|u_T|\, \frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{|f(T,u_T)|\, e^{\frac{\sigma-1}{\sigma}T}}
$$

$$
\begin{aligned}
&+ \frac{\frac{t}{T} e^{\frac{\sigma-1}{\sigma} t}}{|f(T,u_T)|\, e^{\frac{\sigma-1}{\sigma} T}} \sum_{i=1}^{i=m} \frac{\int_0^T (T-\rho)^{\alpha_i-1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} |h_i(\rho, u(\rho))|\, d\rho}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \\
&+ \frac{\frac{t}{T} e^{\frac{\sigma-1}{\sigma} t}}{\sigma^{\beta}\Gamma(\beta)\, e^{\frac{\sigma-1}{\sigma} T}} \int_0^T (T-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} |g(\rho, u(\rho))|\, d\rho \\
&+ \frac{1}{\sigma^{\beta}\Gamma(\beta)} \int_0^t (t-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(t-\rho)} |g(\rho, u(\rho))|\, d\rho.
\end{aligned}
$$

For $u \in \mho$ and $t \in [0, T]$, using (H_3) we get

$$
\begin{aligned}
|h_i(t, u(t))| &= |h_i(t, u(t)) - h_i(t, 0) + h_i(t, 0)| \\
&\leq |h_i(t, u(t)) - h_i(t, 0)| + |h_i(t, 0)| \\
&\leq L_{h_i} R + \|\theta_i\|.
\end{aligned}
$$

Then, using Lemma 13, we obtain

$$
\begin{aligned}
|Bu(t)| &\leq \frac{|u_0|}{|f(0,u_0)|} + \frac{|u_T|}{|f(T,u_T)|\, e^{\frac{\sigma-1}{\sigma} T}} + \frac{1}{|f(T,u_T)|\, e^{\frac{\sigma-1}{\sigma} T}} \sum_{i=1}^{i=m} \frac{(L_{h_i} R + \|\theta_i\|)\, P(\alpha_i, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\alpha_i}} \\
&+ \frac{\|\widehat{g}\|\, \psi(R) P(\beta, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\beta}\, e^{\frac{\sigma-1}{\sigma} T}} + \frac{\|\widehat{g}\|\, \psi(R) P(\beta, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\beta}} \\
&= \varrho < +\infty.
\end{aligned}
$$

Therefore, $\|Bu\| < +\infty$, and consequently $B(\mho)$ is bounded. Hence, B is uniformly bounded.

Step 2. We show that the continuity of B on $\mho$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence where $\lim\limits_{n\to+\infty} u_n = u$ with $u \in \mho$.

For all $t \in [0, T]$, we have

$$
\begin{aligned}
\lim_{n\to+\infty} Bu_n(t) &= \frac{u_0}{f(0,u_0)}\left(1 - \frac{t}{T}\right) e^{\frac{\sigma-1}{\sigma} t} + \frac{u_T \frac{t}{T} e^{\frac{\sigma-1}{\sigma} t}}{f(T,u_T) e^{\frac{\sigma-1}{\sigma} T}} \\
&- \frac{\frac{t}{T} e^{\frac{\sigma-1}{\sigma} t}}{f(T,u_T) e^{\frac{\sigma-1}{\sigma} T}} \sum_{i=1}^{i=m} \frac{\lim_{n\to+\infty} \int_0^T (T-\rho)^{\alpha_i-1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} h_i(\rho, u_n(\rho)) d\rho}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \\
&- \frac{\frac{t}{T} e^{\frac{\sigma-1}{\sigma} t}}{\sigma^{\beta}\Gamma(\beta)\, e^{\frac{\sigma-1}{\sigma} T}} \lim_{n\to+\infty} \int_0^T (T-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} g(\rho, u_n(\rho))\, d\rho \\
&+ \frac{1}{\sigma^{\beta}\Gamma(\beta)} \lim_{n\to+\infty} \int_0^t (t-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(t-\rho)} g(\rho, u_n(\rho))\, d\rho.
\end{aligned}
$$

Using the Lebesgue dominated convergence theorem, we get

$$
\begin{aligned}
\lim_{n\to+\infty} Bu_n(t) &= \frac{u_0}{f(0,u_0)}\left(1 - \frac{t}{T}\right) e^{\frac{\sigma-1}{\sigma} t} + \frac{u_T \frac{t}{T} e^{\frac{\sigma-1}{\sigma} t}}{f(T,u_T) e^{\frac{\sigma-1}{\sigma} T}} \\
&- \frac{\frac{t}{T} e^{\frac{\sigma-1}{\sigma} t}}{f(T,u_T) e^{\frac{\sigma-1}{\sigma} T}} \sum_{i=1}^{i=m} \frac{\int_0^T (T-\rho)^{\alpha_i-1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} \lim_{n\to+\infty} h_i(\rho, u_n(\rho)) d\rho}{\sigma^{\alpha_i}\Gamma(\alpha_i)}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{\sigma^{\beta}\Gamma(\beta)\, e^{\frac{\sigma-1}{\sigma}T}} \int_0^T (T-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(T-\rho)} \lim_{n\to+\infty} g(\rho, u_n(\rho))\, d\rho \\
&+ \frac{1}{\sigma^{\beta}\Gamma(\beta)} \int_0^t (t-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(t-\rho)} \lim_{n\to+\infty} g(\rho, u_n(\rho))\, d\rho \\
&= Bu(t),
\end{aligned}
$$

which yields $\lim_{n\to+\infty} \|Bu_n - Bu\| = 0$. Therefore, B is continuous on $\mho$.

Step 3. We prove that $B(\mho)$ is equicontinuous.

Let $t_1, t_2 \in I = [0, T]$ with $t_1 > t_2$. Then, for any $u \in \mho$, we have

$$
\begin{aligned}
|Bu(t_2) - Bu(t_1)| &= \Big| \frac{u_0}{f(0,u_0)} \left\{ \left(1 - \frac{t_2}{T}\right) e^{\frac{\sigma-1}{\sigma}t_2} - \left(1 - \frac{t_1}{T}\right) e^{\frac{\sigma-1}{\sigma}t_1} \right\} + \frac{u_T \left\{ \frac{t_2}{T} e^{\frac{\sigma-1}{\sigma}t_2} - \frac{t_1}{T} e^{\frac{\sigma-1}{\sigma}t_1} \right\}}{f(T,u_T) e^{\frac{\sigma-1}{\sigma}T}} \\
&\quad - \frac{\left\{ \frac{t_2}{T} e^{\frac{\sigma-1}{\sigma}t_2} - \frac{t_1}{T} e^{\frac{\sigma-1}{\sigma}t_1} \right\}}{f(T,u_T) e^{\frac{\sigma-1}{\sigma}T}} \sum_{i=1}^{i=m} \frac{\int_0^T (T-\rho)^{\alpha_i-1}\, e^{\frac{\sigma-1}{\sigma}(T-\rho)} h_i(\rho, u(\rho)) d\rho}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \\
&\quad - \frac{\left\{ \frac{t_2}{T} e^{\frac{\sigma-1}{\sigma}t_2} - \frac{t_1}{T} e^{\frac{\sigma-1}{\sigma}t_1} \right\}}{\sigma^{\beta}\Gamma(\beta)\, e^{\frac{\sigma-1}{\sigma}T}} \int_0^T (T-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(T-\rho)} g(\rho, u(\rho))\, d\rho \\
&\quad + \frac{1}{\sigma^{\beta}\Gamma(\beta)} \int_0^{t_2} (t_2-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(t_2-\rho)} g(\rho, u(\rho))\, d\rho \\
&\quad - \frac{1}{\sigma^{\beta}\Gamma(\beta)} \int_0^{t_1} (t_1-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(t_1-\rho)} g(\rho, u(\rho))\, d\rho \Big| \\
&\le \frac{|u_0|}{|f(0,u_0)|} \left| \left(1 - \frac{t_2}{T}\right) e^{\frac{\sigma-1}{\sigma}t_2} - \left(1 - \frac{t_1}{T}\right) e^{\frac{\sigma-1}{\sigma}t_1} \right| + \frac{|u_T| \left| \frac{t_2}{T} e^{\frac{\sigma-1}{\sigma}t_2} - \frac{t_1}{T} e^{\frac{\sigma-1}{\sigma}t_1} \right|}{|f(T,u_T)|\, e^{\frac{\sigma-1}{\sigma}T}} \\
&\quad + \frac{\left| \frac{t_2}{T} e^{\frac{\sigma-1}{\sigma}t_2} - \frac{t_1}{T} e^{\frac{\sigma-1}{\sigma}t_1} \right|}{f(T,u_T) e^{\frac{\sigma-1}{\sigma}T}} \sum_{i=1}^{i=m} \frac{(L_{h_i} R + \|\theta_i\|) \int_0^T (T-\rho)^{\alpha_i-1}\, e^{\frac{\sigma-1}{\sigma}(T-\rho)} d\rho}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \\
&\quad + \frac{\left| \frac{t_2}{T} e^{\frac{\sigma-1}{\sigma}t_2} - \frac{t_1}{T} e^{\frac{\sigma-1}{\sigma}t_1} \right| \|\widehat{g}\|\, \psi(R)}{\sigma^{\beta}\Gamma(\beta)\, e^{\frac{\sigma-1}{\sigma}T}} \int_0^T (T-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(T-\rho)} d\rho \\
&\quad + \frac{\|\widehat{g}\|\, \psi(R)}{\sigma^{\beta}\Gamma(\beta)} \int_0^{t_1} \left| (t_2-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(t_2-\rho)} - (t_1-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(t_1-\rho)} \right| d\rho \\
&\quad + \frac{\|\widehat{g}\|\, \psi(R)}{\sigma^{\beta}\Gamma(\beta)} \int_{t_1}^{t_2} (t_2-\rho)^{\beta-1}\, e^{\frac{\sigma-1}{\sigma}(t_2-\rho)} d\rho.
\end{aligned}
$$

Using Lemmas 16 and 14, we obtain

$$
|Bu(t_2) - Bu(t_1)| \to 0 \text{ as } t_1 \to t_2.
$$

Therefore, $Bu(t)$ is equicontinuous on $[0, T]$.

Making use of the Arzela-Ascoli theorem, we have $B(\mho)$ is relatively compact. Thus B is a compact operator and as a consequence B is completely continuous.

Claim (3). $\widetilde{A} M_B + \widetilde{C} < 1$, where $M_B = \sup\{\| Bu \| : u \in \mho\}$.

We have $M_B = \|B(\mho)\| = \sup\{\| Bu \| : u \in \mho\} \leq \varrho$ where ϱ is given by(3.9). So,

$$\widetilde{A}M_B + \widetilde{C} \leq \varrho L_f + \sum_{i=1}^{i=m} \frac{L_{h_i} P(\alpha_i, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}} < 1.$$

Claim (4). $Au B\overline{u} + Cu = u \Rightarrow u \in \mho$ for all $\overline{u} \in \mho$.
Let $\overline{u} \in \mho$ we have

$$\begin{aligned}
\|u\| &\leq \|Au\| \cdot \|B\overline{u}\| + \|Cu\| \\
&= \sup_{t\in[0,T]} \Big[|f(t,u(t))| \Big| \frac{u_0}{f(0,u_0)} \left(1 - \frac{t}{T}\right) e^{\frac{\sigma-1}{\sigma}t} + \frac{u_T \frac{t}{T} e^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T) e^{\frac{\sigma-1}{\sigma}T}} \\
&\quad - \frac{\frac{t}{T} e^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T) e^{\frac{\sigma-1}{\sigma}T}} \sum_{i=1}^{i=m} \frac{\int_0^T (T-\rho)^{\alpha_i - 1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} h_i(\rho, u(\rho)) d\rho}{\sigma^{\alpha_i} \Gamma(\alpha_i)} \\
&\quad - \frac{\frac{t}{T} e^{\frac{\sigma-1}{\sigma}t}}{\sigma^\beta \Gamma(\beta) e^{\frac{\sigma-1}{\sigma}T}} \int_0^T (T-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} g(\rho, u(\rho))\, d\rho \\
&\quad + \frac{1}{\sigma^\beta \Gamma(\beta)} \int_0^t (t-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(t-\rho)} g(\rho, u(\rho))\, d\rho \Big| \\
&\quad + \Big| \sum_{i=1}^{i=m} \frac{1}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \int_0^t (t-\rho)^{\alpha_i - 1} e^{\frac{\sigma-1}{\sigma}(t-\rho)} h_i(\rho, u(\rho)) d\rho \Big| \Big] \\
&\leq \sup_{t\in[0,T]} \Big[\left(L_f \|u\| + \|\zeta\|\right) \varrho + \sum_{i=1}^{i=m} (L_{h_i} \|u\| + \|\theta_i\|) \frac{P(\alpha_i, \frac{1-\sigma}{\sigma}t)}{(1-\sigma)^{\alpha_i}} \Big] \\
&= \left(L_f \|u\| + \|\zeta\|\right) \varrho + \sum_{i=1}^{m} (L_{h_i} \|u\| + \|\theta_i\|) \frac{P(\alpha_i, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}} \\
&= \|u\| \left(\varrho L_f + \sum_{i=1}^{m} \frac{L_{h_i} P(\alpha_i, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}} \right) + \varrho \|\zeta\| + \sum_{i=1}^{m} \frac{\|\theta_i\| P(\alpha_i, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}},
\end{aligned}$$

which yields

$$\|u\| \leq \frac{\varrho \|\zeta\| + \sum_{i=1}^{i=m} \frac{\|\theta_i\| P(\alpha_i, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}}}{1 - \varrho L_f - \sum_{i=1}^{i=m} \frac{L_{h_i} P(\alpha_i, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}}} \leq R.$$

Therefore, $u \in \mho$.

Thus, all the conditions of Dhage fixed point theorem are satisfied; hence, the operator Φ has a fixed point in $\mho$. As a result, the proportional fractional boundary value problem declaired in Eq (1.6) has at least one solution on $[0,T]$. This completes the proof. □

In the following, we present the existence theorem for the given problems in the special cases 1, 2 and 3.

Remark 19. *From Lemma 12, in case $\sigma = 1$, we can replace the formula $\frac{P(\omega, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^\omega}$ by the formula $\frac{T^\omega}{\Gamma(\omega+1)}$ ($\omega \in \{\beta, \alpha_1, ..., \alpha_m\}$). Then, by using Theorem 18 we can conclude the existence results of the given problem with usual Caputo fractional derivative.*

Corollary 20. *Let* $\sigma = 1$ *(i.e.,* ${}^{C}_{p}\mathfrak{D}^{\beta,\sigma}_{0^+} = {}^{C}D^{\beta}_{0^+}$*) and assume that the hypotheses* $(H_1), (H_2)$ *and* (H_3) *hold. If there exists a real number* $R > 0$ *such that*

$$R \geq \frac{\widetilde{\varrho}\,\|\zeta\| + \sum_{i=1}^{i=m} \frac{\|\theta_i\| T^{\alpha_i}}{\Gamma(\alpha_i+1)}}{1 - \widetilde{\varrho} L_f - \sum_{i=1}^{i=m} \frac{L_{h_i} T^{\alpha_i}}{\Gamma(\alpha_i+1)}}, \tag{3.15}$$

where

$$\widetilde{\varrho} L_f + \sum_{i=1}^{i=m} \frac{L_{h_i} T^{\alpha_i}}{\Gamma(\alpha_i + 1)} < 1 \tag{3.16}$$

and

$$\widetilde{\varrho} = \frac{|u_0|}{|f(0,u_0)|} + \frac{1}{|f(T,u_T)|}\Big[\, |u_T| + \sum_{i=1}^{i=m} \frac{\left(L_{h_i} R + \|\theta_i\|\right) T^{\alpha_i}}{\Gamma(\alpha_i + 1)} \Big] + \frac{2\,\|\widehat{g}\|\,\psi(R) T^{\beta}}{\Gamma(\beta + 1)}. \tag{3.17}$$

Then, the identified problem in Eq (1.6) has at least one solution on $[0, T]$.

Corollary 21. *Let* $\sigma \in (0, 1)$ *and assume that the hypothesis* (H_1) *holds. If there exists a real number* $R > 0$ *such that*

$$R \geq \frac{\varrho_0}{1 - \varrho_0}, \tag{3.18}$$

where

$$\varrho_0 = |u_0| + \frac{|u_T|}{e^{\frac{\sigma-1}{\sigma}T}} + \left(\frac{1}{e^{\frac{\sigma-1}{\sigma}T}} + 1 \right) \frac{\|\widehat{g}\|\,\psi(R) P(\beta, \frac{1-\sigma}{\sigma}T)}{(1 - \sigma)^{\beta}} < 1. \tag{3.19}$$

Then, the problem in (1.7) has at least one solution on $[0, T]$.

Corollary 22. *Let* $\sigma = 1$, *and assume that the hypothesis* (H_1) *holds. If there exists a real number* $R > 0$ *such that*

$$R \geq \frac{\varrho_1}{1 - \varrho_1}, \tag{3.20}$$

where

$$\varrho_1 = |u_0| + |u_T| + \frac{2\,\|\widehat{g}\|\,\psi(R) T^{\beta}}{\Gamma(\beta + 1)} < 1. \tag{3.21}$$

Then, the problem (1.8) has at least one solution on $[0, T]$.

4. Uniqueness of the solution

In this section, we discuss the existence and uniqueness of solution for the following problem.

$$\begin{cases} {}^{C}_{p}\mathfrak{D}^{\beta,\sigma}_{0^+} \left(u(t) - \sum_{i=1}^{i=m} J^{\alpha_i,\sigma}_{0^+} h_i(t, u(t)) \right) = g(t, u(t)), \quad t \in [0, T], \\ u(0) = u_0, u(T) = u_T, \end{cases} \tag{4.1}$$

Note that we can write the equivalent equation for problem described by Eq (4.1) as follows.

$$u(t) = B_1 u(t) + C u(t) := \Phi_1 u(t), \tag{4.2}$$

where $B_1 = B$ (with $f(0, u_0) = 1 = f(T, u_T)$) and B and C are given by Eq (3.12) and Eq (3.13), respectively.

The following assumption is essential.

(H_4) Assume $g \in C([0,T]^2 \times \mathbb{R}, \mathbb{R})$ and there exists $L_g > 0$ such that $|g(t,\lambda_1) - g(t,\lambda_2)| \leq L_g |\lambda_1 - \lambda_2|$, for all $(t,\lambda_1,\lambda_2) \in [0,T]\times\mathbb{R}\times\mathbb{R}$ and $|g(t,0)| \leq \Upsilon(t)$, where $\Upsilon(t)$ is positive and continuous function on $[0,T]$, with $\sup_{0\leq t\leq T} \Upsilon(t) = \|\Upsilon\|$.

Theorem 23. *Let $\sigma \in (0,1)$ and assume that (H_2) and (H_4) are satisfied. Then, the problem described by Eq (4.1) has a unique solution on $[0,T]$ if*

$$\Delta < 1, \tag{4.3}$$

where

$$\Delta = \left(\frac{L_g P(\beta, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\beta}} + \sum_{i=1}^{i=m} \frac{L_{h_i} P(\alpha_i, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\alpha_i}} \right) \left(e^{\frac{1-\sigma}{\sigma} T} + 1 \right) \tag{4.4}$$

with P defined as in Eq (2.13).

Proof. Let us set $\overline{\mho} = \{u \in \Sigma, \text{ s.t. } \|u\| \leq r\}$, where $r > 0$ satisfying:

$$r \geq \frac{|u_0| + |u_T| \, e^{\frac{1-\sigma}{\sigma} T} + \overline{\Delta}}{1 - \Delta},$$

where Δ is given by Eq (4.4) and

$$\overline{\Delta} = \left(e^{\frac{1-\sigma}{\sigma} T} + 1 \right) \left(\frac{\|\Upsilon\| \, P(\beta, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\beta}} + \sum_{i=1}^{i=m} \frac{\|\theta_i\| \, P(\alpha_i, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\alpha_i}} \right). \tag{4.5}$$

We will show that $\Phi_1 \overline{\mho} \subset \overline{\mho}$.
For $u \in \overline{\mho}$ and $t \in [0,T]$, we have

$$\begin{aligned} |B_1 u(t)| \;\leq\; & |u_0| + \frac{|u_T|}{e^{\frac{\sigma-1}{\sigma} T}} + \frac{1}{e^{\frac{\sigma-1}{\sigma} T}} \sum_{i=1}^{i=m} \frac{\left(L_{h_i} r + \|\theta_i\|\right) P(\alpha_i, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\alpha_i}} \\ & + \frac{\left(L_g r + \|\Upsilon\|\right) P(\beta, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\beta} \, e^{\frac{\sigma-1}{\sigma} T}} + \frac{\left(L_g r + \|\Upsilon\|\right) P(\beta, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\beta}}. \end{aligned}$$

On the other hand, we have

$$\|Cu\| \leq \sum_{i=1}^{i=m} \frac{\left(L_{h_i} r + \|\theta_i\|\right) P(\alpha_i, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\alpha_i}}.$$

We procure that

$$\begin{aligned} \|\Phi_1 u\| \;&\leq\; \|B_1 u\| + \|Cu\| \\ &\leq\; |u_0| + \frac{|u_T|}{e^{\frac{\sigma-1}{\sigma} T}} + \left(e^{\frac{1-\sigma}{\sigma} T} + 1 \right) \sum_{i=1}^{i=m} \frac{\left(L_{h_i} r + \|\theta_i\|\right) P(\alpha_i, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\alpha_i}} \\ &\quad + \left(e^{\frac{1-\sigma}{\sigma} T} + 1 \right) \frac{\left(L_g r + \|\Upsilon\|\right) P(\beta, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\beta}} \\ &=\; |u_0| + |u_T| \, e^{\frac{1-\sigma}{\sigma} T} + \overline{\Delta} + \Delta r \end{aligned}$$

$$\leq \quad r,$$

which implies that $\Phi_1\overline{\mho} \subset \overline{\mho}$.

Next, we show that the operator Φ_1 is a contraction mapping.

For $u, v \in \Sigma$ and for all $t \in [0, T]$, we have

$$\begin{aligned}
|B_1u(t) - B_1v(t)| &\leq \frac{\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{e^{\frac{\sigma-1}{\sigma}T}} \sum_{i=1}^{i=m} \frac{\int_0^T (T-\rho)^{\alpha_i-1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} |h_i(\rho, u(\rho)) - h_i(\rho, v(\rho))| \, d\rho}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \\
&+ \frac{\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{\sigma^{\beta}\Gamma(\beta)\, e^{\frac{\sigma-1}{\sigma}T}} \int_0^T (T-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} |g(\rho, u(\rho)) - g(\rho, v(\rho))| \, d\rho \\
&+ \frac{1}{\sigma^{\beta}\Gamma(\beta)} \int_0^t (t-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(t-\rho)} |g(\rho, u(\rho)) - g(\rho, u(\rho))| \, d\rho \\
&\leq \frac{1}{e^{\frac{\sigma-1}{\sigma}T}} \sum_{i=1}^{i=m} \frac{L_{h_i} \|u-v\| P(\alpha_i, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}} + \frac{L_g \|u-v\| P(\beta, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\beta} e^{\frac{\sigma-1}{\sigma}T}} \\
&+ \frac{L_g \|u-v\| P(\beta, \frac{1-\sigma}{\sigma}t)}{(1-\sigma)^{\beta}} \\
&\leq \left\{ \frac{1}{e^{\frac{\sigma-1}{\sigma}T}} \sum_{i=1}^{i=m} \frac{L_{h_i} P(\alpha_i, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}} + \frac{L_g P(\beta, \frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\beta}} \left(e^{\frac{1-\sigma}{\sigma}T} + 1\right) \right\} \\
&\times \|u-v\|.
\end{aligned}$$

Taking the supermum over all $t \in [0, T]$ and using the inequality (3.14) yield

$$\begin{aligned}
\|\Phi_1 u - \Phi_1 v\| &\leq \|B_1 u - B_1 v\| + \|Cu - Cv\| \\
&\leq \Delta \|u - v\|.
\end{aligned}$$

From the condition (4.3), we conclude that the operator Φ_1 is a contraction mapping. Hence, the problem (4.1) has a unique solution on $[0, T]$. The proof is completed. □

Now, for $\sigma = 1$, using Remark 19 above, we can easily come by the following result.

Corollary 24. *Let $\sigma = 1$ (i.e., ${}^{C}_{p}\mathfrak{D}^{\beta,\sigma}_{0^+} = {}^{C}D^{\beta}_{0^+}$) and assume that (H_2) and (H_4) are satisfied. Then, the problem describe by Eq (4.1) has a unique solution on $[0, T]$ if*

$$\Delta_1 < 1, \tag{4.6}$$

where

$$\Delta_1 = \frac{2L_g T^{\beta}}{\Gamma(\beta+1)} + \sum_{i=1}^{i=m} \frac{2L_{h_i} T^{\alpha_i}}{\Gamma(\alpha_i+1)}. \tag{4.7}$$

5. Ulam-Hyers stability results

We begin introducing the concept of Ulam-type stability for the problem described by (4.1).

Definition 25. *The solution of the problem described by (4.1) is said to be Ulam-Hyers stable if there exists a real constant $K_g > 0$ such that for given $\varepsilon > 0$ and for each solution $v \in C([0,T],\mathbb{R})$ of the inequality*

$$\left| {}_{p}^{C}\mathfrak{D}_{0^+}^{\beta,\sigma}\Big(u(t) - \sum_{i=1}^{i=m} J_{0^+}^{\alpha_i,\sigma} h_i(t,u(t))\Big) - g(t,u(t)) \right| \leq \varepsilon, \tag{5.1}$$

there exists a solution $u \in C([0,T],\mathbb{R})$ of problem described by Eq (4.1) with

$$|v(t) - u(t)| < K_g\varepsilon, \quad \textit{for} \quad t \in [0,T].$$

Definition 26. *The solution of the problem described by Eq (4.1) is said to be generalized Ulam-Hyers stable if there exists a function $\phi_g \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\phi_g(0) = 0$ such that, for any given $\varepsilon > 0$ and for each solution $v \in C([0,T],\mathbb{R})$ of the inequality (5.1), there exists a solution $u \in C([0,T],\mathbb{R})$ of problem described by (4.1) with*

$$|v(t) - u(t)| < \phi_g(\varepsilon), \quad \textit{for} \quad t \in [0,T].$$

The stability mentioned in Definition 25 can be generalized if the constant ε is replaced by a certain type of functions. Such stability is called the Ulam-Hyers-Rassias stability.

Definition 27. *The solution of the problem described by (4.1) is said to be Ulam-Hyers-Rassias stable with respect to $\phi_g \in C([0,T],\mathbb{R}^+)$ if there exists $K_{g,\phi} > 0$ such that, for any given $\varepsilon > 0$ and for each solution $v \in C([0,T],\mathbb{R})$ of the inequality*

$$\left| {}_{p}^{C}\mathfrak{D}_{0^+}^{\beta,\sigma}\Big(u(t) - \sum_{i=1}^{i=m} J_{0^+}^{\alpha_i,\sigma} h_i(t,u(t))\Big) - g(t,u(t)) \right| \leq \varepsilon\phi_g(t), \tag{5.2}$$

there exists a solution $u \in C([0,T],\mathbb{R})$ of the problem described by (4.1) with

$$|v(t) - u(t)| < K_{g,\phi}\varepsilon\phi_g(t), \quad \textit{for} \quad t \in [0,T].$$

Definition 28. *The solution of the problem described by (4.1) is a generalized Ulam-Hyers-Rassias stable with respect to $\phi_g \in C([0,T],\mathbb{R}^+)$ if there exists $K_{g,\phi} > 0$ such that, for any given $\varepsilon > 0$ and for each solution $v \in C([0,T],\mathbb{R})$ of inequality*

$$\left| {}_{p}^{C}\mathfrak{D}_{0^+}^{\beta,\sigma}\Big(u(t) - \sum_{i=1}^{i=m} J_{0^+}^{\alpha_i,\sigma} h_i(t,u(t))\Big) - g(t,u(t)) \right| \leq \phi_g(t), \tag{5.3}$$

there exists a solution $u \in C([0,T],\mathbb{R})$ of problem (4.1) with

$$|v(t) - u(t)| < K_{g,\phi}\phi_g(t), \quad \textit{for} \quad t \in [0,T].$$

Remark 29. *A function $v \in C([0,T],\mathbb{R})$ is a solution of the inequality (5.2) if and only if there exists a small perturbation $\Xi \in C([0,T],\mathbb{R})$ (dependent on v) such that*
i) $|\Xi(t)| \leq \varepsilon\phi_{\Xi}(t)$, $t \in [0,T]$,
ii) ${}_{p}^{C}\mathfrak{D}_{0^+}^{\beta,\sigma}\Big(u(t) - \sum_{i=1}^{i=m} J_{0^+}^{\alpha_i,\sigma} h_i(t,u(t))\Big) = g(t,u(t)) + \Xi(t)$, $t \in [0,T]$.

Lemma 30. *If $v \in C([0,T],\mathbb{R})$ represents a solution of inequality (5.2), then v is a solution of the following integral inequality*

$$|v(t) - \Phi_1 v(t)| \leq \varepsilon \Pi_{\phi_\Xi}(t) \tag{5.4}$$

where

$$\Pi_{\phi_\Xi}(t) = \frac{t}{T} e^{\frac{1-\sigma}{\sigma}(T-t)} (\mathcal{J}_{0^+}^{\alpha,\sigma} \phi_\Xi)(T) + (\mathcal{J}_{0^+}^{\alpha,\sigma} \phi_\Xi)(t) \tag{5.5}$$

Proof. From Remark 29, we get

$$\begin{aligned}
v(t) &= \frac{u_0}{f(0,u_0)}\left(1-\frac{t}{T}\right)e^{\frac{\sigma-1}{\sigma}t} + \frac{u_T \frac{t}{T} e^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T)e^{\frac{\sigma-1}{\sigma}T}} \\
&\quad - \frac{\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{f(T,u_T)e^{\frac{\sigma-1}{\sigma}T}} \sum_{i=1}^{i=m} \frac{\int_0^T (T-\rho)^{\alpha_i-1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} h_i(\rho, v(\rho)) d\rho}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \\
&\quad - \frac{\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{\sigma^\beta \Gamma(\beta) e^{\frac{\sigma-1}{\sigma}T}} \int_0^T (T-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} \left[g(\rho, v(\rho)) + \Xi(\rho)\right] d\rho \\
&\quad + \frac{1}{\sigma^\beta \Gamma(\beta)} \int_0^t (t-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(t-\rho)} \left[g(\rho, v(\rho)) + \Xi(\rho)\right] d\rho \\
&\quad + \sum_{i=1}^{i=m} \frac{1}{\sigma^{\alpha_i}\Gamma(\alpha_i)} \int_0^t (t-\rho)^{\alpha_i-1} e^{\frac{\sigma-1}{\sigma}(t-\rho)} h_i(\rho, u(\rho)) d\rho,
\end{aligned}$$

this yields that

$$\begin{aligned}
|v(t) - \Phi_1 v(t)| &\leq \Bigg| - \frac{\frac{t}{T}e^{\frac{\sigma-1}{\sigma}t}}{\sigma^\beta \Gamma(\beta) e^{\frac{\sigma-1}{\sigma}T}} \int_0^T (T-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(T-\rho)} \Xi(\rho) d\rho \\
&\quad + \frac{1}{\sigma^\beta \Gamma(\beta)} \int_0^t (t-\rho)^{\beta-1} e^{\frac{\sigma-1}{\sigma}(t-\rho)} \Xi(\rho) d\rho \Bigg| \\
&= \frac{t}{T} e^{\frac{1-\sigma}{\sigma}(T-t)} \left|(\mathcal{J}_{0^+}^{\alpha,\sigma}\Xi)(T) + (\mathcal{J}_{0^+}^{\alpha,\sigma}\Xi)(t)\right| \\
&\leq \frac{t}{T} e^{\frac{1-\sigma}{\sigma}(T-t)} \varepsilon (\mathcal{J}_{0^+}^{\alpha,\sigma}\phi_\Xi)(T) + \varepsilon (\mathcal{J}_{0^+}^{\alpha,\sigma}\phi_\Xi)(t) \\
&= \varepsilon \Pi_{\phi_\Xi}(t),
\end{aligned}$$

which leads to the inequality in (5.4). □

In the following, we present the theorem related the Ulam-Hyers-Rassias stability of the solution of the problem described by Eq (4.1).

Theorem 31. *Let $\sigma \in (0,1)$. Assume that (H_1), (H_2) and (H_4) are satisfied with*

$$\Delta < 1.$$

Then, the solution to the problem described by Eq (4.1) is both Ulam-Hyers-Rassias stable and generalized Ulam-Hyers Rassias stable on $[0,T]$.

Proof. From Lemma 30 we get, for $t \in [0, T]$

$$\begin{aligned} |v(t) - u(t)| &\leq |v(t) - \Phi_1 v(t)| + |\Phi_1 v(t) - u(t)| \\ &= |v(t) - \Phi_1 v(t)| + |\Phi_1 v(t) - \Phi_1 u(t)| \\ &\leq \varepsilon \Pi_{\phi_\Xi}(t) + \Delta \left| v(t) - u(t) \right|. \end{aligned}$$

This yields that

$$|v(t) - u(t)| \leq \frac{\varepsilon \Pi_{\phi_\Xi}(t)}{1 - \Delta}.$$

By setting $K_{g,\phi} = \frac{1}{1-\Delta}$, we end up with

$$|v(t) - u(t)| \leq K_{g,\phi} \varepsilon \Pi_{\phi_\Xi}(t).$$

Hence, the solution of problem described by Eq (4.1) is Ulam-Hyers-Rassias stable with respect to Π_{ϕ_Ξ}. Moreover, if we set $\phi_g(t) = \varepsilon \Pi_{\phi_\Xi}(t)$ with $\phi_g(0) = 0$, then the same solution is generalized Ulam-Hyers-Rassias stable. The proof is completed. □

Corollary 32. *Let* $\sigma = 1$ *(i.e.,* ${}_{p}^{C}\mathfrak{D}_{0^+}^{\beta,\sigma} = {}^{C}D_{0^+}^{\beta}$*) and assume that* (H_2) *and* (H_4) *are satisfied with* $\Delta_1 < 1$ *where* Δ_1 *is given by (4.7). Then, the solution of the problem described by Eq (4.1) is both Ulam-Hyers-Rassias stable and generalized Ulam-Hyers Rassias stable on* $[0, T]$.

Remark 33. *Let* $\sigma \in (0, 1]$. *By using Theorem 31 and Corollary 32, we can conclude that:*

1) *If* $\Pi_{\phi_\Xi}(t) = 1$, *then the solution of the problem described by (4.1) is Ulam-Hyers stable.*
2) *If we set* $\phi_g(\varepsilon) = K_{g,\phi}\varepsilon$ *with* $\phi_g(0) = 0$, *then the solution of the problem described by (4.1) is generalized Ulam-Hyers stable.*

6. Example

Consider the following hybrid proportional fractional integro-differential equation:

$$\begin{cases} {}_{p}^{C}\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{3}{4}} \left(\frac{u(t) - \sum_{i=1}^{i=4} J_{0^+}^{\frac{2i-1}{4},\frac{3}{4}} \frac{\cos|u(t)|}{t+10i}}{f(t,u(t))} \right) = \frac{\epsilon(1-t)\sin|u(t)|}{1+|u(t)|}, t \in [0, 1], \\ u(0) = \frac{1}{10}, \quad u(1) = \frac{1}{20}, \end{cases} \tag{6.1}$$

where the real constant ϵ and the function $f(t, u(t))$ will be fixed later.

Here $T = 1, \beta = \frac{3}{2}, \sigma = \frac{3}{4}, \alpha_i = \frac{2i-1}{4}, h_i(t, u(t)) = \frac{\cos|u(t)|}{t+100i}, (i = 1, ..., 4), g(t, u(t)) = \frac{\epsilon(1-t)\sin|u(t)|}{1+|u(t)|}, u_0 = \frac{1}{10}$ and $u_1 = \frac{1}{20}$.

We can show that

$$|h_i(t, \lambda_1) - h_i(t, \lambda_2)| \leq \frac{1}{10i} \left| \lambda_1 - \lambda_2 \right|, \quad i = 1, ..., 4,$$

$$|h_i(t, 0)| \leq \frac{1}{10i}, \quad i = 1, ..., 4,$$

$$|g(t, u(t))| \leq |\epsilon| (1 - t) \frac{||u||}{1 + ||u||}.$$

Also, for all $(t,\lambda) \in [0,1] \times \mathbb{R}$, we have

$$\begin{aligned} |\partial_\lambda g(t,\lambda)| &= |\epsilon|(1-t)\left|\frac{(1+|\lambda|)\cos|\lambda| - \sin|\lambda|}{(1+|\lambda|)^2}\right| \\ &\leq 2|\epsilon|, \end{aligned}$$

It follows that $L_{h_i} = \frac{1}{10i} = \|\theta_i\|, (i = 1, ..., 4), L_g = 2|\epsilon|, \|\widetilde{g}\| = |\epsilon|$ and $\psi(\|u\|) = \frac{\|u\|}{1+\|u\|}$.
Using Matlab program, we find

$$P(\beta, \frac{1-\sigma}{\sigma}T) = 0.118985157486215,$$

$$P(\alpha_1, \frac{1-\sigma}{\sigma}T) = 0.787212464733354,$$

$$P(\alpha_2, \frac{1-\sigma}{\sigma}T) = 0.415815178677325,$$

$$P(\alpha_3, \frac{1-\sigma}{\sigma}T) = 0.186545461450941,$$

$$P(\alpha_4, \frac{1-\sigma}{\sigma}T) = 0.073797013512809.$$

For illustrating Theorem 18, we take $f(t,u(t)) = \frac{1}{10}\sqrt{1+|u(t)|^2}$ and $\epsilon = 1$.
We can show that

$$|f(t,\lambda_1) - f(t,\lambda_2)| \leq \frac{1}{10}|\lambda_1 - \lambda_2| \text{ and } |f(t,0)| = \frac{1}{10},$$

i.e., $L_f = \frac{1}{10}$ and $\|\zeta\| = \frac{1}{10}$.

We see that the condition (3.7) and (3.8) is followed with a real number $R \in [0.929854125359180, 5.322899777680700]$.

Therefore, all conditions in Theorem 18 are satisfied. We conclude that the problem (6.1) has at least one solution on $[0,1]$.

Remark 34. *If* $R = 0.929854125359180$, *we obtain*

$$\varrho = 2.555690432688444,$$

$$\frac{\varrho\|\zeta\| + \sum_{i=1}^{i=m}\frac{\|\theta_i\|P(\alpha_i,\frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}}}{1 - \varrho L_f - \sum_{i=1}^{i=m}\frac{L_{h_i}P(\alpha_i,\frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}}} = 0.929854125359179 \leq R,$$

and

$$\varrho L_f + \sum_{i=1}^{i=m}\frac{L_{h_i}P(\alpha_i,\frac{1-\sigma}{\sigma}T)}{(1-\sigma)^{\alpha_i}} = 0.481751141209855 < 1,$$

Also, if $R = 5.322899777680700$, we obtain

$$\varrho = 6.153559316370539,$$

$$\frac{\varrho \|\zeta\| + \sum_{i=1}^{i=m} \frac{\|\theta_i\| P(\alpha_i, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\alpha_i}}}{1 - \varrho L_f - \sum_{i=1}^{i=m} \frac{L_{h_i} P(\alpha_i, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\alpha_i}}} = 5.322899777680694 \leq R,$$

and

$$\varrho L_f + \sum_{i=1}^{i=m} \frac{L_{h_i} P(\alpha_i, \frac{1-\sigma}{\sigma} T)}{(1-\sigma)^{\alpha_i}} = 0.841538029578065 < 1.$$

Note that all conditions in Theorem 18 are satisfied. We conclude that the problem (6.1) has at least one solution on $[0, 1]$.

For illustrating Theorems 23 and 31, we take $f(t, \lambda) = 1$ for all $(t, \lambda) \in [0, 1] \times \mathbb{R}$ and $\epsilon = \frac{1}{11\pi}$. Using the above data, we find:

$$\Delta = 0.978416714361031 < 1.$$

In virtue of Theorem 23, problem (6.1) has a unique solution. Furthermore, we can compute

$$K_{g,\phi} = \frac{1}{1 - \Delta} = 46.332148715785415 > 0.$$

Thus, by use of Theorem 31, problem (6.1) is Ulam-Hyers-Rassias stable, and consequently generalized Ulam-Hyers-Rassias stable.

7. Conclusions

In this work, we inspected the existence of solutions to a certain class of hybrid fractional intego-differential equations in the casement of generalized proportional fractional hybrid integro-differential equations supplemented with Dirichlet boundary conditions. The existence of at least one solution was shown with the assistance of the hybrid fixed point theorem for a product of three operators. In addition, for some special cases, the uniqueness of the solutions for the considered class of equations, and their stability in the sense of Ulam, were discussed.

We believe that the results obtained will be very important for the researchers working on the qualitative aspects of the boundary value problems in the frame work of the generalized fractional derivatives mentioned in the article. This is due to the fact that as far as we know this is the first work in which the lower regularized incomplete gamma function is used to discuss some qualitative aspects of solutions to fractional differential equations.

Conflict of interest

The authors declare there is no conflict of interest.

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