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Research article

On existence results of Volterra-type integral equations via C^*-algebra-valued F-contractions

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Abstract: It is a fact that C^*-algebra-valued metric space is more general and hence the results in this space are proper improvements of their corresponding ideas in standard metric spaces. With this motivation, this paper focuses on introducing the concepts of C^*-algebra-valued F-contractions and C^*-algebra-valued F-Suzuki contractions and then investigates novel criteria for the existence of fixed points for such mappings. It is observed that the notions examined herein harmonize and refine a number of existing fixed point results in the related literature. A few of these special cases are highlighted and analyzed as some consequences of our main ideas. Nontrivial comparative illustrations are constructed to support the hypotheses and indicate the preeminence of the obtained key concepts. From application viewpoints, one of our results is applied to discuss new conditions for solving a Volterra-type integral equation.

Keywords: C^*-algebra; C^*-algebra-valued metric space; C^*-algebra-valued F-contraction; C^*-algebra-valued F-Suzuki contraction; integral equation
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1. Introduction

A ten decades ago, the rudiments of metric fixed point theory came up in a work concerning solutions of a class of differential equations. The earliest most celebrated fixed point theorem in this regard was proved by Banach [5]. Thereafter, metric fixed point theory has progressed fruitfully in different directions. (e.g., see [12, 13, 19, 21, 26]). Along the way, Edelstein [9] obtained a modification of the Banach fixed point theorem.

Theorem 1.1. *Let (ϖ, ξ) be a compact metric space (MS) and $\rho : \varpi \longrightarrow \varpi$ be a single-valued mapping. If for all $\jmath, \wp \in \varpi$, $\xi(\rho\jmath, \rho\wp) < \xi(\jmath, \wp)$, then ρ has only one fixed point in ϖ.*

Subsequently, Suzuki [28] came up with an extension of Theorem 1.1 as follows.

Theorem 1.2. *[28] Let (ϖ, ξ) be a compact MS and $\rho : \varpi \longrightarrow \varpi$ be a single-valued mapping. Suppose that for all $\jmath, \wp \in \varpi$, $\jmath \neq \wp$,*

$$\frac{1}{2}\xi(\jmath, \rho\jmath) < \xi(\jmath, \wp) \text{ implies } \xi(\rho\jmath, \rho\wp) < \xi(\jmath, \wp).$$

Then, ρ has exactly one fixed point in ϖ.

Not long ago, a novel type of contraction named the F-contraction was initiated by Wardowski [30] with the corresponding fixed point theorem, via which a significant number of well-known related results were obtained as corollaries. Wardowski set up the idea of an F-contraction in the following fashion.

Definition 1.1. *Let (ϖ, ξ) be a MS. A mapping $\rho : \varpi \longrightarrow \varpi$ is named an F-contraction if there exists a constant $\theta > 0$ and for all $\jmath, \wp \in \varpi$,*

$$\xi(\rho\jmath, \rho\wp) > 0 \text{ implies } \theta + F(\xi(\rho\jmath, \rho\wp)) \leq F(\xi(\jmath, \wp)), \tag{1.1}$$

where the mapping $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ obeys the following.

($F1$) F is strictly increasing, that is, for all $a, b \in \mathbb{R}_+$, $a < b$ implies $F(a) < F(b)$;
($F2$) for each sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of positive numbers,

$$\lim_{n\longrightarrow\infty} \tau_n = 0 \ \Leftrightarrow \lim_{n\longrightarrow\infty} F(\tau_n) = -\infty;$$

($F3$) we can find $\omega \in (0, 1)$ such that $\lim_{n\longrightarrow\infty} \tau^{\omega} F(\tau) = 0$.

Remark 1. *From ($F1$) and (1.1), it can easily be deduced that each F-contraction is continuous.*

Wardowski [30] gave a refined form of the Banach fixed point theorem as:

Theorem 1.3. *[30] Let (ϖ, ξ) be a complete MS and $\rho : \varpi \longrightarrow \varpi$ be an F-contraction. Then, ρ has only one fixed point in ϖ.*

For some examples of F-contractions and related fixed point theorems, please consult [4, 14] and the references therein. For a recent survey of F-contractions with corresponding results, we refer to [15].

Piri and Kumam combined the main ideas in [28, 30] and launched the next concept.

Definition 1.2. *[22] Let (ϖ, ξ) be a MS. A mapping $\rho : \varpi \longrightarrow \varpi$ is called an F-Suzuki contraction, if there exists $\theta > 0$ such that for all $\jmath, \wp \in \varpi$, $\rho\jmath \neq \rho\wp$,*

$$\frac{1}{2}\xi(\jmath, \rho\jmath) < \xi(\jmath, \wp) \text{ implies } \theta + F(\xi(\rho\jmath, \rho\wp)) \leq F(\xi(\jmath, \wp)),$$

where, in addition to $(F1)$, the mapping F satisfies the following.

$(F2^*)$ $\inf F = -\infty$;
$(F3^*)$ F is continuous on $(0, \infty)$.

It is interesting to know that C^*-algebra (see [20]) has been applied several times to explain physical phenomena in quantum field theory and statistical mechanics and later emerged as a diverse area of research. On the other hand, the study of new spaces and their properties has been an interesting research topic among the mathematical community. In this direction, by using the set of all nonnegative elements of a unital C^*-algebra (C^*-Ag) instead of the set of real numbers, Ma et al. [17] initiated the notion of C^*-Ag-valued MS and established related fixed point theorems. Following [17], a number of extensions of the concept of C^*-Ag-valued MS have been presented (see, e.g., [18, 23, 25]).

In what follows, we collect specific concepts of C^*-Ag and C^*-Ag-valued MS.

Definition 1.3. *[20] Let $\mathcal{A}$ be a unital algebra with the unit I. An involution on $\mathcal{A}$ is a conjugate linear mapping $\jmath \longmapsto \jmath^*$ with $\jmath^{**} = \jmath$ and $(\jmath\wp)^* = \wp^*\jmath^*$, for all $\jmath, \wp \in \mathcal{A}$. The pair $(\mathcal{A}, *)$ is named a $*$-algebra. A Banach $*$-algebra is a $*$-algebra $\mathcal{A}$ together with a submultiplicative norm : $\|\jmath^*\| = \|\jmath\|$, for all $\jmath \in \mathcal{A}$, where a norm $\|.\|$ on an algebra $\mathcal{A}$ is named submultiplicative if $\|\jmath\wp\| \leq \|\jmath\|\|\wp\|$, for all $\jmath, \wp \in \mathcal{A}$. A C^*-Ag is a Banach $*$-algebra : $\|\jmath^*\jmath\| = \|\jmath\|^2$, for all $\jmath \in \mathcal{A}$.*

Herein, $\mathcal{A}$ depicts a unital C^*-Ag with the unit I. Also, we let $\mathcal{A}_a = \{\jmath \in \mathcal{A} : \jmath = \jmath^*\}$ and take the zero element in $\mathcal{A}$ by $0_{\mathcal{A}}$. A member $\jmath \in \mathcal{A}$ is named positive, written $\jmath \succeq 0_{\mathcal{A}}$, if $\jmath \in \mathcal{A}_a$ and $\sigma(\jmath) \subset \mathbb{R}_+ = [0, \infty)$, where $\sigma(\jmath) = \{\lambda \in \mathbb{C} : \lambda I - \jmath \text{ is not invertible}\}$ is the spectrum of $\jmath$. Employing nonnegative members, we can define a partial ordering $\preceq$ on $\mathcal{A}_a$ as: $\jmath \preceq \wp \Leftrightarrow \wp - \jmath \succeq 0_{\mathcal{A}}$. Hereunder, by $\mathcal{A}_+$, we represent the set $\{\jmath \in \mathcal{A} : \jmath \succeq 0_{\mathcal{A}}\}$ and $|\jmath|_{\mathcal{A}} = (\jmath^*\jmath)^{\frac{1}{2}}$(cf. [17]).

Remark 2. *When $\mathcal{A}$ is a unital C^*-Ag, then, for any $\jmath \in \mathcal{A}_+$, we get $\jmath \preceq I \Leftrightarrow \|\jmath\| \leq 1$ (cf. [20]).*

Ma et al. [17] launched the idea of a C^*-Ag-valued MS in the following fashion.

Definition 1.4. *[17] Let ϖ be a nonempty set. Assume that the mapping $\xi : \varpi^2 \longrightarrow \mathcal{A}$ obeys the following:*

(c1) $0_{\mathcal{A}} \preceq \xi(\jmath, \wp)$ and $\xi(\jmath, \wp) = 0_{\mathcal{A}}$ if and only if $\jmath = \wp$;
(c2) $\xi(\jmath, \wp) = \xi(\wp, \jmath)$, for all $\jmath, \wp \in \varpi$;
(c3) $\xi(\jmath, \wp) \preceq \xi(\jmath, z) + \xi(z, \wp)$, for all $\jmath, \wp, z \in \varpi$.

Then, ξ is named a C^*-Ag-valued metric, and $(\varpi, \mathcal{A}, \xi)$ is named as a C^*-Ag-valued MS.

Definition 1.5. *[17] Let $(\varpi, \mathcal{A}, \xi)$ be a C^*-Ag-valued MS. Suppose that $\{\jmath_n\}_{n\in\mathbb{N}} \subset \varpi$ and $u \in \varpi$. If for any $\epsilon > 0$, we have $n_0 \in \mathbb{N}$: for all $n > n_0$, $\|\xi(\jmath_n, u)\| \leq \epsilon$, then $\{\jmath_n\}_{n\in\mathbb{N}}$ is said to be convergent to u with respect to $\mathcal{A}$. In this case, we write $\lim_{n\longrightarrow\infty} \jmath_n = u$.*
If for any $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$: $n, m > n_0$, $\|\xi(\jmath_n, \jmath_m)\| \leq \epsilon$, then the sequence $\{\jmath_n\}_{n\in\mathbb{N}}$ is said to be Cauchy with respect to $\mathcal{A}$. $(\varpi, \mathcal{A}, \xi)$ is a complete C^*-Ag-valued MS, if every Cauchy sequence in ϖ is convergent with respect to $\mathcal{A}$.

It is clear that if ϖ is a Banach space (Bs), then $(\varpi, \mathcal{A}, \xi)$ is a complete C^*-Ag-valued MS, if $\xi(\jmath, \wp) = (\|\jmath - \wp\|)I$ for all $\jmath, \wp \in \varpi$, where I is the identity operator ([17]).

Example 1.1. *[17] Let $\varpi = L^\infty(\Omega)$ and $H = L^2(\Omega)$, where Ω is a Lebesgue measurable set. Obviously, $B(H)$ is a C^*-Ag with the usual operator norm. Define $\xi : \varpi^2 \longrightarrow B(H)$ by*

$$\xi(f, g) = \pi_{|f-g|} \ (f, g \in \varpi),$$

where $\pi_l : H \longrightarrow H$ is the multiplication operator given by $\pi_l(\varphi) = l\varphi$, for $\varphi \in H$. Then, ξ is a C^*-Ag-valued metric, and $(\varpi, \mathcal{A}, \xi)$ is a complete C^*-Ag-valued MS.

Example 1.2. *[29] Let $\varpi = \mathbb{C}$ be the set of all complex numbers and $\mathbb{A} = M_n(\mathbb{C})$ be the C^*-Ag of complex numbers. If $B = [a_{ij}] \in \mathbb{A}$, then $B^* = [\overline{a_{ji}}]$ is a nonzero element of $\mathbb{A}$. Define the norm $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$, where $\|.\|_\infty$ is the usual l^∞-norm on $\mathbb{C}^\infty$. Let $\xi : \varpi \times \varpi \longrightarrow \mathbb{A}$ be given by*

$$\xi(\jmath, \wp) = \begin{pmatrix} \eta|\ell - \wp| & 0 & 0 \\ \ell & \kappa|\ell - \wp| & 0 \\ 0 & 0 & \lambda|\ell - \wp| \end{pmatrix},$$

where $\eta, \kappa, \lambda > 0$, and $\ell, \wp \in \mathbb{C}$. Here, ξ is a C^*-Ag-valued metric but not a standard metric. Now, consider a self-mapping $\rho : \varpi \longrightarrow \varpi$ defined by $\rho\ell = \frac{\ell}{2}$ for all $\ell \in \varpi$. We see that ρ fulfils all the assumptions of [17, Theorem 2.1] and possesses a unique fixed point $\ell = 0$. However, the mapping ρ is not a Banach contraction.

For more examples of C^*-Ag-valued MS, we refer to [17, 18, 29].

Remark 3. *Example 1.2 illustrates the fact that neither the fixed point concepts nor the topological properties in C^*-Ag MS can be deduced from their corresponding ones in MS. In addition, the C^*-Ag-valued metric is not metrizable unless $\mathbb{A} = \mathbb{R}$. In other words, Example 1.2 is a counter example to the claims of [1, 8, 16] and some references therein.*

Mat et al. [17] presented a modified version of the Banach contraction and corresponding fixed point theorem as follows.

Definition 1.6. *[17] Let $(\varpi, \mathcal{A}, \xi)$ be a C^*-Ag-valued MS. A mapping $\rho : \varpi \longrightarrow \varpi$ is named a C^*-Ag-valued contractive mapping on ϖ, if there exists an $A \in \mathcal{A}$ with $\|A\| < 1$ such that for all $\jmath, \wp \in \varpi$,*

$$\xi(\rho\jmath, \rho\wp) \preceq A^* \xi(\jmath, \wp) A. \tag{1.2}$$

Theorem 1.4. *[17] Let $(\varpi, \mathcal{A}, \xi)$ be a complete C^*-Ag-valued MS and $\rho : \varpi \longrightarrow \varpi$ a C^*-Ag-valued contractive mapping on ϖ. Then, ρ has exactly one fixed point in ϖ.*

We recall that the introduction of F-contractions and Suzuki contractions witnessed tremendous improvements in metric fixed point theory and applications. However, since the initiation of C^*-Ag-valued MS (see [17, 18]) and the related fixed point theorem, corresponding ideas of C^*-Ag-valued Suzuki-type and Wardowski-type contractions ([28, 30]) are yet to be examined, leaving gaps in the literature. Consequently, this paper focuses on establishing the notions of

C^*-Ag-valued F-contractions and C^*-algebra-valued F-Suzuki contractions and then investigates sufficient conditions for the existence of fixed points for such mappings. We note that the concepts studied herein unify and improve a significant number of existing fixed point theorems. Some of these particular cases include the main results of Ma et al. (Theorem 1.4), Wardowski (Theorem 1.3), Suzuki (Theorem 1.2), Piri and Kumam [22, Theorem 2.1] and related references therein. From application viewpoints, one of our results is availed to put forward novel criteria for solvability of a Volterra-type integral equation.

2. Main results

We start this section by introducing the concept of a C^*-Ag-valued F-contraction in the following manner.

Definition 2.1. *Let $F : \mathcal{A}_+ \longrightarrow \mathcal{A}$ be a mapping satisfying the following assumptions:*

$(A1)$ F is $\preceq$-increasing, that is, for all $a, b \in \mathcal{A}_+$ with $b - a \succeq 0_{\mathcal{A}}$, we have $F(b) - F(a) \succeq 0_{\mathcal{A}}$;
$(A2)$ for every sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}_+$,

$$\lim_{n\longrightarrow\infty} A_n = 0_{\mathcal{A}} \text{ if and only if } \lim_{n\longrightarrow\infty} \|F(A_n)\| = \infty;$$

$(A3)$ there exists $\omega \in (0, 1)$ such that $\lim\limits_{A\longrightarrow 0_{\mathcal{A}}} A^{\omega} F(A) = 0_{\mathcal{A}}$.

We denote the family of mappings obeying $(A1) - (A3)$ by $\mathcal{A}_{\mathbb{F}}$.

Remark 4. *Note that if we take $\mathcal{A} = \mathbb{R}$, then Definition 2.1 reduces to Definition 1.3 due to Wardowski [30].*

Definition 2.2. *Let $(\varpi, \mathcal{A}, \xi)$ be a C^*-Ag-valued MS. A mapping $\rho : \varpi \longrightarrow \varpi$ is named a C^*-Ag-valued F-contraction if there exists an $A \in \mathcal{A}_+$ with $\|A\| < 1$ such that for all $\jmath, \wp \in \varpi$,*

$$\xi(\rho\jmath, \rho\wp) \succ 0_{\mathcal{A}} \text{ implies } A + F(\xi(\rho\jmath, \rho\wp)) \preceq F(A^*\xi(\jmath, \wp)A), \tag{2.1}$$

where $F \in \mathcal{A}_{\mathbb{F}}$.

Remark 5. *It is pertinent to point out that every C^*-Ag-valued contractive mapping (in the sense of Ma et al. [17]), is a C^*-Ag-valued F-contraction. To see this, assume that ρ is a C^*-Ag-valued contractive mapping on $(\varpi, \mathcal{A}, \xi)$, that is, (1.2) holds. Then, if $A = 0_{\mathcal{A}}$ in (1.2), $\xi(\rho\jmath, \rho\wp) = 0_{\mathcal{A}}$ and there is nothing to show. So, for $A \neq 0_{\mathcal{A}}$, passing to logarithm in (1.2), gives*

$$(-\ln(I))I + \ln(\xi(\rho\jmath, \rho\wp))I \preceq \ln(A^*\xi(\jmath, \wp)A)I, \tag{2.2}$$

for all $\jmath, \wp \in \varpi$ with $\xi(\rho\jmath, \rho\wp) \succ 0_{\mathcal{A}}$. Setting $A = (-\ln(I))I$ and $F(a) = (\ln(a))I$ in (2.2), we have that ρ is a C^*-Ag-valued F-contraction.

Moreover, it is clear that a C^*-Ag-valued F-contraction generalizes the notion of an F-contraction (in the sense of Wardowski [30]), by swapping the set of real numbers with $\mathcal{A}$.

Example 2.1. *Let $H = L^2(\Omega)$, where Ω is a Lebesgue measurable set. By $B(H)$, we represent the set of all bounded linear operators on a Hilbert space H with the usual operator norm. Then, the mapping $F : B(H)_+ \longrightarrow B(H)$ defined by $F(a) = (\ln(a))A$ obeys $(A1) - (A3)$.*

Following Remark 5, every mapping $\rho : \varpi \longrightarrow \varpi$ obeying (2.1) is a C^*-Ag-valued F-contraction, where for all $\jmath, \wp \in \varpi$,

$$\begin{aligned} \xi(\rho\jmath, \rho\wp) &\preceq e^{-I}(A^*\xi(\jmath, \wp)A) \\ &\preceq A^*\xi(\jmath, \wp)A. \end{aligned} \tag{2.3}$$

Obviously, for each $\jmath, \wp \in \varpi$ with $\rho\jmath = \rho\wp$, (2.3) is also obeyed, which means alternatively that ρ is a C^*-Ag-valued contractive mapping on ϖ.

Example 2.2. *Let ϖ be a Bs and $\xi(\jmath, \wp) = (\|\jmath - \wp\|)I$ for all $\jmath, \wp \in \varpi$. Then, $(\varpi, \mathcal{A}, \xi)$ is a C^*-Ag-valued MS. The mapping $F : \mathcal{A}_+ \longrightarrow \mathcal{A}$ defined by $F(a) = (\ln(a) + a)I$ obeys $(A1) - (A3)$, and the inequality* (2.1) *becomes*

$$\xi(\rho\jmath, \rho\wp)[A^*\xi(\jmath, \wp)A]^{-1}e^{-[\xi(\jmath,\wp)-\xi(\rho\jmath,\rho\wp)]} \preceq (e^{-A})I,$$

for all $\jmath, \wp \in \varpi$.

Consistent with the notion of continuity in MS, we give the analogue concept in C^*-Ag-valued MS as follows.

Definition 2.3. *Let $(\varpi, \mathcal{A}, \xi)$ be a C^*-Ag-valued MS. A mapping $\rho : \varpi \longrightarrow \varpi$ is said to be continuous with respect to $\mathcal{A}$, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\jmath, \wp \in \varpi$,*

$$\|\xi(\jmath, \wp)\| < \delta \text{ implies } \|\xi(\rho\jmath, \rho\wp)\| < \epsilon.$$

Remark 6. *From $(A1)$ and* (2.1)*, it is easy to deduce that every C^*-Ag-valued F-contraction ρ is continuous, since for all $\jmath, \wp \in \varpi$,*

$$\begin{aligned} \|\xi(\rho\jmath, \rho\wp)\| &\leq \|e^{-A}(A^*\xi(\jmath, \wp)A)\| \\ &\leq \|A^*\xi(\jmath, \wp)A\| \\ &\leq \|A^*A\|\|\xi(\jmath, \wp)\| \\ &= \|A\|^2\|\xi(\jmath, \wp)\| < \|\xi(\jmath, \wp)\|. \end{aligned}$$

Now, we state and prove our first fixed point theorem as follows.

Theorem 2.1. *Let $(\varpi, \mathcal{A}, \xi)$ be a complete C^*-Ag-valued MS and $\rho : \varpi \longrightarrow \varpi$ a C^*-Ag-valued F-contraction. Then, ρ has a unique fixed point in ϖ.*

Proof. Choose $\jmath_0 \in \varpi$ and define a sequence $\{\jmath_n\}_{n\in\mathbb{N}}$ by $\jmath_1 = \rho\jmath_0,\ \jmath_2 = \rho\jmath_1 = \rho^2\jmath_0, \cdots, \jmath_{n+1} = \rho\jmath_n = \rho^{n+1}\jmath_0,\ n = 1, 2, 3, \cdots$. It is obvious that if we have $n \in \mathbb{N}$ with $\xi(\jmath_n, \rho\jmath_n) = 0_{\mathcal{A}}$, then the proof is finished. Whence, suppose that

$$0_{\mathcal{A}} \prec \xi(\jmath_n, \rho\jmath_n) = \xi(\rho\jmath_{n-1}, \rho\jmath_n),\ n \in \mathbb{N}. \tag{2.4}$$

For convenience, we denote the element $\xi(\jmath_{n+1}, \jmath_n)$ by $\eta_n, n = 0, 1, 2, \cdots$. Notice that in a C^*-Ag, if $p, q \in \mathbb{A}_+$ and $p \preceq q$, then for any $r \in \mathbb{A}$, we have $r^*pr, r^*qr \in \mathbb{A}_+$ [20]. Whence, from (2.1), $A + F(\xi(\rho\jmath_{n-1}, \rho\jmath_n)) \preceq F(A^*\xi(\jmath_{n-1}, \jmath_n)A)$, that is,

$$F(\xi(\rho\jmath_{n-1}, \rho\jmath_n)) \preceq F(A^*\xi(\jmath_{n-1}, \jmath_n)A) - A.$$

Repeating this procedure gives

$$\begin{aligned}
F(\xi(\rho\jmath_{n-1},\rho\jmath_n)) &\preceq F(A^*\xi(\jmath_{n-1},\jmath_n)A)-A\\
&= F(A^*\xi(\rho\jmath_{n-2},\rho\jmath_{n-1})A)\\
&\preceq F((A^*)^2\xi(\jmath_{n-1},\jmath_n)A^2)-2A\\
&= F((A^*)^2\xi(\rho\jmath_{n-3},\rho\jmath_{n-2})A^2)\\
&\preceq F((A^*)^3\xi(\jmath_{n-3},\jmath_{n-2})A^3)-3A\\
&\vdots\\
&\preceq F((A^*)^n\eta_0A^n)-nA,
\end{aligned}$$

that is,

$$F(\eta_n)\preceq F((A^*)^n\eta_0A^n)-nA. \tag{2.5}$$

From (2.5), we have $\|F(\eta_n)\|\longrightarrow\infty$ as $n\longrightarrow\infty$. Whence, by $(A2)$,

$$\lim_{n\longrightarrow\infty}\eta_n=0_{\mathcal{A}}. \tag{2.6}$$

From $(A3)$, we have $\omega\in(0,1)$ with

$$\lim_{n\longrightarrow\infty}\|\eta_n^{\omega}F(\eta_n)\|=0_{\mathcal{A}}. \tag{2.7}$$

By (2.5), the following holds for all $n\in\mathbb{N}$:

$$\begin{aligned}
&\eta_n^{\omega}F(\eta_n)-\eta_n^{\omega}\left[F((A^*)^n\eta_0A^n)\right]\\
\preceq\ &\eta_n^{\omega}\left[F((A^*)^n\eta_0A^n)-nA\right]-\eta_n^{\omega}\left[F((A^*)^n\eta_0A^n)\right]\\
=\ &-\eta_n^{\omega}nA\preceq 0_{\mathcal{A}}.
\end{aligned} \tag{2.8}$$

Taking the limit as $n\longrightarrow\infty$ in (2.8) and applying (2.6) and (2.7), we get

$$\lim_{n\longrightarrow\infty}\|n\eta_n^{\omega}A\|=0. \tag{2.9}$$

Whence, from (2.9), we have $n_0\in\mathbb{N}$ with $n\eta_n^{\omega}A\preceq I$, for all $n\geq n_0$. It follows that

$$\|\eta_n\|\leq\frac{(\|A\|)^{-1}}{n^{\frac{1}{\omega}}},\ n\geq n_0. \tag{2.10}$$

So, for $m,n\in\mathbb{N}$ with $m<n+1$, we have

$$\begin{aligned}
\xi(\jmath_{n+1},\jmath_m) &= \eta_m\preceq\eta_n+\eta_{n-1}+\cdots+\eta_m\\
&\preceq \sum_{i=m}^{n}\eta_i\preceq\left\|\sum_{i=m}^{\infty}\eta_i\right\|I\\
&\preceq \sum_{i=m}^{\infty}\|\eta_i\|I=\sum_{i=m}^{\infty}\left\|\frac{A^{-1}}{i^{\frac{1}{\omega}}}\right\|I
\end{aligned}$$

$$\leq \|A\|^{-1} \sum_{i=m}^{\infty} \left\| \frac{1}{i^{\frac{1}{\omega}}} \right\| I \leq \sum_{i=m}^{\infty} \left\| \frac{1}{i^{\frac{1}{\omega}}} \right\| I.$$

Given that the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\omega}}}$ is convergent, $\|\eta_m\| \longrightarrow 0$ as $n \longrightarrow \infty$, establishing that $\{\jmath_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\varpi, \mathcal{A}, \xi)$. The completeness of this space yields $u \in \varpi$ with $\lim_{n\longrightarrow\infty} \jmath_n = \lim_{n\longrightarrow\infty} \rho\jmath_{n-1} = u$. Since ρ is continuous with respect to $\mathcal{A}$,

$$\begin{aligned} \|\xi(\rho u, u)\| &= \lim_{n\longrightarrow\infty} \|\xi(\rho\jmath_n, \jmath_n)\| \\ &= \lim_{n\longrightarrow\infty} \|\xi(\jmath_{n+1}, \jmath_n)\| \\ &= \lim_{n\longrightarrow\infty} \|\xi(u, u)\| = 0, \end{aligned}$$

showing that $\xi(\rho u, u) = 0_{\mathcal{A}}$.
To see that ρ has only one fixed point, let $v \in \varpi$ be with $\rho u = u \neq v = \rho v$. Then, from (2.1),

$$\begin{aligned} F(\xi(u, v)) &= F(\xi(\rho u, \rho v)) \leq A + F(\xi(\rho u, \rho v)) \\ &\leq F(A^*\xi(u, v)A). \end{aligned} \tag{2.11}$$

Given that F is $\preceq$-increasing, (2.11) implies that $\xi(u, v) \preceq A^*\xi(u, v)A$, from which the submultiplicativity of $\|.\|$ yields

$$\begin{aligned} \|\xi(u, v)\| &\leq \|A^*\xi(u, v)A\| \\ &\leq \|A^*A\|\|\xi(u, v)\| \\ &= \|A\|^2\|\xi(u, v)\| < \|\xi(u, v)\|, \end{aligned}$$

a contradiction. Whence, $u = v$. □

Before presenting our second fixed point theorem, we give the following definition.

Definition 2.4. *Let $(\varpi, \mathcal{A}, \xi)$ be a C^*-Ag-valued MS. A mapping $\rho : \varpi \longrightarrow \varpi$ is named a C^*-Ag-valued F-Suzuki contraction, if there exists $A \in \mathcal{A}_+$ with $\|A\| < \frac{1}{2}$ such that for all $\jmath, \wp \in \varpi$, $\jmath \neq \wp$,*

$$A\xi(\jmath, \rho\jmath) \preceq \xi(\jmath, \wp) \text{ implies } A + F(\xi(\rho\jmath, \rho\wp)) \preceq F(A^*\xi(\jmath, \wp)A), \tag{2.12}$$

where $F \in \mathcal{A}_{\mathbb{F}}$.

Theorem 2.2. *Let $(\varpi, \mathcal{A}, \xi)$ be a complete C^*-Ag-valued MS and $\rho : \varpi \longrightarrow \varpi$ be a C^*-Ag-valued F-Suzuki contraction. Then ρ has a unique fixed point in ϖ.*

Proof. Choose $\jmath_0 \in \varpi$ and define a sequence $\{\jmath_n\}_{n\in\mathbb{N}}$ by $\jmath_1 = \rho\jmath_0$, $\jmath_2 = \rho\jmath_1 = \rho^2\jmath_0, \cdots, \jmath_{n+1} = \rho\jmath_n = \rho^{n+1}\jmath_0$, $n \in \mathbb{N}$. If we have $n \in \mathbb{N}$ with $\xi(\jmath_n, \rho\jmath_n) = 0_{\mathcal{A}}$, then we are done with the proof. Whence, presume that $0_{\mathcal{A}} \prec \xi(\jmath_n, \rho\jmath_n)$, for all $n \in \mathbb{N}$. Whence, for all $n \in \mathbb{N}$,

$$A\xi(\jmath_n, \rho\jmath_n) \preceq \xi(\jmath_n, \rho\jmath_n). \tag{2.13}$$

For convenience, we denote the element $\xi(\jmath_n, \rho\jmath_n)$ by α_n, $n \geq 0$. Recall that in a C^*-Ag, if $p, q \in \mathcal{A}_+$ and $p \preceq q$, then for any $r \in \mathcal{A}$, we have $r^*pr, r^*qr \in \mathcal{A}_+$ [20]. Thus, from (2.12),

$$A + F(\xi(\rho\jmath_n, \rho^2\jmath_n)) \preceq F(A^*\xi(\jmath_n, \rho\jmath_n)A),$$

from which we get

$$\begin{aligned} F(\xi(\jmath_{n+1},\rho\jmath_{n+1})) &\preceq F(A^{*}\xi(\jmath_n,\rho\jmath_n)A)-A \\ &\preceq F((A^{*})^{2}\xi(\jmath_{n-1},\rho\jmath_{n-1})(A)^{2})-2A \\ &\preceq F((A^{*})^{3}\xi(\jmath_{n-2},\rho\jmath_{n-2})(A)^{3})-3A \\ &\vdots \\ &\preceq F((A^{*})^{n}\xi(x_0,\rho\jmath_0)(A)^{n})-nA, \end{aligned}$$

that is,

$$F(\alpha_n) \preceq F((A^{*})^{n}\alpha_0 A^{n})-nA. \tag{2.14}$$

From (2.14), we have

$$\begin{aligned} \lim_{n\longrightarrow\infty}\|F(\alpha_n)\| &\leq \lim_{n\longrightarrow\infty}\|F((A^{*})^{n}\alpha_0A^{n})\| \\ &\quad +\|A\|\lim_{n\longrightarrow\infty} n=\infty. \end{aligned}$$

Therefore, applying ($A2$) gives

$$\lim_{n\longrightarrow\infty}\alpha_n = 0_{\mathcal{A}}. \tag{2.15}$$

Now, we claim that $\{\jmath_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\varpi,\mathcal{A},\xi)$. Assume on the contrary that we have $\epsilon>0$ and sequences $\{r(n)\}_{n\in\mathbb{N}}$ and $\{l(n)\}_{n\in\mathbb{N}}$ of points of $\mathbb{N}$ with $r(n)>l(n)$,

$$\|\xi(\jmath_{r(n)},\jmath_{l(n)})\| \geq \frac{\epsilon}{2}=\epsilon^{*}, \|\xi(\jmath_{l(n)-1},\jmath_{l(n)})\| < \frac{\epsilon}{2}=\epsilon^{*},\ n\in\mathbb{N}. \tag{2.16}$$

Then, we have

$$\begin{aligned} \epsilon^{*} \leq \|\xi(\jmath_{r(n)},\jmath_{l(n)})\| &\leq \|\xi(\jmath_{r(n)},\jmath_{r(n)-1})+\xi(\jmath_{r(n)-1},\jmath_{l(n)})\| \\ &\leq \|\xi(\jmath_{r(n)},\jmath_{r(n)-1})\|+\|\xi(\jmath_{r(n)-1},\jmath_{l(n)})\| \\ &\leq \|\xi(\jmath_{r(n)},\jmath_{r(n)-1})\|+\epsilon^{*} \\ &= \|\xi(\jmath_{r(n)-1},\rho\jmath_{r(n)-1})\|+\epsilon^{*}, \end{aligned} \tag{2.17}$$

that is,

$$\epsilon^{*} \leq \|\xi(\jmath_{r(n)},\jmath_{l(n)})\| \leq \|\alpha_{r(n)-1}\|+\epsilon^{*}. \tag{2.18}$$

Passing to the limit in (2.17) as $n\longrightarrow\infty$, and using (2.15), gives

$$\lim_{n\longrightarrow\infty}\|\xi(\jmath_{r(n)},\jmath_{l(n)})\|=\epsilon^{*}. \tag{2.19}$$

From (2.17) and (2.19), we can find an $n_0\in\mathbb{N}$ such that

$$A\xi(\jmath_{r(n)},\rho\jmath_{r(n)}) \preceq A\epsilon^{*} \preceq \xi(\jmath_{r(n)},\jmath_{l(n)}),\ n\geq n_0.$$

Whence, by hypothesis of the theorem,

$$A+F(\xi(\rho\jmath_{r(n)},\rho\jmath_{l(n)})) \preceq F(A^{*}\xi(\jmath_{r(n)},\jmath_{l(n)})A),\ n\geq n_0. \tag{2.20}$$

By (2.12), we depict (2.20) as

$$A + F(\xi(\jmath_{r(n)+1}, \jmath_{l(n)+1})) \leq F(A^{*}\xi(\jmath_{r(n)}, \jmath_{l(n)})A), \ n \geq n_0, \tag{2.21}$$

that is,

$$\begin{aligned} F(\xi(\jmath_{r(n)+1}, \jmath_{l(n)+1})) &\leq F(A^{*}\xi(\jmath_{r(n)}, \jmath_{l(n)})A) - A. \\ &\leq F(A^{*}\xi(\jmath_{r(n)}, \jmath_{l(n)})A), \ n \geq n_0. \end{aligned} \tag{2.22}$$

From ($A1$) and (2.22), we have

$$\xi(\jmath_{r(n)+1}, \jmath_{l(n)+1}) \leq A^{*}\xi(\jmath_{r(n)}, \jmath_{l(n)})A.$$

Whence,

$$\begin{aligned} \epsilon^{*} \leq \|\xi(\jmath_{r(n)+1}, \jmath_{l(n)+1})\| &\leq \|A^{*}\xi(\jmath_{r(n)}, \jmath_{l(n)})A\| \\ &\leq \|A^{*}A\|\|\xi(\jmath_{r(n)}, \jmath_{l(n)})\| \\ &= \|A\|^2\|\xi(\jmath_{r(n)}, \jmath_{l(n)})\| \\ &< \frac{\|\xi(\jmath_{r(n)}, \jmath_{l(n)})\|}{2}. \end{aligned} \tag{2.23}$$

Letting $n \longrightarrow \infty$ in (2.23) and applying (2.19), yields $\epsilon^{*} < \frac{\epsilon^{*}}{2}$, a contradiction. Consequently, $\{\jmath_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\varpi, \mathcal{A}, \xi)$. The completeness of this space implies that we have $u \in \varpi$:

$$\|\xi(\jmath_n, u)\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{2.24}$$

Now, we affirm that for all $n \in \mathbb{N}$,

$$\frac{A}{2}\xi(\jmath_n, \rho\jmath_n) \leq A\xi(\jmath_n, u) \text{ or } \frac{A}{2}\xi(\rho\jmath_n, \rho^2\jmath_n) \leq \xi(\rho\jmath_n, u). \tag{2.25}$$

Again, suppose that we can find $m \in \mathbb{N}$:

$$\frac{A}{2}\xi(\jmath_m, \rho\jmath_m) \geq A\xi(\jmath_m, u) \text{ and } \frac{A}{2}\xi(\rho\jmath_m, \rho^2\jmath_m) \geq A\xi(\rho\jmath_m, u). \tag{2.26}$$

Whence,

$$\begin{aligned} 2A\xi(\jmath_m, u) &\leq A\xi(\jmath_m, \rho\jmath_m) \\ &\leq A\left[\xi(\jmath_m, u) + \xi(u, \rho\jmath_m)\right], \end{aligned}$$

from which we have

$$A\xi(\jmath_m, u) \leq A\xi(u, \rho\jmath_m). \tag{2.27}$$

From (2.26) and (2.27), we have

$$A\xi(\jmath_m, u) \leq A\xi(u, \rho\jmath_m) \leq \frac{A}{2}\xi(\rho\jmath_m, \rho^2\jmath_m). \tag{2.28}$$

From (2.26), (2.27) and the assumption of the theorem,

$$\begin{aligned} A\xi(\jmath_m, u) &\preceq A\xi(u, \rho\jmath_m) \preceq \frac{A}{2}\xi(\rho\jmath_m, \rho^2\jmath_m) \\ &\preceq A\xi(\rho\jmath_m, \rho^2\jmath_m) \\ &\preceq \xi(\jmath_m, \rho\jmath_m). \end{aligned} \tag{2.29}$$

Consequently, from (2.12),

$$\begin{aligned} F(\xi(\rho\jmath_m, \rho^2\jmath_m)) &\preceq F(A^*\xi(\jmath_m, \rho\jmath_m)A) - A \\ &\preceq F(A^*\xi(\jmath_m, \rho\jmath_m)A). \end{aligned}$$

Thus, applying $(A1)$ yields

$$\xi(\rho\jmath_m, \rho^2\jmath_m) \preceq A^*\xi(\jmath_m, \rho\jmath_m)A. \tag{2.30}$$

Then, using (2.26), (2.29) and (2.30) leads to

$$\begin{aligned} \|\xi(\rho\jmath_m, \rho^2\jmath_m)\| &\leq \|A^*\xi(\jmath_m, \rho\jmath_m)A\| \\ &\leq \|A^*A\|\|\xi(\jmath_m, \rho\jmath_m)\| \\ &\leq \|A\|^2\|\xi(\jmath_m, \rho\jmath_m)\| \\ &\leq \|A\|\,[\|\xi(\jmath_m, u) + \xi(u, \rho\jmath_m)\|] \\ &\leq \|A\xi(\jmath_m, u)\| + \|A\xi(u, \rho\jmath_m)\| \\ &\leq \left\|\frac{A}{2}\xi(\rho\jmath_m, \rho^2\jmath_m)\right\| + \left\|\frac{A}{2}\xi(\rho\jmath_m, \rho^2\jmath_m)\right\| \\ &\leq \|A\|\|\xi(\rho\jmath_m, \rho^2\jmath_m)\| \\ &< \frac{\|\xi(\rho\jmath_m, \rho^2\jmath_m)\|}{2}, \end{aligned}$$

a contradiction, showing that (2.25) is true. Thus, for every $n \in \mathbb{N}$, we either have

$$\begin{aligned} F(\xi(\rho\jmath_n, \rho u)) &\preceq F(A^*\xi(\jmath_n, u)A) - A \\ &\preceq F(A^*\xi(\jmath_n, u)A), \end{aligned} \tag{2.31}$$

or

$$\begin{aligned} F(\xi(\rho^2\jmath_n, \rho u)) &\preceq F(A^*\xi(\rho\jmath_n, \rho u)A) - A \\ &\preceq F(A^*\xi(\rho\jmath_n, \rho u)A). \end{aligned} \tag{2.32}$$

Using (2.24), (2.31) and $(A1)$, we obtain that $\|F(\xi(\rho\jmath_n, \rho u))\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Again, by $(A1)$, (2.31) gives

$$\xi(\rho\jmath_n, \rho u) \preceq A^*\xi(\jmath_n, u)A. \tag{2.33}$$

Since (2.24), (2.33) and continuity of the norm $\|.\|$ leads to

$$\lim_{n\longrightarrow\infty} \|\xi(\rho\jmath_n, \rho u)\| \leq \|A\|^2 \lim_{n\longrightarrow\infty} \|\xi(\jmath_n, u)\| = 0,$$

it follows therefore that

$$\begin{aligned}\|\xi(u,\rho u)\| &= \lim_{n\longrightarrow\infty}\|\xi(\jmath_{n+1},\rho u)\| \\ &= \lim_{n\longrightarrow\infty}\|\xi(\rho\jmath_n,\rho u)\| = 0.\end{aligned}$$

Similarly, from (2.24) and (2.32), $\|F(\xi(\rho^2\jmath_n,\rho u))\| \longrightarrow \infty$ as $n \longrightarrow \infty$, and by $(A2)$, $\xi(\rho^2\jmath_n,\rho u) \longrightarrow 0_{\mathcal{A}}$ as $n \longrightarrow \infty$. Consequently,

$$\begin{aligned}\|\xi(u,\rho u)\| &= \lim_{n\longrightarrow\infty}\|\xi(\jmath_{n+1},\rho u)\| \\ &= \lim_{n\longrightarrow\infty}\|\xi(\rho^2\jmath_n,\rho u)\| = 0,\end{aligned}$$

proving that u is a fixed point of ρ.
Now, we show that the fixed point of ρ is unique. For this, suppose that $u, v \in \varpi$ are two fixed point of ρ with $u \neq v$. Then, $0_{\mathcal{A}} = A\xi(u,\rho u) \preceq \xi(u,v)$. Whence, by hypothesis of the theorem, we have

$$\begin{aligned}F(\xi(u,v)) = F(\xi(\rho u,\rho v)) &\preceq A + F(\xi(\rho u,\rho v)) \\ &\preceq F(A^*\xi(u,v)A).\end{aligned}$$

Thus, applying $(A1)$, yields $\xi(u,v) \preceq A^*\xi(u,v)A$, and the submultiplicativity of $\|.\|$ leads to

$$\begin{aligned}\|\xi(u,v)\| &\leq \|A^*\xi(u,v)A\| \leq \|A^*A\|\|\xi(u,v)\| \\ &\leq \|A\|^2\|\xi(u,v)\| < \|\xi(u,v)\|,\end{aligned}$$

a contradiction, showing that $u = v$. □

Remark 7. *(i) In Theorems 2.1 and 2.2, the completeness of ϖ is necessary. For instance, consider Theorem 2.1, let $\varpi = (0,1)$ and $\xi : \varpi \times \varpi \longrightarrow B(H)$ be defined by $\xi(\jmath,\wp) = (|\jmath - \wp|)I$, for all $\jmath,\wp \in \varpi$. Then $(\varpi, B(H), \xi)$ is a C^*-Ag-valued MS, but $(\varpi, B(H), \xi)$ is not complete. However, take $\rho\jmath = \frac{\jmath}{3}$, for all $\jmath \in \varpi$. It is clear that ρ is a C^*-Ag-valued F-contraction on ϖ, but ρ has no fixed point in ϖ.*

(ii) In Theorems 2.1 and 2.2, if $\|A\| = 1$ (or $\|A\| = \frac{1}{2}$), the mapping ρ may not have a fixed point. For example, let $\varpi = l^\infty(\mathbb{N}) = \{(\jmath_1, \jmath_2, \jmath_3, \cdots) : \jmath_n \in \mathbb{C}\}$ and $\sup_n |\jmath_n| < \infty$, for $\jmath = (\jmath_1, \jmath_2, \jmath_3, \cdots \in l^\infty(\mathbb{N}))$, and let $\|\jmath_n\|_\infty = \sup_n |\jmath_n|$. Define $\xi : \varpi \times \varpi \longrightarrow M_2(\mathbb{C})$ by $\xi(\jmath,\wp) = (\|\jmath - \wp\|)I$. Then $(\varpi, M_2(\mathbb{C}), \xi)$ is a complete C^*-Ag-valued MS. Define $\rho : \varpi \longrightarrow \varpi$ by

$$\rho(\jmath_1, \jmath_2, \jmath_3, \cdots) = \left(1 + \jmath_2, \frac{1}{2} + \jmath_3, \frac{1}{2^4} + \jmath_4, \cdots\right).$$

Then ρ is obviously a C^*-Ag-valued F-contraction. But for each $\lambda \in \mathbb{C}$, $\jmath = (\lambda, \lambda - 1, \lambda - \frac{3}{2}, \cdots, \lambda - \sum_{i=0}^{n-1}\left(\frac{1}{2^i}\right), \cdots)$ is a fixed point of ρ, which shows that ρ has many fixed point in ϖ.

In the following, we construct an example to support the hypotheses of Theorem 2.1.

Example 2.3. *Let the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ be given as:*

$$\alpha_1 = 1 \times 1, \alpha_2 = 1 \times 1 + 2 \times 2, \cdots,$$

$$\alpha_n = 1\times 1 + 2\times 2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Let $\varpi = \{\alpha_n : n \in \mathbb{N}\}$, $\mathcal{A} = M_2(\mathbb{C})$ be the set of all invertible 2×2 matrices with elements in $\mathbb{C}$, and for all $\jmath, \wp \in \varpi$, define $\xi : \varpi \times \varpi \longrightarrow \mathcal{A}$ by $\xi(\jmath, \wp) = (\|\jmath - \wp\|)I$. Then, $(\varpi, M_2(\mathbb{C}), \xi)$ is a complete C^*-Ag-valued MS. Define the mapping $\rho : \varpi \longrightarrow \varpi$ by

$$\rho(\alpha_n) = \alpha_{n-1}, \text{ if } n > 1$$

and

$$\rho(\alpha_1) = \alpha_1, \text{ if } n = 1.$$

Note that if we set $A = I \in \mathcal{A}$, then

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \frac{\xi(\rho(\alpha_n), \rho(\alpha_1))}{\xi(\alpha_n, \alpha_1)} &= \lim_{n\longrightarrow\infty}\left(\frac{\|\alpha_{n-1} - 1\|}{\|\alpha_n - 1\|}\right)I \\
&= \lim_{n\longrightarrow\infty}\left\|\frac{n(n-1)(2n-1)-6}{n(n+1)(2n+1)-6}\right\| I \\
&= \lim_{n\longrightarrow\infty}\left\|\frac{2 - \frac{1}{n} - \frac{2}{n} + \frac{1}{n^2} - \frac{6}{n^3}}{2 + \frac{1}{n} + \frac{2}{n} + \frac{1}{n^2} + \frac{6}{n^3}}\right\| I = I.
\end{aligned}$$

Whence, ρ is not a C^*-Ag-valued contractive mapping, that is, Theorem 1.4 is not applicable to this example. On the other hand, defining the mapping $F : M_2(\mathbb{C})_+ \longrightarrow M_2(\mathbb{C})$ by $F(a) = (\ln(a) + a)A$, we reach a conclusion that ρ is a C^*-Ag-valued F-contraction. To see this, take $A = \omega I$, where $\omega \in (0,1)$. Then, $\|A\| = \omega < 1$. Next, notice that for all $m, n \in \mathbb{N}$, $\rho(\alpha_m) \neq \rho(\alpha_n) \Leftrightarrow (m > 2 \wedge n = 1) \vee (m > n > 1)$. Now, for each $m \in \mathbb{N},\ m > 2$, we have

$$\begin{aligned}
&\frac{\xi(\rho(\alpha_m), \rho(\alpha_1))}{\xi(\alpha_m, \alpha_1)} e^{\xi(\rho(\alpha_m),\rho(\alpha_1)) - \xi(\alpha_m,\alpha_1)} \\
&= \frac{\xi(\rho(\alpha_m), \rho(\alpha_1))}{\xi(\alpha_m, \alpha_1)} e^{-\left[\xi(\alpha_m,\alpha_1) - \xi(\rho(\alpha_m),\rho(\alpha_1))\right]} \\
&= (\|\alpha_{m-1} - \alpha_1\|A)(\|\alpha_m - \alpha_1\|A)^{-1}(e)^{-[\|\alpha_m - \alpha_1\| + \|\alpha_{m-1} - \alpha_1\|]A} I \\
&= \left\|\frac{2m^3 - 3m^2 + m - 6}{2m^3 + 3m^2 + m - 6}\right\| I e^{-\|m^2\|A} \preceq (e^{-\|m^2\|A})I \preceq (e^{-A})I.
\end{aligned}$$

Consequently, all the conditions of Theorem 2.1 are obeyed, and whence ρ has a unique fixed point $\alpha_1 = \rho(\alpha_1)$.

3. Applications to Volterra-type integral equations

Integral equations come up in many situations in mathematical physics, bio-mathematics, control theory, critical point theory for non-smooth energy functionals, differential variational inequalities, fuzzy set arithmetic and traffic problems, to mention but a few. In particular, Volterra-type integral equations are known to be of great importance in investigating dynamical systems and stochastic processes. Some instances are in the fields of oscillation problems, sweeping processes, granular systems, control problems and so on.

Fixed point results for contractive mappings are usually examined and enjoy great applications in the theory of differential and integral equations (e.g., see [6, 7, 24, 27, 31]). As an application of Theorem 2.1, we study in this section simple and new criteria for the existence of a unique solution to the following Volterra-type integral equation:

$$\jmath(t) = \int_0^t M(t, s, \jmath(s))ds + p(t),\ t \in [0, \Omega] = \gimel. \tag{3.1}$$

Let $\varpi = C(\gimel, \mathbb{R}) \subseteq L^\infty(\gimel, \mathbb{R})$, $H = L^2(\gimel)$ and $A = B(H)$. For an arbitrary $u \in \varpi$, let

$$\|u\|_\omega = \sup_{t\in\gimel}\{|u(t)|e^{-\omega t} : \omega \in (0, 1)\}.$$

Notice that $\|.\|_\omega$ is a norm equivalent to the Chebyshev norm, and $(\varpi, \|.\|_\omega)$ equipped with the metric ξ_ω defined by

$$\xi_\omega(\jmath, \wp) = \sup_{t\in\gimel}\left\{(\jmath(t) - \wp(t))e^{-\omega t}\right\}, \tag{3.2}$$

for all $\jmath, \wp \in \varpi$, is a Bs. Whence, $(\varpi, B(H), \xi_\omega)$ is a complete C^*-Ag-valued MS. Note that problem (3.1) represents an integral reformulation of physical phenomena such as the motion of a spring that is under the influence of a frictional force or a damping force. For some related results modeling practical phenomena, employing integral/differential equations, see [2, 3, 10, 11] and the references therein.

We study solvability conditions of (3.1) under the following hypotheses.

Theorem 3.1. *Suppose that the following conditions are satisfied:*

$(C1)$ $M : \gimel \times \gimel \times \mathbb{R} \longrightarrow \mathbb{R}$ *and* $p : \gimel \longrightarrow \mathbb{R}$ *are continuous mappings;*

$(C2)$ *for all* $t, s \in \gimel$ *and* $u, v \in \mathbb{R}$,

$$|M(t, s, u) - M(t, s, v)| \leq \omega e^{-\omega}|\omega(u - v)|.$$

Then, (3.1) *has a unique solution in* ϖ.

Proof. Put $A = \omega I$, then $A \in B(H)_+$, and $\|A\| = \omega < 1$. Define the operator $\rho : \varpi \longrightarrow \varpi$ by

$$\rho\jmath(t) = \int_0^t M(t, s, \jmath(s))ds + p(t),\ t \in \gimel.$$

Then, for all $\jmath, \wp \in \varpi$, we have

$$\begin{aligned} |\rho\jmath(t) - \rho\wp(t)| &\leq \int_0^t |M(t, s, \jmath(s)) - M(t, s, \wp(s))|ds \\ &\leq \int_0^t \omega e^{-\omega}|\omega(\jmath(s) - \wp(s))|ds \\ &= \int_0^t \omega e^{-\omega}\omega|\jmath(s) - \wp(s)|e^s e^{-s}ds \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^t e^s \omega e^{-\omega}\omega |\jmath(s) - \wp(s)| e^{-s} ds \\
&\leq \omega^2 e^{-\omega} \|\jmath - \wp\|_\omega \int_0^t e^s ds \\
&\leq \omega^2 e^{-\omega} \|\jmath - \wp\|_\omega e^t.
\end{aligned}$$

This implies that $|\rho\jmath(t) - \rho\wp(t)|e^{-t} \leq e^{-\omega}\omega(\|\jmath - \wp\|_\omega)\omega$. Equivalently,

$$\xi_\omega(\rho\jmath, \rho\wp) \preceq e^{-\omega}\omega\xi_\omega(\jmath, \wp)\omega. \tag{3.3}$$

Passing to the logarithm in (3.3) yields

$$\ln(\xi_\omega(\rho\jmath, \rho\wp)) \preceq \ln\left[e^{-\omega}\omega\xi_\omega(\jmath, \wp)\omega\right],$$

from which we have

$$\omega + \ln(\xi_\omega(\rho\jmath, \rho\wp)) \preceq \ln(\omega\xi_\omega(\jmath, \wp)\omega). \tag{3.4}$$

Now, by defining the function $F : B(H)_+ \longrightarrow B(H)$ by $F(a) = (\ln(a))I$, (3.4) becomes

$$\|A\| + \|F(\xi_\omega(\rho\jmath, \rho\wp)) \leq \|F(\|A^*\|\xi_\omega(\jmath, \wp)\|A\|)\|\|.$$

Consequently, Theorem 2.1 applies to the operator ρ, which has a unique fixed point in ϖ, corresponding to the unique solution of (3.1). □

The following example is constructed to illustrate an application of Theorem 3.1.

Example 3.1. *Consider the Volterra-type integral equation:*

$$\jmath(t) = \frac{t}{1+t^2} + \int_0^t \frac{\jmath(s)}{36 + (\jmath(s))^2} ds, \ t \in [0, \Omega], \Omega > 0. \tag{3.5}$$

From (3.1) *and* (3.5)*, we observe that* $p(t) = \frac{t}{1+t^2}$ *and* $M(t, s, \jmath(s)) = \frac{\jmath(s)}{36+(\jmath(s))^2}$ *are continuous; that is, Condition* $(C1)$ *of Theorem 3.1 is obeyed. Furthermore,*

$$\begin{aligned}
|M(t, s, \jmath(s)) - M(t, s, \wp(s))| &\leq \frac{1}{36}|\jmath(s) - \wp(s)| \\
&\leq (1)e^{-1}|\jmath(s) - \wp(s)| = \omega e^{-\omega}|\jmath(s) - \wp(s)|.
\end{aligned}$$

Thus, Condition $(C2)$ *is obeyed. Consequently, Theorem 3.1 can be employed to infer that* (3.1) *has a solution in* $\varpi = ([0, \Omega], \mathbb{R})$.

4. Conclusions

Following the fact that a C^*-Ag-valued MS is more general, and concepts in this space are proper improvements of the equivalent results in standard metric spaces, the notions proposed in this paper are significant extensions of the ideas of F-contractions, Suzuki contractions and similar fixed point results in the literature. However, the main idea of this paper, being discussed in the setting of C^*-Ag-valued MS, is rudimentary. Whence, it can be fine-tuned when examined in the context of some generalized C^*-Ag-valued MS such as C^*-Ag-valued b-MS, C^*-Ag-valued-R-MS and some other pseudo-MS. In addition, the investigated single-valued mappings can be extended to set-valued mappings, and whence the integral equations investigated herein can be upgraded as integral inclusions, compatible with multi-valued versions.

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Conflict of interest

The authors declare that there is no competing interest.

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