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Research article

New stability criteria for systems with an interval time-varying delay

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Abstract: This paper studies the stability analysis of systems with an interval time-varying delay. First, some new integral inequalities are introduced. Second, based on these new integral inequalities, some less conservative stability criteria are introduced in terms of the linear matrix inequalities. Finally, the merits of the stability criteria are shown via two numerical examples.

Keywords: integral inequality; time-varying delay; stability; linear matrix inequality (LMI)
Mathematics Subject Classification: 34K20, 34D20, 34K25

1. Introduction

Time-delay occurs in many practical systems, and it may cause poor performance or even instability. Therefore, the stability analysis of time-delay systems has attracted considerable attention over the past two decades [1–11]. The authors of [1–4] studied linear time-delay systems. The authors of [5–11] studied nonlinear time-delay systems. As is well known, the Lyapunov-Krasovskii functional (LKF) method is an effective method for stability analysis of time-delay systems. Researches often carried out their studies in two steps. One step is to construct an appropriate LKF, and the other is to estimate the derivative of the LKF. For the first one, there are many types of LKFs, such as integral delay partitioning-based LKFs [12], the augmented LKFs [13] and delay partitioning-based LKFs [14,15]. Based on the delay-partitioning method, a new stability criterion for systems with an interval time-varying delay is introduced in [15]. In [15], the time-varying delay satisfies $0 \leq h_1 \leq d(t) \leq h_2$, but only the delay interval $[0, h_1]$ is divided into m segments, i.e., the delay interval $[h_1, h_2]$ is ignored. This motivated our present research.

Sometimes in order to contain more information about the time-delay, some quadratic terms of the time-delay are introduced [16]. Therefore, it is necessary to study the negative-determination of quadratic functions. In [17], a new inequality is proposed for the quadratic polynomials by introducing free matrix variables. However, these free matrix variables increase computational complexity.

In recent years, several inequalities have been introduced to estimate the integral terms in the derivative of LKFs, such as the Jensen inequality [18,19], Wirtinger inequality [20], auxiliary inequality [21], Bessel inequality [22] and free matrix inequality [23]. By using the Jensen inequality, Wirtinger inequality and Bessel inequality to estimate the integral term in the derivative of the LKF, the term $-\frac{1}{\alpha}\zeta_1^T(t)Q\zeta_1(t) - \frac{1}{1-\alpha}\zeta_2^T(t)Q\zeta_2(t)$ is obtained, where $\alpha \in (0,1)$, $\zeta_1(t)$ and $\zeta_2(t)$ are two real column vectors and Q is a positive symmetric matrix. This term is usually handled by a reciprocally convex lemma [24] and some improved reciprocally convex lemmas [25–29]. The advantage of these lemmas lie in changing the non convex term into a convex expression. However, it is shown that these lemmas increase the number of matrix variables which may increase computational complexity. This motivated our present research.

In this paper, some new integral inequalities have been introduced without using any reciprocally convex method. Based on these new integral inequalities and a new delay partitioning approach, some new stability criteria are obtained for systems with a interval time-varying delay. It is worth noting that not only the interval $[0, h_1]$ but also the interval $[h_1, h_2]$ is partitioned. The merits of the presented criteria are demonstrated through two numerical examples. The contributions of our paper are as follows:

• Different from Refs. [19–24], some new integral inequalities are introduced to deal with the integral term $-h_{12}\int_{t-h_2}^{t-h_1} \dot{y}^T(s)R\dot{y}(s)ds$ without using any reciprocally convex method.

• In this paper, not only the interval $[0, h_1]$ but also the interval $[h_1, h_2]$ is partitioned. A new LKF is introduced based on this new delay-partitioning approach. It can be seen that the relationship among some state vectors $x^T(t-h_1), x^T(t-h_1-\frac{1}{m_2}h_{12}), \cdots, x^T(t-h_1-\frac{m_2-1}{m_2}h_{12})$ and $x^T(t-h_2)$ are considered sufficiently, which may yield less conservative results.

In this paper, the set S^n denotes the set of symmetric matrices, the set S_+^n denotes the set of symmetric positive definite matrices and $Sym\{A\}$ denotes $A + A^T$.

2. Preliminary

Consider the following systems with a time-varying delay

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-h(t)) \\ x(t) = \phi(t), \quad t \in [-h_2, 0], \end{cases} \tag{2.1}$$

where $x(t) \in R^n$ is the state vector and $A, B, \in R^{n\times n}$ are constant matrices. The time-varying delay $h(t)$ satisfies

$$0 \le h_1 \le h(t) \le h_2, \quad h_{12} = h_2 - h_1, \tag{2.2}$$

$$\dot{h}(t) \le u. \tag{2.3}$$

Lemma 2.1. For any matrix $R \in S_+^n$, scalars h_1 and h_2 with $h_1 \le h_2$, scalar function $0 \le h_1 \le h(t) \le h_2$, a vector valued function $y(t) : [h_1, h_2] \longrightarrow R^n$ such that the following inequality holds for every integer

$M \geq 0$:

$$-h_{12}\int_{t-h_2}^{t-h_1} y^T(s)Ry(s)ds \leq -\sum_{i=0}^{M}\alpha^i(h(t)-h_1)\int_{t-h(t)}^{t-h_1} y^T(s)Ry(s)ds - \sum_{i=0}^{M}\beta^i(h_2-h(t))\int_{t-h_2}^{t-h(t)} y^T(s)Ry(s)ds, \tag{2.4}$$

where $\alpha = \frac{h_2-h(t)}{h_{12}}, \beta = 1-\alpha$.

Proof. According to $0 \leq h_1 \leq h(t) \leq h_2$, for every integer $M \geq 0$, define $\alpha = \frac{h_2-h(t)}{h_{12}}$; we have

$$\begin{aligned}
h_{12} &= (h(t)-h_1) + (h_2-h(t)) \\
&= (h(t)-h_1) + \alpha h_{12} \\
&= (h(t)-h_1) + \alpha[(h(t)-h_1) + (h_2-h(t))] \\
&= (h(t)-h_1) + \alpha(h(t)-h_1) + \alpha^2 h_{12} \\
&= (h(t)-h_1) + \alpha(h(t)-h_1) + \alpha^2[(h(t)-h_1) + (h_2-h(t))] \\
&= (h(t)-h_1) + \alpha(h(t)-h_1) + \alpha^2(h(t)-h_1) + \alpha^3 h_{12} \\
&= \cdots \\
&= (h(t)-h_1) + \alpha(h(t)-h_1) + \alpha^2(h(t)-h_1) + \cdots + \alpha^{M-1}(h(t)-h_1) + \alpha^M h_{12} \\
&\geq (h(t)-h_1) + \alpha(h(t)-h_1) + \alpha^2(h(t)-h_1) + \cdots + \alpha^{M-1}(h(t)-h_1) + \alpha^M(h(t)-h_1).
\end{aligned} \tag{2.5}$$

Similarly, the following inequality can be obtained easily:

$$h_{12} \geq (h_2-h(t)) + \beta(h_2-h(t)) + \beta^2(h_2-h(t)) + \cdots + \beta^{M-1}(h_2-h(t)) + \beta^M(h_2-h(t)). \tag{2.6}$$

Then, based on (2.5) and (2.6), we can obtain:

$$\begin{aligned}
-h_{12}\int_{t-h_2}^{t-h_1} y^T(s)Ry(s)ds &= -h_{12}\int_{t-h(t)}^{t-h_1} y^T(s)Ry(s)ds - h_{12}\int_{t-h_2}^{t-h(t)} y^T(s)Ry(s)ds \\
&\leq -\sum_{i=0}^{M}\alpha^i(h(t)-h_1)\int_{t-h(t)}^{t-h_1} y^T(s)Ry(s)ds \\
&\quad - \sum_{i=0}^{M}\beta^i(h_2-h(t))\int_{t-h_2}^{t-h(t)} y^T(s)Ry(s)ds.
\end{aligned} \tag{2.7}$$

Lemma 2.2. [18]. For a matrix $R \in S_+^n$ and scalars a and b satisfying $a < b$, a vector function $y : [a,b] \longrightarrow R^n$ such that the following inequality holds:

$$(b-a)\int_a^b y^T(s)Ry(s)ds \geq \int_a^b y^T(s)dsR\int_a^b y(s)ds. \tag{2.8}$$

Lemma 2.3. [20]. For a matrix $R \in S_+^n$ and scalars a and b satisfying $a < b$, any continuously differentiable function $y : [a,b] \longrightarrow R^n$ such that the following inequality holds:

$$\int_a^b \dot{y}^T(s)R\dot{y}(s)ds \geq \frac{1}{b-a}(\Omega_0^T R\Omega_0 + 3\Omega_1^T R\Omega_1), \tag{2.9}$$

where

$$\Omega_0 = y(b) - y(a),$$
$$\Omega_1 = y(b) + y(a) - \frac{2}{b-a}\int_a^b y(s)ds.$$

Lemma 2.4. [23]. Let $g(y) = a_0 + a_1 y + a_2 y^2$, where $y \in [h_1, h_2]$ and $a_0, a_1, a_2 \in R$. For a given nonnegative integer N, if the following conditions hold for $i = 1, 2, \cdots, 2^N$:
(i)$g(h_1) < 0$,
(ii)$g(h_2) < 0$,
(iii)$\frac{h_{12}}{2^{N+1}}\dot{g}(\frac{i-1}{2^N}h_{12} + h_1) + g(\frac{i-1}{2^N}h_{12} + h_1) < 0$, $i = 1, 2, \cdots, 2^N$,
then $g(y) < 0$.

Lemma 2.5. [29]. Let $f(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3$, where $y \in [h_1, h_2]$ and $a_0, a_1, a_2, a_3 \in S^n$, if there exist constant matrices $F_i \in R^{n\times n}$ such that for $i = 1, 2$:

$$\begin{bmatrix} h_i(a_1 + F_1 + F_1^T) + a_0 & \frac{h_i}{2}a_2 - F_1 + h_i F_2^T \\ * & h_i a_3 - F_2 - F_2^T \end{bmatrix} < 0,$$

then $f(y) < 0$.

Based on Lemma 2.1 and Lemma 2.2, a novel integral inequality is obtained as follows.

Lemma 2.6. For any matrix $R \in S_+^n$ and scalars h_1 and h_2 with $h_1 \le h_2$, a nonnegative integer M and scalar function $0 \le h_1 \le h(t) \le h_2$, any continuously differentiable function $y : [a, b] \longrightarrow R^n$ such that the following inequality holds:

$$\begin{aligned} &- h_{12}\int_{t-h_2}^{t-h_1} \dot{y}^T(s)R\dot{y}(s)ds \\ \le &- \sum_{i=0}^{M} \alpha^i [y(t-h_1) - y(t-h(t))]^T R[y(t-h_1) - y(t-h(t))] \\ &- \sum_{i=0}^{M} \beta^i [y(t-h(t)) - y(t-h_2)]^T R[y(t-h(t)) - y(t-h_2)], \end{aligned} \tag{2.10}$$

where $\alpha = \frac{h_2 - h(t)}{h_{12}}, \beta = 1 - \alpha$.

Remark 1. When $M = 1$, Lemma 2.6 is reduced to the method in [15]. So the method in [15] is a special case of Lemma 2.6 in this paper.

Based on Lemma 2.1 and Lemma 2.3, another improved integral inequality is obtained as follows.

Lemma 2.7. For any matrix $R \in S_+^n$, scalars h_1 and h_2 with $h_1 \le h_2$, a nonnegative integer M and scalar function $0 \le h_1 \le h(t) \le h_2$, any continuously differentiable function $y : [a, b] \longrightarrow R^n$ such that the following inequality holds:

$$-h_{12}\int_{t-h_2}^{t-h_1} \dot{y}^T(s)R\dot{y}(s)ds \le -\sum_{i=0}^{M} \alpha^i(\bar{\Omega}_1^T R\bar{\Omega}_1 + 3\bar{\Omega}_2^T R\bar{\Omega}_2) - \sum_{i=0}^{M} \beta^i(\bar{\Omega}_3^T R\bar{\Omega}_3 + 3\bar{\Omega}_4^T R\bar{\Omega}_4), \tag{2.11}$$

where $\alpha = \frac{h_2 - h(t)}{h_{12}}, \beta = 1 - \alpha$.

$$\bar{\Omega}_1 = y(t-h_1) - y(t-h(t))),$$

$$\bar{\Omega}_2 = y(t-h_1) + y(t-h(t))) - \frac{2}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} y(s)ds,$$
$$\bar{\Omega}_3 = y(t-h(t)) - y(t-h_2),$$
$$\bar{\Omega}_4 = y(t-h(t)) + y(t-h_2) - \frac{2}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} y(s)ds.$$

3. Results

Theorem 3.1. For given scalars $h_1 > 0$ and $h_2 > 0$, u and nonnegative integers M, m_1 and m_2, System (2.1) is asymptotically stable if there exist matrices $P \in S_+^{(m_1+m_2+1)n}$, $Q_1 \in S_+^{m_1 n}$, $Q_2 \in S_+^{m_2 n}$, $Q_3 \in S_+^{(m_1-1)n}$, $Q_4 \in S_+^{(m_2-1)n}$, $Q_5 \in S_+^n$, $R_i \in S_+^n$, $i = 1, 2, \cdots, m_1$ and $Z_k \in S_+^n$, $k = 1, 2, \cdots, m_2$ such that the following linear matrix inequalities (LMIs) hold for $k = 1, 2, \cdots, m_2$:

$$\begin{aligned}\varpi_k =& Sym\left\{\Pi_{11}^T P \Pi_{12}\right\} + \Pi_{21}^T Q_1 \Pi_{21} - \Pi_{22}^T Q_1 \Pi_{22} + \Pi_{23}^T Q_2 \Pi_{23} - \Pi_{24}^T Q_2 \Pi_{24} + \Pi_{31}^T Q_3 \Pi_{31} - \Pi_{32}^T Q_3 \Pi_{32} \\ &+ \Pi_{33}^T Q_4 \Pi_{33} - \Pi_{34}^T Q_4 \Pi_{34} + e_{m_1+i}^T Q_5 e_{m_1+i} - (1-u) e_{2(m_1+m_2)+2}^T Q_5 e_{2(m_1+m_2)+2} \\ &+ \sum_{i=1}^{m_1} (\frac{h_1}{m_1})^2 e_0 R_i e_0 + \sum_{k=1}^{m_2} (\frac{h_{12}}{m_2})^2 e_0 Z_k e_0 - \sum_{i=1}^{m_1} \Sigma_i^T \bar{R}_i \Sigma_i - \sum_{k=1}^{m_2} \hat{\Sigma}_k^T \bar{Z}_k \hat{\Sigma}_k + \hat{\Sigma}_k^T \bar{Z}_k \hat{\Sigma}_k \\ &- \sum_{j=0}^{M} \alpha^j \Pi_{41k}^T \bar{Z}_k \Pi_{41k} - \sum_{j=0}^{M} \beta^j \hat{\Pi}_{42k}^T \bar{Z}_k \hat{\Pi}_{42k} < 0,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\Pi_{11} &= \begin{bmatrix} e_1^T & e_{m_1+m_2+2}^T & \cdots & e_{2m_1+m_2+1}^T & e_{2m_1+m_2+2}^T & \cdots & e_{2m_1+2m_2+1}^T \end{bmatrix}^T, \\
\Pi_{12} &= \begin{bmatrix} e_0^T & e_1^T - e_2^T & \cdots & e_{m_1}^T - e_{m_1+1}^T & e_{m_1+2}^T - e_{m_1+3}^T & \cdots & e_{m_1+m_2}^T - e_{m_1+m_2+1}^T \end{bmatrix}^T, \\
\Pi_{21} &= \begin{bmatrix} e_1^T & e_2^T & \cdots & e_m^T \end{bmatrix}^T, \\
\Pi_{22} &= \begin{bmatrix} e_2^T & e_3^T & \cdots & e_{m_1+1}^T \end{bmatrix}^T, \\
\Pi_{23} &= \begin{bmatrix} e_{m_1+1}^T & e_{m_1+2}^T & \cdots & e_{m_1+m_2}^T \end{bmatrix}^T, \\
\Pi_{23} &= \begin{bmatrix} e_{m_1+2}^T & e_{m_1+3}^T & \cdots & e_{m_1+m_2+1}^T \end{bmatrix}^T, \\
\Pi_{31} &= \begin{bmatrix} e_{m_1+m_2+2}^T & e_{m_1+m_2+3}^T & \cdots & e_{2m_1+m_2}^T \end{bmatrix}^T, \\
\Pi_{32} &= \begin{bmatrix} e_{m_1+m_2+3}^T & e_{m_1+m_2+4}^T & \cdots & e_{2m_1+m_2+1}^T \end{bmatrix}^T, \\
\Pi_{33} &= \begin{bmatrix} e_{2m_1+m_2+2}^T & e_{2m_1+m_2+3}^T & \cdots & e_{2m_1+2m_2}^T \end{bmatrix}^T, \\
\Pi_{34} &= \begin{bmatrix} e_{2m_1+m_2+3}^T & e_{2m_1+m_2+4}^T & \cdots & e_{2m_1+2m_2+1}^T \end{bmatrix}^T, \\
\Sigma_i &= \begin{bmatrix} e_i^T - e_{i+1}^T & e_i^T + e_{i+1}^T - 2e_{m_1+m_2+1+i}^T \end{bmatrix}^T, \\
\hat{\Sigma}_k &= \begin{bmatrix} e_{m_1+1+k}^T - e_{m_1+2+k}^T & e_{m_1+1+k}^T + e_{m_1+2+k}^T - 2e_{2m_1+m_2+1+k}^T \end{bmatrix}^T, \\
\Pi_{41k} &= \begin{bmatrix} e_{m_1+k}^T - e_{2(m_1+m_2)+2}^T & e_{m_1+k}^T + e_{2(m_1+m_2)+2}^T - 2e_{2(m_1+m_2)+3}^T \end{bmatrix}^T, \\
\Pi_{42k} &= \begin{bmatrix} e_{2(m_1+m_2)+2}^T - e_{m_1+1+k}^T & e_{2(m_1+m_2)+2}^T + e_{m_1+1+k}^T - 2e_{2(m_1+m_2)+4}^T \end{bmatrix}^T, \\
e_0 &= Ae_1 + Be_{2m_1+2m_2+2},
\end{aligned}$$

$$\bar{R}_i = diag(R_i, 3R_i),\ i = 1, 2, \cdots, m_1,$$
$$\bar{Z}_k = diag(Z_k, 3Z_k),\ k = 1, 2, \cdots, m_2,$$
$$e_i = \begin{bmatrix} 0_{n\times(i-1)n} & I_n & 0_{n\times(2(m_1+m_2)+4-i)n} \end{bmatrix}, \text{for } i = 1, 2, \cdots, 2(m_1+m_2)+4.$$

Proof. Let integers $m_1 > 0$, $m_2 > 0$, $[0, h_1]$ and $[h_1, h_2]$ be divisible into m_1 and m_2 segments, i.e., $[0, h_1] = \bigcup_{i=1}^{m_1}[\frac{i-1}{m_1}h_1, \frac{i}{m_1}h_1]$ and $[h_1, h_2] = \bigcup_{k=1}^{m_2}[h_1 + \frac{k-1}{m_2}h_{12}, h_1 + \frac{k}{m_2}h_{12}]$. Thus, for any $t \geq 0$, there exists an integer $k \in \{1, 2, \cdots, m_2\}$ such that $h(t) \in [h_1 + \frac{k-1}{m_2}h_{12}, h_1 + \frac{k}{m_2}h_{12}]$. Then, we introduce the following LKF candidate:

$$V(x(t))|_k = V(x(t))|_{h(t)\in[h_1+\frac{k-1}{m_2}h_{12}, h_1+\frac{k}{m_2}h_{12}]}, \tag{3.2}$$

$$V(x(t))|_k = V_1(x(t)) + V_2(x(t)) + V_3(x(t)), \tag{3.3}$$

where

$$V_1(x(t)) = \eta^T(t)P\eta(t), \tag{3.4}$$

$$\begin{aligned} V_2(x(t)) = & \int_{t-\frac{h_1}{m_1}}^{t} \Upsilon_3^T(s)Q_1\Upsilon_3(s)ds + \int_{t-\frac{h_{12}}{m_2}}^{t} \Upsilon_4^T(s)Q_2\Upsilon_4(s)ds \\ & \int_{t-\frac{h_1}{m_1}}^{t} \Upsilon_5^T(\theta)Q_3\Upsilon_5(\theta)ds + \int_{t-\frac{h_{12}}{m_2}}^{t} \Upsilon_6^T(\theta)Q_4\Upsilon_6(\theta)ds \\ & + \int_{t-h(t)}^{t-h_1-\frac{k-1}{m_2}h_{12}} x^T(s)Q_5x(s)ds, \end{aligned} \tag{3.5}$$

$$\begin{aligned} V_3(x(t)) = & \sum_{i=1}^{m_1}\frac{h_1}{m_1}\int_{-\frac{i}{m_1}h_1}^{-\frac{i-1}{m_1}h_1}\int_{t+u}^{t}\dot{x}^T(s)R_i\dot{x}(s)dsdu \\ & + \sum_{k=1}^{m_2}\frac{h_{12}}{m_2}\int_{-h_1-\frac{k}{m_2}h_{12}}^{-h_1-\frac{k-1}{m_2}h_{12}}\int_{t+u}^{t}\dot{x}^T(s)Z_k\dot{x}(s)dsdu, \end{aligned} \tag{3.6}$$

where

$$\eta(t) = \begin{bmatrix} x^T(t) & \Upsilon_1(t) & \Upsilon_2(t) \end{bmatrix}^T,$$
$$\Upsilon_1(t) = \begin{bmatrix} \int_{t-\frac{1}{m_1}h_1}^{t} x^T(s)ds & \int_{t-\frac{2}{m_1}h_1}^{t-\frac{1}{m_1}h_1} x^T(s)ds & \cdots & \int_{t-h_1}^{t-\frac{m_1-1}{m_1}h_1} x^T(s)ds \end{bmatrix}^T,$$
$$\Upsilon_2(t) = \begin{bmatrix} \int_{t-h_1-\frac{1}{m_2}h_{12}}^{t-h_1} x^T(s)ds & \int_{t-h_1-\frac{2}{m_2}h_{12}}^{t-h_1-\frac{1}{m_2}h_{12}} x^T(s)ds & \cdots & \int_{t-h_2}^{t-h_1-\frac{m_2-1}{m_2}h_{12}} x^T(s)ds \end{bmatrix}^T,$$
$$\Upsilon_3(s) = \begin{bmatrix} x^T(s) & x^T(s-\frac{1}{m_1}h_1) & \cdots & x^T(s-\frac{m_1-1}{m_1}h_1) \end{bmatrix}^T,$$
$$\Upsilon_4(s) = \begin{bmatrix} x^T(s-h_1) & x^T(s-h_1-\frac{1}{m_2}h_{12}) & \cdots & x^T(s-h_1-\frac{m_2-1}{m_2}h_{12}) \end{bmatrix}^T,$$

$$\Upsilon_5(\theta)=\left[\begin{array}{cccc}\int_{\theta-\frac{1}{m_1}h_1}^{\theta}x^T(s)ds & \int_{\theta-\frac{2}{m_1}h_1}^{\theta-\frac{1}{m_1}h_1}x^T(s)ds & \cdots & \int_{\theta-\frac{m_1-1}{m_1}h_1}^{\theta-\frac{m_1-2}{m_1}h_1}x^T(s)ds\end{array}\right]^T,$$

$$\Upsilon_6(\theta)=\left[\begin{array}{cccc}\int_{\theta-h_1-\frac{1}{m_2}h_{12}}^{\theta-h_1}x^T(s)ds & \int_{\theta-h_1-\frac{2}{m_2}h_{12}}^{\theta-h_1-\frac{1}{m_2}h_{12}}x^T(s)ds & \cdots & \int_{\theta-h_1-\frac{m_2-1}{m_2}h_{12}}^{\theta-h_1-\frac{m_2-2}{m_2}h_{12}}x^T(s)ds\end{array}\right]^T.$$

Taking the derivative of $V(x(t))|_k$ yields

$$\begin{aligned}\dot{V}_1(x(t))&=2\eta^T(t)P\dot{\eta}(t)\\&=\zeta^T(t)(Sym\left\{\Pi_{11}^TP\Pi_{12}\right\})\zeta(t),\end{aligned}\tag{3.7}$$

where

$$\zeta^T(t)=\left[\begin{array}{cccccccc}\Upsilon_3^T(t) & \Upsilon_4^T(t) & x^T(t-h_2) & \frac{h_1}{m_1}\Upsilon_1^T(t) & \frac{h_{12}}{m_2}\Upsilon_2^T(t) & x^T(t-h(t)) & v_{1k}^T(t) & v_{2k}^T(t)\end{array}\right]^T,$$

$$v_{1k}(t)=\frac{1}{h(t)-h_1-\frac{k-1}{m_2}h_{12}}\int_{t-h(t)}^{t-h_1-\frac{k-1}{m_2}h_{12}}x(s)ds,$$

$$v_{2k}(t)=\frac{1}{h_1+\frac{k}{m_2}h_{12}-h(t)}\int_{t-h_1-\frac{k}{m_2}h_{12}}^{t-h(t)}x(s)ds,$$

$$\begin{aligned}\dot{V}_2(x(t))=&\Upsilon_3^T(t)Q_1\Upsilon_3^T(t)-\Upsilon_3^T(t-\frac{h_1}{m_1})Q_1\Upsilon_3(t-\frac{h_1}{m_1})\\&+\Upsilon_4^T(t)Q_2\Upsilon_4^T(t)-\Upsilon_4^T(t-\frac{h_{12}}{m_2})Q_2\Upsilon_4(t-\frac{h_{12}}{m_2})\\&+\Upsilon_5^T(t)Q_3\Upsilon_5^T(t)-\Upsilon_5^T(t-\frac{h_1}{m_1})Q_3\Upsilon_5(t-\frac{h_1}{m_1})\\&+\Upsilon_6^T(t)Q_4\Upsilon_6^T(t)-\Upsilon_6^T(t-\frac{h_{12}}{m_2})Q_4\Upsilon_6(t-\frac{h_{12}}{m_2})\\&+x^T(t-h_1-\frac{k-1}{m_2}h_{12})Q_5x(t-h_1-\frac{k-1}{m_2}h_{12})\\&-(1-\dot{h}(t))x^T(t-h(t))Q_5x(t-h(t))\\\leq&\zeta^T(t)\hat{\varpi}_1\zeta(t),\end{aligned}\tag{3.8}$$

$$\dot{V}_3(x(t))=\sum_{i=1}^{m_1}(\frac{h_1}{m_1})^2\dot{x}^T(t)R_i\dot{x}(t)+\sum_{k=1}^{m_2}(\frac{h_{12}}{m_2})^2\dot{x}^T(t)Z_k\dot{x}(t)-\hat{\varpi}_2-\hat{\varpi}_3,\tag{3.9}$$

$$\hat{\varpi}_2=\sum_{i=1}^{m_1}\frac{h_1}{m_1}\int_{t-\frac{i}{m_1}h_1}^{t-\frac{i-1}{m_1}h_1}\dot{x}^T(s)R_i\dot{x}(s)dsd,$$

$$\hat{\varpi}_3=\sum_{k=1}^{m_2}\frac{h_{12}}{m_2}\int_{t-h_1-\frac{k}{m_2}h_{12}}^{t-h_1-\frac{k-1}{m_2}h_{12}}\dot{x}^T(s)Z_k\dot{x}(s)ds.$$

Based on Lemma 2.3, we obtain:

$$\hat{\varpi}_2 \geq \zeta^T(t)(\sum_{i=1}^{m_1} \Sigma_i^T \bar{R}_i \Sigma_i)\zeta(t), \tag{3.10}$$

$$\hat{\varpi}_3 \geq \zeta^T(t)(\sum_{k=1}^{m_2} \hat{\Sigma}_k^T \bar{Z}_k \hat{\Sigma}_k - \hat{\Sigma}_k^T \bar{Z}_k \hat{\Sigma}_k)\zeta(t) + \frac{h_{12}}{m_2}\int_{t-h_1-\frac{k}{m_2}h_{12}}^{t-h_1-\frac{k-1}{m_2}h_{12}} \dot{x}^T(s) Z_k \dot{x}(s)ds. \tag{3.11}$$

Then, based on Lemma 2.7, we have:

$$\frac{h_{12}}{m_2}\int_{t-h_1-\frac{k}{m_2}h_{12}}^{t-h_1-\frac{k-1}{m_2}h_{12}} \dot{x}^T(s) Z_k \dot{x}(s)ds \geq \zeta^T(t)(\sum_{j=0}^{M} \alpha^j \Pi_{41k}^T \bar{Z}_k \Pi_{41k} + \sum_{j=0}^{M} \beta^j \hat{\Pi}_{42k}^T \bar{Z}_k \hat{\Pi}_{42k})\zeta(t). \tag{3.12}$$

According to (3.7)–(3.12), we can obtain:

$$\dot{V}(x(t))|_k \leq \zeta^T(t)\varpi_k\zeta(t), \tag{3.13}$$

where ϖ_k, $k = 1, 2, \cdots, m_2$ are defined in Theorem 3.1. For $h(t) \in [h_1, h_2]$, if (3.1) holds, then we have $\zeta^T(t)\varpi_k\zeta(t) < 0$, i.e., $\dot{V}(x(t))|_k < 0$. Thus, System (2.1) is asymptotically stable. This completes the proof.

Remark 3.2. Compared with the method in [15], not only the delay interval $[0, h_1]$ but also the delay interval $[h_1, h_2]$ is decomposed into several subintervals equally. The purpose of such a method is to make the constructed LKF includes more information of some state vectors. For example, the LKF includes the term $\int_{t-\frac{h_1}{m_1}}^{t} \Upsilon_3^T(s) Q_1 \Upsilon_3(s)ds$, so the relationship among some state vectors $x^T(t), x^T(t - \frac{1}{m}h_1), \cdots, x^T(t - \frac{m-1}{m}h_1)$ and $x^T(t - h_1)$ are considered sufficiently, which may yield less conservative results.

Remark 3.3. Theorem 3.1 presents an M-dependent stability criterion for System (1) based on Lemma 2.7 and a new LKF. Clearly, the left-hand side of the inequality (3.1) is $M-$degree polynomial matrices on $h(t)$. When $M = 1$, the inequality (3.1) is linear. Thus, under this condition, Theorem 3.1 can be solved by using the LMI toolbox. When $M \geq 2$, it is nonlinear on $h(t)$, which is not solvable directly. Thus, we employ Lemma 2.4 and Lemma 2.5 to transform it into LMIs for $M = 2$ and $M = 3$, respectively.

For $M = 1$, Theorem 3.1 is transformed into the following corollary easily.

Corollary 3.4. For given scalars $h_1 > 0, h_2 > 0$, u and $\alpha \in [0, 1]$ and nonnegative integers m_1 and m_2, System (1) is asymptotically stable if there exist matrices $P \in S_+^{(m_1+m_2+1)n}$, $Q_1 \in S_+^{m_1 n}$, $Q_2 \in S_+^{m_2 n}$, $Q_3 \in S_+^{(m_1-1)n}$, $Q_4 \in S_+^{(m_2-1)n}$, $Q_5 \in S_+^n$, $R_i \in S_+^n$, $i = 1, 2, \cdots, m_1$ and $Z_k \in S_+^n$, $k = 1, 2, \cdots, m_2$ such that the following LMIs hold for $k = 1, 2, \cdots, m_2$:

$$\begin{aligned}
\varpi_{1k} =& Sym\left\{\Pi_{11}^T P \Pi_{12}\right\} + \Pi_{21}^T Q_1 \Pi_{21} - \Pi_{22}^T Q_1 \Pi_{22} + \Pi_{23}^T Q_2 \Pi_{23} - \Pi_{24}^T Q_2 \Pi_{24} + \Pi_{31}^T Q_3 \Pi_{31} - \Pi_{32}^T Q_3 \Pi_{32} \\
&+ \Pi_{33}^T Q_4 \Pi_{33} - \Pi_{34}^T Q_4 \Pi_{34} + e_{m_1+i}^T Q_5 e_{m_1+i} - (1-u) e_{2(m_1+m_2)+2}^T Q_5 e_{2(m_1+m_2)+2} \\
&+ \sum_{i=1}^{m_1} (\frac{h_1}{m_1})^2 e_0 R_i e_0 + \sum_{k=1}^{m_2} (\frac{h_{12}}{m_2})^2 e_0 Z_k e_0 - \sum_{i=1}^{m_1} \Sigma_i^T \bar{R}_i \Sigma_i - \sum_{k=1}^{m_2} \hat{\Sigma}_k^T \bar{Z}_k \hat{\Sigma}_k + \hat{\Sigma}_k^T \bar{Z}_k \hat{\Sigma}_k \\
&- (1+\alpha)\Pi_{41k}^T \bar{Z}_k \Pi_{41k} - (2-\alpha)\hat{\Pi}_{42k}^T \bar{Z}_k \hat{\Pi}_{42k} < 0.
\end{aligned} \tag{3.14}$$

From Corollary 3.4, if u is unknown, a corollary is obtained by eliminating Q_5.

Corollary 3.5. For given scalars $h_1 > 0, h_2 > 0$ and $\alpha \in [0,1]$ and nonnegative integers m_1 and m_2, System (2.1) is asymptotically stable if there exist matrices $P \in S_+^{(m_1+m_2+1)n}$, $Q_1 \in S_+^{m_1 n}$, $Q_2 \in S_+^{m_2 n}$, $Q_3 \in S_+^{(m_1-1)n}$, $Q_4 \in S_+^{(m_2-1)n}$, $R_i \in S_+^n$, $i = 1, 2, \cdots, m_1$ and $Z_k \in S_+^n$, $k = 1, 2, \cdots, m_2$ such that the following LMIs hold for $k = 1, 2, \cdots, m_2$:

$$\begin{aligned}
&Sym\left\{\Pi_{11}^T P \Pi_{12}\right\} + \Pi_{21}^T Q_1 \Pi_{21} - \Pi_{22}^T Q_1 \Pi_{22} + \Pi_{23}^T Q_2 \Pi_{23} - \Pi_{24}^T Q_2 \Pi_{24} \\
&+ \Pi_{31}^T Q_3 \Pi_{31} - \Pi_{32}^T Q_3 \Pi_{32} + \Pi_{33}^T Q_4 \Pi_{33} - \Pi_{34}^T Q_4 \Pi_{34} \\
&+ \sum_{i=1}^{m_1} (\frac{h_1}{m_1})^2 e_0 R_i e_0 + \sum_{k=1}^{m_2} (\frac{h_{12}}{m_2})^2 e_0 Z_k e_0 - \sum_{i=1}^{m_1} \Sigma_i^T \bar{R}_i \Sigma_i - \sum_{k=1}^{m_2} \hat{\Sigma}_k^T \bar{Z}_k \hat{\Sigma}_k + \hat{\Sigma}_k^T \bar{Z}_k \hat{\Sigma}_k \\
&- (1+\alpha) \Pi_{41k}^T \bar{Z}_k \Pi_{41k} - (2-\alpha) \hat{\Pi}_{42k}^T \bar{Z}_k \hat{\Pi}_{42k} < 0.
\end{aligned} \tag{3.15}$$

For $M = 2$, by Lemma 2.4, Theorem 3.1 is transformed into the following corollary easily.

Corollary 3.6. For given scalars $h_1 > 0, h_2 > 0$ and u and nonnegative integers m_1 and m_2, System (2.1) is asymptotically stable if there exist matrices $P \in S_+^{(m_1+m_2+1)n}$, $Q_1 \in S_+^{m_1 n}$, $Q_2 \in S_+^{m_2 n}$, $Q_3 \in S_+^{(m_1-1)n}$, $Q_4 \in S_+^{(m_2-1)n}$, $Q_5 \in S_+^n$, $R_i \in S_+^n$, $i = 1, 2, \cdots, m_1$ and $Z_k \in S_+^n$, $k = 1, 2, \cdots, m_2$ such that the following LMIs hold for $k = 1, 2, \cdots, m_2$:

$$\Phi_{0k} \le 0, \tag{3.16}$$

$$\Phi_{2k} + \Phi_{1k} + \Phi_{0k} \le 0, \tag{3.17}$$

$$(\frac{1}{2^N}\bar{\rho}_j + \bar{\rho}_j^2)\Phi_{2k} + (\frac{1}{2^{N+1}}\bar{\rho}_j + \bar{\rho}_j^2)\Phi_{1k} + \Phi_{0k} \le 0, \tag{3.18}$$

where

$$\begin{aligned}
\Phi_{0k} &= \varphi_k - \Pi_{41k}^T \bar{Z}_k \Pi_{41k} - 3\Pi_{42k}^T \bar{Z}_k \hat{\Pi}_{42k}, \\
\Phi_{1k} &= -\Pi_{41k}^T \bar{Z}_k \Pi_{41k} + 3\Pi_{42k}^T \bar{Z}_k \hat{\Pi}_{42k}, \\
\Phi_{2k} &= -\Pi_{41k}^T \bar{Z}_k \Pi_{41k} - 3\Pi_{42k}^T \bar{Z}_k \hat{\Pi}_{42k},
\end{aligned}$$

$$\begin{aligned}
\varphi_k =& Sym\left\{\Pi_{11}^T P \Pi_{12}\right\} + \Pi_{21}^T Q_1 \Pi_{21} - \Pi_{22}^T Q_1 \Pi_{22} + \Pi_{23}^T Q_2 \Pi_{23} - \Pi_{24}^T Q_2 \Pi_{24} + \Pi_{31}^T Q_3 \Pi_{31} - \Pi_{32}^T Q_3 \Pi_{32} \\
&+ \Pi_{33}^T Q_4 \Pi_{33} - \Pi_{34}^T Q_4 \Pi_{34} + e_{m_1+i}^T Q_5 e_{m_1+i} - (1-u) e_{2(m_1+m_2)+2}^T Q_5 e_{2(m_1+m_2)+2} \\
&+ \sum_{i=1}^{m_1} (\frac{h_1}{m_1})^2 e_0 R_i e_0 + \sum_{k=1}^{m_2} (\frac{h_{12}}{m_2})^2 e_0 Z_k e_0 - \sum_{i=1}^{m_1} \Sigma_i^T \bar{R}_i \Sigma_i - \sum_{k=1}^{m_2} \hat{\Sigma}_k^T \bar{Z}_k \hat{\Sigma}_k + \hat{\Sigma}_k^T \bar{Z}_k \hat{\Sigma}_k,
\end{aligned} \tag{3.19}$$

$\bar{\rho}_j = \frac{j-1}{2^N}$, $j = 1, 2, \cdots, 2^N$.

Proof. For $M = 2$, based on (3.13), we can obtain:

$$\dot{V}(x(t))|_k \le \zeta^T(t) \varpi_{2k} \zeta(t), \tag{3.20}$$

where

$$\begin{aligned}\varpi_{2k} =& Sym\left\{\Pi_{11}^T P\Pi_{12}\right\} + \Pi_{21}^T Q_1\Pi_{21} - \Pi_{22}^T Q_1\Pi_{22} + \Pi_{23}^T Q_2\Pi_{23} - \Pi_{24}^T Q_2\Pi_{24} + \Pi_{31}^T Q_3\Pi_{31} - \Pi_{32}^T Q_3\Pi_{32} \\ &+ \Pi_{33}^T Q_4\Pi_{33} - \Pi_{34}^T Q_4\Pi_{34} + e_{m_1+i}^T Q_5 e_{m_1+i} - (1-u)e_{2(m_1+m_2)+2}^T Q_5 e_{2(m_1+m_2)+2} \\ &+ \sum_{i=1}^{m_1}(\frac{h_1}{m_1})^2 e_0 R_i e_0 + \sum_{k=1}^{m_2}(\frac{h_{12}}{m_2})^2 e_0 Z_k e_0 - \sum_{i=1}^{m_1}\Sigma_i^T\bar{R}_i\Sigma_i - \sum_{k=1}^{m_2}\hat{\Sigma}_k^T\bar{Z}_k\hat{\Sigma}_k + \hat{\Sigma}_k^T\bar{Z}_k\hat{\Sigma}_k \\ &- (1+\alpha+\alpha^2)\Pi_{41k}^T\bar{Z}_k\Pi_{41k} - (1+\beta+\beta^2)\hat{\Pi}_{42k}^T\bar{Z}_k\hat{\Pi}_{42k} < 0.\end{aligned} \tag{3.21}$$

Then, (3.20) can be rewritten as

$$\dot{V}(x(t))|_k \leq \zeta^T(t)(\alpha^2\Phi_{2k} + \alpha\Phi_{1k} + \Phi_{0k})\zeta(t). \tag{3.22}$$

From Lemma 2.4, if (3.16)–(3.18) hold, then we obtain $\alpha^2\Phi_{2k} + \alpha\Phi_{1k} + \Phi_{0k} < 0$, i.e., $\dot{V}(x(t))|_k < 0$. This completes the proof.

From Corollary 3.6, if u is unknown, the following corollary can be obtained by eliminating Q_5.

Corollary 3.7. For given scalars $h_1 > 0$ and $h_2 > 0$ and nonnegative integers m_1 and m_2, System (2.1) is asymptotically stable if there exist matrices $P \in S_+^{(m_1+m_2+1)n}$, $Q_1 \in S_+^{m_1 n}$, $Q_2 \in S_+^{m_2 n}$, $Q_3 \in S_+^{(m_1-1)n}$, $Q_4 \in S_+^{(m_2-1)n}$, $R_i \in S_+^n$, $i = 1, 2, \cdots, m_1$ and $Z_k \in S_+^n$, $k = 1, 2, \cdots, m_2$ such that the following LMIs hold for $k = 1, 2, \cdots, m_2$:

$$\bar{\Phi}_{0k} \leq 0, \tag{3.23}$$

$$\Phi_{2k} + \Phi_{1k} + \bar{\Phi}_{0k} \leq 0, \tag{3.24}$$

$$(\frac{1}{2^N}\bar{\rho}_j + \bar{\rho}_j^2)\Phi_{2k} + (\frac{1}{2^{N+1}}\bar{\rho}_j + \bar{\rho}_j^2)\Phi_{1k} + \bar{\Phi}_{0k} \leq 0, \tag{3.25}$$

where Φ_{1k}, Φ_{2k} and $\bar{\rho}_j = \frac{j-1}{2^N}$, $j = 1, 2, \cdots, 2^N$ are defined in Corollary 3.6.

$$\bar{\Phi}_{0k} = \bar{\varphi}_k - \Pi_{41k}^T\bar{Z}_k\Pi_{41k} - 3\Pi_{42k}^T\bar{Z}_k\hat{\Pi}_{42k},$$

$$\begin{aligned}\bar{\varphi}_k =& Sym\left\{\Pi_{11}^T P\Pi_{12}\right\} + \Pi_{21}^T Q_1\Pi_{21} - \Pi_{22}^T Q_1\Pi_{22} + \Pi_{23}^T Q_2\Pi_{23} - \Pi_{24}^T Q_2\Pi_{24} \\ &+ \Pi_{31}^T Q_3\Pi_{31} - \Pi_{32}^T Q_3\Pi_{32} + \Pi_{33}^T Q_4\Pi_{33} - \Pi_{34}^T Q_4\Pi_{34} \\ &+ \sum_{i=1}^{m_1}(\frac{h_1}{m_1})^2 e_0 R_i e_0 + \sum_{k=1}^{m_2}(\frac{h_{12}}{m_2})^2 e_0 Z_k e_0 - \sum_{i=1}^{m_1}\Sigma_i^T\bar{R}_i\Sigma_i - \sum_{k=1}^{m_2}\hat{\Sigma}_k^T\bar{Z}_k\hat{\Sigma}_k + \hat{\Sigma}_k^T\bar{Z}_k\hat{\Sigma}_k.\end{aligned} \tag{3.26}$$

For $M = 3$, by Lemma 2.5, Theorem 3.1 is transformed into the following corollary easily.

Corollary 3.8. For given scalars $h_1 > 0, h_2 > 0$ and u and nonnegative integers m_1 and m_2, System (2.1) is asymptotically stable if there exist matrices $P \in S_+^{(m_1+m_2+1)n}$, $Q_1 \in S_+^{m_1 n}$, $Q_2 \in S_+^{m_2 n}$, $Q_3 \in S_+^{(m_1-1)n}$, $Q_4 \in S_+^{(m_2-1)n}$, $Q_5 \in S_+^n$, $R_i \in S_+^n$, $i = 1, 2, \cdots, m_1$, $Z_k \in S_+^n$, $k = 1, 2, \cdots, m_2$ and $F_1, F_2 \in R^{(2(m_1+m_2)+4)\times(2(m_1+m_2)+4)}$ such that the following LMIs hold for $k = 1, 2, \cdots, m_2$:

$$\begin{bmatrix} \hat{\Phi}_{0k} & -F_1 \\ * & -F_2 - F_2^T \end{bmatrix} < 0, \tag{3.27}$$

$$\begin{bmatrix} \hat{\Phi}_{1k} + F_1 + F_1^T + \hat{\Phi}_{0k} & \frac{1}{2}\hat{\Phi}_{2k} - F_1 + F_2^T \\ * & \hat{\Phi}_{3k} - F_2 - F_2^T \end{bmatrix} < 0, \tag{3.28}$$

where φ_k is defined in Corollary 3.4.

$$\begin{aligned} \hat{\Phi}_{0k} &= \varphi_k - \Pi_{41k}^T \bar{Z}_k \Pi_{41k} - 4\Pi_{42k}^T \bar{Z}_k \hat{\Pi}_{42k}, \\ \hat{\Phi}_{1k} &= -\Pi_{41k}^T \bar{Z}_k \Pi_{41k} + 6\Pi_{42k}^T \bar{Z}_k \hat{\Pi}_{42k}, \\ \hat{\Phi}_{2k} &= -\Pi_{41k}^T \bar{Z}_k \Pi_{41k} - 4\Pi_{42k}^T \bar{Z}_k \hat{\Pi}_{42k}, \\ \hat{\Phi}_{3k} &= -\Pi_{41k}^T \bar{Z}_k \Pi_{41k} + \Pi_{42k}^T \bar{Z}_k \hat{\Pi}_{42k}. \end{aligned}$$

Proof. For $M = 3$, based on (3.13), we can obtain:

$$\dot{V}(x(t)) \leq \zeta^T(t)\varpi_{3k}\zeta(t), \tag{3.29}$$

where

$$\varpi_{3k} = \varphi_k - (1 + \alpha + \alpha^2 + \alpha^3)\Pi_{41k}^T \bar{Z}_k \Pi_{41k} - (1 + \beta + \beta^2 + \beta^3)\hat{\Pi}_{42k}^T \bar{Z}_k \hat{\Pi}_{42k}. \tag{3.30}$$

Then, (3.29) can be rewritten as

$$\dot{V}(x(t)) \leq \zeta^T(t)(\alpha^3\hat{\Phi}_{3k} + \alpha^2\hat{\Phi}_{2k} + \alpha\hat{\Phi}_{1k} + \hat{\Phi}_{0k})\zeta(t). \tag{3.31}$$

From Lemma 2.4, if (3.27) and (3.28) hold, then we obtain $\alpha^3\hat{\Phi}_{3k} + \alpha^2\hat{\Phi}_{2k} + \alpha\hat{\Phi}_{1k} + \hat{\Phi}_{0k} < 0$, i.e., $\dot{V}(x(t))|_k < 0$. This completes the proof.

From Corollary 3.8, if u is unknown, a corollary is obtained by eliminating Q_5.

Corollary 3.9. For given scalars $h_1 > 0$ and $h_2 > 0$ and nonnegative integers m_1 and m_2, System (2.1) is asymptotically stable if there exist matrices $P \in S_+^{(m_1+m_2+1)n}$, $Q_1 \in S_+^{m_1 n}$, $Q_2 \in S_+^{m_2 n}$, $Q_3 \in S_+^{(m_1-1)n}$, $Q_4 \in S_+^{(m_2-1)n}$, $R_i \in S_+^n$, $i = 1, 2, \cdots, m_1$, $Z_k \in S_+^n$, $k = 1, 2, \cdots, m_2$ and $F_1, F_2 \in R^{(2(m_1+m_2)+4)\times(2(m_1+m_2)+4)}$ such that the following LMIs hold for $k = 1, 2, \cdots, m_2$:

$$\begin{bmatrix} \hat{\hat{\Phi}}_{0k} & -F_1 \\ * & -F_2 - F_2^T \end{bmatrix} < 0, \tag{3.32}$$

$$\begin{bmatrix} \hat{\Phi}_{1k} + F_1 + F_1^T + \hat{\hat{\Phi}}_{0k} & \frac{1}{2}\hat{\Phi}_{2k} - F_1 + F_2^T \\ * & \hat{\Phi}_{3k} - F_2 - F_2^T \end{bmatrix} < 0, \tag{3.33}$$

where $\bar{\varphi}_k$ is defined in Corollary 3.7 and $\hat{\Phi}_{ik}$, $i = 1, 2, 3$ are defined in Corollary 3.8.

$$\hat{\hat{\Phi}}_{0k} = \bar{\varphi}_k - \Pi_{41k}^T \bar{Z}_k \Pi_{41k} - 4\Pi_{42k}^T \bar{Z}_k \hat{\Pi}_{42k}.$$

4. Numerical examples

In this section, two examples are given to demonstrate the advantages of the proposed criteria.

Example 4.1. Consider System (2.1) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Our purpose was to find the upper bounds of h_2 for given h_1, m_1, m_2 and u. The upper bounds of h_2 calculated by Corollary 3.6, Corollary 3.8 and methods in [2–5] are listed in Table 1. From Table 1, it can be seen that the stability criteria presented in this paper are less conservative than those in [2– 5]. For $h_1 = 2$ and $u = 0.3$, the upper bounds of h_2 calculated by Corollary 3.6 in this paper are 3.28 ($m_1 = 2, m_2 = 2$) and 3.33 ($m_1 = 3, m_2 = 3$). Therefore, the conservativeness of the obtained results will be reduced with the increase of m_1 and m_2.

Table 1. Upper bound of h_2 for different values of h_1 and u.

h_1	Method	$u = 0.3$	$u = 0.5$	$u = 0.9$	Number of variables
2	[2]	2.69	2.50	2.50	18
	[3]	3.01	2.56	2.56	85
	[4]	3.02	2.69	2.69	101
	[5]	3.21	2.76	2.76	399
	Corollary 3.6 ($m_1 = 2, m_2 = 2$)	3.28	2.83	2.83	96
	Corollary 3.6 ($m_1 = 3, m_2 = 3$)	3.33	2.92	2.92	188
	Corollary 3.8 ($m_1 = 2, m_2 = 2$)	3.31	2.90	2.90	252
	Corollary 3.8 ($m_1 = 3, m_2 = 3$)	3.36	2.94	2.94	460
3	[2]	3.25	3.25	3.25	18
	[3]	3.34	3.34	3.34	85
	[4]	3.41	3.41	3.41	101
	[5]	3.49	3.49	3.49	399
	Corollary 3.6 ($m_1 = 2, m_2 = 2$)	3.62	3.62	3.62	96
	Corollary 3.6 ($m_1 = 3, m_2 = 3$)	3.65	3.65	3.65	188
	Corollary 3.8 ($m_1 = 2, m_2 = 2$)	3.64	3.64	3.64	252
	Corollary 3.8 ($m_1 = 3, m_2 = 3$)	3.67	3.67	3.67	460

Example 4.2. Consider System (2.1) with

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}.$$

Our purpose was to find the upper bounds of h_2 for given h_1, m_1 and m_2 and an unknown u. The upper bounds of h_2 calculated by Corollary 3.7, Corollary 3.9 and methods in [20–22, 25, 28] are listed in Table 2. From Table 2, it can be seen that the stability criteria presented in this paper were less conservative than those in [20–22,25,28]. For $h_1 = 1$, the upper bounds of h_2 calculated by Corollary 3.7 in this paper are 3.46 ($m_1 = 2, m_2 = 2$) and 3.49 ($m_1 = 3, m_2 = 3$). Therefore, the conservativeness of the obtained results will be reduced with the increase of m_1 and m_2.

Table 2. Upper bound of h_2 for different values of h_1.

h_1	0.0	0.4	0.7	1.0	Number of variables
[20]	1.59	2.01	2.41	2.62	49
[21]	1.64	2.13	2.70	2.96	96
[25]	1.86	2.28	2.69	2.89	93
[22]	2.39	2.76	3.15	3.41	627
[28]	2.54	2.90	3.23	3.44	424
Corollary 3.7 ($m_1 = 2, m_2 = 2$)	2.58	2.93	3.26	3.46	93
Corollary 3.7 ($m_1 = 3, m_2 = 3$)	2.65	2.98	3.29	3.49	185
Corollary 3.9 ($m_1 = 2, m_2 = 2$)	2.60	2.95	3.27	3.47	249
Corollary 3.9 ($m_1 = 3, m_2 = 3$)	2.68	2.30	3.31	3.50	457

5. Conclusions

In this paper, some new integral inequalities were introduced without using any reciprocally convex method. Some less conservative stability criteria were obtained based on these new integral inequalities and a new delay-partitioning approach. Finally, two numerical examples were provided to show the effectiveness of the presented method. Furthermore, how to decompose the delay interval needs to be further studied.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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