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Research article

Uniform stability result of laminated beams with thermoelasticity of type III

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Abstract: In this work, we study the effect of heat conduction theories pioneered by Green and Naghdi, popularly called thermoelasticity of type III, on the stability of laminated Timoshenko beams. Without the structural (interfacial slip) damping or any other forms of damping mechanisms, we establish an exponential stability result depending on the equality of wave velocities of the system. Our work shows that the thermal effect is strong enough to stabilize the system exponentially without any additional internal or boundary dampings. The result extends some of the developments in literature where structural damping (in addition to some internal or boundary dampings) is necessary to bring about exponential stability.

Keywords: laminated beams; thermoelasticity type III; uniform stability; multiplier technique; structural damping; wave speeds
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1. Introduction

Our objective in this present study is to consider laminated Timoshenko beams with thermoelasticity of type III and establish an exponential stability result without structural (interfacial slip) damping or any other internal or boundary dampings. The heat conduction governing thermoelasticity of type III is popularly called the Green and Naghdi theory. In contrast with the classical Fourier's law of heat conduction, the Green and Naghdi theory propounds a finite speed of heat propagation. See [1–4] for detailed historical reviews of the theory.

Towards the end of the nineteenth century, Hansen and Spies [5] introduced a model describing the vibrations in a structure made up of two uniform layered beams stuck together by a thin adhesive layer such that interfacial slip is possible when the beams are in continuous contact. Mathematically, the model comprises of the following three differential equations

$$\begin{cases}\rho u_{tt} + G(w - u_x)_x = 0,\\ I_\rho(3v_{tt} - w_{tt}) - D(3v_{xx} - w_{xx}) - G(w - u_x) = 0,\\ 3I_\rho v_{tt} - 3Dv_{xx} + 3G(w - u_x) + 4\gamma v + 4\beta v_t = 0.\end{cases} \tag{1.1}$$

The first two equations in (1.1), derived based on the assumptions of Timoshenko beam theory, are coupled with the third equation describing the slip dynamics. The dependent variables u, v, and w represent the transverse displacement, the proportionality to the volume of slip along the interface, and the rotation angle. The constants $\rho, G, I_\rho, D, \gamma$, and β are positive parameters representing the density, the shear stiffness, the mass moment of inertia, the flexural rigidity, the adhesive stiffness, and the adhesive damping, respectively.

The first stability result concerning (1.1) was carried out by Wang et al. [6]. They opined that the structural damping is not strong enough to exponentially stabilize the system. As a result, they only proved an exponential stability result for (1.1) by adding additional boundary damping terms. However, the structural damping was demonstrated to stabilize the system exponentially, provided the wave velocities of the system are equal. The result was independently established by Apalara in [7] and Alves and Monteiro in [8]. When the three equations in System (1.1) are damped, the system is exponentially stable regardless of the wave velocities or any other relationship between the coefficients ρ, G, I_ρ, and D of the system. In this regard, we cite, among others, Raposo [9], Liu and Zhao [10], Lo and Tatar [11]. For other results, we invite the reader to see [12–14]. However, Mustafa [15,16] showed that if the damping terms are only on one or two of the equations of the system, then an exponential stability result is only possible if some relationship exists among the coefficients of the system. A similar result can be found in [17–31] and references therein.

In the present study, we consider the laminated beams system given by (1.1) with thermoelasticity of type III acting on the effective rotation angle, but with negligible structural damping ($\beta = 0$). That is,

$$\begin{cases}\rho u_{tt} + G(w - u_x)_x = 0,\\ I_\rho(3v_{tt} - w_{tt}) - D(3v_{xx} - w_{xx}) - G(w - u_x) + \alpha\Theta_x = 0,\\ 3I_\rho v_{tt} - 3Dv_{xx} + 3G(w - u_x) + 4\gamma v = 0,\\ \rho_2\Theta_t + q_x + \alpha(3v_{tx} - w_{tx}) = 0.\end{cases} \tag{1.2}$$

The parameter $\alpha \neq 0$ denotes the thermoelastic coefficient. The dependent variable Θ represents the empirical temperature. The variable q denotes the heat flux with constitutive law given by

$$q = -\kappa\omega_x - k\omega_{xt}, \tag{1.3}$$

where ω is the thermal displacement given by (see [32])

$$\omega(x, t) = \int_0^t \Theta(x, s)ds + \omega(x, 0). \tag{1.4}$$

From (1.4), it follows that the time derivative of the thermal displacement yields the empirical temperature, that is, $\omega_t = \Theta$. The coefficients $k > 0$, $\rho_2 > 0$, and $\kappa > 0$ represent thermal and elastic properties. We differentiate the fourth equation in (1.2) with respect to time, perform simple computations using (1.3) while bearing in mind that $\omega_t = \Theta$ and then supplement the resulting system with suitable initial and boundary conditions. Thus, we obtain the following laminated beams system with thermoelasticity of type III

$$\begin{cases} \rho u_{tt} + G(w - u_x)_x = 0, \\ I_\rho(3v_{tt} - w_{tt}) - D(3v_{xx} - w_{xx}) - G(w - u_x) + \alpha\Theta_x = 0, \\ 3I_\rho v_{tt} - 3Dv_{xx} + 3G(w - u_x) + 4\gamma v = 0, \\ \rho_2\Theta_{tt} - \kappa\Theta_{xx} - k\Theta_{txx} + \alpha(3v_{ttx} - w_{ttx}) = 0, \\ u(x,0) = u_0,\ u_t(x,0) = u_1,\ w(x,0) = w_0,\ w_t(x,0) = w_1, \\ v(x,0) = v_0,\ v_t(x,0) = v_1,\ \Theta(x,0) = \Theta_0,\ \Theta_t(x,0) = \Theta_1, \\ u_x(0,t) = w(0,t) = v(0,t) = \Theta_x(0,t) = 0, \\ u(1,t) = w_x(1,t) = v_x(1,t) = \Theta(1,t) = 0, \end{cases} \tag{1.5}$$

where $x \in (0,1)$ and $t \geq 0$. Liu et al. [33] considered (1.5) in the presence of structural damping (i.e., $4\beta v_t$ added to the third equation) and established exponential stability of the system provided that the wave velocities of the system are equal and assuming a positive thermoelastic coefficient ($\alpha > 0$).

Remark 1.1. (i) *In the result of Liu et al. [33], the structural damping* ($4\beta v_t$) *freely provided the needed negative term for* v_t^2*; it was very crucial and unavoidable in the proof of their stability result.*

(ii) *In the coupling, they assumed that* $\alpha > 0$.

It is very important to mention here that, as far as the coupling is concerned, in the current work α is not necessarily positive; it is only required to be different from zero.

The novelty of this paper lies in addressing the two weaknesses presented by the result of Liu et al. [33], as encapsulated in Remark 1.1. We consider System (1.5) without structural damping (i.e., without the term $4\beta v_t$ in the third equation) and establish an exponential stability result under the condition of equal wave velocities of the system. The work is more challenging than that of Liu et al. [33] due to the absence of structural damping. In other words, instead of two dampings in [33], we have only one dissipation source via heat conduction. Furthermore, our result covers a broader range of α; we only assumed that α is different from zero instead of applying $\alpha > 0$, as in [33]. Consequently, we extend the result obtained by Liu et al. [33] and some other results in the literature.

It is paramount to discuss the dissipative nature of System (1.5). It is not obvious at the level of the energy that System (1.5) is dissipative. In fact, the energy functional $\mathcal{E}_0(t)$ of System (1.5) given as

$$\begin{aligned} \mathcal{E}_0(t) =& \mathcal{E}_0(u, w, v, \Theta, t) = \frac{1}{2}\Big[\rho\|u_t\|^2 + I_\rho\|3v_t - w_t\|^2 + 3I_\rho\|v_t\|^2 + \rho_2\|\Theta_t\|^2 \\ &+ 4\gamma\|v\|^2 + \kappa\|\Theta_x\|^2 + G\|w - u_x\|^2 + D\|3v_x - w_x\|^2 + 3D\|v_x\|^2\Big] \end{aligned} \tag{1.6}$$

has the derivative

$$\frac{d}{dt}\mathcal{E}_0(t) = -k\|\Theta_{tx}\|^2 - \alpha\Theta_x(3v_t - w_t) + \alpha\Theta_{tx}(3v_{tt} - w_{tt}), \quad t \geq 0, \tag{1.7}$$

which is not necessarily decreasing (because $\alpha \neq 0$ is any real number). To overcome this problem, as in [34], we introduce a new variable

$$\theta(x,t) = \int_0^t \Theta(x,s)ds + \frac{1}{\kappa}\zeta(x) \tag{1.8}$$

where $\zeta(x)$ is the solution of

$$\begin{cases} -\zeta'' = k\Theta_0'' - \rho_2\Theta_1 - \alpha(3v_1' - w_1') \text{ in } (0,1), \\ \zeta'(0) = \zeta(1) = 0. \end{cases} \tag{1.9}$$

Here, the superscripts denote the first and second derivatives in the variable x, respectively. By integrating the fourth equation in (1.5) and using (1.8) and (1.9), System (1.5) becomes

$$\begin{cases} \rho u_{tt} + G(w - u_x)_x = 0, \\ I_\rho(3v_{tt} - w_{tt}) - D(3v_{xx} - w_{xx}) - G(w - u_x) + \alpha\theta_{tx} = 0, \\ 3I_\rho v_{tt} - 3Dv_{xx} + 3G(w - u_x) + 4\gamma v = 0, \\ \rho_2\theta_{tt} - \kappa\theta_{xx} + \alpha(3v_{tx} - w_{tx}) - k\theta_{txx} = 0, \\ u(x,0) = u_0,\ u_t(x,0) = u_1,\ w(x,0) = w_0,\ w_t(x,0) = w_1, \\ v(x,0) = v_0,\ v_t(x,0) = v_1,\ \theta(x,0) = \theta_0,\ \theta_t(x,0) = \theta_1, \\ u_x(0,t) = w(0,t) = v(0,t) = \theta_x(0,t) = 0, \\ u(1,t) = w_x(1,t) = v_x(1,t) = \theta(1,t) = 0, \end{cases} \tag{1.10}$$

for $x \in (0,1)$, $t \geq 0$. Using the perturbed energy method, we consider (1.10) and proceed to show an exponential decay result provided that

$$\frac{G}{\rho} = \frac{D}{I_\rho} \tag{1.11}$$

holds. Throughout this paper, $\|.\|$ and $\langle\cdot,\cdot\rangle$ represent the usual $L^2(0,1)-$norm and inner product, respectively. The letter "$c > 0$" represents a generic constant. Concerning the well-posedness result for (1.10), we refer the reader to [33].

2. Stability result

In this section, we state and prove our stability result using the multiplier method. To achieve this task, we need to introduce some auxiliary functionals together with the classical energy defined by (2.2) in order to construct a suitable Lyapunov functional (of course, equivalent to the energy). In the lemma below, we establish the dissipativity of System (1.10) via the energy functional $\mathcal{E}$.

Lemma 2.1. *$\mathcal{E}(u,w,v,\theta,t)$ is the energy given by (1.6), and it satisfies*

$$\frac{d}{dt}\mathcal{E}(t) = -k\|\theta_{tx}\|^2 \leq 0, \quad t \geq 0, \tag{2.1}$$

where

$$\begin{aligned} \mathcal{E}(t) = &\mathcal{E}(u,w,v,\theta,t) = \frac{1}{2}\Big[\rho\|u_t\|^2 + I_\rho\|3v_t - w_t\|^2 + 3I_\rho\|v_t\|^2 + \rho_2\|\theta_t\|^2 \\ &+ 4\gamma\|v\|^2 + \kappa\|\theta_x\|^2 + G\|w - u_x\|^2 + D\|3v_x - w_x\|^2 + 3D\|v_x\|^2\Big]. \end{aligned} \tag{2.2}$$

Proof. We multiply $(1.10)_1$ by u_t and integrate (by parts) over $(0,1)$; then, the use of the boundary conditions leads to

$$\frac{1}{2}\frac{d}{dt}\left[\rho\|u_t\|^2 + G\|w-u_x\|^2\right] = G\langle w_t, w-u_x\rangle. \tag{2.3}$$

Similarly, by multiplying the second, third, and fourth equations in (1.10) by $3v_t - w_t, v_t$, and θ_t, respectively, we get

$$\frac{1}{2}\frac{d}{dt}\left[I_\rho\|3v_t-w_t\|^2 + D\|3v_x-w_x\|^2\right] = G\langle 3v_t-w_t, w-u_x\rangle + \alpha\langle 3v_{tx}-w_{tx}, \theta_t\rangle, \tag{2.4}$$

$$\frac{1}{2}\frac{d}{dt}\left[3I_\rho\|v_t\|^2 + 3D\|v_x\|^2 + 4\gamma\|v\|^2\right] = -3G\langle v_t, w-u_x\rangle, \tag{2.5}$$

$$\frac{1}{2}\frac{d}{dt}\left[\rho_2\|\theta_t\|^2 + \kappa\|\theta_x\|^2\right] = -k\|\theta_{tx}\|^2 - \alpha\langle 3v_{tx}-w_{tx}, \theta_t\rangle. \tag{2.6}$$

Summing up (2.3)–(2.6), we end up with (2.1). □

The next seven lemmas have been created to produce the needed auxiliary functionals and their derivatives.

Lemma 2.2. *Let* (1.11) *hold; the functional*

$$\mathcal{D}_1(t) := \rho D\langle u_t, 3v_x - w_x\rangle - I_\rho G\langle 3v_t - w_t, w-u_x\rangle, \quad t \geq 0,$$

along the solution of (1.10)*, satisfies for any* $\varepsilon_1 > 0$

$$\frac{d}{dt}\mathcal{D}_1(t) \leq -\frac{G^2}{2}\|w-u_x\|^2 + \varepsilon_1\|v_t\|^2 + c\left(1+\frac{1}{\varepsilon_1}\right)\|3v_t-w_t\|^2 + c\|\theta_{tx}\|^2, \quad t \geq 0. \tag{2.7}$$

Proof. Clearly, taking the derivative of $\mathcal{D}_1$ and incorporating the boundary conditions in (1.10) yields

$$\begin{aligned}\frac{d}{dt}\mathcal{D}_1(t) = {} & \rho D\langle u_{tt}, 3v_x - w_x\rangle - I_\rho G\langle 3v_{tt}-w_{tt}, w-u_x\rangle - I_\rho G\langle w_t, 3v_t-w_t\rangle \\ & + (I_\rho G - \rho D)\langle u_{xt}, 3v_t - w_t\rangle.\end{aligned}$$

Using the first two equations in (1.10), the identity $-w_t = (3v_t - w_t) - 3v_t$, and Eq (1.11), we end up with

$$\frac{d}{dt}\mathcal{D}_1(t) = -G^2\|w-u_x\|^2 + I_\rho G\|3v_t-w_t\|^2 - 3I_\rho G\langle v_t, 3v_t-w_t\rangle + \alpha G\langle\theta_{tx}, w-u_x\rangle. \tag{2.8}$$

The last two terms on the right of Eq (2.8) is estimated using Young's inequality; thus, for any $\varepsilon_1 > 0$, we have

$$-3I_\rho G\langle v_t, 3v_t - w_t\rangle \leq \varepsilon_1\|v_t\|^2 + \frac{9I_\rho^2G^2}{4\varepsilon_1}\|3v_t - v_t\|^2, \tag{2.9}$$

$$\alpha G\langle\theta_{tx}, w-u_x\rangle \leq \frac{G^2}{2}\|w-u_x\|^2 + \frac{\alpha^2}{2}\|\theta_{tx}\|^2. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8) and taking $\delta_1 = \dfrac{G^2}{2}$ yields (2.7). □

Lemma 2.3. *The derivative of the functional*

$$\mathcal{D}_2(t) := \alpha\rho_2 \left\langle 3v_t - w_t, \int_0^x \theta_t(s,t)ds \right\rangle - \alpha\kappa\langle 3v - w, \theta_x\rangle, \quad t \geq 0,$$

along solutions of (1.10) *and for any* $\varepsilon_2, \varepsilon_3 > 0$, *satisfies*

$$\begin{aligned}\frac{d}{dt}\mathcal{D}_2(t) \leq &-\frac{\alpha^2}{2}\|3v_t - w_t\|^2 + \varepsilon_2\|3v_x - w_x\|^2 + \varepsilon_3\|w - u_x\|^2 \\ &+ c\left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right)\|\theta_{tx}\|^2, \quad t \geq 0.\end{aligned} \tag{2.11}$$

Proof. Performing the same routine derivative of $\mathcal{D}_2$, integrating by parts, and incorporating the boundary conditions in (1.10), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{D}_2(t) = &-\alpha^2\|3v_t - w_t\|^2 + \frac{\alpha^2\rho_2}{I_\rho}\|\theta_t\|^2 + k\alpha\langle 3v_t - w_t, \theta_{tx}\rangle + \kappa\alpha\langle 3v_x - w_x, \theta_t\rangle \\ &- \frac{\alpha\rho_2 D}{I_\rho}\langle 3v_x - w_x, \theta_t\rangle + \frac{\alpha\rho_2 G}{I_\rho}\left\langle w - u_x, \int_0^x \theta_t(s,t)ds \right\rangle, \quad t \geq 0.\end{aligned} \tag{2.12}$$

Now, some terms on the right of Eq (2.12) will be estimated. For any $\varepsilon_2 > 0$, we use Young's inequality and Poincaré's inequality with the constant c_p, to perform the estimation as follows:

$$k\alpha\langle 3v_t - w_t, \theta_{tx}\rangle \leq \frac{\alpha^2}{2}\|3v_t - v_t\|^2 + \frac{k^2}{2}\|\theta_{tx}\|^2, \tag{2.13}$$

$$\kappa\alpha\langle 3v_x - w_x, \theta_t\rangle \leq \frac{\varepsilon_2}{2}\|3v_x - v_x\|^2 + \frac{\kappa^2\alpha^2 c_p}{2\varepsilon_2}\|\theta_{tx}\|^2, \tag{2.14}$$

$$-\frac{\alpha\rho_2 D}{I_\rho}\langle 3v_x - w_x, \theta_t\rangle \leq \frac{\varepsilon_2}{2}\|3v_x - v_x\|^2 + \frac{\rho_2^2\alpha^2 D^2 c_p}{2\varepsilon_2 I_\rho^2}\|\theta_{tx}\|^2. \tag{2.15}$$

To estimate the last term (on the right) of Eq (2.12), using Young's inequality for any $\varepsilon_3 > 0$, we get

$$\begin{aligned}\frac{\alpha\rho_2 G}{I_\rho}\left\langle w - u_x, \int_0^x \theta_t(s,t)ds \right\rangle &\leq \varepsilon_3\|w - u_x\|^2 + \frac{\alpha^2\rho_2^2 G^2}{4\varepsilon_3 I_\rho^2}\left\|\int_0^x \theta_t(s,t)ds\right\|^2 \\ &\leq \varepsilon_3\|w - u_x\|^2 + \frac{\alpha^2\rho_2^2 G^2 c_p}{4\varepsilon_3 I_\rho^2}\|\theta_{tx}\|^2,\end{aligned} \tag{2.16}$$

where we have applied the Cauchy-Schwarz inequality and Poincaré's inequality to have

$$\left\|\int_0^x \theta_t(s,t)ds\right\|^2 \leq \left\|\int_0^1 \theta_t(x,t)dx\right\|^2 \leq \|\theta_t\|^2 \leq c_p\|\theta_{tx}\|^2.$$

We end up with the estimate (2.11) by merely substituting (2.13)–(2.16) into (2.12). □

Remark 2.4. *The next lemma caters for the missing structural (interfacial slip) damping.*

Lemma 2.5. *Let* (1.11) *hold; the functional*

$$\mathcal{D}_3(t) := \frac{\rho D}{G}\langle u_t, v_x\rangle - I_\rho\langle v_t, w - u_x\rangle, \quad t \geq 0,$$

along the solution of (1.10) *satisfies*

$$\frac{d}{dt}\mathcal{D}_3(t) \leq -2I_\rho\|v_t\|^2 + \frac{\gamma}{2}\|v\|^2 + c\|w - u_x\|^2 + c\|3v_t - w_t\|^2, \quad t \geq 0. \tag{2.17}$$

Proof. As done in previous lemmas, by differentiating, integrating by parts and then using boundary conditions as usual we get

$$\frac{d}{dt}\mathcal{D}_3(t) = -I_\rho\langle v_t, w_t\rangle + G\|w - u_x\|^2 + \frac{4\gamma}{3}\langle v, w - u_x\rangle + \left(\frac{D\rho}{G} - I_\rho\right)\langle u_t, v_{tx}\rangle.$$

Using (1.11) and the identity $w_t = -(3v_t - w_t) + 3v_t$ leads to

$$\frac{d}{dt}\mathcal{D}_3(t) = -3I_\rho\|v_t\|^2 + G\|w - u_x\|^2 + I_\rho\langle v_t, 3v_t - w_t\rangle + \frac{4\gamma}{3}\langle v, w - u_x\rangle. \tag{2.18}$$

By Young's inequality, we have

$$\begin{aligned} I_\rho\langle v_t, 3v_t - w_t\rangle &\leq I_\rho\|v_t\|^2 + \frac{I_\rho}{4}\|3v_t - v_t\|^2, \\ \frac{4\gamma}{3}\langle v, w - u_x\rangle &\leq \frac{\gamma}{2}\|v\|^2 + \frac{8\gamma}{9}\|w - u_x\|^2. \end{aligned}$$

Consequently, we arrive at (2.17). □

Lemma 2.6. *The derivative of the functional*

$$\mathcal{D}_4(t) := \rho_2\langle \theta_t, \theta\rangle - \alpha\langle \theta_x, 3v - w\rangle + \frac{k}{2}\|\theta_x\|^2, \quad t \geq 0,$$

along the solution of (1.10) *satisfies*

$$\frac{d}{dt}\mathcal{D}_4(t) \leq -\frac{\kappa}{2}\|\theta_x\|^2 + \frac{D}{8}\|3v_x - w_x\|^2 + c\|\theta_{tx}\|^2, \quad t \geq 0. \tag{2.19}$$

Proof. Applying the fourth equation in (1.10) with the boundary conditions, it follows that

$$\frac{d}{dt}\mathcal{D}_4(t) = -\kappa\|\theta_x\|^2 + \rho_2\|\theta_t\|^2 + \alpha\langle \theta_t, 3v_x - w_x\rangle, \quad t \geq 0.$$

Indeed, applying Young's and Poincaré's inequalities, the estimate (2.19) is reached. □

Lemma 2.7. *The functional*

$$\mathcal{D}_5(t) := -\rho\langle u_t, u\rangle, \quad t \geq 0,$$

along the solution of (1.10) *satisfies*

$$\frac{d}{dt}\mathcal{D}_5(t) \leq -\rho\|u_t\|^2 + \frac{D}{8}\|3v_x - w_x\|^2 + \gamma\|v\|^2 + c\|w - u_x\|^2, \quad \forall t > 0. \tag{2.20}$$

Proof. After differentiating $\mathcal{D}_5(t)$, we integrate by parts and apply boundary conditions; then, we administer the identity $u_x = -(w - u_x) - (3v - w) + 3v$. Consequently, we deduce

$$\frac{d}{dt}\mathcal{D}_5(t) = -\rho\|u_t\|^2 + G\|w - u_x\|^2 + G\langle 3v - w, w - u_x\rangle - 3G\langle v, w - u_x\rangle. \tag{2.21}$$

The last two terms on the right of Eq (2.21) are estimated to get

$$G\langle 3v - w, w - u_x\rangle \leq \frac{D}{8}\|3v_x - w_x\|^2 + \frac{2G^2}{D}\|w - u_x\|^2, \tag{2.22}$$

$$-3G\langle v, w - u_x\rangle \leq \gamma\|v\|^2 + \frac{9G^2}{4\gamma}\|w - u_x\|^2, \tag{2.23}$$

courtesy of Poincaré's and Young's inequalities. As usual, we substitute (2.22) and (2.23) into (2.21) to get (2.20). □

Lemma 2.8. *The functional*

$$\mathcal{D}_6(t) := I_\rho\langle 3v - w, 3v_t - w_t\rangle, \quad t \geq 0,$$

along the solution of (1.10) *satisfies*

$$\frac{d}{dt}\mathcal{D}_6(t) \leq -\frac{D}{2}\|3v_x - w_x\|^2 + c\|3v_t - w_t\|^2 + c\|w - u_x\|^2 + c\|\theta_{tx}\|^2, \quad t \geq 0. \tag{2.24}$$

Proof. By using the second equation in (1.10), it is obvious that for all $t \geq 0$,

$$\frac{d}{dt}\mathcal{D}_6(t) = -D\|3v_x - w_x\|^2 + I_\rho\|3v_t - w_t\|^2 + G\langle 3v - w, w - u_x\rangle - \alpha\langle 3v - w, \theta_{tx}\rangle.$$

Then, application of Young's and Poincaré's inequalities brings about the estimate (2.24). □

Lemma 2.9. *The functional*

$$\mathcal{D}_7(t) := 3I_\rho\langle v_t, v\rangle, \quad t \geq 0,$$

along solutions of (1.10) *satisfies*

$$\frac{d}{dt}\mathcal{D}_7(t) \leq -3D\|v_x\|^2 - 3\gamma\|v\|^2 + 3I_\rho\|v_t\|^2 + c\|w - u_x\|^2, \quad t \geq 0. \tag{2.25}$$

Proof. By exploiting (1.10) together with Poincaré's and Young's inequalities, (2.25) is deduced. □

Having introduced all the needed auxiliary functionals, we now define

$$\mathcal{P}(t) := \mathcal{N}\mathcal{E}(t) + \mathcal{N}_1\mathcal{D}_1(t) + \mathcal{N}_2\mathcal{D}_2(t) + 2\mathcal{D}_3(t) + \mathcal{D}_4(t) + \mathcal{D}_5(t) + \mathcal{D}_6(t) + \mathcal{D}_7(t), \tag{2.26}$$

which is a Lyapunov functional, where $\mathcal{N} > 0$, $\mathcal{N}_1 > 0$, and $\mathcal{N}_2 > 0$ are constants to be carefully chosen later. Of course, we can verify (applying Poincaré's and Young's inequalities) that $\mathcal{P}$ is equivalent to the energy functional $\mathcal{E}$ given that $\mathcal{N}$ is sufficiently large. Summarily, for a large $\mathcal{N}$,

$$\ell_1\mathcal{E}(t) \leq \mathcal{P}(t) \leq \ell_2\mathcal{E}(t), \qquad \forall t \geq 0, \tag{2.27}$$

for some $\ell_1,\ \ell_2 > 0$.

Next, our stability result is stated and proved.

Theorem 2.10. *Let* (1.11) *hold; then, there exist* $\mu > 0$ *and* $\nu > 0$ *such that*

$$\mathcal{E}(t) \le \mu e^{-\nu t}, \quad t \ge 0. \tag{2.28}$$

Proof. Differentiating $\mathcal{P}$ and incorporating (2.1), (2.7), (2.11), (2.17), (2.19), (2.20), (2.24), and (2.25), we end up with

$$\begin{aligned}\mathcal{P}'(t) \le &-\left[k\mathcal{N} - c\mathcal{N}_1 - c\mathcal{N}_2\left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right) - c\right]\|\theta_{tx}\|^2 - \left[\frac{G^2}{2}\mathcal{N}_1 - \varepsilon_3\mathcal{N}_2 - c\right]\|w - u_x\|^2\\ &-\left[\frac{\alpha}{2}\mathcal{N}_2 - c\mathcal{N}_1\left(1 + \frac{1}{\varepsilon_1}\right) - c\right]\|3v_t - w_t\|^2 - \left[\frac{D}{4} - \varepsilon_2\mathcal{N}_2\right]\|3v_x - w_x\|^2 - \frac{\kappa}{2}\|\theta_x\|^2\\ &-\left[I_\rho - \varepsilon_1\mathcal{N}_1\right]\|v_t\|^2 - \rho\|u_t\|^2 - 3D\|v_x\|^2 - \gamma\|v\|^2.\end{aligned}$$

It is clear that by taking

$$\varepsilon_3 = \frac{G^2\mathcal{N}_1}{4\mathcal{N}_2},\ \varepsilon_2 = \frac{D}{8\mathcal{N}_2},\ \varepsilon_1 = \frac{I_\rho}{2\mathcal{N}_1},$$

we obtain

$$\begin{aligned}\mathcal{P}'(t) \le &-\left[k\mathcal{N} - c\mathcal{N}_1 - c\mathcal{N}_2\left(1 + \mathcal{N}_2 + \frac{\mathcal{N}_2}{\mathcal{N}_1}\right) - c\right]\|\theta_{tx}\|^2 - \left[\frac{G^2}{4}\mathcal{N}_1 - c\right]\|w - u_x\|^2 - 3D\|v_x\|^2 - \gamma\|v\|^2\\ &-\left[\frac{\alpha}{2}\mathcal{N}_2 - c\mathcal{N}_1\left(1 + \mathcal{N}_1\right) - c\right]\|3v_t - w_t\|^2 - \frac{D}{8}\|3v_x - w_x\|^2 - \frac{\kappa}{2}\|\theta_x\|^2 - \frac{I_\rho}{2}\|v_t\|^2 - \rho\|u_t\|^2.\end{aligned}$$

Taking $\mathcal{N}_1, \mathcal{N}_2$, and $\mathcal{N}$ large enough, sequentially, we end up with

$$\begin{aligned}\mathcal{P}'(t) \le -\zeta\Big[&\|3v_t - w_t\|^2 + \|w - u_x\|^2 + \|u_t\|^2 + \|3v_x - w_x\|^2 + \|v_x\|^2 + \|v\|^2 + \|v_t\|^2\\ &+ \|\theta_{tx}\|^2 + \|\theta_x\|^2\Big], \quad t \ge 0,\end{aligned} \tag{2.29}$$

for some $\zeta > 0$. Using (1.6), (2.27) and (2.29), for some $\nu > 0$, we get

$$\mathcal{P}'(t) \le -\nu\mathcal{P}(t), \quad \forall t \ge 0. \tag{2.30}$$

Now, (2.30) is integrated over $(0, t)$ to get

$$\mathcal{P}(t) \le \mathcal{P}(0)e^{-\nu t}, \quad \forall t \ge 0, \tag{2.31}$$

which implies (2.28) with $\mu = \frac{\ell_2}{\ell_1}\mathcal{E}(0)$ due to (2.27). □

3. Conclusions and open problems

In this work, we establish that the thermal effect exhibited by heat conduction of type III is strong enough to exponentially stabilize the laminated beams system without any additional internal or boundary damping mechanism. For example, instead of the two dampings used in [33], we used only one dissipation source to achieve the exponential stability result. Furthermore, our result covers a broader range of α, assuming that it is only different from zero instead of $\alpha > 0$, as in [33]. As usual, we assume that the system's wave propagation velocities are equal. It is an intriguing open problem to prove that the system has no exponential stability except that the condition of equal wave velocities is imposed. In addition, the polynomial stability of the system when the wave velocities are not equal is an enthralling problem to consider.

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Conflict of interest

The authors declare no potential conflicts of interests.

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