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Research article

Positive radial solutions for a boundary value problem associated to a system of elliptic equations with semipositone nonlinearities

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Abstract: In this paper we use the fixed point index theory to study the existence of positive radial solutions for a system of boundary value problems with semipositone second order elliptic equations. Some appropriate concave and convex functions are utilized to characterize coupling behaviors of our nonlinearities.

Keywords: system of elliptic equations; boundary value problem; positive radial solution; fixed point index
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1. Introduction

Let $\Omega = \{z \in \mathbb{R}^n : R_1 < |z| < R_2, R_1, R_2 > 0\}$. In this work we study the existence of positive radial solutions for the following system of boundary value problems with semipositone second order elliptic equations:

$$\begin{cases} \Delta\varphi + k(|z|)f(\varphi, \phi) = 0, \ z \in \Omega, \\ \Delta\phi + k(|z|)g(\varphi, \phi) = 0, \ z \in \Omega, \\ \alpha\varphi + \beta\dfrac{\partial\varphi}{\partial n} = 0, \ \alpha\phi + \beta\dfrac{\partial\phi}{\partial n} = 0, \ |z| = R_1, \\ \gamma\varphi + \delta\dfrac{\partial\varphi}{\partial n} = 0, \ \gamma\phi + \delta\dfrac{\partial\phi}{\partial n} = 0, \ |z| = R_2, \end{cases} \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta, k, f, g$ satisfy the conditions:

(H1) $\alpha, \beta, \gamma, \delta \geq 0$ with $\rho \equiv \gamma\beta + \alpha\gamma + \alpha\delta > 0$;

(H2) $k \in C([R_1, R_2], \mathbb{R}^+)$, and k is not vanishing on $[R_1, R_2]$;

(H3) $f, g \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$, and there is a positive constant M such that

$$f(u, v), g(u, v) \geq -M, \ \forall u, v \in \mathbb{R}^+.$$

Elliptic equations have attracted a lot of attention in the literature since they are closely related to many mathematical and physical problems, for instance, incineration theory of gases, solid state physics, electrostatic field problems, variational methods and optimal control. The existence of solutions for this type of equation in annular domains has been discussed in the literature, see for example, [1–18] and the references therein. In [1] the authors used the fixed point index to study positive solutions for the elliptic system:

$$\begin{cases} \Delta u + a(|x|) f(u, v) = 0, \\ \Delta v + b(|x|) g(u, v) = 0, \end{cases}$$

with one of the following boundary conditions

$$\begin{aligned} & u = v = 0, |x| = R_1, |x| = R_2, \\ & u = v = 0, |x| = R_1, \frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} = 0, |x| = R_2, \\ & \frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} = 0, |x| = R_1, u = v = 0, |x| = R_2. \end{aligned}$$

In [2] the authors used the method of upper and lower solutions to establish the existence of positive radial solutions for the elliptic equation

$$\begin{cases} -\Delta u = f(|x|, u, |\nabla u|), \ x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega = \left\{ x \in \mathbb{R}^N : \ |x| < 1 \right\}, N \geq 2$, and $f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function.

However, we note that in most of the papers on nonlinear differential equations the nonlinear term is usually assumed to be nonnegative. In recent years boundary value problems for semipositone equations ($f(t, x) \geq -M, M > 0$) has received some attention (see [19–32]), and these equations describe and solve many natural phenomena in engineering and technical problems in real life, for example in mechanical systems, suspension bridge design, astrophysics and combustion theoretical models. In [19] the authors used a fixed point theorem to study the system for HIV-1 population dynamics in the fractional sense

$$\begin{cases} D_{0+}^{\alpha} u(t) + \lambda f\left(t, u(t), D_{0+}^{\beta} u(t), v(t)\right) = 0, t \in (0, 1), \\ D_{0+}^{\gamma} v(t) + \lambda g(t, u(t)) = 0, t \in (0, 1), \\ D_{0+}^{\beta} u(0) = D_{0+}^{\beta+1} u(0) = 0, D_{0+}^{\beta} u(1) = \int_0^1 D_{0+}^{\beta} u(s) dA(s), \\ v(0) = v'(0) = 0, v(1) = \int_0^1 v(s) dB(s), \end{cases}$$

where $D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{\gamma}$ are the standard Riemann-Liouville derivatives, and f, g are two semipositone nonlinearities. In [28] the authors used the nonlinear alternative of Leray-Schauder type and the Guo-Krasnosel'skii fixed point theorem to study the existence of positive solutions for a system of nonlinear

Riemann-Liouville fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,v(t))=0, 0<t<1,\ \lambda>0,\\ D_{0+}^{\alpha}v(t)+\lambda g(t,u(t))=0, 0<t<1,\ \lambda>0,\\ u^{(j)}(0)=v^{(j)}(0)=0, 0\leqslant j\leqslant n-2,\\ u(1)=\mu\int_0^1 u(s)ds, v(1)=\mu\int_0^1 v(s)ds,\end{cases}$$

where f,g satisfy some superlinear or sublinear conditions:

$(HZ)_1$ There exist $M>0$ such that $\limsup_{z\to 0}\frac{g(t,z)}{z}<M$ uniformly for $t\in[0,1]$ (sublinear growth condition).

$(HZ)_2$ There exists $[\theta_1,\theta_2]\subset(0,1)$ such that $\liminf_{z\to+\infty}\frac{f(t,z)}{z}=+\infty$ and $\liminf_{z\to+\infty}\frac{g(t,z)}{z}=+\infty$ uniformly for $t\in[\theta_1,\theta_2]$ (superlinear growth condition).

Inspired by the aforementioned work, in particular [31–34], we study positive radial solutions for (1.1) when the nonlinearities f,g satisfy the semipositone condition (H3). Moreover, some appropriate concave and convex functions are utilized to characterize coupling behaviors of our nonlinearities. Note that our conditions (H4) and (H6) (see Section 3) are more general than that in $(HZ)_1$ and $(HZ)_2$.

2. Preliminaries

Using the methods in [1, 4], we transform (1.1) into a system of ordinary differential equations involving Sturm-Liouville boundary conditions. Let $\varphi=\varphi(r),\phi=\phi(r),r=|z|=\sqrt{\sum_{i=1}^{n}z_i^2}$. Then (1.1) can be expressed by the following system of ordinary differential equations:

$$\begin{cases} \varphi''(r)+\dfrac{n-1}{r}\varphi'(r)+k(r)f(\varphi(r),\phi(r))=0,\ R_1<r<R_2,\\ \phi''(r)+\dfrac{n-1}{r}\phi'(r)+k(r)g(\varphi(r),\phi(r))=0,\ R_1<r<R_2,\\ \alpha\varphi(R_1)-\beta\varphi'(R_1)=0,\ \gamma\varphi(R_2)+\delta\varphi'(R_2)=0,\\ \alpha\phi(R_1)-\beta\phi'(R_1)=0,\ \gamma\phi(R_2)+\delta\phi'(R_2)=0.\end{cases}\tag{2.1}$$

Then if we let $s=-\int_r^{R_2}\left(1/t^{n-1}\right)\mathrm{d}t, t=(m-s)/m, m=-\int_{R_1}^{R_2}\left(1/t^{n-1}\right)\mathrm{d}t$, (2.1) can be transformed into the system

$$\begin{cases} \varphi''(t)+h(t)f(\varphi(t),\phi(t))=0,\quad 0<t<1,\\ \phi''(t)+h(t)g(\varphi(t),\phi(t))=0,\quad 0<t<1,\\ \alpha\varphi(0)-\beta\varphi'(0)=0,\quad \gamma\varphi(1)+\delta\varphi'(1)=0,\\ \alpha\phi(0)-\beta\phi'(0)=0,\quad \gamma\phi(1)+\delta\phi'(1)=0,\end{cases}\tag{2.2}$$

where $h(t)=m^2r^{2(n-1)}(m(1-t))k(r(m(1-t)))$. Consequently, (2.2) is equivalent to the following system of integral equations

$$\begin{cases} \varphi(t)=\displaystyle\int_0^1 G(t,s)h(s)f(\varphi(s),\phi(s))\mathrm{d}s,\\ \phi(t)=\displaystyle\int_0^1 G(t,s)h(s)g(\varphi(s),\phi(s))\mathrm{d}s,\end{cases}\tag{2.3}$$

where

$$G(t,s)=\frac{1}{\rho}\begin{cases}(\gamma+\delta-\gamma t)(\beta+\alpha s), & 0\le s\le t\le 1,\\ (\gamma+\delta-\gamma s)(\beta+\alpha t), & 0\le t\le s\le 1,\end{cases} \tag{2.4}$$

and ρ is defined in (H1).

Lemma 2.1. *Suppose that (H1) holds. Then*

(i)

$$\frac{\rho}{(\gamma+\delta)(\beta+\alpha)}G(t,t)G(s,s)\le G(t,s)\le G(s,s),\ t,s\in[0,1];$$

(ii)

$$G(t,s)\le G(t,t),\ t,s\in[0,1].$$

Proof. (i) In $G(t,s)$, we fix the second variable s, we have

$$G(t,s)=\frac{1}{\rho}\begin{cases}(\gamma+\delta-\gamma t)(\beta+\alpha s)\le(\gamma+\delta-\gamma s)(\beta+\alpha s), & 0\le s\le t\le 1,\\ (\gamma+\delta-\gamma s)(\beta+\alpha t)\le(\gamma+\delta-\gamma s)(\beta+\alpha s), & 0\le t\le s\le 1.\end{cases}$$

This implies that

$$G(t,s)\le G(s,s),t,s\in[0,1].$$

When $t\ge s$, we have

$$\frac{\frac{1}{\rho}(\gamma+\delta-\gamma t)(\beta+\alpha s)}{\rho\cdot\frac{1}{\rho}\cdot\frac{1}{\rho}(\gamma+\delta-\gamma t)(\beta+\alpha t)(\gamma+\delta-\gamma s)(\beta+\alpha s)}\ge\frac{1}{(\beta+\alpha)(\gamma+\delta)}.$$

When $t\le s$, we have

$$\frac{\frac{1}{\rho}(\gamma+\delta-\gamma s)(\beta+\alpha t)}{\rho\cdot\frac{1}{\rho}\cdot\frac{1}{\rho}(\gamma+\delta-\gamma t)(\beta+\alpha t)(\gamma+\delta-\gamma s)(\beta+\alpha s)}\ge\frac{1}{(\beta+\alpha)(\gamma+\delta)}.$$

Combining the above we obtain

$$\frac{G(t,s)}{G(t,t)G(s,s)}\ge\frac{\rho}{(\beta+\alpha)(\gamma+\delta)}.$$

(ii) In $G(t,s)$ we fix the first variable t, and we obtain

$$G(t,s)=\frac{1}{\rho}\begin{cases}(\gamma+\delta-\gamma t)(\beta+\alpha s)\le(\gamma+\delta-\gamma t)(\beta+\alpha t), & 0\le s\le t\le 1,\\ (\gamma+\delta-\gamma s)(\beta+\alpha t)\le(\gamma+\delta-\gamma t)(\beta+\alpha t), & 0\le t\le s\le 1.\end{cases}$$

Thus

$$G(t,s)\le G(t,t),t,s\in[0,1].$$

□

Lemma 2.2. *Suppose that (H1) holds. Let* $\vartheta(t) = G(t,t)h(t), t \in [0,1]$. *Then*

$$\kappa_1\vartheta(s) \le \int_0^1 G(t,s)h(s)\vartheta(t)\mathrm{d}t \le \kappa_2\vartheta(s),$$

where

$$\kappa_1 = \frac{\rho}{(\gamma+\delta)(\beta+\alpha)}\int_0^1 G(t,t)\vartheta(t)\mathrm{d}t,\ \kappa_2 = \int_0^1 \vartheta(t)\mathrm{d}t.$$

Proof. From (H1) and Lemma 2.1(i) we have

$$\int_0^1 G(t,s)h(s)\vartheta(t)\mathrm{d}t \le \int_0^1 G(s,s)h(s)\vartheta(t)\mathrm{d}t = \kappa_2\vartheta(s)$$

and

$$\int_0^1 G(t,s)h(s)\vartheta(t)\mathrm{d}t \ge \int_0^1 \frac{\rho}{(\gamma+\delta)(\beta+\alpha)}G(t,t)G(s,s)h(s)\vartheta(t)\mathrm{d}t = \kappa_1\vartheta(s).$$

Note we study (2.3) to obtain positive solutions for (1.1). However here the nonlinear terms f, g can be sign-changing (see (H3)). Therefore we study the following auxiliary problem:

$$u(t) = \int_0^1 G(t,s)h(s)\widetilde{f}(u(s))\mathrm{d}s, \tag{2.5}$$

where G is in (2.4) and $\widetilde{f}$ satisfies the condition:

(H2′) $\widetilde{f} \in C(\mathbb{R}^+,\mathbb{R})$, and there exists a positive constant M such that

$$\widetilde{f}(u) \ge -M,\ \forall u \in \mathbb{R}^+.$$

Let $w(t) = M\int_0^1 G(t,s)h(s)\mathrm{d}s, \forall t \in [0,1]$. Then w is a solution of the following boundary value problem:

$$\begin{cases} u''(t) + h(t)M = 0, 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0. \end{cases} \tag{2.6}$$

□

Lemma 2.3. *(i) If* u^* *satisfies* (2.5)*, then* $u^* + w$ *is a solution of the equation:*

$$u(t) = \int_0^1 G(t,s)h(s)\widetilde{F}(u(s) - w(s))\mathrm{d}s, \tag{2.7}$$

where

$$\widetilde{F}(u) = \begin{cases} \widetilde{f}(u) + M, u \ge 0, \\ \widetilde{f}(0) + M, u < 0. \end{cases} \tag{2.8}$$

(ii) If u^{**} *satisfies* (2.7) *with* $u^{**}(t) \ge w(t), t \in [0,1]$*, then* $u^{**} - w$ *is a positive solution for* (2.5).

Proof. We omit its proof since it is immediate. □

Let $E = C[0,1]$, $\|u\| = \max_{t\in[0,1]} |u(t)|$. Then $(E, \|\cdot\|)$ is a Banach space. Define a set on E as follows:

$$P = \{u \in E : u(t) \geq 0, \forall t \in [0,1]\},$$

and note P is a cone on E. Note, $E^2 = E \times E$ is also a Banach space with the norm: $\|(u,v)\| = \|u\| + \|v\|$, and $P^2 = P \times P$ a cone on E^2. In order to obtain positive radial solutions for (1.1), combining with (2.5)–(2.7), we define the following operator equation:

$$A(\varphi, \phi) = (\varphi, \phi), \tag{2.9}$$

where $A(\varphi,\phi) = (A_1, A_2)(\varphi,\phi)$, $A_i (i = 1,2)$ are

$$\begin{cases} A_1(\varphi,\phi)(t) = \displaystyle\int_0^1 G(t,s)h(s)\mathcal{F}_1(\varphi(s) - w(s), \phi(s) - w(s))\mathrm{d}s, \\ A_2(\varphi,\phi)(t) = \displaystyle\int_0^1 G(t,s)h(s)\mathcal{F}_2(\varphi(s) - w(s), \phi(s) - w(s))\mathrm{d}s, \end{cases} \tag{2.10}$$

and

$$\mathcal{F}_1(\varphi,\phi) = \begin{cases} f(\varphi,\phi) + M, \varphi, \phi \geq 0, \\ f(0,\phi) + M, \varphi < 0, \phi \geq 0, \\ f(\varphi, 0) + M, \varphi \geq 0, \phi < 0, \\ f(0,0) + M, \varphi, \phi < 0, \end{cases}$$

$$\mathcal{F}_2(\varphi,\phi) = \begin{cases} g(\varphi,\phi) + M, \varphi, \phi \geq 0, \\ g(0,\phi) + M, \varphi < 0, \phi \geq 0, \\ g(\varphi, 0) + M, \varphi \geq 0, \phi < 0, \\ g(0,0) + M, \varphi, \phi < 0. \end{cases}$$

Lemma 2.4. *Define* $P_0 = \left\{\varphi \in P : \varphi(t) \geq \frac{\rho}{(\gamma+\delta)(\beta+\alpha)} G(t,t)\|\varphi\|, t \in [0,1]\right\}$. *Then* $A_i(P \times P) \subset P_0, i = 1,2$.

Proof. We only prove it for A_1. If $\varphi, \phi \in P$, note the non-negativity of $\mathcal{F}_1$(denoted by $\mathcal{F}_1(\cdot,\cdot)$), from Lemma 2.1(i) we have

$$\int_0^1 \frac{\rho}{(\gamma+\delta)(\beta+\alpha)} G(t,t)G(s,s)h(s)\mathcal{F}_1(\cdot,\cdot)\mathrm{d}s \leq A_1(\varphi,\phi)(t) \leq \int_0^1 G(s,s)h(s)\mathcal{F}_1(\cdot,\cdot)\mathrm{d}s.$$

This implies that

$$A_1(\varphi,\phi)(t) \geq \int_0^1 \frac{\rho}{(\gamma+\delta)(\beta+\alpha)} G(t,t)G(s,s)h(s)\mathcal{F}_1(\cdot,\cdot)\mathrm{d}s \geq \frac{\rho}{(\gamma+\delta)(\beta+\alpha)} G(t,t)\|A_1(\varphi,\phi)\|.$$

□

Remark 2.1. *(i)* $w(t) = M\int_0^1 G(t,s)h(s)\mathrm{d}s \in P_0$;

(ii) Note (see Corollary 1.5.1 in [35]):

If $k(x, y, u) : \widetilde{G} \times \widetilde{G} \times \mathbb{R} \to \mathbb{R}$ is continuous ($\widetilde{G}$ is a bounded closed domain in $\mathbb{R}^n$), then K is a completely continuous operator from $C(\widetilde{G})$ into itself, where

$$K\psi(x) = \int_{\widetilde{G}} k(x, y, \psi(y))\mathrm{d}y.$$

Note that $G(t, s), h(s), \mathcal{F}_i (i = 1, 2)$ are continuous, and also A_i, A are completely continuous operators,$i = 1, 2$.

From Lemma 2.3 if there exists $(\varphi, \phi) \in P^2\backslash\{(0, 0)\}$ such that (2.9) holds with $(\varphi, \phi) \geq (w, w)$, then $\varphi(t), \phi(t) \geq w(t), t \in [0, 1]$, and $(\varphi - w, \phi - w)$ is a positive solution for (2.3), i.e., we obtain positive radial solutions for (1.1). Note that $\varphi, \phi \in P_0$, and from Lemma 2.1(ii) we have

$$\varphi(t) - w(t) \geq \frac{\rho}{(\gamma + \delta)(\beta + \alpha)} G(t, t)\|\varphi\| - M \int_0^1 G(t, t)h(s)\mathrm{d}s,$$

$$\phi(t) - w(t) \geq \frac{\rho}{(\gamma + \delta)(\beta + \alpha)} G(t, t)\|\phi\| - M \int_0^1 G(t, t)h(s)\mathrm{d}s.$$

Hence, if

$$\|\varphi\|, \|\phi\| \geq \frac{M(\gamma + \delta)(\beta + \alpha)}{\rho} \int_0^1 h(s)\mathrm{d}s,$$

we have $(\varphi, \phi) \geq (w, w)$. As a result, we only need to seek fixed points of (2.9), when their norms are greater than $\frac{M(\gamma+\delta)(\beta+\alpha)}{\rho} \int_0^1 h(s)\mathrm{d}s$.

Let E be a real Banach space. A subset $X \subset E$ is called a retract of E if there exists a continuous mapping $r : E \to X$ such that $r(x) = x,\ x \in X$. Note that every cone in E is a retract of E. Let X be a retract of real Banach space E. Then, for every relatively bounded open subset U of X and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on ∂U, there exists an integer $i(A, U, X)$ satisfying the following conditions:

(i) Normality: $i(A, U, X) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in \overline{U}$.

(ii) Additivity: $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ whenever U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\overline{U}\backslash (U_1 \cup U_2)$.

(iii) Homotopy invariance: $i(H(t, \cdot), U, X)$ is independent of t $(0 \leq t \leq 1)$ whenever $H : [0, 1]\times\overline{U} \to X$ is completely continuous and $H(t, x) \neq x$ for any $(t, x) \in [0, 1] \times \partial U$.

(iv) Permanence: $i(A, U, X) = i(A, U \cap Y, Y)$ if Y is a retract of X and $A(\overline{U}) \subset Y$.

Then $i(A, U, X)$ is called the fixed point index of A on U with respect to X.

Lemma 2.5. *(see [35, 36]). Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $A : \overline{\Omega} \cap P \to P$ is a continuous compact operator. If there exists $\omega_0 \in P\backslash\{0\}$ such that*

$$\omega - A\omega \neq \lambda\omega_0, \forall\lambda \geq 0, \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed point index on P.

Lemma 2.6. *(see [35, 36]). Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A : \overline{\Omega} \cap P \to P$ is a continuous compact operator. If*

$$\omega - \lambda A\omega \neq 0, \forall\lambda \in [0, 1], \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 1$.

3. Main results

Denote $O_{M,h} = \frac{M(\gamma+\delta)(\beta+\alpha)}{\rho}\int_0^1 h(s)\mathrm{d}s$, $B_\zeta = \{u \in E : \|u\| < \zeta\}, \zeta > 0, B^2_\zeta = B_\zeta \times B_\zeta$. We list our assumptions as follows:

(H4) There exist $p, q \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

(i) p is a strictly increasing concave function on $\mathbb{R}^+$;

(ii) $\liminf_{v\to\infty} \frac{f(u,v)}{p(v)} \geq 1$, $\liminf_{u\to\infty} \frac{g(u,v)}{q(u)} \geq 1$;

(iii) there exists $e_1 \in (\kappa_1^{-2}, \infty)$ such that $\liminf_{z\to\infty} \frac{p(\mathcal{L}_{G,h}q(z))}{z} \geq e_1\mathcal{L}_{G,h}$, where $\mathcal{L}_{G,h} = \max_{t,s\in[0,1]} G(t,s)h(s)$.

(H5) There exists $Q_i \in \left(0, \frac{O_{M,h}}{\kappa_2}\right)$ such that

$$\mathcal{F}_i(u-w, v-w) \leq Q_i, u, v \in [0, O_{M,h}], i = 1, 2.$$

(H6) There exist $\zeta, \eta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

(i) ζ is a strictly increasing convex function on $\mathbb{R}^+$;

(ii) $\limsup_{v\to\infty} \frac{f(u,v)}{\zeta(v)} \leq 1$, $\limsup_{u\to\infty} \frac{g(u,v)}{\eta(u)} \leq 1$;

(iii) there exists $e_2 \in (0, \kappa_2^{-2})$ such that $\limsup_{z\to\infty} \frac{\zeta(\mathcal{L}_{G,h}\eta(z))}{z} \leq e_2\mathcal{L}_{G,h}$.

(H7) There exists $\widetilde{Q}_i \in \left(\frac{O_{M,h}}{\kappa_2\mathcal{L}_G}, \infty\right)$ such that

$$\mathcal{F}_i(u-w, v-w) \geq \widetilde{Q}_i, u, v \in [0, O_{M,h}], i = 1, 2,$$

where $\mathcal{L}_G = \max_{t\in[0,1]} \frac{\rho}{(\gamma+\delta)(\beta+\alpha)}G(t,t)$.

Remark 3.1. *Condition (H4) implies that f grows $p(v)$-superlinearly at ∞ uniformly on $u \in \mathbb{R}^+$, g grows $q(u)$-superlinearly at ∞ uniformly on $v \in \mathbb{R}^+$; condition (H6) implies that f grows $\zeta(v)$-sublinearly at ∞ uniformly on $u \in \mathbb{R}^+$, g grows $\eta(u)$-sublinearly at ∞ uniformly on $v \in \mathbb{R}^+$.*

Theorem 3.1. *Suppose that (H1)–(H5) hold. Then* (1.1) *has at least one positive radial solution.*

Proof. **Step 1.** When $\varphi, \phi \in \partial B_{O_{M,h}} \cap P$, we have

$$(\varphi, \phi) \neq \lambda A(\varphi, \phi), \lambda \in [0, 1]. \tag{3.1}$$

Suppose the contrary i.e., if (3.1) is false, then there exist $\varphi_0, \phi_0 \in \partial B_{O_{M,h}} \cap P$ and $\lambda_0 \in [0, 1]$ such that

$$(\varphi_0, \phi_0) = \lambda_0 A(\varphi_0, \phi_0).$$

This implies that

$$\varphi_0, \phi_0 \in P_0 \tag{3.2}$$

and

$$\|\varphi_0\| \leq \|A_1(\varphi_0, \phi_0)\|, \ \|\phi_0\| \leq \|A_2(\varphi_0, \phi_0)\|. \tag{3.3}$$

From (H5) we have

$$A_i(\varphi_0, \phi_0)(t) = \int_0^1 G(t,s)h(s)\mathcal{F}_i(\varphi_0(s) - w(s), \phi_0(s) - w(s))\mathrm{d}s \leq \int_0^1 \vartheta(s)Q_i\mathrm{d}s < O_{M,h}, i = 1, 2.$$

Thus

$$\|A_1(\varphi_0, \phi_0)\| + \|A_2(\varphi_0, \phi_0)\| < 2\mathcal{O}_{M,h} = \|\varphi_0\| + \|\phi_0\| (\varphi_0, \phi_0 \in \partial B_{\mathcal{O}_{M,h}} \cap P),$$

which contradicts (3.3), and thus (3.1) holds. From Lemma 2.6 we have

$$i(A, B^2_{\mathcal{O}_{M,h}} \cap P^2, P^2) = 1. \tag{3.4}$$

Step 2. There exists a sufficiently large $R > \mathcal{O}_{M,h}$ such that

$$(\varphi, \phi) \neq A(\varphi, \phi) + \lambda(\varrho_1, \varrho_1), \varphi, \phi \in \partial B_R \cap P, \lambda \geq 0, \tag{3.5}$$

where $\varrho_1 \in P_0$ is a given element. Suppose the contrary. Then there are $\varphi_1, \phi_1 \in \partial B_R \cap P, \lambda_1 \geq 0$ such that

$$(\varphi_1, \phi_1) = A(\varphi_1, \phi_1) + \lambda_1(\varrho_1, \varrho_1). \tag{3.6}$$

This implies that

$$\varphi_1(t) = A_1(\varphi_1, \phi_1)(t) + \lambda_1\varrho_1(t),\ \phi_1(t) = A_2(\varphi_1, \phi_1)(t) + \lambda_1\varrho_1(t), t \in [0, 1].$$

From Lemma 2.4 and $\varrho_1 \in P_0$ we have

$$\varphi_1, \phi_1 \in P_0. \tag{3.7}$$

Note that $\|\varphi_1\| = \|\phi_1\| = R > \mathcal{O}_{M,h}$, and thus $\varphi_1(t) \geq w(t), \phi_1(t) \geq w(t), t \in [0, 1]$.

By (H4)(ii) we obtain

$$\liminf_{\phi\to\infty} \frac{\mathcal{F}_1(\varphi, \phi)}{p(\phi)} = \liminf_{\phi\to\infty} \frac{f(\varphi, \phi) + M}{p(\phi)} \geq 1,\ \liminf_{\varphi\to\infty} \frac{\mathcal{F}_2(\varphi, \phi)}{q(\varphi)} = \liminf_{\varphi\to\infty} \frac{g(\varphi, \phi) + M}{q(\varphi)} \geq 1.$$

This implies that there exist $c_1, c_2 > 0$ such that

$$\mathcal{F}_1(\varphi, \phi) \geq p(\phi) - c_1,\ \mathcal{F}_2(\varphi, \phi) \geq q(\varphi) - c_2,\ \varphi, \phi \in \mathbb{R}^+.$$

Therefore, we have

$$\begin{aligned}
\varphi_1(t) &= A_1(\varphi_1, \phi_1)(t) + \lambda_1\varrho_1(t) \\
&\geq A_1(\varphi_1, \phi_1)(t) \\
&\geq \int_0^1 G(t, s)h(s)[p(\phi_1(s) - w(s)) - c_1]\mathrm{d}s \\
&\geq \int_0^1 G(t, s)h(s)p(\phi_1(s) - w(s))\mathrm{d}s - c_1\kappa_2
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
\phi_1(t) &= A_2(\varphi_1, \phi_1)(t) + \lambda_1\varrho_1(t) \\
&\geq A_2(\varphi_1, \phi_1)(t) \\
&\geq \int_0^1 G(t, s)h(s)[q(\varphi_1(s) - w(s)) - c_2]\mathrm{d}s \\
&\geq \int_0^1 G(t, s)h(s)q(\varphi_1(s) - w(s))\mathrm{d}s - c_2\kappa_2.
\end{aligned} \tag{3.9}$$

Consequently, we have

$$\begin{aligned}\phi_1(t)-w(t) &\geq \int_0^1 G(t,s)h(s)q(\varphi_1(s)-w(s))\mathrm{d}s - c_2\kappa_2 - w(t)\\ &\geq \int_0^1 G(t,s)h(s)q(\varphi_1(s)-w(s))\mathrm{d}s - (c_2+M)\kappa_2.\end{aligned}$$

From (H4)(iii), there is a $c_3 > 0$ such that

$$p(\mathcal{L}_{G,h}q(z)) \geq e_1\mathcal{L}_{G,h}z - \mathcal{L}_{G,h}c_3, z \in \mathbb{R}^+.$$

Combining with (H4)(i), we have

$$\begin{aligned}p(\phi_1(t)-w(t)) &\geq p(\phi_1(t)-w(t)+(c_2+M)\kappa_2) - p((c_2+M)\kappa_2)\\ &\geq p\left(\int_0^1 G(t,s)h(s)q(\varphi_1(s)-w(s))\mathrm{d}s\right) - p((c_2+M)\kappa_2)\\ &= p\left(\int_0^1 \frac{G(t,s)h(s)}{\mathcal{L}_{G,h}}\mathcal{L}_{G,h}q(\varphi_1(s)-w(s))\mathrm{d}s\right) - p((c_2+M)\kappa_2)\\ &\geq \int_0^1 p\left(\frac{G(t,s)h(s)}{\mathcal{L}_{G,h}}\mathcal{L}_{G,h}q(\varphi_1(s)-w(s))\right)\mathrm{d}s - p((c_2+M)\kappa_2)\\ &\geq \int_0^1 \frac{G(t,s)h(s)}{\mathcal{L}_{G,h}}p\left(\mathcal{L}_{G,h}q(\varphi_1(s)-w(s))\right)\mathrm{d}s - p((c_2+M)\kappa_2)\\ &\geq \int_0^1 \frac{G(t,s)h(s)}{\mathcal{L}_{G,h}}(e_1\mathcal{L}_{G,h}(\varphi_1(s)-w(s)) - \mathcal{L}_{G,h}c_3)\mathrm{d}s - p((c_2+M)\kappa_2)\\ &\geq e_1\int_0^1 G(t,s)h(s)(\varphi_1(s)-w(s))\mathrm{d}s - p((c_2+M)\kappa_2) - c_3\kappa_2.\end{aligned}$$

Substituting this inequality into (3.8) we have

$$\begin{aligned}\varphi_1(t)-w(t) \geq& \int_0^1 G(t,s)h(s)\left[e_1\int_0^1 G(s,\tau)h(\tau)(\varphi_1(\tau)-w(\tau))\mathrm{d}\tau - p((c_2+M)\kappa_2) - c_3\kappa_2\right]\mathrm{d}s\\ &- (c_1+M)\kappa_2\\ \geq& e_1\int_0^1\int_0^1 G(t,s)h(s)G(s,\tau)h(\tau)(\varphi_1(\tau)-w(\tau))\mathrm{d}\tau\mathrm{d}s\\ &- p((c_2+M)\kappa_2)\kappa_2 - c_3\kappa_2^2 - (c_1+M)\kappa_2.\end{aligned}$$

Multiply by $\vartheta(t)$ on both sides of the above and integrate over $[0,1]$ and use Lemma 2.2 to obtain

$$\begin{aligned}\int_0^1(\varphi_1(t)-w(t))\vartheta(t)\mathrm{d}t \geq& e_1\int_0^1\vartheta(t)\int_0^1\int_0^1 G(t,s)h(s)G(s,\tau)h(\tau)(\varphi_1(\tau)-w(\tau))\mathrm{d}\tau\mathrm{d}s\mathrm{d}t\\ &- p((c_2+M)\kappa_2)\kappa_2^2 - c_3\kappa_2^3 - (c_1+M)\kappa_2^2\\ \geq& e_1\kappa_1^2\int_0^1(\varphi_1(t)-w(t))\vartheta(t)\mathrm{d}t - p((c_2+M)\kappa_2)\kappa_2^2 - c_3\kappa_2^3 - (c_1+M)\kappa_2^2.\end{aligned}$$

From this inequality we have

$$\int_0^1 (\varphi_1(t) - w(t))\vartheta(t)\mathrm{d}t \le \frac{p((c_2 + M)\kappa_2)\kappa_2^2 + c_3\kappa_2^3 + (c_1 + M)\kappa_2^2}{e_1\kappa_1^2 - 1}$$

and thus

$$\begin{aligned}\int_0^1 \varphi_1(t)\vartheta(t)\mathrm{d}t &\le \frac{p((c_2 + M)\kappa_2)\kappa_2^2 + c_3\kappa_2^3 + (c_1 + M)\kappa_2^2}{e_1\kappa_1^2 - 1} + \int_0^1 w(t)\vartheta(t)\mathrm{d}t \\ &\le \frac{p((c_2 + M)\kappa_2)\kappa_2^2 + c_3\kappa_2^3 + (c_1 + M)\kappa_2^2}{e_1\kappa_1^2 - 1} + M\kappa_2^2.\end{aligned}$$

Note that (3.7), $\varphi_1 \in P_0$, and we have

$$\|\varphi_1\| \le \frac{p((c_2 + M)\kappa_2)\kappa_2^2 + c_3\kappa_2^3 + (c_1 + M)\kappa_2^2}{\kappa_1(e_1\kappa_1^2 - 1)} + \frac{M\kappa_2^2}{\kappa_1}.$$

On the other hand, multiply by $\vartheta(t)$ on both sides of (3.8) and integrate over $[0, 1]$ and use Lemma 2.2 to obtain

$$\begin{aligned}\kappa_1 \int_0^1 \vartheta(t)p(\phi_1(t) - w(t))\mathrm{d}t &\le \int_0^1 \varphi_1(t)\vartheta(t)\mathrm{d}t + c_1\kappa_2^2 \\ &\le \frac{p((c_2 + M)\kappa_2)\kappa_2^2 + c_3\kappa_2^3 + (c_1 + M)\kappa_2^2}{e_1\kappa_1^2 - 1} + M\kappa_2^2 + c_1\kappa_2^2.\end{aligned}$$

From Remark 2.1 we have $w \in P_0$, note that $\|\phi_1\| = R > \frac{M(\gamma+\delta)(\beta+\alpha)}{\rho} \int_0^1 h(s)\mathrm{d}s \ge \|w\|$ and $\phi_1 \in P_0$, then $\phi_1 - w \in P_0$. By the concavity of p we have

$$\begin{aligned}\|\phi_1 - w\| &\le \kappa_1^{-1} \int_0^1 (\phi_1(t) - w(t))\vartheta(t)\mathrm{d}t = \frac{\|\phi_1 - w\|}{\kappa_1 p(\|\phi_1 - w\|)} \int_0^1 \frac{\phi_1(t) - w(t)}{\|\phi_1 - w\|} p(\|\phi_1 - w\|)\vartheta(t)\mathrm{d}t \\ &\le \frac{\|\phi_1 - w\|}{\kappa_1 p(\|\phi_1 - w\|)} \int_0^1 p\left(\frac{\phi_1(t) - w(t)}{\|\phi_1 - w\|}\|\phi_1 - w\|\right)\vartheta(t)\mathrm{d}t \\ &\le \frac{\|\phi_1 - w\|}{\kappa_1^2 p(\|\phi_1 - w\|)} \left[\frac{p((c_2 + M)\kappa_2)\kappa_2^2 + c_3\kappa_2^3 + (c_1 + M)\kappa_2^2}{e_1\kappa_1^2 - 1} + M\kappa_2^2 + c_1\kappa_2^2\right].\end{aligned}$$

This implies that

$$p(\|\phi_1 - w\|) \le \frac{1}{\kappa_1^2}\left[\frac{p((c_2 + M)\kappa_2)\kappa_2^2 + c_3\kappa_2^3 + (c_1 + M)\kappa_2^2}{e_1\kappa_1^2 - 1} + M\kappa_2^2 + c_1\kappa_2^2\right].$$

From (H4)(i) we have

$$\begin{aligned}p(\|\phi_1\|) &= p(\|\phi_1 - w + w\|) \le p(\|\phi_1 - w\| + \|w\|) \le p(\|\phi_1 - w\|) + p(\|w\|) \\ &\le \frac{1}{\kappa_1^2}\left[\frac{p((c_2 + M)\kappa_2)\kappa_2^2 + c_3\kappa_2^3 + (c_1 + M)\kappa_2^2}{e_1\kappa_1^2 - 1} + M\kappa_2^2 + c_1\kappa_2^2\right] + p(\|w\|) \\ &\le \frac{1}{\kappa_1^2}\left[\frac{p((c_2 + M)\kappa_2)\kappa_2^2 + c_3\kappa_2^3 + (c_1 + M)\kappa_2^2}{e_1\kappa_1^2 - 1} + M\kappa_2^2 + c_1\kappa_2^2\right] + p(M\kappa_2) \\ &< +\infty.\end{aligned}$$

Therefore, there exists $O_{\phi_1} > 0$ such that $\|\phi_1\| \le O_{\phi_1}$.

We have prove the boundedness of φ_1, ϕ_1 when (3.6) holds, i.e., when $\varphi_1, \phi_1 \in \partial B_R \cap P$, there exist a positive constant to control the norms of φ_1, ϕ_1. Now we choose a sufficiently large

$$R_1 > \max\left\{O_{M,h}, O_{\phi_1}, \frac{p((c_2+M)\kappa_2)\kappa_2^2 + c_3\kappa_2^3 + (c_1+M)\kappa_2^2}{\kappa_1(e_1\kappa_1^2 - 1)} + \frac{M\kappa_2^2}{\kappa_1}\right\}.$$

Then when $\varphi_1, \phi_1 \in \partial B_{R_1} \cap P$, (3.6) is not satisfied, and thus (3.5) holds. From Lemma 2.5 we have

$$i(A, B_{R_1}^2 \cap P^2, P^2) = 0. \tag{3.10}$$

Combining (3.4) with (3.10) we have

$$i(A, (B_{R_1}^2 \backslash \overline{B}_{O_{M,h}}^2) \cap P^2, P^2) = i(A, B_{R_1}^2 \cap P^2, P^2) - i(A, B_{O_{M,h}}^2 \cap P^2, P^2) = 0 - 1 = -1.$$

Then the operator A has at least one fixed point (denoted by (φ^*, ϕ^*)) on $(B_{R_1}^2 \backslash \overline{B}_{O_{M,h}}^2) \cap P^2$ with $\varphi^*(t), \phi^*(t) \ge w(t), t \in [0,1]$. Therefore, $(\varphi^* - w, \phi^* - w)$ is a positive solution for (2.2), and (1.1) has at least one positive radial solution. □

Theorem 3.2. *Suppose that (H1)–(H3), (H6) and (H7) hold. Then* (1.1) *has at least one positive radial solution.*

Proof. **Step 1.** When $\varphi, \phi \in \partial B_{O_{M,h}} \cap P$, we have

$$(\varphi, \phi) \ne A(\varphi, \phi) + \lambda(\varrho_2, \varrho_2), \lambda \ge 0, \tag{3.11}$$

where $\varrho_2 \in P$ is a given element. Suppose the contrary. Then there exist $\varphi_2, \phi_2 \in \partial B_{O_{M,h}} \cap P, \lambda_2 \ge 0$ such that

$$(\varphi_2, \phi_2) = A(\varphi_2, \phi_2) + \lambda_2(\varrho_2, \varrho_2).$$

This implies that

$$\|\varphi_2\| \ge \varphi_2(t) \ge A_1(\varphi_2, \phi_2)(t) + \lambda_2\varrho_2(t) \ge A_1(\varphi_2, \phi_2)(t), t \in [0,1],$$

$$\|\phi_2\| \ge \phi_2(t) \ge A_2(\varphi_2, \phi_2)(t) + \lambda_2\varrho_2(t) \ge A_2(\varphi_2, \phi_2)(t), t \in [0,1].$$

Then we have

$$\|\varphi_2\| + \|\phi_2\| \ge \|A_1(\varphi_2, \phi_2)\| + \|A_2(\varphi_2, \phi_2)\|. \tag{3.12}$$

From (H7) we have

$$\begin{aligned}
\|A_i(\varphi_2, \phi_2)\| &= \max_{t\in[0,1]} A_i(\varphi_2, \phi_2)(t) \\
&\ge \max_{t\in[0,1]} \frac{\rho}{(\gamma+\delta)(\beta+\alpha)} G(t,t) \int_0^1 G(s,s)h(s)\mathcal{F}_i(\varphi_2(s) - w(s), \phi_2(s) - w(s))\mathrm{d}s \\
&\ge \mathcal{L}_G \int_0^1 G(s,s)h(s)\widetilde{Q}_i \mathrm{d}s = \widetilde{Q}_i\kappa_2\mathcal{L}_G, i = 1, 2.
\end{aligned}$$

By the condition on $\widetilde{Q}_i$ we have

$$\|A_1(\varphi_2,\phi_2)\| + \|A_2(\varphi_2,\phi_2)\| > 2\mathcal{O}_{M,h} = \|\varphi_2\| + \|\phi_2\|,$$

and this contradicts (3.12), so (3.11) holds. By Lemma 2.5 we have

$$i(A, B^2_{\mathcal{O}_{M,h}} \cap P^2, P^2) = 0. \tag{3.13}$$

Step 2. There exists a sufficiently large $R > \mathcal{O}_{M,h}$ such that

$$(\varphi,\phi) \neq \lambda A(\varphi,\phi), \varphi,\phi \in \partial B_R \cap P, \lambda \in [0,1]. \tag{3.14}$$

Suppose the contrary. Then there exist $\varphi_3, \phi_3 \in \partial B_R \cap P, \lambda_3 \in [0,1]$ such that

$$(\varphi_3,\phi_3) = \lambda_3 A(\varphi_3,\phi_3). \tag{3.15}$$

Combining with Lemma 2.4 we have

$$\varphi_3, \phi_3 \in P_0. \tag{3.16}$$

Note that $\varphi_3, \phi_3 \in \partial B_R \cap P$, and then $\varphi_3(t) - w(t), \phi_3(t) - w(t) \geq 0, t \in [0,1]$. Hence, from (H6) we have

$$\limsup_{\phi\to\infty} \frac{\mathcal{F}_1(\varphi,\phi)}{\zeta(\phi)} = \limsup_{\phi\to\infty} \frac{f(\varphi,\phi)+M}{\zeta(\phi)} \leq 1,\ \limsup_{\varphi\to\infty} \frac{\mathcal{F}_2(\varphi,\phi)}{\eta(\varphi)} = \limsup_{\varphi\to\infty} \frac{g(\varphi,\phi)+M}{\eta(\varphi)} \leq 1.$$

This implies that there exists $\widetilde{M} > 0$ such that

$$\mathcal{F}_1(\varphi,\phi) \leq \zeta(\phi),\ \mathcal{F}_2(\varphi,\phi) \leq \eta(\varphi), \varphi,\phi \geq \widetilde{M}. \tag{3.17}$$

By similar methods as in Theorem 3.1, choosing $R > \widetilde{M}$, and from (3.15) we obtain

$$\varphi_3(t) = \lambda_3 A_1(\varphi_3,\phi_3)(t) \leq \int_0^1 G(t,s)h(s)\zeta(\phi_3(s) - w(s))\mathrm{d}s \tag{3.18}$$

and

$$\phi_3(t) = \lambda_3 A_2(\varphi_3,\phi_3)(t) \leq \int_0^1 G(t,s)h(s)\eta(\varphi_3(s) - w(s))\mathrm{d}s. \tag{3.19}$$

From (H6)(iii), there exists $c_4 > 0$ such that

$$\zeta(\mathcal{L}_{G,h}\eta(z)) \leq e_2\mathcal{L}_{G,h}z + c_4\mathcal{L}_{G,h}, z \in \mathbb{R}^+.$$

By the convexity of ζ we have

$$\begin{aligned}
\zeta(\phi_3(t) - w(t)) &\leq \zeta\left(\int_0^1 G(t,s)h(s)\eta(\varphi_3(s) - w(s))\mathrm{d}s\right) \\
&\leq \int_0^1 \zeta\left[G(t,s)h(s)\eta(\varphi_3(s) - w(s))\right]\mathrm{d}s \\
&= \int_0^1 \zeta\left[\frac{G(t,s)h(s)}{\mathcal{L}_{G,h}}\mathcal{L}_{G,h}\eta(\varphi_3(s) - w(s))\right]\mathrm{d}s \\
&\leq \int_0^1 \frac{G(t,s)h(s)}{\mathcal{L}_{G,h}}\zeta\left[\mathcal{L}_{G,h}\eta(\varphi_3(s) - w(s))\right]\mathrm{d}s \\
&\leq \int_0^1 \frac{G(t,s)h(s)}{\mathcal{L}_{G,h}}[e_2\mathcal{L}_{G,h}(\varphi_3(s) - w(s)) + c_4\mathcal{L}_{G,h}]\mathrm{d}s \\
&\leq \int_0^1 G(t,s)h(s)[e_2(\varphi_3(s) - w(s)) + c_4]\mathrm{d}s.
\end{aligned} \tag{3.20}$$

Substituting this inequality into (3.18) we have

$$\begin{aligned}\varphi_3(t) &\leq \int_0^1 G(t,s)h(s)\int_0^1 G(s,\tau)h(\tau)[e_2(\varphi_3(\tau)-w(\tau))+c_4]\mathrm{d}\tau\mathrm{d}s \\ &\leq e_2\int_0^1\int_0^1 G(t,s)h(s)G(s,\tau)h(\tau)(\varphi_3(\tau)-w(\tau))\mathrm{d}\tau\mathrm{d}s+c_4\kappa_2^2.\end{aligned} \tag{3.21}$$

Consequently, we have

$$\begin{aligned}\varphi_3(t)-w(t) &\leq \int_0^1 G(t,s)h(s)\int_0^1 G(s,\tau)h(\tau)[e_2(\varphi_3(\tau)-w(\tau))+c_4]\mathrm{d}\tau\mathrm{d}s \\ &\leq e_2\int_0^1\int_0^1 G(t,s)h(s)G(s,\tau)h(\tau)(\varphi_3(\tau)-w(\tau))\mathrm{d}\tau\mathrm{d}s+c_4\kappa_2^2.\end{aligned} \tag{3.22}$$

Multiply by $\vartheta(t)$ on both sides of (3.22) and integrate over $[0, 1]$ and use Lemma 2.2 to obtain

$$\int_0^1(\varphi_3(t)-w(t))\vartheta(t)\mathrm{d}t \leq e_2\kappa_2^2\int_0^1(\varphi_3(t)-w(t))\vartheta(t)\mathrm{d}t+c_4\kappa_2^3,$$

and we have

$$\int_0^1(\varphi_3(t)-w(t))\vartheta(t)\mathrm{d}t \leq \frac{c_4\kappa_2^3}{1-e_2\kappa_2^2}.$$

Note that (3.16), $w \in P_0$, and

$$\|\varphi_3-w\| \leq \frac{c_4\kappa_2^3}{\kappa_1(1-e_2\kappa_2^2)}.$$

By the triangle inequality we have

$$\|\varphi_3\| = \|\varphi_3-w+w\| \leq \|\varphi_3-w\|+\|w\| \leq \frac{c_4\kappa_2^3}{\kappa_1(1-e_2\kappa_2^2)}+M\kappa_2.$$

On the other hand, from (3.20) we have

$$\begin{aligned}\zeta(\phi_3(t)-w(t)) &\leq \int_0^1 G(t,s)h(s)[e_2(\varphi_3(s)-w(s))+c_4]\mathrm{d}s \\ &\leq \int_0^1 \vartheta(s)[e_2(\varphi_3(s)-w(s))+c_4]\mathrm{d}s \\ &\leq \frac{c_4e_2\kappa_2^3}{1-e_2\kappa_2^2}+c_4\kappa_2.\end{aligned}$$

Note that $\frac{c_4e_2\kappa_2^3}{1-e_2\kappa_2^2}+c_4\kappa_2$ is independent to R, and using (H6)(i) there exists $O_{\phi_3}>0$ such that

$$\|\phi_3-w\| \leq O_{\phi_3},$$

and then

$$\|\phi_3\| = \|\phi_3-w+w\| \leq \|\phi_3-w\|+\|w\| \leq O_{\phi_3}+M\kappa_2.$$

Therefore, when $\varphi_3, \phi_3 \in \partial B_R \cap P$, we obtain there is a positive constant to control the norms of φ_3, ϕ_3. Then if we choose

$$R_2 > \left\{ O_{M,h}, O_{\phi_3} + M\kappa_2, \widetilde{M}, \frac{c_4\kappa_2^3}{\kappa_1(1 - e_2\kappa_2^2)} + M\kappa_2 \right\},$$

then (3.14) holds, and from Lemma 2.6 we have

$$i(A, B_{R_2}^2 \cap P^2, P^2) = 1. \tag{3.23}$$

From (3.13) and (3.23) we have

$$i(A, (B_{R_2}^2 \backslash \overline{B}_{O_{M,h}}^2) \cap P^2, P^2) = i(A, B_{R_2}^2 \cap P^2, P^2) - i(A, B_{O_{M,h}}^2 \cap P^2, P^2) = 1 - 0 = 1.$$

Then the operator A has at least one fixed point (denoted by (u^{**}, v^{**})) on $(B_{R_2}^2 \backslash \overline{B}_{O_{M,h}}^2) \cap P^2$ with $u^{**}(t), v^{**}(t) \geq w(t), t \in [0, 1]$. Therefore, $(u^{**} - w, v^{**} - w)$ is a positive solution for (2.2), and (1.1) has at least one positive radial solution.

We now provide some examples to illustrate our main results. Let $\alpha = \beta = \gamma = \delta = 1$, and $k(|z|) = e^{|z|}, z \in \mathbb{R}^n$. Then (H1) and (H2) hold. □

Example 3.1. *Let* $p(\phi) = \phi^{\frac{4}{5}}, q(\varphi) = \varphi^2, \varphi, \phi \in \mathbb{R}^+$. *Then* $\liminf_{z\to\infty} \frac{p(\mathcal{L}_{G,h}q(z))}{z} = \liminf_{z\to\infty} \frac{\mathcal{L}_{G,h}^{\frac{4}{5}} z^{\frac{8}{5}}}{z} \geq \infty$, *and (H4)(i), (iii) hold. If we choose*

$$f(\varphi, \phi) = \frac{1}{\beta_1\kappa_2(|\sin\varphi| + 1)}\phi - M, \ g(\varphi, \phi) = \frac{O_{M,h}^{1-\beta_3}}{\beta_2\kappa_2(|\cos\phi| + 1)}\varphi^{\beta_3} - M, \beta_1, \beta_2 > 1, \beta_3 > 2,$$

then (H3) holds, and when $\varphi, \phi \in [0, O_{M,h}]$, *we have*

$$\mathcal{F}_1(\varphi, \phi) = f(\varphi, \phi) + M \leq \frac{O_{M,h}}{\beta_1\kappa_2} := Q_1, \ \mathcal{F}_2(\varphi, \phi) = g(\varphi, \phi) + M \leq \frac{O_{M,h}^{1-\beta_3}}{\beta_2\kappa_2} O_{M,h}^{\beta_3} = \frac{O_{M,h}}{\beta_2\kappa_2} := Q_2.$$

Hence, (H5) holds. Also we have

$$\liminf_{\phi\to\infty} \frac{f(\varphi, \phi)}{p(\phi)} = \liminf_{\phi\to\infty} \frac{\frac{1}{\beta_1\kappa_2(|\sin\varphi|+1)}\phi - M}{\phi^{\frac{4}{5}}} = \infty, \ \liminf_{\varphi\to\infty} \frac{g(\varphi, \phi)}{q(\varphi)} = \liminf_{\varphi\to\infty} \frac{\frac{O_{M,h}^{1-\beta_3}}{\beta_2\kappa_2(|\cos\phi|+1)}\varphi^{\beta_3} - M}{\varphi^2} = \infty.$$

Then (H4)(ii) holds. As a result, all the conditions in Theorem 3.1 hold, and (1.1) *has at least one positive radial solution.*

Example 3.2. *Let* $\zeta(\phi) = \phi^2, \eta(\varphi) = \varphi^{\frac{2}{5}}$, $\varphi, \phi \in \mathbb{R}^+$. *Then* $\limsup_{z\to\infty} \frac{\zeta(\mathcal{L}_{G,h}\eta(z))}{z} = \limsup_{z\to\infty} \frac{\mathcal{L}_{G,h}^2 z^{\frac{4}{5}}}{z} = 0 \leq e_2\mathcal{L}_{G,h}$, *and (H7)(i), (iii) hold. If we choose*

$$f(\varphi, \phi) = \widetilde{Q}_1 + (\phi + |\cos\varphi|)^{\alpha_1} - M, \ g(\varphi, \phi) = \widetilde{Q}_2 + (|\sin\phi| + \varphi)^{\alpha_2} - M, \varphi, \phi \in \mathbb{R}^+,$$

where $\alpha_1 \in (0, 2), \alpha_2 \in \left(0, \frac{2}{5}\right)$. *Then (H3) holds. Moreover, we have*

$$\mathcal{F}_1(\varphi, \phi) = f(\varphi, \phi) + M \geq \widetilde{Q}_1, \ \mathcal{F}_2(\varphi, \phi) = g(\varphi, \phi) + M \geq \widetilde{Q}_2,$$

and

$$\limsup_{\phi\to\infty}\frac{\widetilde{Q}_1 - M + (\phi + |\cos\varphi|)^{\alpha_1}}{\phi^2} = 0, \limsup_{\varphi\to\infty}\frac{\widetilde{Q}_2 - M + (|\sin\phi| + \varphi)^{\alpha_2}}{\varphi^{\frac{2}{5}}} = 0.$$

Therefore, (H6) *and* (H7) *(ii) hold. As a result, all the conditions in Theorem 3.2 hold, and* (1.1) *has at least one positive radial solution.*

Remark 3.2. *Note that condition* $(HZ)_2$ *is often used to study various kinds of semipositone boundary value problems (for example, see [19, 22, 23, 26, 28–30]). However, in Example 3.1 we have*

$$\liminf_{\phi\to+\infty}\frac{f(\varphi,\phi)}{\varphi} = \liminf_{\phi\to+\infty}\frac{\frac{1}{\beta_1\kappa_2(|\sin\varphi|+1)}\phi - M}{\phi} = \frac{1}{2\beta_1\kappa_2}, \forall\varphi\in\mathbb{R}^+.$$

Comparing with $(HZ)_2$ *we see that our theory gives new results for boundary value problem with semipositone nonlinearities.*

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Conflict of interest

The authors declare no conflict of interest.

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