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Research article

Existence and data dependence results for neutral fractional order integro-differential equations

Veliappan Vijayaraj[1], **Chokkalingam Ravichandran**[1], **Thongchai Botmart**[2,*], **Kottakkaran Sooppy Nisar**[3] **and Kasthurisamy Jothimani**[4]

[1] Department of Mathematics, Kongunadu Arts and Science College (Autonomous), Coimbatore 641029, Tamil Nadu, India

[2] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

[3] Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawaser 11991, Saudi Arabia

[4] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632014, Tamil Nadu, India

* **Correspondence:** Email: thongbo@kku.ac.th.

Abstract: We assess the multi-derivative nonlinear neutral fractional order integro-differential equations with Atangana-Baleanu fractional derivative of the Riemann-Liouville sense. We discuss results about the existence and difference solution on some data, based on the Prabhakar fractional integral operator $\varepsilon^{\alpha}_{\delta,\eta,\mathcal{V};c+}$ with generalized Mittag-Leffler function. The results are obtained by using Krasnoselskii's fixed point theorem and the Gronwall-Bellman inequality.

Keywords: fractional calculus; fixed point techniques; Gronwall-Bellman inequality
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1. Introduction

Fractional differential equations (FDEs) appeared as an excellent mathematical tool for, modeling of many physical phenomena appearing in various branches of science and engineering, such as viscoelasticity, statistical mechanics, dynamics of particles, etc. Fractional calculus is a recently developing work in mathematics which studies derivatives and integrals of functions of fractional order [26].

The most used fractional derivatives are the Riemann-Liouville (RL) and Caputo derivatives. These derivatives contain a non-singular derivatives but still conserves the most important peculiarity of the fractional operators [1,2,10,11,23,24]. Atangana and Baleanu described a derivative with a generalized Mittag-leffler (ML) function. This derivative is often called the Atangana-Baleanu (AB) fractional derivative. The AB-derivative in the senses of Riemman-Liouville and Caputo are denoted by ABR-derivative and ABC-derivative, respectively.

The AB fractional derivative is a nonlocal fractional derivative with nonsingular kernel which is connected with various applications [3, 5, 6, 8, 9, 13–16]. Using the advantage of the non-singular ML kernal present in the AB fractional derivatives, operators, many authors from various branches of applied mathematics have developed and studied mathematical models involving AB fractional derivatives [18, 22, 29–32, 35–37].

Mohamed et al. [25] considered a system of multi-derivatives for Caputo FDEs with an initial value problem, examined the existence and uniqueness results and obtained numerical results. Sutar et al. [32, 33] considered multi-derivative FDEs involving the ABR derivative and examined existence, uniqueness and dependence results. Kucche et al. [12, 19–21, 34] enlarged the work of multi-derivative fractional differential equations involving the Caputo fractional derivative and studied the existence, uniqueness and continuous dependence of the solution.

Inspired by the preceding work, we perceive the multi-derivative nonlinear neutral fractional integro-differential equation with AB fractional derivative of the Riemann-Liouville sense of the problem:

$$\frac{d\mathcal{V}}{d\jmath} +^{\star}_{0} D^{\delta}_{\jmath}[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]$$
$$= \varphi\left(\jmath, \mathcal{V}(\jmath), \int_0^{\jmath} \mathcal{K}(\jmath, \theta, \mathcal{V}(\theta))d\theta, \int_0^{T} \chi(\jmath, \theta, \mathcal{V}(\theta))d\theta\right), \jmath \in \mathscr{I} \tag{1.1}$$
$$\mathcal{V}(0) = \mathcal{V}_0 \in \mathscr{R}, \tag{1.2}$$

where ${}^{\star}_{0}D^{\delta}_{\jmath}$ denotes the ABR fractional derivative of order $\delta \in (0, 1)$, and $\varphi \in \mathscr{C}(\mathscr{I} \times \mathscr{R} \times \mathscr{R} \times \mathscr{R}, \mathscr{R})$ is a non-linear function. Let $\mathcal{P}_1\mathcal{V}(\jmath) = \int_0^{\jmath} \mathcal{K}(\jmath, \theta, \mathcal{V}(\theta))d\theta$ and $\mathcal{P}_2\mathcal{V}(\jmath) = \int_0^{T} \chi(\jmath, \theta, \mathcal{V}(\theta))d\theta$. Now, (1.1) becomes,

$$\frac{d\mathcal{V}}{d\jmath} +^{\star}_{0} D^{\delta}_{\jmath}[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] = \varphi(\jmath, \mathcal{V}(\jmath), \mathcal{P}_1\mathcal{V}(\jmath), \mathcal{P}_2\mathcal{V}(\jmath)), \jmath \in \mathscr{I}, \tag{1.3}$$
$$\mathcal{V}(0) = \mathcal{V}_0 \in \mathscr{R}. \tag{1.4}$$

In this work, we derive a few supplemental results using the characteristics of the fractional integral operator $\varepsilon^{\alpha}_{\delta,\eta,\mathcal{V};c+}$. The existence results are obtained by Krasnoselskii's fixed point theorem and the uniqueness and data dependence results are obtained by the Gronwall-Bellman inequality.

2. Preliminaries

Definition 2.1. *[14] The Sobolev space* $\mathrm{H}^{\mathfrak{q}}(X)$ *is defined as*
$\mathrm{H}^{\mathfrak{q}}(X) = \left\{\varphi \in \mathrm{L}^2(X) : D^{\beta}\varphi \in \mathrm{L}^2(X), \forall\, |\beta| \le \mathfrak{q}\right\}$. *Let* $\mathfrak{q} \in [1, \infty)$ *and* X *be open,* $X \subset \mathbb{R}$.

Definition 2.2. *[11, 17] The generalized ML function* $E^{\alpha}_{\delta,\beta}(u)$ *for complex* δ, β, α *with* $Re(\delta) > 0$ *is defined by*

$$E^{\alpha}_{\delta,\beta}(u) = \sum_{t=0}^{\infty} \frac{(\alpha)_t}{\alpha(\delta t + \beta)} \frac{u^t}{t!},$$

and the Pochhammer symbol is $(\alpha)_t$, *where* $(\alpha)_0 = 1, (\alpha)_t = \alpha(\alpha+1)...(\alpha+t-1)$, $t = 1, 2....$, *and* $E^1_{\delta,\beta}(u) = E_{\delta,\beta}(u), E^1_{\delta,1}(u) = E_\delta(u)$.

Definition 2.3. *[4] The ABR fractional derivative of* $\mathcal{V}$ *of order* δ *is*

$$ {}^{\star}_{0}D^{\delta}_{\jmath}[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] = \frac{B(\delta)}{1-\delta}\frac{d}{d\jmath}\int_0^{\jmath} E_\delta\left[-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right]\mathcal{V}(\theta)d\theta, $$

where $\mathcal{V} \in \mathrm{H}^1(0,1)$, $\delta \in (0,1)$, $B(\delta) > 0$. *Here,* E_δ *is a one parameter ML function, which shows* $B(0) = B(1) = 1$.

Definition 2.4. *[4] The ABC fractional derivative of* $\mathcal{V}$ *of order* δ *is*

$$ {}^{\star}_{0}D^{\delta}_{\jmath}[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] = \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left[-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right]\mathcal{V}'(\theta)d\theta, $$

where $\mathcal{V} \in \mathrm{H}^1(0,1)$, $\delta \in (0,1)$, *and* $B(\delta) > 0$. *Here,* E_δ *is a one parameter ML function, which shows* $B(0) = B(1) = 1$.

Lemma 2.5. *[4] If* $L\{g(\jmath); b\} = \bar{G}(b)$, *then* $L\left\{{}^{\star}_{0}D^{\delta}_{\jmath}g(\jmath); b\right\} = \frac{B(\delta)}{1-\delta}\frac{b^\delta \bar{G}(b)}{b^\delta + \frac{\delta}{1-\delta}}$.

Lemma 2.6. *[26]* $L\left[\jmath^{m\delta+\beta-1}E^{(m)}_{\delta,\beta}\left(\pm a\jmath^\delta\right); b\right] = \frac{m!b^{\delta-\beta}}{(b^\delta \pm a)^{m+1}}, E^m(\jmath) = \frac{d^m}{d\jmath^m}E(\jmath)$.

Definition 2.7. *[17, 27] The operator* $\varepsilon^{\alpha}_{\delta,\eta,\mathcal{V};c+}$ *on class* $\mathrm{L}(m,n)$ *is*

$$ \left(\varepsilon^{\alpha}_{\delta,\eta,\mathcal{V};c+}\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] = \int_0^t (\jmath-\theta)^{\alpha-1} E^{\alpha}_{\delta,\eta}\left[\mathcal{V}(\jmath-\theta)^\delta\right]\Theta(\theta)d\theta, \jmath \in [c,d], $$

where $\delta, \eta, \mathcal{V}, \alpha \in \mathbb{C}\,(Re(\delta), Re(\eta) > 0)$, *and* $n > m$.

Lemma 2.8. *[17, 27] The operator* $\varepsilon^{\alpha}_{\delta,\eta,\mathcal{V};c+}$ *is bounded on* $C[m,n]$, *such that* $\left\|\left(\varepsilon^{\alpha}_{\delta,\eta,\mathcal{V};c+}\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\right\| \leq \mathcal{P}\,\|\Theta\|$, *where*

$$ P = (n-m)^{Re(\eta)} \sum_{t=0}^{\infty} \frac{|(\alpha)_t|}{|\alpha(\delta t + \eta)|\,[Re(\delta)t + Re(\eta)]} \frac{\left|\mathcal{V}(n-m)^{Re(\delta)}\right|^t}{t!}. $$

Here, $\delta, \eta, \mathcal{V}, \alpha \in \mathbb{C}\,(Re(\delta), Re(\eta) > 0)$, *and* $n > m$.

Lemma 2.9. *[17, 27] The operator* $\varepsilon^{\alpha}_{\delta,\eta,\mathcal{V};c+}$ *is invertible in the space* $\mathrm{L}(m,n)$ *and* $\varphi \in \mathrm{L}(m,n)$ *its left inversion is given by*

$$ \left(\left[\varepsilon^{\alpha}_{\delta,\eta,\mathcal{V};c+}\right]^{-1}\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] = \left(D^{\eta+\varsigma}_{c+}\varepsilon^{-\alpha}_{\delta,\eta,\mathcal{V};c+}\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))], \jmath \in (m,n], $$

where $\delta, \eta, \mathcal{V}, \alpha \in \mathbb{C}\,(Re(\delta), Re(\eta) > 0)$, *and* $n > m$.

Lemma 2.10. *[17, 27] Let* $\delta, \eta, \mathcal{V}, \alpha \in \mathbb{C}\,(Re(\delta), Re(\eta) > 0)$, $n > m$ *and suppose that the integral equation is*

$$ \int_0^{\jmath} (\jmath-\theta)^{\alpha-1} E^{\alpha}_{\delta,\eta}\left[\mathcal{V}(\jmath-\theta)^\delta\right]\Theta(\theta)d\theta = \varphi(\jmath), \jmath \in (m,n], $$

is solvable in the space $\mathrm{L}(m,n)$.*Then, its unique solution* $\Theta(\jmath)$ *is given by*

$$ \Theta(\jmath) = \left(D^{\eta+\varsigma}_{c+}\varepsilon^{-\alpha}_{\delta,\eta,\mathcal{V};c+}\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))], \jmath \in (m,n]. $$

Lemma 2.11. *[7] (Krasnoselskii's fixed point theorem) Let A be a Banach space and X be bounded, closed, convex subset of A. Let $\mathscr{F}_1, \mathscr{F}_2$ be maps of S into A such that $\mathscr{F}_1\mathcal{V} + \mathscr{F}_2\varphi \in X \; \forall \; \mathcal{V}, \varphi \in U$. The equation $\mathscr{F}_1\mathcal{V} + \mathscr{F}_2\mathcal{V} = \mathcal{V}$ has a solution on S, and $\mathscr{F}_1$, $\mathscr{F}_2$ is a contraction and completely continuous.*

Lemma 2.12. *[28] (Gronwall-Bellman inequality) Let $\mathcal{V}$ and φ be continuous and non-negative functions defined on $\mathscr{I}$. Let $\mathcal{V}(\jmath) \leq \mathcal{A} + \int_a^{\jmath} \varphi(\theta)\mathcal{V}(\theta)d\theta, \jmath \in \mathscr{I}$; here, $\mathcal{A}$ is a non-negative constant.*

$$\mathcal{V}(\jmath) \leq \mathcal{A}exp\left(\int_a^{\jmath} \varphi(\theta)d\theta\right), \jmath \in \mathscr{I}.$$

In this part, we need some fixed-point-techniques-based hypotheses for the results:

(H1) Let $\mathcal{V} \in C[0, T]$, function $\varphi \in (C[0, T] \times \mathscr{R} \times \mathscr{R} \times \mathscr{R}, \mathscr{R})$ is a continuous function, and there exist $+^{ve}$ constants ζ_1, ζ_2 and ζ.
$\|\varphi(\jmath, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3) - \varphi(\jmath, \varphi_1, \varphi_2, \varphi_3)\| \leq \zeta_1 (\|\mathcal{V}_1 - \varphi_1\| + \|\mathcal{V}_2 - \varphi_2\| + \|\mathcal{V}_3 - \varphi_3\|)$ for all $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \varphi_1, \varphi_2, \varphi_3$ in Y, $\zeta_2 = max_{\mathcal{V}\in\mathscr{R}} \|f(\jmath, 0, 0, 0)\|$, and $\zeta = max\{\zeta_1, \zeta_2\}$.

(H2) $\mathcal{P}_1$ is a continuous function, and there exist $+^{ve}$ constants $\mathscr{C}_1, \mathscr{C}_2$ and $\mathscr{C}$.
$\|\mathcal{P}_1(\jmath, \theta, \mathcal{V}_1) - \mathcal{P}_1(\jmath, \theta, \varphi_1)\| \leq \mathscr{C}_1 (\|\mathcal{V}_1 - \varphi_1\|) \; \forall \; \mathcal{V}_1, \varphi_1$ in Y,
$\mathscr{C}_2 = max_{(\jmath,\theta)\in D} \|\mathcal{P}_1(\jmath, \theta, 0)\|$, and $\mathscr{C} = max\{\mathscr{C}_1, \mathscr{C}_2\}$.

(H3) $\mathcal{P}_2$ is a continuous function and there are $+^{ve}$ constants $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}$.
$\|\mathcal{P}_2(\jmath, \theta, \mathcal{V}_1) - \mathcal{P}_2(\jmath, \theta, \varphi_1)\| \leq \mathcal{D}_1 (\|\mathcal{V}_1 - \varphi_1\|)$ for all $\mathcal{V}_1, \varphi_1$ in Y,
$\mathcal{D}_2 = max_{(\jmath,\theta)\in D} \|\mathcal{P}_2(\jmath, \theta, 0)\|$ and $\mathcal{D} = max\{\mathcal{D}_1, \mathcal{D}_2\}$.

(H4) Let $x \in c[0, I]$, function $u \in (c[0, I] \times \mathscr{R}, \mathscr{R})$ is a continuous function, and there is a $+^{ve}$ constant $k > 0$, such that $\|u(\jmath, x) - u(\jmath, y)\| \leq k \|x - y\|$. Let $Y = C[\mathscr{R}, X]$ be the set of continuous functions on $\mathscr{R}$ with values in the Banach space X.

Lemma 2.13. *If* $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *are satisfied the following estimates,*
$\|\mathcal{P}_1\mathcal{V}(\jmath)\| \leq \jmath(\mathscr{C}_1 \|\mathcal{V}\| + \mathscr{C}_2), \|\mathcal{P}_1\mathcal{V}(\jmath) - \mathcal{P}_1\varphi(\jmath)\| \leq \mathscr{C}\jmath \|\mathcal{V} - \varphi\|$, *and*
$\|\mathcal{P}_2\mathcal{V}(\jmath)\| \leq \jmath(\mathcal{D}_1 \|\mathcal{V}\| + \mathcal{D}_2), \|\mathcal{P}_2\mathcal{V}(\jmath) - \mathcal{P}_2\varphi(\jmath)\| \leq \mathcal{D}\jmath \|\mathcal{V} - \varphi\|$.

3. Equivalent neutral fractional integral equation

Theorem 3.1. *The function $\varphi \in \mathscr{C}(\mathscr{I} \times \mathscr{R} \times \mathscr{R} \times \mathscr{R}, \mathscr{R})$ and $\mathcal{V} \in \mathscr{C}(\mathscr{I})$ is a solution for the problem of Eqs (1.3) and (1.4), iff $\mathcal{V}$ is a solution of the fractional equation*

$$\begin{aligned}\mathcal{V}(\jmath) = \mathcal{V}_0 - \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left[-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right][\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]d\theta \\ + \int_0^{\jmath} \varphi(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta))\, d\theta, \jmath \in \mathscr{I}. \qquad (3.1)\end{aligned}$$

Proof. (1) By using Definition 2.3 and Eq (1.3), we get

$$\frac{d}{d\jmath}\left(\mathcal{V}(\jmath) + \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left[-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right][\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]d\theta\right) = \varphi(\jmath, \mathcal{V}(\jmath), \mathcal{P}_1\mathcal{V}(\jmath), \mathcal{P}_2\mathcal{V}(\jmath)).$$

Integrating both sides of the above equation with limits 0 to $\jmath$, we get

$$\mathcal{V}(\jmath) + \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left[-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right][\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]d\theta - \mathcal{V}(0)$$
$$= \int_0^{\jmath} \varphi\left(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta)\right) d\theta, \jmath \in \mathscr{I}.$$

Conversely, with differentiation on both sides of Eq (3.1) with respect to $\jmath$, we get

$$\frac{d\mathcal{V}}{d\jmath} + \frac{B(\delta)}{1-\delta}\frac{d}{d\jmath}\int_0^{\jmath} E_\delta\left[-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right][\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]d\theta = \varphi\left(\jmath, \mathcal{V}(\jmath), \mathcal{P}_1\mathcal{V}(\jmath), \mathcal{P}_2\mathcal{V}(\jmath)\right), \jmath \in \mathscr{I}.$$

Using Definition 2.3, we get Eq (1.3) and substitute $\jmath = 0$ in Eq (3.1), we get Eq (1.4). □

Proof. (2) In Equation (1.3), taking the Laplace Transform on both sides, we get

$$L\left[\mathcal{V}'(\jmath); b\right] + L\left[{}^{\star}_{0}D^{\delta}_{\jmath}; b\right][\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] = L\left[\varphi\left(\jmath, \mathcal{V}(\jmath), \mathcal{P}_1\mathcal{V}(\jmath), \mathcal{P}_2\mathcal{V}(\jmath)\right); b\right].$$

Now, using the Laplace Transform formula for the AB fractional derivative of the RL sense, as given in Lemma 2.5, we get

$$b\bar{X}(b) - [\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] - \mathcal{V}(0) + \frac{B(\delta)}{1-\delta}\frac{b^\delta\bar{X}(b)}{b^\delta + \frac{\delta}{1-\delta}} = \bar{G}(b),$$

$\bar{X}(b) = [\mathcal{V}(\jmath); b]$ and $\bar{G}(b) = L\left[\varphi\left(\jmath, \mathcal{V}(\jmath), P_1\mathcal{V}(\jmath), P_2\mathcal{V}(\jmath)\right); b\right]$. Using Eq (1.4), we get

$$\bar{X}(b) = \mathcal{V}_0\frac{1}{b} - \frac{B(\delta)}{1-\delta}\frac{b^{\delta-1}\bar{X}(b)}{b^\delta + \frac{\delta}{1-\delta}}[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] + \frac{1}{b}\bar{G}(b). \tag{3.2}$$

In Eq (3.2) applying the inverse Laplace Transform on both sides using Lemma 2.6 and the convolution theorem, we get

$$L^{-1}[\bar{X}(b); \jmath]$$
$$= \mathcal{V}_0 L^{-1}\left[\frac{1}{b}; \jmath\right] - \frac{B(\delta)}{1-\delta}\left(L^{-1}\left[\frac{b^{\delta-1}}{b^\delta + \frac{\delta}{1-\delta}}\right][\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] * L^{-1}[\bar{X}(b); \jmath]\right)$$
$$+ L^{-1}[\bar{G}(b); \jmath] * L^{-1}\left[\frac{1}{b}; \jmath\right]$$
$$= \mathcal{V}_0 - \frac{B(\delta)}{1-\delta}\left(E_\delta\left[-\frac{\delta}{1-\delta}\jmath^\delta\right][\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\right) + \varphi\left(\jmath, \mathcal{V}(\jmath), \mathcal{P}_1\mathcal{V}(\jmath), \mathcal{P}_2\mathcal{V}(\jmath)\right)$$
$$= \mathcal{V}_0 - \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left[-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right][\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]d\theta$$
$$+ \int_0^{\jmath} \varphi\left(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta)\right) d\theta.$$

$$\mathcal{V}(\jmath) = \mathcal{V}_0 - \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left[-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right][\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]d\theta$$
$$+ \int_0^{\jmath} \varphi\left(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta)\right) d\theta. \tag{3.3}$$

□

Theorem 3.2. *Let $\delta \in (0,1)$. Define the operator $\mathscr{F}$ on $\mathscr{C}(\mathscr{I})$:*

$$(\mathscr{F}\mathcal{V})(\jmath) = \mathcal{V}_0 - \frac{B(\delta)}{1-\delta}\left(\varepsilon^{1}_{\delta,1,\frac{-\delta}{1-\delta};0+}\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))], \mathcal{V} \in \mathscr{C}(\mathscr{I}). \tag{3.4}$$

(*A*) *$\mathscr{F}$ is a bounded linear operator on $\mathscr{C}(\mathscr{I})$.*

(*B*) *$\mathscr{F}$ satisfying the hypotheses.*

(*C*) *$\mathscr{F}(X)$ is equicontinuous, and X is a bounded subset of $\mathscr{C}(\mathscr{I})$.*

(*D*) *$\mathscr{F}$ is invertible, function $\varphi \in \mathscr{C}(\mathscr{I})$, and the operator equation $\mathscr{F}\mathcal{V} = \varphi$ has a unique solution in $\mathscr{C}(\mathscr{I})$.*

Proof. (A) From Definition 2.7 and Lemma 2.8, the fractional integral operator $\varepsilon^{1}_{\delta,1,\frac{-\delta}{1-\delta};0^+}$ is a bounded linear operator on $\mathscr{C}(\mathscr{I})$, such that

$$\left\|\varepsilon^{1}_{\delta,1,\frac{-\delta}{1-\delta};0+}\right\| \|[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\| \leq \mathcal{P}\|\mathcal{V}\|, \jmath \in \mathscr{I}, where$$

$$\mathcal{P} = T\sum_{n=0}^{\infty}\frac{(1)_n}{\alpha(\delta n+1)(\delta n+1)}\frac{\left|\frac{-\delta}{1-\delta}T^{\delta}\right|^n}{n!} = T\sum_{n=0}^{\infty}\frac{\left(\frac{\delta}{1-\delta}\right)^n T^{\delta^n}}{\alpha(\delta n+2)} = TE_{\delta,2}\left(\frac{\delta}{1-\delta}T^{\delta}\right),$$

and we have

$$\|\mathscr{F}\mathcal{V}\| = \left|\frac{B(\delta)}{1-\delta}\right|\left\|\varepsilon^{1}_{\delta,1,\frac{-\delta}{1-\delta};0+}\right\| \|[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\| \leq \mathcal{P}\frac{B(\delta)}{1-\delta}\|\mathcal{V}\|, \forall\mathcal{V} \in \mathscr{C}(\mathscr{I}). \tag{3.5}$$

Thus, $\mathscr{F}\mathcal{V} = \varphi$ is a bounded linear operator on $\mathscr{C}(\mathscr{I})$.
(B) We consider $\mathcal{V}, \varphi \in \mathscr{C}(\mathscr{I})$. By using linear operator $\mathscr{F}$ and bounded operator $\varepsilon^{1}_{\delta,1,\frac{-\delta}{1-\delta};0^+}$, for any $\jmath \in \mathscr{I}$,

$$\begin{aligned}|(\mathscr{F}\mathcal{V})(\jmath) - (\mathscr{F}\varphi)(\jmath)| &= |\mathscr{F}(\mathcal{V} - \varphi)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]| \\ &\leq \frac{B(\delta)}{1-\delta}\left\|\left(\varepsilon^{1}_{\delta,1,\frac{-\delta}{1-\delta};0+}\mathcal{V} - \varphi\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\right\| \\ &\leq \mathcal{P}\frac{B(\delta)}{1-\delta}\|\mathcal{V} - \varphi\|.\end{aligned}$$

Where, $P = TE_{\delta,2}\left(\frac{\delta}{1-\delta}T^{\delta}\right)$, then the operator $\mathscr{F}$ is satisfied the hypotheses with constant $P\frac{B(\delta)}{1-\delta}$.
(C) Let $U = \{\mathcal{V} \in \mathscr{C}(\mathscr{I}) : \|\mathcal{V}\| \leq R\}$ be a bounded and closed subset of $\mathscr{C}(\mathscr{I})$, $\mathcal{V} \in U$, and $\jmath_1, \jmath_2 \in \mathscr{I}$ with $\jmath_1 \leq \jmath_2$.

$$\begin{aligned}&|(\mathscr{F}\mathcal{V})(\jmath_1) - (\mathscr{F}\mathcal{V})(\jmath_2)| \\ &= \left|\frac{B(\delta)}{1-\delta}\left(\varepsilon^{1}_{\delta,1,\frac{-\delta}{1-\delta};0+}\right)[\mathcal{V}(\jmath_1) - u(l_1, x(l))] - \frac{B(\delta)}{1-\delta}\left(\varepsilon^{1}_{\delta,1,\frac{-\delta}{1-\delta};0+}\right)[\mathcal{V}(\jmath_2) - u(l_2, x(l))]\right|\end{aligned}$$

$$\leq \frac{B(\delta)}{1-\delta}\left|\int_0^{\jmath_1}\left\{E_{\delta}\left[-\frac{\delta}{1-\delta}(\jmath_1-\theta)^{\delta}\right] - E_{\delta}\left[-\frac{\delta}{1-\delta}(\jmath_2-\theta)^{\delta}\right]\right\}[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]d\theta\right|$$

$$
\begin{aligned}
&+\frac{B(\delta)}{1-\delta}\left|\int_{\jmath_1}^{\jmath_2} E_\delta\left[-\frac{\delta}{1-\delta}(\jmath_2-\theta)^\delta\right][\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]d\theta\right|\\
&\leq \frac{B(\delta)}{1-\delta}\sum_{n=0}^{\infty}\left|\left(\frac{-\delta}{1-\delta}\right)^n\right|\frac{1}{\alpha(n\delta+1)}\int_0^{\jmath_1}\left|(\jmath_1-\theta)^{n\delta}-(\jmath_2-\theta)^{n\delta}\right|\|[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]\|\,d\theta\\
&+\frac{B(\delta)}{1-\delta}\sum_{n=0}^{\infty}\left|\left(\frac{-\delta}{1-\delta}\right)^n\right|\frac{1}{\alpha(n\delta+1)}\int_{\jmath_1}^{\jmath_2}\left|(\jmath_2-\theta)^{n\delta}\right|\|[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]\|\,d\theta\\
&\leq \frac{LB(\delta)}{1-\delta}\sum_{n=0}^{\infty}\left(\frac{\delta}{1-\delta}\right)^n\frac{1}{\alpha(n\delta+1)}\int_0^{\jmath_1}(\jmath_2-\theta)^{n\delta}-(\jmath_1-\theta)^{n\delta}d\theta\\
&+\frac{LB(\delta)}{1-\delta}\sum_{n=0}^{\infty}\left(\frac{\delta}{1-\delta}\right)^n\frac{1}{\alpha(n\delta+1)}\int_{\jmath_1}^{\jmath_2}(\jmath_2-\theta)^{n\delta}d\theta\\
&\leq \frac{RB(\delta)}{1-\delta}\sum_{n=0}^{\infty}\left(\frac{\delta}{1-\delta}\right)^n\frac{1}{\alpha(n\delta+1)}\left\{-(\jmath_2-\jmath_1)^{n\delta+1}+\jmath_2^{n\delta+1}-\jmath_1^{n\delta+1}+(\jmath_2-\jmath_1)^{n\delta+1}\right\}\\
&\leq \frac{RB(\delta)}{1-\delta}\sum_{n=0}^{\infty}\left(\frac{\delta}{1-\delta}\right)^n\frac{1}{\alpha(n\delta+2)}\left\{\jmath_2^{n\delta+1}-\jmath_1^{n\delta+1}\right\}
\end{aligned}
$$

$$
|(\mathscr{F}\mathcal{V})(\jmath_1)-(\mathscr{F}\mathcal{V})(\jmath_2)|\leq \frac{RB(\delta)}{1-\delta}\sum_{n=0}^{\infty}\left(\frac{\delta}{1-\delta}\right)^n\frac{1}{\alpha(n\delta+2)}\left\{\jmath_2^{n\delta+1}-\jmath_1^{n\delta+1}\right\}. \tag{3.6}
$$

Hence, if $|\jmath_1-\jmath_2|\to 0$ then $|(\mathscr{F}\mathcal{V})(\jmath_1)-(\mathscr{F}\mathcal{V})(\jmath_2)|\to 0$.
$\therefore$ $(\mathscr{F}\mathcal{V})$ is equicontinuous on $\mathscr{I}$.
(D) By Lemmas 2.9 and 2.10, $\varphi\in\mathscr{C}(\mathscr{I})$, and we get

$$
\left(\varepsilon^1_{\delta,1,\frac{-\delta}{1-\delta};0+}\right)^{-1}[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]=\left(D^{1+\beta}_{0^+}\varepsilon^{-1}_{\delta\beta,\frac{-\delta}{1-\delta};0+}\right)[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))],\ \jmath\in(m,n). \tag{3.7}
$$

By Eqs (3.4) and (3.5), we have

$$
\begin{aligned}
\left(\mathscr{F}^{-1}\right)[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]&=\left(\frac{B(\delta)}{1-\delta}\varepsilon^1_{\delta,1,\frac{-\delta}{1-\delta};0+}\right)^{-1}[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]\\
&=\frac{1-\delta}{B(\delta)}\left(D^{1+\beta}_{0^+}\varepsilon^{-1}_{\delta\beta,\frac{-\delta}{1-\delta};0+}\right)[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))],\ \jmath\in(m,n),
\end{aligned}
$$

where $\beta\in\mathbb{C}$ with $Re(\beta)>0$. This shows $\mathscr{F}$ is invertible on $\mathscr{C}(\mathscr{I})$ and

$$
(\mathscr{F}\mathcal{V})[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]=[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))],\ \jmath\in\mathscr{I},
$$

has the unique solution,

$$
\begin{aligned}
\mathcal{V}(\jmath)&=\left(\mathscr{F}^{-1}[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]\right)\\
&=\frac{1-\delta}{B(\delta)}\left(D^{1+\beta}_{0^+}\varepsilon^{-1}_{\delta\beta,\frac{-\delta}{1-\delta};0+}\right)[\mathcal{V}(\jmath)-x(\jmath)),y(\jmath))],\ \jmath\in(m,n).
\end{aligned} \tag{3.8}
$$

□

4. Existence and uniqueness results

Theorem 4.1. *Let $\varphi \in \mathscr{C}(\mathscr{I} \times \mathscr{R} \times \mathscr{R} \times \mathscr{R}, \mathscr{R})$. Then, the ABR derivative ${}^{\star}_{0}D^{\delta}_{\jmath}[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] = \varphi(\jmath, \mathcal{V}(\jmath), \mathcal{P}_1\mathcal{V}(\jmath), \mathcal{P}_2\mathcal{V}(\jmath)),\ \jmath \in \mathscr{I}$, is solvable in $\mathscr{C}(\mathscr{I})$, and the solution in $\mathscr{C}(\mathscr{I})$ is*

$$\mathcal{V}(\jmath) = \frac{1-\delta}{B(\delta)}\left(D^{1+\beta}_{0^+}\varepsilon^{-1}_{\delta,\beta,\frac{-\delta}{1-\delta};0+}\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))],\ \jmath \in \mathscr{I}, \tag{4.1}$$

where $\beta \in \mathbb{C}, Re(\beta) > 0$, and $\hat{\varphi}(\jmath) = \int_0^{\jmath} \varphi(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta))\, d\theta,\ \jmath \in \mathscr{I}$.

Proof. The corresponding fractional equation of the ABR derivative

$${}^{\star}_{0}D^{\delta}_{\jmath}[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] = \varphi(\jmath, \mathcal{V}(\jmath), \mathcal{P}_1\mathcal{V}(\jmath), \mathcal{P}_2\mathcal{V}(\jmath)),\ \jmath \in \mathscr{I},$$

is given by

$$\frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left[-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right][\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]d\theta = \int_0^{\jmath} \varphi(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta))\, d\theta,\ \jmath \in \mathscr{I}.$$

Using operator $\mathscr{F}$ of Eq (3.4), we get

$$(\mathscr{F}\mathcal{V})(s) = \int_0^{\jmath} \varphi(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta))\, d\theta = \hat{\varphi}(\jmath),\ \jmath \in \mathscr{I}. \tag{4.2}$$

Equations (3.7) and (4.2) are solvable, and we get

$$\mathcal{V}(\jmath) = \frac{1-\delta}{B(\delta)}\left(D^{1+\beta}_{0^+}\varepsilon^{-1}_{\delta,\beta,\frac{-\delta}{1-\delta};0+}\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))],\ \jmath \in \mathscr{I}; \beta \in \mathbb{C}, Re(\beta) > 0. \tag{4.3}$$

□

Theorem 4.2. *Let $\varphi \in \mathscr{C}(\mathscr{I} \times R \times R \times R, \mathscr{R})$ satisfy* ($\mathbf{H_1}$)–($\mathbf{H_3}$) *with $L = \sup_{\jmath \in \mathscr{I}} \omega(\jmath)$, where $\omega(\jmath) = \zeta(1 + \mathscr{C}\jmath + \mathcal{D}T)$, if $L = \min\left\{1, \frac{1}{2T}\right\}$. Then problem of (1.3) and (1.4) has a solution in $\mathscr{C}(\mathscr{I})$ provided*

$$\frac{2B(\delta)TE_{\delta,2}\left(\frac{\delta}{1-\delta}\right)T^\delta}{1-\delta} \leq 1. \tag{4.4}$$

Proof. Define

$$R = \frac{\|\mathcal{V}_0\| + N_\varphi T}{1 - LT - \frac{B(\delta)TE_{\delta,2}\left(\frac{\delta}{1-\delta}\right)T^\delta}{1-\delta}},$$

where $N_\varphi = \sup_{\jmath \in \mathscr{I}} \|\varphi(\jmath, 0, 0, 0)\|$. Let $U = \{\mathcal{V} \in \mathscr{C}(\mathscr{I}) : \|\mathcal{V}\| \leq R\}$. Consider $\mathscr{F}_1 : X \to A$ and $\mathscr{F}_2 : X \to A$ given as

$$(\mathscr{F}_1\mathcal{V})(\jmath) = \mathcal{V}_0 + \int_0^{\jmath} \varphi(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta))\, d\theta,\ \jmath \in \mathscr{I},$$
$$(\mathscr{F}_2\mathcal{V})(\jmath) = -(\mathscr{F})[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))],\ \jmath \in \mathscr{I}.$$

Let $\mathcal{V} = \mathscr{F}_1\mathcal{V} + \mathscr{F}_2\mathcal{V}, \mathcal{V} \in \mathscr{C}(\mathscr{I})$ is the fractional Eq (3.1) to the problems (1.3) and (1.4).
Hence, the operators $\mathscr{F}_1$ and $\mathscr{F}_2$ satisfy the Krasnoselskii's fixed point theorem.
Step (i) $\mathscr{F}_1$ is a contraction.
By $(\mathbf{H_1})$–$(\mathbf{H_3})$ on φ, $\forall\ \mathcal{V}, \varphi \in \mathscr{C}(\mathscr{I})$ and $\jmath \in \mathscr{I}$,

$$|\mathscr{F}_1\mathcal{V}(\jmath) - \mathscr{F}_2\varphi(\jmath)| \leq \omega(\jmath)\,|\mathcal{V}(\jmath) - \varphi(\jmath)| \leq R\,\|\mathcal{V} - \varphi\|\,. \tag{4.5}$$

This gives, $\|\mathscr{F}_1\mathcal{V} - \mathscr{F}_2\varphi\| \leq RT\,\|\mathcal{V} - \varphi\|\,, \mathcal{V}, \varphi \in \mathscr{C}(\mathscr{I})$.
Step (ii) $\mathscr{F}_2$ is completely continuous. By using Theorem 3.3 and Ascoli-Arzela theorem, $\mathscr{F}_2 = -\mathscr{F}$ is completely continuous .
Step (iii) $\mathscr{F}_1\mathcal{V} + \mathscr{F}_2\varphi \in U$, for any $\mathcal{V}, \varphi \in U$, using Theorem 3.3, we obtain

$$\begin{aligned}
&\|(\mathscr{F}_1\mathcal{V} + \mathscr{F}_2\varphi)(\jmath)\| \leq \|(\mathscr{F}_1\mathcal{V})(\jmath)\| + \|(\mathscr{F}_2\varphi)(\jmath)\|\\
&\leq \|\mathcal{V}_0\| + \int_0^J \|\varphi\left(\theta, \mathcal{V}\left(\theta\right), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta)\right)\|\,d\theta + \left\|\varepsilon^1_{\delta,1,\frac{-\delta}{1-\delta};0+}\varphi\right\|\\
&\leq \|\mathcal{V}_0\| + \int_0^J \|\varphi\left(\theta, \mathcal{V}\left(\theta\right), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta)\right)\|\,d\theta + \frac{B(\delta)}{1-\delta}TE_{\delta,2}\left(\frac{\delta}{1-\delta}T^\delta\right)\|\varphi\|\\
&\\
&\leq \|\mathcal{V}_0\| + \int_0^J \|\varphi\left(\theta, \mathcal{V}\left(\theta\right), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta)\right) - \varphi(\theta,0,0,0)\|\,d\theta + \int_0^J \|\varphi(\theta,0,0,0)\|\,d\theta\\
&\quad + \frac{B(\delta)}{1-\delta}TE_{\delta,2}\left(\frac{\delta}{1-\delta}T^\delta\right)L\\
&\leq \|\mathcal{V}_0\| + \int_0^J \zeta\left(\|\mathcal{V}\| + \mathscr{C}\jmath\,\|\mathcal{V}\| + \mathcal{D}T\,\|\mathcal{V}\|\right)d\theta + N_\varphi\int_0^J d\theta + \frac{B(\delta)}{1-\delta}TE_{\delta,2}\left(\frac{\delta}{1-\delta}T^\delta\right)L\\
&\leq \|\mathcal{V}_0\| + \zeta(1 + \mathscr{C}\jmath + \mathcal{D}T)\int_0^J \|\mathcal{V}\|\,d\theta + N_\varphi\int_0^J d\theta + \frac{B(\delta)}{1-\delta}TE_{\delta,2}\left(\frac{\delta}{1-\delta}T^\delta\right)L\\
&\leq \|\mathcal{V}_0\| + \omega(\jmath)R\int_0^J d\theta + N_\varphi\int_0^J d\theta + \frac{B(\delta)}{1-\delta}TE_{\delta,2}\left(\frac{\delta}{1-\delta}T^\delta\right)L\\
&\leq \|\mathcal{V}_0\| + LRT + N_\varphi T + \frac{B(\delta)}{1-\delta}TE_{\delta,2}\left(\frac{\delta}{1-\delta}T^\delta\right)L.
\end{aligned} \tag{4.6}$$

By definition of R, we get

$$\|\mathcal{V}_0\| + N_\varphi T = L\left(1 - RT + \frac{B(\delta)TE_{\delta,2}\left(\frac{\delta}{1-\delta}T^\delta\right)}{1-\delta}\right). \tag{4.7}$$

Using the Eq (4.5) in (4.7), we get condition of Eq (4.4).

$$\|(\mathscr{F}_1\mathcal{V} + \mathscr{F}_2\varphi)(\jmath)\| \leq L\left(\frac{2B(\delta)TE_{\delta,2}\left(\frac{\delta}{1-\delta}\right)T^\delta}{1-\delta}\right), \jmath \in \mathscr{I}. \tag{4.8}$$

$\therefore\ \|(\mathscr{F}_1\mathcal{V} + \mathscr{F}_2\varphi)(\jmath)\| \leq L, \jmath \in \mathscr{I}$. This gives, $\mathscr{F}_1\mathcal{V} + \mathscr{F}_2\varphi \in U, \forall\ \mathcal{V}, \varphi \in X$.
From Steps (i)–(iii), all the conditions of Lemma 2.11 follow. □

Theorem 4.3. *By Theorem 4.2, the Eqs (1.3) and (1.4) have a unique solution in $\mathscr{C}(\mathscr{I})$.*

Proof. (1) The problems (1.3) and (1.4) have an operator equation form as:

$$\left(\varepsilon^{1}_{\delta,1,\frac{-\delta}{1-\delta};0+}\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] = \hat{\varphi}(\jmath),\ \jmath \in \mathscr{I}, \tag{4.9}$$

where,

$$\hat{\varphi}(\jmath) = \frac{1-\delta}{B(\delta)}\left(\mathcal{V}_0 - \mathcal{V}(\jmath) + \int_0^{J} \varphi\left(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta)\right) d\theta\right),\ \jmath \in \mathscr{I}.$$

By Theorem 4.2, Eq (4.7) is solvable in $\mathscr{C}(\mathscr{I})$, by Lemma 2.10 we get a unique solution of Eqs (1.3) and (1.4),

$$\mathcal{V}(\jmath) = \left(D^{1+\beta}_{0^+}\varepsilon^{-1}_{\delta,\beta,\frac{-\delta}{1-\delta};0+}\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))],\ \mathcal{V} \in \mathscr{C}(\mathscr{I}).$$

□

Proof. (2) Let $\mathcal{V}, \varphi$ be solutions of Eqs (1.3) and (1.4). By fractional integral operators and $(\mathbf{H_1})$–$(\mathbf{H_3})$, we find, for any $\jmath \in \mathscr{I}$,

$$\begin{aligned}
&|\mathcal{V}(\jmath) - \varphi(\jmath)| \\
&\leq \left|\frac{B(\delta)}{1-\delta}\left(\varepsilon^{1}_{\delta,1,\frac{-\delta}{1-\delta};0+}(\mathcal{V} - \varphi)\right)[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\right| \\
&\quad + \int_0^{J} |\varphi\left(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta)\right) - \varphi\left(\theta, \varphi(\theta), \mathcal{P}_1\varphi(\theta), \mathcal{P}_2\varphi(\theta)\right)| \, d\theta \\
&\leq \left|\frac{B(\delta)}{1-\delta}\int_0^{J} E_\delta\left[-\frac{\delta}{1-\delta}(\jmath-\theta)^{\delta}\right](\mathcal{V}(\theta) - \varphi(\theta))d\theta\right| \\
&\quad + \int_0^{J} \zeta(|\mathcal{V}(\theta) - \varphi(\theta)| + \mathscr{C}\,|\mathcal{V}(\theta) - \varphi(\theta)| + \mathcal{D}\,|\mathcal{V}(\theta) - \varphi(\theta)|)d\theta \\
&\leq \frac{B(\delta)}{1-\delta}\int_0^{J} E_\delta\left(\left|-\frac{\delta}{1-\delta}T^{\delta}\right|\right)|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta + \int_0^{J} \zeta(1 + \mathscr{C} + \mathcal{D})\,|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta \\
&\leq \frac{B(\delta)}{1-\delta}\int_0^{J} E_\delta\left(\frac{\delta}{1-\delta}T^{\delta}\right)|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta + \int_0^{J} [\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\,|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta \\
&\leq \int_0^{J}\left[\frac{B(\delta)}{1-\delta}E_\delta\left(\frac{\delta}{1-\delta}T^{\delta}\right) + [\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\right]|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta
\end{aligned}$$

$$|\mathcal{V}(\jmath) - \varphi(\jmath)| \leq \int_0^{J}\left[\frac{B(\delta)}{1-\delta}E_\delta\left(\frac{\delta}{1-\delta}T^{\delta}\right) + [\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\right]|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta. \tag{4.10}$$

□

5. Estimate solution

Theorem 5.1. *By Theorem 4.2, if $\mathcal{V}(\jmath)$ is a solution of Eqs (1.3) and (1.4), then*

$$|\mathcal{V}(\jmath)| \leq \left\{|\mathcal{V}_0| + N_\varphi T\right\} exp\left(\int_0^{J}\left[\frac{B(\delta)}{1-\delta}E_\delta\left(\frac{\delta}{1-\delta}T^{\delta}\right) + [\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\right]d\theta\right),\ \jmath \in \mathscr{I}, \tag{5.1}$$

where, $N_\varphi = \sup_{\jmath\in\mathscr{I}} |\varphi(\jmath, 0, 0, 0)|$.

Proof. If $\mathcal{V}(\jmath)$ is a solution of Eqs (1.3) and (1.4), for all $\jmath \in \mathscr{I}$,

$$
\begin{aligned}
|\mathcal{V}(\jmath)| & \leq |\mathcal{V}_0| - \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left(\left|-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right|\right)|\mathcal{V}(\theta)|\,d\theta + \int_0^{\jmath}|\varphi(\theta,\mathcal{V}(\theta),\mathcal{P}_1\mathcal{V}(\theta),\mathcal{P}_2\mathcal{V}(\theta))|\,d\theta \\
& \leq |\mathcal{V}_0| - \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left(\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right)|\mathcal{V}(\theta)|\,d\theta \\
& \quad + \int_0^{\jmath}|\varphi(\theta,\mathcal{V}(\theta),\mathcal{P}_1\mathcal{V}(\theta),\mathcal{P}_2\mathcal{V}(\theta)) - \varphi(\theta,0,0,0)|\,d\theta + \int_0^{\jmath}|\varphi(\theta,0,0,0)|\,d\theta \\
& \leq |\mathcal{V}_0| - \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left(\frac{\delta}{1-\delta}T^\delta\right)|\mathcal{V}(\theta)|\,d\theta + \int_0^{\jmath}\zeta\left(|\mathcal{V}(\theta)| + \mathscr{C}\,|\mathcal{V}(\theta)| + \mathcal{D}\,|\mathcal{V}(\theta)|\right)d\theta + N_\varphi \jmath \\
& \leq |\mathcal{V}_0| - \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left(\frac{\delta}{1-\delta}T^\delta\right)|\mathcal{V}(\theta)|\,d\theta + \int_0^{\jmath}\zeta(1+\mathscr{C}+\mathcal{D})\,|\mathcal{V}(\jmath)|\,d\theta + N_\varphi T \\
& \leq |\mathcal{V}_0| - \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left(\frac{\delta}{1-\delta}T^\delta\right)|\mathcal{V}(\theta)|\,d\theta + \int_0^{\jmath}[\mathcal{V}(\jmath) - x(\jmath,y(\jmath))]\,|\mathcal{V}(\theta)|\,d\theta + N_\varphi T \\
& \leq \left\{|\mathcal{V}_0| + N_\varphi T\right\} + \int_0^{\jmath}\left[\frac{B(\delta)}{1-\delta}E_\delta\left(\frac{\delta}{1-\delta}T^\delta\right) + [\mathcal{V}(\jmath) - x(\jmath,y(\jmath))]\right]|\mathcal{V}(\theta)|\,d\theta.
\end{aligned}
$$

By Lemma 2.12, we get

$$
|\mathcal{V}(\jmath)| \leq \left\{|\mathcal{V}_0| + N_\varphi T\right\} exp\left(\int_0^{\jmath}\left[\frac{B(\delta)}{1-\delta}E_\delta\left(\frac{\delta}{1-\delta}T^\delta\right) + [\mathcal{V}(\jmath) - x(\jmath,y(\jmath))]\right]d\theta\right),\ \jmath \in \mathscr{I}. \tag{5.2}
$$

□

6. Data dependence results

We discuss data dependence results for the problem

$$
\frac{d\varphi}{d\jmath} +_0^{\star} D_\jmath^\delta[\mathcal{V}(\jmath) - x(\jmath,y(\jmath))] = \tilde{\varphi}(\jmath,\varphi(\jmath),\mathcal{P}_1\varphi(\jmath),\mathcal{P}_2\varphi(\jmath)),\ \jmath \in \mathscr{I}, \tag{6.1}
$$

$$
\varphi(0) = \varphi_0 \in \mathscr{R}. \tag{6.2}
$$

Theorem 6.1. *Equation (4.2) holds, and* $\xi_k > 0$, *where* $k = 1,2$ *are real numbers such that,*

$$
\begin{aligned}
& |\mathcal{V}_0 - \varphi_0| \leq \xi_1, \\
& |\varphi(\jmath,\mathcal{V}(\jmath),\mathcal{P}_1\mathcal{V}(\jmath),\mathcal{P}_2\mathcal{V}(\jmath)) - \tilde{\varphi}(\jmath,\varphi(\jmath),\mathcal{P}_1\varphi(\jmath),\mathcal{P}_2\varphi(\jmath))| \leq \xi_2,\ \jmath \in \mathscr{I}.
\end{aligned} \tag{6.3}
$$

$\varphi(\jmath)$ is a solution of ABR fractional derivative Eqs (6.1) and (6.2), and $\mathcal{V}(\jmath)$ is a solution of Eqs (1.3) and (1.4).

Proof. Let $\mathcal{V}, \varphi$ are the solution of Eqs (1.3) and (1.4), (6.1) and (6.2) respectively. We find for any

$$
\begin{aligned}
&|\mathcal{V}(\jmath) - \varphi(\jmath)| \\
&\leq |\mathcal{V}_0 - \varphi_0| + \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left(\left|-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right|\right)|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta \\
&\quad + \int_0^{\jmath} |\varphi(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta)) - \tilde{\varphi}(s, \varphi(\theta), \mathcal{P}_1\varphi(\theta), \mathcal{P}_2\varphi(\theta))|\, d\theta \\
&\leq |\mathcal{V}_0 - \varphi_0| + \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left(\left|-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right|\right)|\mathcal{V}(\theta) - \varphi(s)|\, d\theta \\
&\quad + \int_0^{\jmath} |\varphi(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta)) - \varphi(\theta, \varphi(\theta), \mathcal{P}_1\varphi(\theta), \mathcal{P}_2\varphi(\theta))|\, d\theta \\
&\quad + \int_0^{\jmath} |\varphi(\theta, \mathcal{V}(\theta), \mathcal{P}_1\mathcal{V}(\theta), \mathcal{P}_2\mathcal{V}(\theta)) - \tilde{\varphi}(\theta, \varphi(\theta), \mathcal{P}_1\varphi(\theta), \mathcal{P}_2\varphi(\theta))|\, d\theta \\
&\leq \xi_1 + \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left(\frac{\delta}{1-\delta}T^\delta\right)|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta \\
&\quad + \int_0^{\jmath} \zeta\left(|\mathcal{V}(\theta) - \varphi(\theta)| + \mathscr{C}\,|\mathcal{V}(\theta) - \varphi(\theta)| + \mathcal{D}\,|\mathcal{V}(\theta) - \varphi(\theta)|\right) d\theta + \xi_2 \int_0^{\jmath} d\theta \\
&\leq \xi_1 + \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left(\frac{\delta}{1-\delta}T^\delta\right)|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta + \int_0^{\jmath} \zeta\left(1 + \mathscr{C} + \mathcal{D}\right)|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta + \xi_2 \jmath \\
&\leq \xi_1 + \frac{B(\delta)}{1-\delta}\int_0^{\jmath} E_\delta\left(\frac{\delta}{1-\delta}T^\delta\right)|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta + \int_0^{\jmath} [\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\,|\mathcal{V}(\theta) - \varphi(\theta)|\, d\theta + \xi_2 T \\
&|\mathcal{V}(\jmath) - \varphi(\jmath)| \leq \xi_1 + \xi_2 T + \int_0^{\jmath}\left[\frac{B(\delta)}{1-\delta}E_\delta\left(\frac{\delta}{1-\delta}T^\delta\right) + [\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\right]|\mathcal{V}(\jmath) - \varphi(\theta)|\, d\theta.
\end{aligned}
$$

By Lemma 2.12, we get

$$
|\mathcal{V}(\jmath) - \varphi(\jmath)| \leq (\xi_1 + \xi_2 T) exp\left(\int_0^{\jmath}\left[\frac{B(\delta)}{1-\delta}E_\delta\left(\frac{\delta}{1-\delta}T^\delta\right) + [\mathcal{V}(\jmath) - x(\jmath, y(\jmath))]\right] d\theta\right), \jmath \in \mathscr{I}. \quad (6.4)
$$

□

7. Dependence results on parameters

Let any $\lambda, \lambda_0 \in \mathscr{R}$ and

$$
\frac{d\mathcal{V}}{d\jmath} +_0^{\star} D_{\jmath}^{\delta}[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] = \Theta(\jmath, \mathcal{V}(\jmath), \mathcal{P}_1\mathcal{V}(\jmath), \mathcal{P}_2\mathcal{V}(\jmath), \lambda), \jmath \in \mathscr{I}, \tag{7.1}
$$

$$
\mathcal{V}(0) = \mathcal{V}_0 \in \mathscr{R}. \tag{7.2}
$$

$$
\frac{d\mathcal{V}}{d\jmath} +_0^{\star} D_{\jmath}^{\delta}[\mathcal{V}(\jmath) - x(\jmath, y(\jmath)] = \Theta(\jmath, \mathcal{V}(\jmath), \mathcal{P}_1\mathcal{V}(\jmath), \mathcal{P}_2\mathcal{V}(\jmath), \lambda_0), \jmath \in \mathscr{I}, \tag{7.3}
$$

$$
\mathcal{V}(0) = \mathcal{V}_0 \in \mathscr{R}. \tag{7.4}
$$

Theorem 7.1. *Let the function Θ satisfy Theorem 4.2. Suppose there exists $\omega, u \in \mathscr{C}(\mathscr{I}, \mathscr{R}^+)$ such that,*

$$|\Theta(\jmath,\mathcal{V},\mathcal{P}_1\mathcal{V},\mathcal{P}_2\mathcal{V},\lambda)-\Theta(\jmath,\varphi,\mathcal{P}_1\varphi,\mathcal{P}_2\varphi,\lambda)|\leq\omega(\jmath)\,|\mathcal{V}-\varphi|\,,$$
$$|\Theta(\jmath,\mathcal{V},\mathcal{P}_1\mathcal{V},\mathcal{P}_2\mathcal{V},\lambda)-\Theta(\jmath,\mathcal{V},\mathcal{P}_1\mathcal{V},\mathcal{P}_2\mathcal{V},\lambda_0)|\leq u(\jmath)\,|\lambda-\lambda_0|\,.$$

If $\mathcal{V}_1,\mathcal{V}_2$ are the solutions of Eqs (7.1) and (7.3), then

$$|\mathcal{V}_1(\jmath)-\mathcal{V}_2(\jmath)|\leq\mathcal{P}T\,|\lambda-\lambda_0|\,exp\left(\int_0^{\jmath}\left[\frac{B(\delta)}{1-\delta}E_\delta\left(-\frac{\delta}{1-\delta}T^\delta\right)+[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]\right]d\theta\right),\ \jmath\in\mathscr{I},\quad(7.5)$$

where $\mathcal{P}=\sup_{\jmath\in\mathscr{I}}u(\jmath)$.

Proof. Let, for any $\jmath\in\mathscr{I}$,

$$\begin{aligned}
&|\mathcal{V}_1(\jmath)-\mathcal{V}_2(\jmath)|\\
&\leq\frac{B(\delta)}{1-\delta}\left|\int_0^{\jmath}E_\delta\left[-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right](\mathcal{V}_2(\theta)-\mathcal{V}_1(\theta)d\theta)\right|\\
&\quad+\int_0^{\jmath}|\Theta(\theta,\mathcal{V}_1(\theta),\mathcal{P}_1\mathcal{V}_1(\theta),\mathcal{P}_2\mathcal{V}_1(\theta),\lambda)-\Theta(\theta,\mathcal{V}_2(\theta),\mathcal{P}_1\mathcal{V}_2(\theta),\mathcal{P}_2\mathcal{V}_2(\theta),\lambda_0)|\,d\theta\\
&\leq\frac{B(\delta)}{1-\delta}\int_0^{\jmath}E_\delta\left(\left|-\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right|\right)|\mathcal{V}_1(\theta)-\mathcal{V}_2(\theta)|\,d\theta\\
&\quad+\int_0^{\jmath}|\Theta(\theta,\mathcal{V}_1(\theta),\mathcal{P}_1\mathcal{V}_1(\theta),\mathcal{P}_2\mathcal{V}_1(\theta),\lambda)-\Theta(\theta,\mathcal{V}_2(\theta),\mathcal{P}_1\mathcal{V}_2(\theta),\mathcal{P}_2\mathcal{V}_2(\theta),\lambda)|\,d\theta\\
&\quad+\int_0^{\jmath}|\Theta(\theta,\mathcal{V}_2(\theta),\mathcal{P}_1\mathcal{V}_2(\theta),\mathcal{P}_2\mathcal{V}_2(\theta),\lambda)-\Theta(\theta,\mathcal{V}_2(\theta),\mathcal{P}_1\mathcal{V}_2(\theta),\mathcal{P}_2\mathcal{V}_2(\theta),\lambda_0)|\,d\theta\\
&\leq\frac{B(\delta)}{1-\delta}\int_0^{\jmath}E_\delta\left(\frac{\delta}{1-\delta}(\jmath-\theta)^\delta\right)|\mathcal{V}_1(\theta)-\mathcal{V}_2(\theta)|\,d\theta\\
&\quad+\int_0^{\jmath}\zeta\left(|\mathcal{V}_1(\theta)-\mathcal{V}_2(\theta)|+\mathscr{C}\,|\mathcal{V}_1(\theta)-\mathcal{V}_2(\theta)|+\mathcal{D}\,|\mathcal{V}_1(\theta)-\mathcal{V}_2(\theta)|\right)d\theta+\int_0^{\jmath}u(\theta)\,|\lambda-\lambda_0|\,d\theta\\
&\leq\frac{B(\delta)}{1-\delta}\int_0^{\jmath}E_\delta\left(\frac{\delta}{1-\delta}T\delta\right)|\mathcal{V}_1(\theta)-\mathcal{V}_2(\theta)|\,d\theta+\int_0^{\jmath}\zeta(1+\mathscr{C}+\mathcal{D})\,|\mathcal{V}_1(\theta)-\mathcal{V}_2(\theta)|\,d\theta\\
&\quad+P\jmath\,|\lambda-\lambda_0|\\
&\leq\frac{B(\delta)}{1-\delta}\int_0^{\jmath}E_\delta\left(\frac{\delta}{1-\delta}T\delta\right)|\mathcal{V}_1(\theta)-\mathcal{V}_2(\theta)|\,d\theta+\int_0^{\jmath}[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]\,|\mathcal{V}_1(\theta)-\mathcal{V}_2(\theta)|\,d\theta\\
&\quad+PT\,|\lambda-\lambda_0|\\
&\leq\int_0^{\jmath}\left[\frac{B(\delta)}{1-\delta}E_\delta\left(\frac{\delta}{1-\delta}T\delta\right)+[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]\right]|\mathcal{V}_1(\theta)-\mathcal{V}_2(\theta)|\,d\theta+PT\,|\lambda-\lambda_0|\,.
\end{aligned}$$

By Lemma 2.12,

$$|\mathcal{V}_1(\jmath)-\mathcal{V}_2(\jmath)|\leq\mathcal{P}T\,|\lambda-\lambda_0|\,exp\left(\int_0^{\jmath}\left[\frac{B(\delta)}{1-\delta}E_\delta\left(\frac{\delta}{1-\delta}T^\delta\right)+[\mathcal{V}(\jmath)-x(\jmath,y(\jmath))]\right]d\theta\right),\ \jmath\in\mathscr{I}.\quad(7.6)$$

□

8. Example

Consider a nonlinear ABR fractional derivative with neutral integro-differential equations of the form:

$$\frac{d\mathcal{V}}{d\jmath} +_0^\star D_\jmath^{\frac{1}{2}}[\mathcal{V}(\jmath) - x(\jmath, y(\jmath))] = \varphi(\jmath, \mathcal{V}(\jmath), P_1\mathcal{V}(\jmath), P_2\mathcal{V}(\jmath)), \jmath \in \mathscr{I} = [0,2], \tag{8.1}$$

$$\mathcal{V}(0) = 1 \in \mathscr{R}. \tag{8.2}$$

$\varphi : (\mathscr{I} \times \mathscr{R} \times \mathscr{R} \times \mathscr{R}) \to \mathscr{R}$ is a continuous nonlinear function such that,

$$\varphi(\jmath, \mathcal{V}(\jmath), P_1\mathcal{V}(\jmath), P_2\mathcal{V}(\jmath)) = \frac{|\mathcal{V}(\jmath)| + 1}{3} + \mathcal{M}(\jmath) + \mathcal{N}(\jmath), \jmath \in \mathscr{I},$$

and

$$\mathcal{M}(\jmath) = B\left(\frac{1}{2}\right)\left\{\jmath E_{\frac{1}{2},2}\left(-\jmath^{\frac{1}{2}}\right) + E_{\frac{1}{2}}\left(-\jmath^{\frac{1}{2}}\right) - \jmath - 1\right\},$$

$$\mathcal{N}(\jmath) = B\left(\frac{1}{2}\right)\left\{E_{\frac{1}{2},2}\left(-\jmath^{\frac{1}{2}}\right) + \jmath E_{\frac{1}{2}}\left(-\jmath^{\frac{1}{2}}\right) - 1\right\}.$$

We observe that for all $\mathcal{V}, \varphi \in \mathscr{R}$ and $\jmath \in \mathscr{I}$,

$$\begin{aligned}
&|\varphi(\jmath, \mathcal{V}, P_1\mathcal{V}, P_2\mathcal{V}) - \varphi(\jmath, \varphi, P_1\varphi, P_2\varphi)| \\
&\quad = \left|\left(\frac{|\mathcal{V}(\jmath)| + 1}{3} + \mathcal{M}(\jmath) + \mathcal{N}(\jmath)\right) - \left(\frac{|\varphi(\jmath)| + 1}{3} + M(\jmath) + N(\jmath)\right)\right| \\
&\quad \leq \frac{1}{3}|\mathcal{V} - \varphi|.
\end{aligned} \tag{8.3}$$

The function φ satisfies (H_1)–(H_4) with constant $\frac{1}{3}$. From Theorem 4.2, we have $\delta = \frac{1}{2}$ and T=2 which is substitute in Eq (4.2), and we get

$$B\left(\frac{1}{2}\right) < \frac{1}{8E_{\frac{1}{2},2}\left(2^{\frac{1}{2}}\right)}. \tag{8.4}$$

If the function $B(\delta)$ satisfies Eq (8.4), then Eqs (8.1) and (8.2) have a unique solution.

$$\mathcal{V}(\jmath) = \frac{\jmath}{3} + 1, \jmath \in [0,2]. \tag{8.5}$$

9. Conclusions

In this research article, we explored multi-derivative nonlinear neutral fractional integro-differential equations involving the ABR fractional derivative. The elementary results of the existence, uniqueness and dependence solution on various data are based on the Prabhakar fractional integral operator $\varepsilon^{\alpha}_{\delta,\eta,\mathcal{V};c+}$ involving a generalized ML function. The existence results are obtained by Krasnoselskii's fixed point theorem, and the uniqueness and data dependence results are obtained by the Gronwall-Bellman inequality with continuous functions.

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Conflict of interest

The authors declare no conflict of interest.

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