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Research article

Positive solution for a class of nonlinear fourth-order boundary value problem

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Abstract: In this paper, we are concerned with the existence of positive solutions for boundary value problems of nonlinear fourth-order differential equations

$$u^{(4)} + c(x)u = \lambda a(x)f(u), \quad x \in (0,1),$$
$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

where $a(x)$ may change signs. The proof of main results is based on Leray-Schauder's fixed point theorem and the properties of Green's function of the fourth-order differential operator $L_c u = u^{(4)} + c(x)u$.

Keywords: fourth-order differential operator; positive solution; boundary value problem; Leray-Schauder's fixed point theorem
Mathematics Subject Classification: 34A08, 34B15, 35J05

1. Introduction

Nonlinear mathematical models [1,2] were widely used in many fields. In particular, boundary value problems of nonlinear differential equations have received extensive attention and have been intensively studied in the past thirty years, see [3,4]. We point out that boundary value problems for second order differential equations, see, for example [5–9] and the references therein. While studies about boundary value problems of nonlinear fourth-order differential equations are much more less. One of the earliest papers about boundary value problems of nonlinear fourth-order differential operator is [10] from R. Ma and H. Wang, there they concerned the following problem

$$y^{(4)} - h(x)f(y(x)) = 0, \quad x \in (0,1)$$

with boundary condition

$$y(0) = y(1) = y''(0) = y''(1) = 0$$

or

$$y(0) = y'(1) = y''(0) = y'''(1) = 0.$$

By the fixed point theorem in cone, they proved the existence of positive solutions under the conditions that f is either superlinear or sublinear. In another paper [11], the author obtained the positive solution of the following problem

$$\begin{aligned} &u^{(4)} + \beta u'' - \alpha u = f(t, u), \quad x \in (0, 1) \\ &u(0) = u(1) = u''(0) = u''(1) = 0 \end{aligned}$$

by the fixed point theorem in cone. R.Vrabel [12] studied the upper solution and lower solution of the problem

$$\begin{aligned} &y^{(4)}(x) + \lambda y''(x) = h(x, y(x)), \quad x \in (0, 1) \\ &y(0) = y(1) = y''(0) = y''(1) = 0. \end{aligned}$$

There are many other papers we will not list but we find that they have a common point, that is, the fourth-order differential operators they dealt with can be resolved into composition of two second-order positive linear operators. And therefore, the corresponding Green's function for fourth-order linear operator is the form of the product of two Green's functions for second-order linear operators.

In a recently paper [13], Drábet discussed the existence of positive solutions for the following fourth-order linear problem

$$\begin{aligned} &u^{(4)} + c(x)u = h(x), \quad x \in (0, 1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned}$$

Obviously, the fourth-order differential operator can not be resolution into composition of two second-order positive linear operators. For more results on nonlinear fourth-order differential operator problems we can refer to [14, 15].

Based on the above literature inspiration. We now consider the fourth-order nonlinear equation with Dirichlet boundary conditions

$$u^{(4)} + c(x)u = \lambda a(x) f(u), \tag{1.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $c(x), a(x)$ satisfy some conditions that we will give bellow, especially, $a(x)$ may change signs.

We make the following assumptions throughout the paper:

(A1) $-\pi^4 < c(x) < c_0$,

(A2) $f : \mathbb{R}^+ \to \mathbb{R}$ is continuous, and $f(0) > 0$,

(A3) $a : [0, 1] \to \mathbb{R}$ is continuous with $a(x) \not\equiv 0$, and there exists a constant $K > 0$ such that

$$\int_0^1 G(x, y) a^+(y) dy \geq K \int_0^1 G(x, y) a^-(y) dy$$

for every $x \in (0,1)$, where a^+(resp. a^-) is the positive (resp. negative) part of a, c_0 is the constant given in [13], and $G(x,y)$ is the Green's function of L_c with boundary conditions (1.2).

Our main result is as follows:

Theorem 1.1. *Let (A1)–(A3) hold. Then there exists a positive number* λ^* *such that* (1.1) *and* (1.2) *have a positive solution for* $0 < \lambda < \lambda^*$.

Remark 1.1. *Since the fourth-order differential operator can not be resolution into composition of two second-order positive linear operators, as a result, the Green's function have no explicit expression. So the method or technic used in [10–12] does not work. To deal with the new case and the difficult it brings, we are inspired by the method to second-order elliptic boundary value problems in [8], and the result that the fourth-order operator* $u^{(4)} + c(x)u$ *is strictly inverse positive in [13, 16]. Thanks to the existence and its properties of the Green's function given in [17–19], we obtain the existence of a positive solution to the problems* (1.1) *and* (1.2).

2. Preliminaries

In this section we present two important lemmas. The main method we use is the fixed point theorem of Leray-Schauder type. We refer interested readers to the literature [20, 21].

Set

$$W = \{u \in C^4([0,1]) : u(0) = u(1) = u''(0) = u''(1) = 0\},$$

and let the linear operator $L_c : W \to C([0,1])$ defined by

$$L_c u = u^{(4)} + c(x)u.$$

Then the boundary value problems (1.1) and (1.2) are equivalent to the operator equation

$$L_c u = \lambda a(x) f(u).$$

Lemma 2.1. *Let (A1) hold. Then* L_c *is strictly inverse positive, and therefore it has a positive Green's function.*

Proof. L_c is strictly inverse positive, we can refer to [13, 16] and the reference therein. From the definition of L_c is strictly inverse positive there and the well-known truth that

$$L_c u = h(x)$$

is equivalent to

$$u(x) = \int_0^1 G(x,y)h(y))dy,$$

we can get the positiveness of the Green's function $G(x,y)$ immediately.

Lemma 2.2. *Let (A1)–(A3) hold, and let* $0 < \delta < 1$. *Then there exists a positive number* $\bar{\lambda}$ *such that, for* $0 < \lambda < \bar{\lambda}$*, the problem*

$$u^{(4)} + c(x)u = \lambda a(x)^+ f(u) \tag{2.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0 \tag{2.2}$$

has a positive solution $\tilde{u}_\lambda$ *with* $|\tilde{u}_\lambda| \to 0$ *as* $\lambda \to 0$*, and*

$$\tilde{u}_\lambda(x) \geq \lambda\delta f(0)p(x), \quad x \in (0,1),$$

where $p(x) = \int_0^1 G(x,y)a^+(y)dy$.

Proof. It follows from Lemma 2.1 that L_c is strictly inverse positive, and therefore it has a positive Green's function $G(x,y)$. For each $u \in C([0,1])$, let

$$Au(x) = \lambda \int_0^1 G(x,y)a^+(y)f(u(y))dy, \quad x \in [0,1].$$

Then the fixed points of A are solutions of problems (2.1) and (2.2). We now verify the condition of Leray-Schauder fixed point theorem to show that A has a fixed point for λ small.

Firstly, $A : C([0,1]) \to C([0,1])$ is completely continuous by the assumptions and Arzela-Ascoli theorem.

Secondly, we find a bounded open set Ω with $0 \in \Omega$ in $C([0,1])$, such that for $u \in C(\bar{\Omega})$ and $\theta \in (0,1)$, if $u = \theta Au$, then $u \bar{\in} \partial\Omega$.

By $(A2)$, the function $g(s) = \frac{f(s)}{f(0)}$ is continuous and $g(0) = 1$, since $0 < \delta < 1$, we can choose $\varepsilon > 0$ such that

$$f(s) > \delta f(0) \qquad s \in [0,\varepsilon].$$

Also we have

$$|Au|_0 \leq \lambda|p|_0\tilde{f}(|u|_0) \leq \lambda|p|_0\tilde{f}(\varepsilon), \qquad u \in [0,\varepsilon],$$

where $\tilde{f}(t) = \max_{0\leq s\leq t} f(s)$, and $|\cdot|_0$ is the usual norm in $C([0,1])$.

Suppose $\lambda < \frac{1}{2|p|_0\tilde{f}(\varepsilon)} =: \bar{\lambda}$, then there exists a $A_\lambda \in (0,\varepsilon)$ such that

$$\frac{\tilde{f}(A_\lambda)}{A_\lambda} = \frac{1}{2\lambda|p|_0}.$$

Let $\Omega = \{u \in C([0,1]) : |u|_0 < A_\lambda\}$ and $\theta \in (0,1)$ such that $u = \theta Au$. Then we have

$$|u|_0 \leq |Au|_0 \leq \lambda|p|_0\tilde{f}(|u|_0),$$

or

$$\frac{\tilde{f}(|u|_0)}{|u|_0} \geq \frac{1}{\lambda|p|_0}.$$

So $u \neq A_\lambda$, which means $u \bar{\in} \partial\Omega$.

By the Leray-Schauder fixed point theorem, A has a fixed point $\tilde{u_\lambda}$ in Ω for $0 < \lambda < \bar{\lambda}$, that is, problems (2.1) and (2.2) have a positive solution $\tilde{u}_\lambda$ with $\tilde{u}_\lambda \leq A_\lambda < \varepsilon$. Notice that $A_\lambda \to 0$ as $\lambda \to 0$, $|\tilde{u}_\lambda| \to 0$ as $\lambda \to 0$ and

$$\tilde{u}_\lambda(x) = A\tilde{u}_\lambda(x) \geq \lambda\delta f(0)p(x), \quad x \in (0,1).$$

The proof is completed.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $q(x) = \int_0^1 G(x,y)a^-(y)dy$, recall that $p(x) = \int_0^1 G(x,y)a^+(y)dy$. By $(A2)$,

$$q(x) \le \frac{1}{K}p(x).$$

From the proof of Lemma 2.2 that $g(0) = 1$, there is a $\alpha \in (0,1)$ and we can choose $1 < \sigma < K$, such that $f(s) < \sigma f(0)$, and $\gamma = \frac{\sigma}{K} \in (0,1)$, then we have

$$q(x)f(s) \le \gamma f(0)p(x) \tag{3.1}$$

for $s \in [0,\alpha]$, $x \in (0,1)$. Fix $\delta \in (0,1)$ and let $\lambda^* > 0$ be such that

$$|\tilde{u}_\lambda|_0 + \lambda\delta f(0)|p|_0 \le \alpha \tag{3.2}$$

for $0 < \lambda < \lambda^*$, where $\tilde{u}_\lambda$ is the solution of (2.1) and (2.2) given by Lemma 2.2, and

$$|f(s) - f(t)| \le \frac{\delta - \gamma}{2} \cdot f(0) \tag{3.3}$$

for $s,\ t \in [-\alpha,\alpha]$ with $|s-t| \le \lambda^*\delta f(0)|p|_0$.

Let $0 < \lambda < \lambda^*$, we look for a solution $u_\lambda = \tilde{u}_\lambda + v_\lambda$. Since $\tilde{u}_\lambda$ is the solution of (2.1) and (2.2), then v_λ solves

$$\begin{aligned}
&L_c v_\lambda = \lambda a^+[f(\tilde{u}_\lambda + v_\lambda) - f(\tilde{u}_\lambda)] - \lambda a^- f(\tilde{u}_\lambda + v_\lambda), \quad x \in (0,1),\\
&v_\lambda(0) = v_\lambda(1) = v_\lambda''(0) = v_\lambda''(1) = 0.
\end{aligned}$$

For each $w \in C([0,1])$, let $v = Aw$ be the solution of

$$\begin{aligned}
&L_c v = \lambda a^+[f(\tilde{u}_\lambda + w) - f(\tilde{u}_\lambda)] - \lambda a^- f(\tilde{u}_\lambda + w), \quad x \in (0,1),\\
&v(0) = v(1) = v''(0) = v''(1) = 0,
\end{aligned}$$

where the operator A is as in Lemma 2.2, we have

$$\begin{aligned}
Aw(x) \quad = \quad &\lambda\int_0^1 G(x,y)a^+(y)[f(\tilde{u}_\lambda(y) + w(y)) - f(\tilde{u}_\lambda(y))]dy\\
&-\lambda\int_0^1 G(x,y)a^-(y)f(\tilde{u}_\lambda(y) + w(y))dy, \quad x \in [0,1],
\end{aligned}$$

and A is completely continuous.

Let

$$\bar{\Omega}' = \{v \in C([0,1]); |v|_0 \le \lambda\delta f(0)|p|_0\},$$

if $v \in C(\bar{\Omega}')$ and $\theta \in (0,1)$, such that $v = \theta Av$, that is

$$\begin{aligned} v(x) &= \lambda\theta \int_0^1 G(x,y)a^+(y)[f(\tilde{u}_\lambda(y)+v(y)) - f(\tilde{u}_\lambda(y))]dy \\ &\quad -\lambda\theta \int_0^1 G(x,y)a^-(y)f(\tilde{u}_\lambda(y)+v(y))dy, \quad x \in [0,1], \end{aligned}$$

we are going to show that

$$|v|_0 \neq \lambda\delta f(0)|p|_0.$$

Suppose the contrary that $|v|_0 = \lambda\delta f(0)|p|_0$. Then by (3.2) and (3.3), we get

$$|\tilde{u}_\lambda + v|_0 \leq |\tilde{u}_\lambda|_0 + |v|_0 \leq \alpha,$$

and

$$|f(\tilde{u}_\lambda + v) - f(\tilde{u}_\lambda)|_0 \leq \frac{\delta-\gamma}{2} \cdot f(0),$$

together with (3.1) implies that

$$\begin{aligned} |v(x)| &\leq \lambda \cdot \frac{\delta-\gamma}{2} \cdot f(0)p(x) + \lambda\gamma f(0)p(x) \\ &= \lambda \cdot \frac{\delta+\gamma}{2} \cdot f(0)p(x), \qquad x \in [0,1], \end{aligned} \tag{3.4}$$

and

$$|v|_0 \leq \lambda \cdot \frac{\delta+\gamma}{2} \cdot f(0)|p|_0 < \lambda\delta f(0)|p|_0,$$

a contradiction.

By the Leray-Schauder fixed point theorem, A has a fixed point v_λ in $\bar{\Omega}'$ with $|v_\lambda|_0 \leq \lambda\delta f(0)|p|_0$. Hence v_λ satisfies (3.4), and using Lemma 2.2, we obtain

$$\begin{aligned} u_\lambda(x) &= \tilde{u}_\lambda(x) + v_\lambda(x) \geq \tilde{u}_\lambda(x) - |v_\lambda(x)| \\ &\geq \lambda\delta f(0)p(x) - \lambda \cdot \frac{\delta+\gamma}{2} \cdot f(0)p(x) = \lambda \cdot \frac{\delta-\gamma}{2} \cdot f(0)p(x) > 0. \end{aligned}$$

We have proved that u_λ is a positive solution of (1.1) and (1.2).

4. Conclusions

In this paper, we mainly study the existence of solutions to a class of nonlinear fourth-order Dilrichlet boundary value problems through Leray-Schauder's fixed point theorem, and show the asymptotic behavior of the solution as λ changes. In the future, we can try to construct such solutions, give the properties of the solutions, or study numerical solutions for such problems.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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