AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 997–1013.
DOI:10.3934/math.2023048
Received: 31 August 2022
Revised: 29 September 2022
Accepted: 07 October 2022
Published: 14 October 2022

Research article

Existence and multiplicity of standing wave solutions for perturbed fractional p-Laplacian systems involving critical exponents

Shulin Zhang[1,2,*]

[1] School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China

[2] School of Mathematics, Xuzhou Vocational Technology Academy of Finance and Economics, Xuzhou 221116, China

* **Correspondence:** Email: zhangshulin0228@126.com.

Abstract: In this paper, we investigate the existence of standing wave solutions to the following perturbed fractional p-Laplacian systems with critical nonlinearity

$$\begin{cases} \varepsilon^{ps}(-\Delta)_p^s u + V(x)|u|^{p-2}u = K(x)|u|^{p_s^*-2}u + F_u(x,u,v), \ x \in \mathbb{R}^N, \\ \varepsilon^{ps}(-\Delta)_p^s v + V(x)|v|^{p-2}v = K(x)|v|^{p_s^*-2}v + F_v(x,u,v), \ x \in \mathbb{R}^N. \end{cases}$$

Under some proper conditions, we obtain the existence of standing wave solutions $(u_\varepsilon, v_\varepsilon)$ which tend to the trivial solutions as $\varepsilon \to 0$. Moreover, we get m pairs of solutions for the above system under some extra assumptions. Our results improve and supplement some existing relevant results.

Keywords: perturbed fractional p-Laplacian systems; critical growth; variational methods; standing wave solutions
Mathematics Subject Classification: 35B33, 35J50, 35J60

1. Introduction

In this paper, we discuss the existence and multiplicity of standing wave solutions for the following perturbed fractional p-Laplacian systems with critical nonlinearity

$$\begin{cases} \varepsilon^{ps}(-\Delta)_p^s u + V(x)|u|^{p-2}u = K(x)|u|^{p_s^*-2}u + F_u(x,u,v), \ x \in \mathbb{R}^N, \\ \varepsilon^{ps}(-\Delta)_p^s v + V(x)|v|^{p-2}v = K(x)|v|^{p_s^*-2}v + F_v(x,u,v), \ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where ε is a positive parameter, $N > ps$, $s \in (0,1)$, $p_s^* = \frac{Np}{N-ps}$ and $(-\Delta)_p^s$ is the fractional p-Laplacian operator, which is defined as

$$(-\Delta)_p^s u(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy, \ x \in \mathbb{R}^N,$$

where $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$. The functions $V(x), K(x)$ and $F(x,u,v)$ satisfy the following conditions:

(V_0) $V \in C(\mathbb{R}^N, \mathbb{R}), \min_{x \in \mathbb{R}^N} V(x) = 0$ and there is a constant $b > 0$ such that the set
$V^b := \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure;

(K_0) $K \in C(\mathbb{R}^N, \mathbb{R}), 0 < \inf K \le \sup K < \infty$;

(F_1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ and $F_s(x,s,t), F_t(x,s,t) = o(|s|^{p-1} + |t|^{p-1})$ uniformly in $x \in \mathbb{R}^N$ as $|s| + |t| \to 0$;

(F_2) there exist $C_0 > 0$ and $p < \kappa < p_s^*$ such that
$|F_s(x,s,t)|, |F_t(x,s,t)| \le C_0(1 + |s|^{\kappa-1} + |t|^{\kappa-1})$;

(F_3) there exist $l_0 > 0$, $d > p$ and $\mu \in (p, p_s^*)$ such that $F(x,s,t) \ge l_0(|s|^d + |t|^d)$ and
$0 < \mu F(x,s,t) \le F_s(x,s,t)s + F_t(x,s,t)t$ for all $(x,s,t) \in \mathbb{R}^N \times \mathbb{R}^2$;

(F_4) $F_s(x,-s,t) = -F_s(x,s,t)$ and $F_t(x,s,-t) = -F_t(x,s,t)$ for all $(x,s,t) \in \mathbb{R}^N \times \mathbb{R}^2$.

Conditions $(V_0), (K_0)$, suggested by Ding and Lin [11] in studying perturbed Schrödinger equations with critical nonlinearity, and then was used in [28, 32, 33].

In recent years, a great deal of attention has been focused on the study of standing wave solutions for perturbed fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \tag{1.2}$$

where $s \in (0,1)$, $N > 2\text{s}$ and $\varepsilon > 0$ is a small parameter. It is well known that the solution of (1.2) is closely related to the existence of solitary wave solutions for the following eqation

$$i\varepsilon\omega_t - \varepsilon^2(-\Delta)^s\omega - V(x)\omega + f(\omega) = 0, (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where i is the imaginary unit. $(-\Delta)^s$ is the fractional Laplacian operator which arises in many areas such as physics, phase transitions, chemical reaction in liquids, finance and so on, see [1, 6, 18, 22, 27]. Additionally, Eq (1.2) is a fundamental equation of fractional quantum mechanics. For more details, please see [17, 18].

Equation (1.2) was also investigated extensively under various hypotheses on the potential and the nonlinearity. For example, Floer and Weinstein [12] first considered the existence of single-peak solutions for $N = 1$ and $f(t) = t^3$. They obtained a single-peak solution which concentrates around any given nondegenerate critical point of V. Jin, Liu and Zhang [16] constructed a localized bound-state solution concentrating around an isolated component of the positive minimum point of V, when the nonlinear term $f(u)$ is a general critical nonlinearity. More related results can be seen in [5, 7, 10, 13, 14, 26, 43] and references therein. Recently, Zhang and Zhang [46] obtained the multiplicity and concentration of positive solutions for a class of fractional unbalanced double-phase problems by topological and variational methods. Related to (1.2) with $s = 1$, see [31, 39] for quasilinear Schrödinger equations.

On the other hand, fractional p-Laplacian operator can be regarded as an extension of fractional Laplacian operator. Many researchers consider the following equation

$$\varepsilon^{ps}(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u). \tag{1.3}$$

When $f(x,u) = A(x)|u|^{p_s^*-2}u + h(x,u)$, Li and Yang [21] obtained the existence and multiplicity of weak solutions by variational methods. When $f(x,u) = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u$, under suitable assumptions on nonlinearity and weight functions, Lou and Luo [19] established the existence and multiplicity of positive solutions via variational methods. With regard to the p-fractional Schrödinger-Kirchhoff, Song and Shi [29] considered the following equation with electromagnetic fields

$$\begin{cases} \varepsilon^{ps}M([u]_s^p, A_\varepsilon)(-\Delta)_{p,A_\varepsilon}^s u + V(x)|u|^{p-2}u = |u|^{p_s^*-2}u + h(x,|u|^p)|u|^{p-2}u, x \in \mathbb{R}^N, \\ u(x) \to 0, \text{ as } \to \infty. \end{cases} \tag{1.4}$$

They obtained the existence and multiplicity solutions for (1.4) by using the fractional version of concentration compactness principle and variational methods, see also [24, 25, 34, 35, 38, 41] and references therein. Related to (1.3) with $s = 1$, see [15, 23].

Recently, from a mathematical point of view, (fractional) elliptic systems have been the focus for many researchers, see [2, 8, 9, 20, 30, 37, 42, 44, 45]. As far as we know, there are few results concerned with the (fractional) p-Laplacian systems with a small parameter. In this direction, we cite the work of Zhang and Liu [40], who studied the following p-Laplacian elliptic systems

$$\begin{cases} -\varepsilon^p\Delta_p u + V(x)|u|^{p-2}u = K(x)|u|^{p^*-2}u + H_u(u,v),\ x \in \mathbb{R}^N, \\ -\varepsilon^p\Delta_p v + V(x)|v|^{p-2}v = K(x)|v|^{p^*-2}v + H_v(u,v),\ x \in \mathbb{R}^N. \end{cases} \tag{1.5}$$

By using variational methods, they proved the existence of nontrivial solutions for (1.5) provided that ε is small enough. In [36], Xiang, Zhang and Wei investigated the following fractional p-Laplacian systems without a small parameter

$$\begin{cases} (-\Delta)_p^s u + a(x)|u|^{p-2}u = H_u(x,u,v),\ x \in \mathbb{R}^N, \\ (-\Delta)_q^s v + b(x)|v|^{p-2}v = H_v(x,u,v),\ x \in \mathbb{R}^N. \end{cases} \tag{1.6}$$

Under some suitable conditions, they obtained the existence of nontrivial and nonnegative solutions for (1.6) by using the mountain pass theorem.

Motivated by the aforementioned works, it is natural to ask whether system (1.5) has a nontrivial solution when the p-Laplacian operator is replaced by the fractional p-Laplacian operator. As far as we know, there is no related work in this direction so far. In this paper, we give an affirmative answer to this question considering the existence and multiplicity of standing wave solutions for (1.1).

Now, we present our results of this paper.

Theorem 1.1. Assume that (V_0), (K_0) and (F_1)–(F_3) hold. Then for any $\tau > 0$, there is $\Gamma_\tau > 0$ such that if $\varepsilon < \Gamma_\tau$, system (1.1) has at least one solution $(u_\varepsilon, v_\varepsilon) \to (0,0)$ in W as $\varepsilon \to 0$, where W is stated later, satisfying:

$$\frac{\mu-p}{\mu p}\Big[\int\int_{\mathbb{R}^{2N}}\varepsilon^{ps}\Big(\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^p}{|x-y|^{N+ps}}+\frac{|v_\varepsilon(x)-v_\varepsilon(y)|^p}{|x-y|^{N+ps}}\Big)dxdy + \int_{\mathbb{R}^N}V(x)(|u_\varepsilon|^p+|v_\varepsilon|^p)dx\Big]\le\tau\varepsilon^N$$

and

$$\frac{s}{N}\int_{\mathbb{R}^N}K(x)(|u_\varepsilon|^{p_s^*}+|v_\varepsilon|^{p_s^*})dx+\frac{\mu-p}{p}\int_{\mathbb{R}^N}F(x,u_\varepsilon,v_\varepsilon)dx\le\tau\varepsilon^N.$$

Theorem 1.2. Let (V_0), (K_0) and (F_1)–(F_4) hold. Then for any $m\in\mathbb{N}$ and $\tau>0$ there is $\Gamma_{m\tau}>0$ such that if $\varepsilon<\Gamma_{m\tau}$, system (1.1) has at least m pairs of solutions $(u_\varepsilon, v_\varepsilon)$, which also satisfy the above estimates in Theorem 1.1. Moreover, $(u_\varepsilon, v_\varepsilon)\to(0,0)$ in W as $\varepsilon\to 0$.

Remark 1.1. On one hand, our results extend the results in [40], in which the authors considered the existence of solutions for perturbed p-Laplacian system, i.e., system (1.1) with $s=1$. On the other hand, our results also extend the results in [21] to a class of perturbed fractional p-Laplacian system (1.1).

Remark 1.2. Compared with the results obtained by [12–16], when $\varepsilon\to 0$, the solutions of Theorems 1.1 and 1.2 are close to trivial solutions.

In this paper, our goal is to prove the existence and multiplicity of standing wave solutions for (1.1) by variational approach. The main difficulty lies on the lack of compactness of the energy functional associated to system (1.1) because of unbounded domain $\mathbb{R}^N$ and critical nonlinearity. To overcome this difficulty, we adopt some ideas used in [11] to prove that $(PS)_c$ condition holds.

The rest of this article is organized as follows. In Section 2, we introduce the working space and restate the system in a equivalent form by replacing ε^{-ps} with λ. In Section 3, we study the behavior of $(PS)_c$ sequence. In Section 4, we complete the proof of Theorems 2.1 and 2.2, respectively.

2. Preliminaries

To obtain the existence and multiplicity of standing wave solutions of system (1.1) for small ε, we rewrite (1.1) in a equivalent form. Let $\lambda=\varepsilon^{-ps}$, then system (1.1) can be expressed as

$$\begin{cases}(-\Delta)_p^s u+\lambda V(x)|u|^{p-2}u=\lambda K(x)|u|^{p_s^*-2}u+\lambda F_u(x,u,v),\ x\in\mathbb{R}^N,\\ (-\Delta)_p^s v+\lambda V(x)|v|^{p-2}v=\lambda K(x)|v|^{p_s^*-2}v+\lambda F_v(x,u,v),\ x\in\mathbb{R}^N,\end{cases}\tag{2.1}$$

for $\lambda\to+\infty$.

We introduce the usual fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N):=\{u\in L^p(\mathbb{R}^N):[u]_{s,p}<\infty\}$$

equipped with the norm

$$\|u\|_{s,p}=(|u|^p+[u]_{s,p}^p)^{\frac{1}{p}},$$

where $|\cdot|_p$ is the norm in $L^p(\mathbb{R}^N)$ and

$$[u]_{s,p}=\Big(\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\Big)^{\frac{1}{p}}$$

is the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$. In this paper, we continue to work in the following subspace of $W^{s,p}(\mathbb{R}^N)$ which is defined by

$$W_\lambda := \{u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda V(x)|u|^p dx < \infty, \lambda > 0\}$$

with the norm

$$\|u\|_\lambda = ([u]_{s,p}^p + \int_{\mathbb{R}^N} \lambda V(x)|u|^p dx)^{\frac{1}{p}}.$$

Notice that the norm $\|\cdot\|_{s,p}$ is equivalent to $\|\cdot\|_\lambda$ for each $\lambda > 0$. It follows from (V_0) that W_λ continuously embeds in $W^{s,p}(\mathbb{R}^N)$. For the fractional system (2.1), we shall work in the product space $W = W_\lambda \times W_\lambda$ with the norm $\|(u,v)\|^p = \|u\|_\lambda^p + \|v\|_\lambda^p$ for any $(u,v) \in W$.

We recall that $(u,v) \in W$ is a weak solution of system (2.1) if

$$\begin{aligned}
&\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}} dxdy + \lambda \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\phi dx \\
&\quad + \int\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+ps}} dxdy + \lambda \int_{\mathbb{R}^N} V(x)|v|^{p-2}v\psi dx \\
&= \lambda \int_{\mathbb{R}^N} K(x)(|u|^{p_s^*-2}u\phi + |v|^{p_s^*-2}v\psi)dx + \lambda \int_{\mathbb{R}^N} (F_u(x,u,v)\phi + F_v(x,u,v)\psi)dx
\end{aligned}$$

for all $(\phi, \psi) \in W$.

Note that the energy functional associated with (2.1) is defined by

$$\begin{aligned}
\Phi_\lambda(u,v) &= \frac{1}{p}\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \frac{1}{p}\int_{\mathbb{R}^N} \lambda V(x)|u|^p dx + \frac{1}{p}\int\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N} \lambda V(x)|v|^p dx - \frac{\lambda}{p_s^*}\int_{\mathbb{R}^N} K(x)(|u|^{p_s^*} + |v|^{p_s^*})dx - \lambda \int_{\mathbb{R}^N} F(x,u,v)dx \\
&= \frac{1}{p}\|(u,v)\|^p - \frac{\lambda}{p_s^*}\int_{\mathbb{R}^N} K(x)(|u|^{p_s^*} + |v|^{p_s^*})dx - \lambda \int_{\mathbb{R}^N} F(x,u,v)dx.
\end{aligned}$$

Clearly, it is easy to check that $\Phi_\lambda \in C^1(W, \mathbb{R})$ and its critical points are weak solution of system (2.1).

In order to prove Theorem 1.1 and 1.2, we only need to prove the following results.

Theorem 2.1. Assume that (V_0), (K_0) and (F_1)–(F_3) hold. Then for any $\tau > 0$, there is $\Lambda_\tau > 0$ such that if $\lambda \geq \Lambda_\tau$, system (2.1) has at least one solution $(u_\lambda, v_\lambda) \to (0,0)$ in W as $\lambda \to \infty$, satisfying:

$$\begin{aligned}
&\frac{\mu - p}{\mu p}\Big[\int\int_{\mathbb{R}^{2N}} \Big(\frac{|u_\lambda(x)-u_\lambda(y)|^p}{|x-y|^{N+ps}} + \frac{|v_\lambda(x)-v_\lambda(y)|^p}{|x-y|^{N+ps}}\Big)dxdy \\
&\qquad + \int_{\mathbb{R}^N} \lambda V(x)(|u_\lambda|^p + |v_\lambda|^p)dx\Big] \leq \tau \lambda^{1-\frac{N}{ps}}
\end{aligned} \tag{2.2}$$

and

$$\frac{s}{N}\int_{\mathbb{R}^N} K(x)(|u_\lambda|^{p_s^*} + |v_\lambda|^{p_s^*})dx + \frac{\mu - p}{p}\int_{\mathbb{R}^N} F(x,u_\lambda,v_\lambda)dx \leq \tau \lambda^{-\frac{N}{ps}}. \tag{2.3}$$

Theorem 2.2. Assume that (V_0), (K_0) and (F_1)–(F_4) hold. Then for any $m \in \mathbb{N}$ and $\tau > 0$ there is $\Lambda_{m\tau} > 0$ such that if $\lambda \geq \Lambda_{m\tau}$, system (2.1) has at least m pairs of solutions (u_λ, v_λ), which also satisfy the estimates in Theorem 2.1. Moreover, $(u_\lambda, v_\lambda) \to (0,0)$ in W as $\lambda \to \infty$.

3. Behaviors of $(PS)_c$ sequences

In this section, we are focused on the compactness of the functional Φ_λ.

Recall that a sequence $\{(u_n, v_n)\} \subset W$ is a $(PS)_c$ sequence at level c, if $\Phi_\lambda(u_n, v_n) \to c$ and $\Phi'_\lambda(u_n, v_n) \to 0$. Φ_λ is said to satisfy the $(PS)_c$ condition if any $(PS)_c$ sequence contains a convergent subsequence.

Proposition 3.1. Assume that the conditions $(V_0), (K_0)$ and (F_1)–(F_3) hold. Then there exists a constant $\alpha > 0$ independent of λ such that, for any $(PS)_c$ sequence $\{(u_n, v_n)\} \subset W$ for Φ_λ with $(u_n, v_n) \rightharpoonup (u, v)$, either $(u_n, v_n) \to (u, v)$ or $c - \Phi_\lambda(u, v) \geq \alpha\lambda^{1-\frac{N}{ps}}$.

Corollary 3.1. Under the assumptions of Proposition 3.1, Φ_λ satisfies the $(PS)_c$ condition for all $c < \alpha\lambda^{1-\frac{N}{ps}}$.

The proof of Proposition 3.1 consists of a series of lemmas which will occupy the rest of this section.

Lemma 3.1. Assume that $(V_0), (K_0)$ and (F_3) are satisfied. Let $\{(u_n, v_n)\} \subset W$ be a $(PS)_c$ sequence for Φ_λ. Then $c \geq 0$ and $\{(u_n, v_n)\}$ is bounded in W.

Proof. Let $\{(u_n, v_n)\}$ be a $(PS)_c$ sequence for Φ_λ, we obtain that

$$\Phi_\lambda(u_n, v_n) \to c,\ \Phi'_\lambda(u_n, v_n) \to 0,\ n \to \infty.$$

By (K_0) and (F_3), we deduce that

$$\begin{aligned} c + o(1)\|(u_n, v_n)\| &= \Phi_\lambda(u_n, v_n) - \frac{1}{\mu}\langle \Phi'_\lambda(u_n, v_n), (u_n, v_n)\rangle \\ &= (\frac{1}{p} - \frac{1}{\mu})\|(u_n, v_n)\|^p + \lambda(\frac{1}{\mu} - \frac{1}{p_s^*})\int_{\mathbb{R}^N} K(x)(|u|^{p_s^*} + |v|^{p_s^*})dx \\ &\quad + \lambda\int_{\mathbb{R}^N}\Big[\frac{1}{\mu}\Big(F_u(x, u_n, v_n)u_n + F_v(x, u_n, v_n)v_n\Big) - F(x, u_n, v_n)\Big]dx \\ &\geq (\frac{1}{p} - \frac{1}{\mu})\|(u_n, v_n)\|^p, \end{aligned} \tag{3.1}$$

which implies that there exists $M > 0$ such that

$$\|(u_n, v_n)\|^p \leq M.$$

Thus, $\{(u_n, v_n)\}$ is bounded in W. Taking the limit in (3.1), we show that $c \geq 0$. This completes the proof. □

From the above lemma, there exists $(u, v) \in W$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in W. Furthermore, passing to a subsequence, we have $u_n \to u$ and $v_n \to v$ in $L^\gamma_{loc}(\mathbb{R}^N)$ for any $\gamma \in [p, p_s^*)$ and $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. Clearly, (u, v) is a critical point of Φ_λ.

Lemma 3.2. Let $\{(u_n, v_n)\}$ be stated as in Lemma 3.1 and $\gamma \in [p, p_s^*)$. Then there exists a subsequence $\{(u_{n_j}, v_{n_j})\}$ such that for any $\varepsilon > 0$, there is $r_\varepsilon > 0$ with

$$\limsup_{j\to\infty}\int_{B_j\setminus B_r}|u_{n_j}|^\gamma dx \leq \varepsilon,\ \limsup_{j\to\infty}\int_{B_j\setminus B_r}|v_{n_j}|^\gamma dx \leq \varepsilon,$$

for all $r \geq r_\varepsilon$, where, $B_r := \{x \in \mathbb{R}^N : |x| \leq r\}$.

Proof. The proof is similar to the one of Lemma 3.2 of [11]. We omit it here. □

Let $\sigma : [0,\infty) \to [0,1]$ be a smooth function satisfying $\sigma(t) = 1$ if $t \le 1$, $\sigma(t) = 0$ if $t \ge 2$. Define $\overline{u}_j(x) = \sigma(\frac{2|x|}{j})u(x)$, $\overline{v}_j(x) = \sigma(\frac{2|x|}{j})v(x)$. It is clear that

$$\|u - \overline{u}_j\|_\lambda \to 0 \text{ and } \|v - \overline{v}_j\|_\lambda \to 0 \text{ as } j \to \infty. \tag{3.2}$$

Lemma 3.3. Let $\{(u_{n_j}, v_{n_j})\}$ be stated as in Lemma 3.2, then

$$\lim_{j\to\infty}\int_{\mathbb{R}^N}\Big[F_u(x,u_{n_j},v_{n_j}) - F_u(x,u_{n_j}-\overline{u}_j, v_{n_j}-\overline{v}_j) - F_u(x,\overline{u}_j,\overline{v}_j)\Big]\phi dx = 0$$

and

$$\lim_{j\to\infty}\int_{\mathbb{R}^N}\Big[F_v(x,u_{n_j},v_{n_j}) - F_v(x,u_{n_j}-\overline{u}_j, v_{n_j}-\overline{v}_j) - F_v(x,\overline{u}_j,\overline{v}_j)\Big]\psi dx = 0$$

uniformly in $(\phi,\psi) \in W$ with $\|(\phi,\psi)\| \le 1$.

Proof. By (3.2) and the local compactness of Sobolev embedding, we know that for any $r > 0$,

$$\lim_{j\to\infty}\int_{B_r}\Big[F_u(x,u_{n_j},v_{n_j}) - F_u(x,u_{n_j}-\overline{u}_j, v_{n_j}-\overline{v}_j) - F_u(x,\overline{u}_j,\overline{v}_j)\Big]\phi dx = 0, \tag{3.3}$$

uniformly for $\|\phi\| \le 1$. For any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$\limsup_{j\to\infty}\int_{B_j\backslash B_r}|\overline{u}_j|^\gamma dx \le \int_{\mathbb{R}^N\backslash B_r}|u|^\gamma dx \le \varepsilon,$$

for all $r \ge r_\varepsilon$, see [Lemma 3.2, 11]. From (F_1) and (F_2), we obtain

$$|F_u(x,u,v)| \le C_0(|u|^{p-1} + |v|^{p-1} + |u|^{\kappa-1} + |v|^{\kappa-1}). \tag{3.4}$$

Thus, from (3.3), (3.4) and the Hölder inequality, we have

$$\begin{aligned}
&\limsup_{j\to\infty}\int_{\mathbb{R}^N}\Big[F_u(x,u_{n_j},v_{n_j}) - F_u(x,u_{n_j}-\overline{u}_j, v_{n_j}-\overline{v}_j) - F_u(x,\overline{u}_j,\overline{v}_j)\Big]\phi dx\\
&\le \limsup_{j\to\infty}\int_{B_j\backslash B_r}\Big[F_u(x,u_{n_j},v_{n_j}) - F_u(x,u_{n_j}-\overline{u}_j, v_{n_j}-\overline{v}_j) - F_u(x,\overline{u}_j,\overline{v}_j)\Big]\phi dx\\
&\le C_1\limsup_{j\to\infty}\int_{B_j\backslash B_r}\Big[(|u_{n_j}|^{p-1} + |\overline{u}_j|^{p-1} + |v_{n_j}|^{p-1} + |\overline{v}_j|^{p-1})\Big]\phi dx\\
&\quad + \le C_2\limsup_{j\to\infty}\int_{B_j\backslash B_r}\Big[(|u_{n_j}|^{\kappa-1} + |\overline{u}_j|^{\kappa-1} + |v_{n_j}|^{\kappa-1} + |\overline{v}_j|^{\kappa-1})\Big]\phi dx\\
&\le C_1\limsup_{j\to\infty}\Big[|u_{n_j}|^{p-1}_{L^p(B_j\backslash B_r)} + |\overline{u}_j|^{p-1}_{L^p(B_j\backslash B_r)} + |v_{n_j}|^{p-1}_{L^p(B_j\backslash B_r)} + |\overline{v}_j|^{p-1}_{L^p(B_j\backslash B_r)}\Big]|\phi|_p\\
&\quad + C_2\limsup_{j\to\infty}\Big[|u_{n_j}|^{\kappa-1}_{L^\kappa(B_j\backslash B_r)} + |\overline{u}_j|^{\kappa-1}_{L^\kappa(B_j\backslash B_r)} + |v_{n_j}|^{\kappa-1}_{L^\kappa(B_j\backslash B_r)} + |\overline{v}_j|^{\kappa}_{L^\kappa(B_j\backslash B_r)}\Big]|\phi|_\kappa\\
&\le C_3\varepsilon^{\frac{p-1}{p}} + C_4\varepsilon^{\frac{\kappa-1}{\kappa}},
\end{aligned}$$

where C_1, C_2, C_3 and C_4 are positive constants. Similarly, we can deduce that the other equality also holds. □

Lemma 3.4. Let $\{(u_{n_j}, v_{n_j})\}$ be stated as in Lemma 3.2, the following facts hold:

(i) $\Phi_\lambda(u_{n_j} - \overline{u}_j, v_{n_j} - \overline{v}_j) \to c - \Phi_\lambda(u, v)$;
(ii) $\Phi'_\lambda(u_{n_j} - \overline{u}_j, v_{n_j} - \overline{v}_j) \to 0$ in W^{-1} (the dual space of W).

Proof. (i) We have

$$\begin{aligned}&\Phi_\lambda(u_{n_j} - \overline{u}_j, v_{n_j} - \overline{v}_j)\\ &= \Phi_\lambda(u_{n_j}, v_{n_j}) - \Phi_\lambda(\overline{u}_j, \overline{v}_j)\\ &\quad + \frac{\lambda}{p_s^*}\int_{\mathbb{R}^N} K(x)\Big(|u_{n_j}|^{p_s^*} - |u_{n_j} - \overline{u}_j|^{p_s^*} - |\overline{u}_j|^{p_s^*} + |v_{n_j}|^{p_s^*} - |v_{n_j} - \overline{v}_j|^{p_s^*} - |\overline{v}_j|^{p_s^*}\Big)dx\\ &\quad + \lambda\int_{\mathbb{R}^N}\Big(F(x, u_{n_j}, v_{n_j}) - F(x, u_{n_j} - \overline{u}_j, v_{n_j} - \overline{v}_j) - F(x, \overline{u}_j, \overline{v}_j)\Big)dx.\end{aligned}$$

Using (3.2) and the Brézis-Lieb Lemma [4], it is easy to get

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} K(x)\Big(|u_{n_j}|^{p_s^*} - |u_{n_j} - \overline{u}_j|^{p_s^*} - |\overline{u}_j|^{p_s^*} + |v_{n_j}|^{p_s^*} - |v_{n_j} - \overline{v}_j|^{p_s^*} - |\overline{v}_j|^{p_s^*}\Big)dx = 0$$

and

$$\lim_{j\to\infty}\int_{\mathbb{R}^N}\Big(F(x, u_{n_j}, v_{n_j}) - F(x, u_{n_j} - \overline{u}_j, v_{n_j} - \overline{v}_j) - F(x, \overline{u}_j, \overline{v}_j)\Big)dx = 0.$$

Using the fact that $\Phi_\lambda(u_{n_j}, v_{n_j}) \to c$ and $\Phi_\lambda(\overline{u}_j, \overline{v}_j) \to \Phi_\lambda(u, v)$ as $j \to \infty$, we have

$$\Phi_\lambda(u_{n_j} - \overline{u}_j, v_{n_j} - \overline{v}_j) \to c - \Phi_\lambda(u, v).$$

(ii) We observe that for any $(\phi, \psi) \in W$ satisfying $\|(\phi, \psi)\| \le 1$,

$$\begin{aligned}&\langle\Phi'_\lambda(u_{n_j} - \overline{u}_j, v_{n_j} - \overline{v}_j), (\phi, \psi)\rangle\\ &= \langle\Phi'_\lambda(u_{n_j}, v_{n_j}), (\phi, \psi)\rangle - \langle\Phi'_\lambda(\overline{u}_j, \overline{v}_j), (\phi, \psi)\rangle\\ &\quad + \lambda\int_{\mathbb{R}^N} K(x)\Big[\Big(|u_{n_j}|^{p_s^*-2}u_{n_j} - |u_{n_j} - \overline{u}_j|^{p_s^*-2}(u_{n_j} - \overline{u}_j) - |\overline{u}_j|^{p_s^*-2}\overline{u}_j\Big)\phi\\ &\quad + \Big(|v_{n_j}|^{p_s^*-2}v_{n_j} - |v_{n_j} - \overline{v}_j|^{p_s^*-2}(v_{n_j} - \overline{v}_j) - |\overline{v}_j|^{p_s^*-2}\overline{v}_j\Big)\psi\Big]dx\\ &\quad + \lambda\int_{\mathbb{R}^N}\Big[\Big(F_u(x, u_{n_j}, v_{n_j}) - F_u(x, u_{n_j} - \overline{u}_j, v_{n_j} - \overline{v}_j) - F_u(x, \overline{u}_j, \overline{v}_j)\Big)\phi\\ &\quad + \Big(F_v(x, u_{n_j}, v_{n_j}) - F_v(x, u_{n_j} - \overline{u}_j, v_{n_j} - \overline{v}_j) - F_v(x, \overline{u}_j, \overline{v}_j)\Big)\psi\Big]dx.\end{aligned}$$

It follows from a standard argument that

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} K(x)\Big(|u_{n_j}|^{p_s^*-2}u_{n_j} - |u_{n_j} - \overline{u}_j|^{p_s^*-2}(u_{n_j} - \overline{u}_j) - |\overline{u}_j|^{p_s^*-2}\overline{u}_j\Big)\phi dx = 0$$

and

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} K(x)\Big(|v_{n_j}|^{p_s^*-2}v_{n_j} - |v_{n_j} - \overline{v}_j|^{p_s^*-2}(v_{n_j} - \overline{v}_j) - |\overline{v}_j|^{p_s^*-2}\overline{v}_j\Big)\psi dx = 0$$

uniformly in $\|(\phi,\psi)\| \leq 1$. By Lemma 3.3, we obtain $\Phi'_\lambda(u_{n_j}-\overline{u}_j, v_{n_j}-\overline{v}_j) \to 0$. We complete this proof. □

Set $u_j^1 = u_{n_j}-\overline{u}_j$, $v_j^1 = v_{n_j}-\overline{v}_j$, then $u_{n_j}-u = u_j^1+(\overline{u}_j-u)$, $v_{n_j}-v = v_j^1+(\overline{v}_j-v)$. From (3.2), we have $(u_{n_j}, v_{n_j}) \to (u,v)$ if and only if $(u_j^1, v_j^1) \to (0,0)$. By Lemma 3.4, one has along a subsequence that $\Phi_\lambda(u_j^1, v_j^1) \to c-\Phi_\lambda(u,v)$ and $\Phi'_\lambda(u_j^1, v_j^1) \to 0$.

Note that $\langle \Phi'_\lambda(u_j^1, v_j^1), (u_j^1, v_j^1)\rangle = 0$, by computation, we get

$$\begin{aligned}\int\int_{\mathbb{R}^{2N}} \frac{|u_j^1(x)-u_j^1(y)|^p}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N} \lambda V(x)|u_j^1|^p dx + \int\int_{\mathbb{R}^{2N}} \frac{|v_j^1(x)-v_j^1(y)|^p}{|x-y|^{N+ps}}dxdy \\ + \int_{\mathbb{R}^N} \lambda V(x)|v_j^1|^p dx - \lambda\int_{\mathbb{R}^N} K(x)(|u_j^1|^{p_s^*}+|v_j^1|^{p_s^*})dx - \lambda\int_{\mathbb{R}^N} F(x,u_j^1,v_j^1)dx = 0\end{aligned} \tag{3.5}$$

Hence, by (F_3) and (3.5), we have

$$\begin{aligned}&\Phi_\lambda(u_j^1, v_j^1) - \frac{1}{p}\langle \Phi'_\lambda(u_j^1, v_j^1), (u_j^1, v_j^1)\rangle \\ &= (\frac{1}{p}-\frac{1}{p_s^*})\lambda\int_{\mathbb{R}^N} K(x)(|u_j^1|^{p_s^*}+|v_j^1|^{p_s^*})dx \\ &\quad + \lambda\int_{\mathbb{R}^N} \Big[\frac{1}{p}\Big(F_u(x,u_j^1,v_j^1)u_j^1 + F_u(x,u_j^1,v_j^1)v_j^1\Big) - F(x,u_j^1,v_j^1)\Big]dx \\ &\geq \frac{\lambda s K_{min}}{N}\int_{\mathbb{R}^N}\Big(|u_j^1|^{p_s^*}+|v_j^1|^{p_s^*}\Big)dx,\end{aligned}$$

where $K_{min} = \inf_{x\in\mathbb{R}^N} K(x) > 0$. So, it is easy to see that

$$|u_j^1|_{p_s^*}^{p_s^*} + |v_j^1|_{p_s^*}^{p_s^*} \leq \frac{N(c-\Phi_\lambda(u,v))}{\lambda s K_{min}} + o(1). \tag{3.6}$$

Denote $V_b(x) = \max\{V(x), b\}$, where b is the positive constant from assumption of (V_0). Since the set V^b has finite measure and $(u_j^1, v_j^1) \to (0,0)$ in $L^p_{loc}\times L^p_{loc}$, we obtain

$$\int_{\mathbb{R}^N} V(x)(|u_j^1|^p + |v_j^1|^p)dx = \int_{\mathbb{R}^N} V_b(x)(|u_j^1|^p+|v_j^1|^p)dx + o(1). \tag{3.7}$$

By (K_0), (F_1) and (F_2), we can find a constant $C_b > 0$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N} K(x)(|u_j^1|^{p_s^*}+|v_j^1|^{p_s^*})dx + \int_{\mathbb{R}^N}(F_u(x,u_j^1,v_j^1)u_j^1 + F_v(x,u_j^1,v_j^1)v_j^1)dx \\ &\leq b(|u_j^1|_p^p + |v_j^1|_p^p) + C_b(|u_j^1|_{p_s^*}^{p_s^*}+|v_j^1|_{p_s^*}^{p_s^*}).\end{aligned} \tag{3.8}$$

Let S is fractional Sobolev constant which is defined by

$$S|u|_{p_s^*}^p \leq \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy \text{ for all } u\in W^{s,p}(\mathbb{R}^N). \tag{3.9}$$

Proof of Proposition 3.1. Assume that $(u_{n_j}, v_{n_j}) \nrightarrow (u,v)$, then $\liminf_{j\to\infty}\|(u_j^1, v_j^1)\| > 0$ and $c-\Phi_\lambda(u,v) > 0$.

From (3.5), (3.7), (3.8) and (3.9), we deduce

$$
\begin{aligned}
S(|u_j^1|_{p_s^*}^p + |v_j^1|_{p_s^*}^p) \leq & \int\int_{\mathbb{R}^{2N}} \frac{|u_j^1(x) - u_j^1(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} \lambda V(x)|u_j^1|^p dx + \int\int_{\mathbb{R}^{2N}} \frac{|v_j^1(x) - v_j^1(y)|^p}{|x-y|^{N+ps}} dxdy \\
& + \int_{\mathbb{R}^N} \lambda V(x)|v_j^1|^p dx - \int_{\mathbb{R}^N} \lambda V(x)(|u_j^1|^p + |v_j^1|^p) dx \\
= & \lambda \int_{\mathbb{R}^N} K(x)(|u_j^1|^{p_s^*} + |v_j^1|^{p_s^*}) dx + \lambda \int_{\mathbb{R}^N} (F_u(x, u_j^1, v_j^1) u_j^1 + F_v(x, u_j^1, v_j^1) v_j^1) dx \\
& - \lambda \int_{\mathbb{R}^N} V_b(x)(|u_j^1|^p + |v_j^1|^p) dx \\
\leq & \lambda C_b(|u_j^1|_{p_s^*}^{p_s^*} + |v_j^1|_{p_s^*}^{p_s^*}) + o(1).
\end{aligned}
$$

Thus, by (3.6), we have

$$
S \leq \lambda C_b \Big(|u_j^1|_{p_s^*}^{p_s^*} + |v_j^1|_{p_s^*}^{p_s^*}\Big)^{\frac{p_s^*-p}{p_s^*}} + o(1) \leq \lambda C_b \Big(\frac{N(c - \Phi_\lambda(u,v))}{\lambda s K_{min}}\Big)^{\frac{s}{N}} + o(1),
$$

or equivalently

$$
\alpha \lambda^{1-\frac{N}{ps}} \leq c - \Phi_\lambda(u,v),
$$

where $\alpha = \frac{sK_{min}}{N}(\frac{S}{C_b})^{\frac{N}{ps}}$. The proof is complete. □

4. Proof of the main results

Lemma 4.1. Suppose that (V_0), (K_0), (F_1), (F_2) and (F_3) are satisfied, then the functional Φ_λ satisfies the following mountain pass geometry structure:

(i) there exist positive constants ρ and a such that $\Phi_\lambda(u,v) \geq a$ for $\|(u,v)\| = \rho$;

(ii) for any finite-dimensional subspace $Y \subset W$,

$$
\Phi_\lambda(u,v) \to -\infty, \quad \text{as } (u,v) \in W, \ \|(u,v)\| \to +\infty.
$$

(iii) for any $\tau > 0$ there exists $\Lambda_\tau > 0$ such that each $\lambda \geq \Lambda_\tau$, there exists $\widetilde{\omega}_\lambda \in Y$ with $\|\widetilde{\omega}_\lambda\| > \rho$, $\Phi_\lambda(\widetilde{\omega}_\lambda) \leq 0$ and

$$
\max_{t \geq 0} \Phi_\lambda(t\widetilde{\omega}_\lambda) \leq \tau \lambda^{1-\frac{N}{ps}}.
$$

Proof. (i) From (F_1), (F_2), we have for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$
\frac{1}{p_s^*} \int_{\mathbb{R}^N} K(x)(|u|^{p_s^*} + |v|^{p_s^*}) dx + \int_{\mathbb{R}^N} F(x,u,v) dx \leq \varepsilon |(u,v)|_p^p + C_\varepsilon |(u,v)|_{p_s^*}^{p_s^*}. \tag{4.1}
$$

Thus, combining with (4.1) and Sobolev inequality, we deduce that

$$
\Phi_\lambda(u,v) = \frac{1}{p}\|(u,v)\|^p - \frac{\lambda}{p_s^*} \int_{\mathbb{R}^N} K(x)(|u|^{p_s^*} + |v|^{p_s^*}) dx - \lambda \int_{\mathbb{R}^N} F(x,u,v) dx
$$

$$\geq \frac{1}{p}\|(u,v)\|^p - \lambda\varepsilon C_5\|(u,v)\|^p - \lambda C_6 C_\varepsilon\|(u,v)\|^{p_s^*},$$

where ε is small enough and $C_5, C_6 > 0$, thus (i) is proved because $p_s^* > p$.

(ii) By (F_3), we define the functional $\Psi_\lambda \in C^1(W,\mathbb{R})$ by

$$\Psi_\lambda(u,v) = \frac{1}{p}\|(u,v)\|^p - \lambda l_0 \int_{\mathbb{R}^N}(|u|^d + |v|^d)dx.$$

Then

$$\Phi_\lambda(u,v) \leq \Psi_\lambda(u,v), \text{ for all } (u,v) \in W.$$

For any finite-dimensional subspace $Y \subset W$, we only need to prove

$$\Psi_\lambda(u,v) \to -\infty, \text{ as } (u,v) \in Y, \ \|(u,v)\| \to +\infty.$$

In fact, we have

$$\Psi_\lambda(u,v) = \frac{1}{p}\|(u,v)\|^p - \lambda l_0|(u,v)|_d^d.$$

Since all norms in a finite dimensional space are equivalent and $p < d < p_s^*$, thus (ii) holds.

(iii) From Corollary 3.1, for λ large and c small enough, Φ_λ satisfies $(PS)_c$ condition. Thus, we will find a special finite dimensional-subspace by which we construct sufficiently small minimax levels for Φ_λ when λ large enough.

Recall that

$$\inf\Big\{\int_{\mathbb{R}^{2N}}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}dxdy : \varphi \in C_0^\infty(\mathbb{R}^N), |\varphi|_d = 1\Big\} = 0, \ p < d < p_s^*,$$

see [40] for this proof. For any $0 < \varepsilon < 1$, we can take $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ with $|\varphi_\varepsilon|_d = 1$, supp $\varphi_\varepsilon \subset B_{r_\varepsilon}(0)$ and $[\varphi_\varepsilon]_{p,s}^p < \varepsilon$.

Let

$$\overline{\omega}_\lambda(x) := (\omega_\lambda(x), \omega_\lambda(x)) = (\varphi_\varepsilon(\lambda^{\frac{1}{ps}}x), \varphi_\varepsilon(\lambda^{\frac{1}{ps}}x)).$$

For $t \geq 0$, (F_3) imply that

$$\begin{aligned}
\Phi_\lambda(t\overline{\omega}_\lambda) &\leq \frac{2t^p}{p}\int\int_{\mathbb{R}^{2N}}\frac{|\omega_\lambda(x)-\omega_\lambda(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{2t^p}{p}\int_{\mathbb{R}^N}\lambda V(x)|\omega_\lambda|^p dx - \lambda\int_{\mathbb{R}^N}F(x,t\omega_\lambda,t\omega_\lambda)dx \\
&\leq \lambda^{1-\frac{N}{ps}}\Big\{\frac{2t^p}{p}\int\int_{\mathbb{R}^{2N}}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{2t^p}{p}\int_{\mathbb{R}^N}V(\lambda^{-\frac{1}{ps}}x)|\varphi_\varepsilon|^p dx - 2l_0t^d\int_{\mathbb{R}^N}|\varphi_\varepsilon|^d dx\Big\} \\
&\leq \lambda^{1-\frac{N}{ps}}\frac{2l_0(d-p)}{p}\Big(\frac{\int\int_{\mathbb{R}^{2N}}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^p}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N}V(\lambda^{-\frac{1}{ps}}x)|\varphi_\varepsilon|^p dx}{l_0 d}\Big)^{\frac{d}{d-p}}.
\end{aligned}$$

Indeed, for $t > 0$, define

$$g(t) = \frac{2t^p}{p}\int\int_{\mathbb{R}^{2N}}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{2t^p}{p}\int_{\mathbb{R}^N}\lambda V(\lambda^{-\frac{1}{ps}}x)|\varphi_\varepsilon|^p dx - 2l_0t^d\int_{\mathbb{R}^N}|\varphi_\varepsilon|^d dx.$$

It is easy to show that $t_0 = (\frac{\int\int_{\mathbb{R}^{2N}} \frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^p}{|x-y|^{N+ps}}dxdy+\int_{\mathbb{R}^N} V(\lambda^{-\frac{1}{ps}}x)|\varphi_\varepsilon|^p dx}{l_0 d})^{\frac{1}{d-p}}$ is a maximum point of g and

$$\max_{t\geq 0} g(t) = g(t_0) = \frac{2l_0(d-p)}{p}\Big(\frac{\int\int_{\mathbb{R}^{2N}} \frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^p}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N} V(\lambda^{-\frac{1}{ps}}x)|\varphi_\varepsilon|^p dx}{l_0 d}\Big)^{\frac{d}{d-p}}.$$

Since $V(0) = 0$ and supp $\varphi_\varepsilon \subset B_{r_\varepsilon}(0)$, there exists $\Lambda_\varepsilon > 0$ such that

$$V(\lambda^{-\frac{1}{ps}}x) < \frac{\varepsilon}{|\varphi_\varepsilon|_p^p},\ \forall |x| \leq r_\varepsilon,\ \lambda > \Lambda_\varepsilon.$$

Hence, we have

$$\max_{t\geq 0} \Phi_\lambda(t\varpi_\lambda) \leq \frac{2l_0(d-p)}{p}(\frac{1}{l_0 d})^{\frac{d}{d-p}}(2\varepsilon)^{\frac{d}{d-p}}\lambda^{1-\frac{N}{ps}}, \forall \lambda > \Lambda_\varepsilon.$$

Choose $\varepsilon > 0$ such that

$$\frac{2l_0(d-p)}{p}(\frac{1}{l_0 d})^{\frac{d}{d-p}}(2\varepsilon)^{\frac{d}{d-p}} \leq \tau,$$

and taking $\Lambda_\tau = \Lambda_\varepsilon$, from (ii), we can take $\bar{t}$ large enough and define $\widetilde{\omega}_\lambda = \bar{t}\varpi_\lambda$, then we have

$$\Phi_\lambda(\widetilde{\omega}_\lambda) < 0 \text{ and } \max_{0\leq t\leq 1} \Phi_\lambda(t\widetilde{\omega}_\lambda) \leq \tau\lambda^{1-\frac{N}{ps}}.$$

□

Proof of Theorem 2.1. From Lemma 4.1, for any $0 < \tau < \alpha$, there exists $\Lambda_\tau > 0$ such that for $\lambda \geq \Lambda_\tau$, we have

$$c = \inf_{\eta\in\Gamma_\lambda}\max_{t\in[0,1]} \Phi_\lambda(\eta(t)) \leq \tau\lambda^{1-\frac{N}{ps}},$$

where $\Gamma_\lambda = \{\eta \in C([0,1], W) : \eta(0) = 0, \eta(1) = \widetilde{\omega}_\lambda\}$. Furthermore, in virtue of Corollary 3.1, we obtain that $(PS)_c$ condition hold for Φ_λ at c. Therefore, by the mountain pass theorem, there is $(u_\lambda, v_\lambda) \in W$ such that $\Phi'_\lambda(u_\lambda, v_\lambda) = 0$ and $\Phi_\lambda(u_\lambda, v_\lambda) = c$.

Finally, we prove that (u_λ, v_λ) satisfies the estimates in Theorem 2.1.

Since (u_λ, v_λ) is a critical point of Φ_λ, there holds for $\theta \in [p, p_s^*]$

$$\begin{aligned}\tau\lambda^{1-\frac{N}{ps}} &\geq \Phi_\lambda(u_\lambda, v_\lambda) - \frac{1}{\theta}\langle\Phi'_\lambda(u_\lambda, v_\lambda), (u_\lambda, v_\lambda)\rangle\\ &\geq (\frac{1}{p} - \frac{1}{\theta})\|(u_\lambda, v_\lambda)\|^p + \lambda(\frac{1}{\theta} - \frac{1}{p_s^*})\int_{\mathbb{R}^N} K(x)(|u_\lambda|^{p_s^*} + |v_\lambda|^{p_s^*})dx + \lambda(\frac{\mu}{\theta} - 1)\int_{\mathbb{R}^N} F(x, u_\lambda, v_\lambda)dx.\end{aligned}$$

Taking $\theta = \mu$, we get the estimate (2.2) and taking $\theta = p$ yields the estimate (2.3). □

To obtain the multiplicity of critical points, we will adopt the index theory defined by the Krasnoselski genus.

Proof of Theorem 2.2. Denote the set of all symmetric (in the sense that $-A = A$) and closed subsets of A by $\sum$. For any $A \in \sum$ let gen (A) be the Krasnoselski genus and

$$i(A) = \min_{k\in\Upsilon} \text{gen}(k(A)\bigcap\partial B_\rho),$$

where Υ is the set of all odd homeomorphisms $k \in C(W, W)$ and ρ is the number from Lemma 4.1. Then i is a version of Benci's pseudoindex [3]. (F_4) implies that Φ_λ is even. Set

$$c_{\lambda_j} := \inf_{i(A)\geq j} \sup_{(u,v)\in A} \Phi_\lambda(u, v), \quad 1 \leq j \leq m.$$

If c_{λ_j} is finite and Φ_λ satisfies $(PS)_{c_{\lambda_j}}$ condition, then we know that all c_{λ_j} are critical values for Φ_λ.

Step 1. We show that Φ_λ satisfies $(PS)_{c_{\lambda_j}}$ condition at all levels $c_{\lambda_j} < \tau\lambda^{1-\frac{N}{ps}}$.

To complete the claim, we need to estimate the level c_{λ_j} in special finite-dimensional subspaces.

Similar to proof in Lemma 4.1, for any $m \in \mathbb{N}$, $\varepsilon > 0$ and $j = 1, 2, \cdots, m$, one can choose m functions $\varphi_\varepsilon^j \in C_0^\infty(\mathbb{R}^N)$ with supp $\varphi_\varepsilon^i \bigcap$ supp $\varphi_\varepsilon^j = \emptyset$ if $i \neq j$, $|\varphi_\varepsilon^j|_d = 1$ and $[\varphi_\varepsilon^j]_{p,s}^p < \varepsilon$.

Let $r_\varepsilon^m > 0$ be such that supp $\varphi_\varepsilon^j \subset B_{r_\varepsilon^m}(0)$. Set

$$\overline{\omega}_\lambda^j(x) := (\omega_\lambda^j(x), \omega_\lambda^j(x)) = (\varphi_\varepsilon^j(\lambda^{\frac{1}{ps}}x), \varphi_\varepsilon^j(\lambda^{\frac{1}{ps}}x))$$

and define

$$F_\lambda^m := Span\,\{\overline{\omega}_\lambda^1, \overline{\omega}_\lambda^2, \cdots, \overline{\omega}_\lambda^m\}.$$

Then $i(F_\lambda^m) = \dim F_\lambda^m = m$. Observe that for each $\widetilde{\omega} = \sum_{j=1}^m t_j\overline{\omega}_\lambda^j \in F_\lambda^m$,

$$\Phi_\lambda(\widetilde{\omega}) = \sum_{j=1}^m \Phi_\lambda(t_j\overline{\omega}_\lambda^j)$$

and for $t_j > 0$

$$\begin{aligned}\Phi_\lambda(t_j\overline{\omega}_\lambda^j) &\leq \frac{2t_j^p}{p}\int\int_{\mathbb{R}^{2N}} \frac{|\omega_\lambda^j(x) - \omega_\lambda^j(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{2t_j^p}{p}\int_{\mathbb{R}^N} \lambda V(x)|\omega_\lambda^j|^p dx - \lambda\int_{\mathbb{R}^N} F(x, t_j\omega_\lambda^j, t_j\omega_\lambda^j)dx \\ &\leq \lambda^{1-\frac{N}{ps}}\Big\{\frac{2t_j^p}{p}\int\int_{\mathbb{R}^{2N}} \frac{|\varphi_\varepsilon^j(x) - \varphi_\varepsilon^j(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{2t_j^p}{p}\int_{\mathbb{R}^N} V(\lambda^{-\frac{1}{ps}}x)|\varphi_\varepsilon^j|^p dx - 2l_0t_j^d\int_{\mathbb{R}^N}|\varphi_\varepsilon^j|^d dx\Big\}.\end{aligned}$$

Set

$$\eta_\varepsilon := \max\{|\varphi_\varepsilon^j|_p^p : j = 1, 2, \cdots, m\}.$$

Since $V(0) = 0$ and supp $\varphi_\varepsilon^j \subset B_{r_\varepsilon^m}(0)$, there exists $\Lambda_{m\varepsilon} > 0$ such that

$$V(\lambda^{-\frac{1}{ps}}x) < \frac{\varepsilon}{\eta_\varepsilon}, \ \forall |x| \leq r_\varepsilon^m, \ \lambda > \Lambda_{m\varepsilon}.$$

Consequently, there holds

$$\sup_{\widetilde{w}\in F_\lambda^m} \Phi_\lambda(\widetilde{w}) \leq ml_0(2\varepsilon)^{\frac{d}{d-p}}\lambda^{1-\frac{N}{ps}}, \forall \lambda > \Lambda_{m\varepsilon}.$$

Choose $\varepsilon > 0$ small that $ml_0(2\varepsilon)^{\frac{d}{d-p}} < \tau$. Thus for any $m \in N$ and $\tau \in (0, \alpha)$, there exists $\Lambda_{m\tau} = \Lambda_{m\varepsilon}$ such that $\lambda > \Lambda_{m\tau}$, we can choose a m-dimensional subspace F_λ^m with $\max \Phi_\lambda(F_\lambda^m) \leq \tau\lambda^{1-\frac{N}{ps}}$ and

$$c_{\lambda_1} \leq c_{\lambda_2} \leq \cdots \leq \sup_{\widetilde{w}\in F_\lambda^m} \Phi_\lambda(\widetilde{w}) \leq \tau\lambda^{1-\frac{N}{ps}}.$$

From Corollary 3.1, we know that Φ_λ satisfies the (PS) condition at all levels c_{λ_j}. Then all c_{λ_j} are critical values.

Step 2. We prove that (2.1) has at least m pairs of solutions by the mountain-pass theorem.

By Lemma 4.1, we know that Φ_λ satisfies the mountain pass geometry structure. From step 1, we note that Φ_λ also satisfies $(PS)_{c_{\lambda_j}}$ condition at all levels $c_{\lambda_j} < \tau\lambda^{1-\frac{N}{ps}}$. By the usual critical point theory, all c_{λ_j} are critical levels and Φ_λ has at least m pairs of nontrivial critical points satisfying

$$a \leq \Phi_\lambda(u, v) \leq \tau\lambda^{1-\frac{N}{ps}}.$$

Thus, (2.1) has at least m pairs of solutions. Finally, as in the proof of Theorem 2.1, we know that these solutions satisfy the estimates (2.2) and (2.3). □

5. Conclusions

In this paper, we have obtained the existence and multiplicity of standing wave solutions for a class of perturbed fractional p-Laplacian systems involving critical exponents by variational methods. In the next work, we will extend the study to the case of perturbed fractional p-Laplacian systems with electromagnetic fields.

Acknowledgments

The author is grateful to the referees and the editor for their valuable comments and suggestions.

Conflict of interest

The author declares no conflict of interest.

References

1. D. Applebaum, Lévy processes-from probability theory to finance and quantum groups, *Notices of the American Mathematical Society*, **51** (2004), 1336–1347.
2. H. Alsulami, M. Kirane, S. Alhodily, T. Saeed, N. Nyamoradi, Existence and multiplicity of sulutions to fractional p-Laplacian system with concave-convex nonlinearities, *Bull. Math. Sci.*, **10** (2020), 2050007. https://doi.org/10.1142/S1664360720500071
3. V. Benci, On critical point theory of indefinite functionals in the presence of symmetries, *Trans. Amer. Math. Soc.*, **274** (1982), 533–572. https://doi.org/10.1090/S0002-9947-1982-0675067-X
4. H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.*, **88** (1983), 486–490. https://doi.org/10.1090/S0002-9939-1983-0699419-3
5. J. Byeon, Mountain pass solutions for singularly perturbed nonlinear Dirichlet problems, *J. Differ. Equations*, **217** (2005), 257–281. https://doi.org/10.1016/j.jde.2005.07.008
6. L. Caffarelli, Nonlocal equations, drifts and games, In: *Nonlinear partial differential equations*, Berlin, Heidelberg: Springer, 2012, 37–52. https://doi.org/10.1007/978-3-642-25361-4_3

7. G. Chen, Y. Zhang, Concentration phenomenon for fractional nonlinear Schrödinger equations, *Commun. Pure Appl. Anal.*, **13** (2014), 2359–2376. https://doi.org/10.3934/cpaa.2014.13.2359
8. W. Chen, S. Deng, The Nehari manifold for a fractional p-Laplacian system involving concave-convex nonlinearities, *Nonlinear Anal. Real*, **27** (2016), 80–92. https://doi.org/10.1016/j.nonrwa.2015.07.009
9. W. Chen, M. Squassina, Critical nonlocal systems with concave-convex powers, *Adv. Nonlinear Stud.*, **16** (2016), 821–842. https://doi.org/10.1515/ans-2015-5055
10. M. del Pino, P. L. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var.*, **4** (1996), 121–137. https://doi.org/10.1007/BF01189950
11. Y. Ding, F. Lin, Solutions of perturbed Schrödinger equations with critical nonlinearity, *Calc. Var.*, **30** (2007), 231–249. https://doi.org/10.1007/s00526-007-0091-z
12. A. Floer, A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equations with a bounded potential, *J. Funct. Anal.*, **69** (1986), 397–408. https://doi.org/10.1016/0022-1236(86)90096-0
13. M. M. Fall, F. Mahmoudi, E. Valdinoci, Ground states and concentration phenomena for the fractional Schrödinger equation, *Nonlinearity*, **28** (2015), 1937–1961. https://doi.org/10.1088/0951-7715/28/6/1937
14. C. F. Gui, Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method, *Commun. Part. Diff. Eq.*, **21** (1996), 787–820. https://doi.org/10.1080/03605309608821208
15. Y. He, G. Li, The existence and concentration of weak solutions to a class of p-Laplacian type problems in unbounded domains, *Sci. China Math.*, **57** (2014), 1927–1952. https://doi.org/10.1007/s11425-014-4830-2
16. H. Jin, W. Liu, J. Zhang, Singularly perturbed fractional Schrödinger equation involving a general critical nonlinearity, *Adv. Nonlinear Stud.*, **18** (2018), 487–499. https://doi.org/10.1515/ans-2018-2015
17. N. Laskin, Fractional quantum mechanics and Lévy path integrals, *Phys. Lett. A*, **268** (2000), 298–305. https://doi.org/10.1016/S0375-9601(00)00201-2
18. N. Laskin, Fractional Schröding equation, *Phys. Rev. E*, **66** (2002), 056108. https://doi.org/10.1103/PhysRevE.66.056108
19. Q. Lou, H. Luo, Multiplicity and concentration of positive solutions for fractional p-Laplacian problem involving concave-convex nonlinearity, *Nonlinear Anal. Real*, **42** (2018), 387–408. https://doi.org/10.1016/j.nonrwa.2018.01.013
20. G. Lu, Y. Shen, Existence of solutions to fractional p-Laplacian system with homogeneous nonlinearities of critical sobolev growth, *Adv. Nonlinear Stud.*, **20** (2020), 579–597. https://doi.org/10.1515/ans-2020-2098
21. Q. Li, Z. Yang, Existence and multiplicity of solutions for perturbed fractional p-Laplacian equations with critical nonlinearity in $\mathbb{R}^N$, *Appl. Anal.*, in press. https://doi.org/10.1080/00036811.2022.2045969

22. A. Mellet, S. Mischler, C. Mouhot, Fractional diffusion limit for collisional kinetic equations, *Arch. Rational Mech. Anal.*, **199** (2011), 493–525. https://doi.org/10.1007/s00205-010-0354-2

23. J. M. do Ó, On existence and concentration of positive bound states of p-Laplacian equations in $\mathbb{R}^N$ involving critical growth, *Nonlinear Anal. Theor.*, **62** (2005), 777–801. https://doi.org/10.1016/j.na.2005.03.093

24. P. Pucci, M. Xiang, B. Zhang, Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$, *Calc. Var.*, **54** (2015), 2785–2806. https://doi.org/10.1007/s00526-015-0883-5

25. P. Pucci, M. Xiang, B. Zhang, Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations, *Adv. Nonlinear Anal.*, **5** (2016), 27–55. https://doi.org/10.1515/anona-2015-0102

26. P. H. Rabinowitz, On a class of nonlinear Schröding equations, *Z. Angew. Math. Phys.*, **43** (1992), 270–291. https://doi.org/10.1007/BF00946631

27. L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Commun. Pure Appl. Math.*, **60** (2007), 67–112. https://doi.org/10.1002/cpa.20153

28. B. Sirakov, Standing wave solutions of the nonlinear Schrödinger equations in $\mathbb{R}^N$, *Annali di Matematica*, **181** (2002), 73–83. https://doi.org/10.1007/s102310200029

29. Y. Song, S. Shi, Existence and multiplicity solutions for the p-Fractional Schrödinger-Kirchhoff equations with electromagnetic fields and critical nonlinearity, *Complex Var. Elliptic Equ.*, **64** (2019), 1163–1183. https://doi.org/10.1080/17476933.2018.1511707

30. Y. Shen, Existence of solutions to elliptic problems with fractional p-Laplacian and multiple critical nonlinearities in the entire space, *Nonlinear Anal.*, **202** (2021), 112102. https://doi.org/10.1016/j.na.2020.112102

31. W. Wang, X. Yang, F. Zhao, Existence and concentration of ground state solutions for a subcubic quasilinear problem via Pohozaev manifold, *J. Math. Anal. Appl.*, **424** (2015), 1471–1490. https://doi.org/10.1016/j.jmaa.2014.12.013

32. L. Wang, X. Fang, Z. Han, Existence of standing wave solutions for coupled quasilinear Schrodinger systems with critical exponents in $\mathbb{R}^N$, *Electron. J. Qual. Theory Differ. Equ.*, **2017** (2017), 12. https://doi.org/10.14232/ejqtde.2017.1.12

33. J. Wang, The existence of semiclassical states for some p-Laplacian equation with critical exponent, *Acta Math. Appl. Sin. Engl. Ser.*, **33** (2017), 417–434. https://doi.org/10.1007/s10255-017-0671-4

34. M. Xiang, G. Molica Bisci, G. Tian, B. Zhang, Infinitely many solutions for the stationary Kirchhoff problems involving the fractional p-Laplacian, *Nonlinearity*, **29** (2016), 357–374. https://doi.org/10.1088/0951-7715/29/2/357

35. M. Xiang, B. Zhang, X. Zhang, A nonhomogeneous fractional p-Kirchhoff type problem involving critical exponent in $\mathbb{R}^N$, *Adv. Nonlinear Stud.*, **17** (2017), 611–640. https://doi.org/10.1515/ans-2016-6002

36. M. Xiang, B. Zhang, Z. Wei, Existence of solutions to a class of quasilinear Schrödinger systems involving the fractional p-Laplacian, *Electron. J. Qual. Theory Differ. Equ.*, **2016** (2016), 107. https://doi.org/10.14232/ejqtde.2016.1.107

37. J. Xie, X. Huang, Y. Chen, Existence of multiple solutions for a fractional p-Laplacian system with concave-convex term, *Acta Math. Sci.*, **38** (2018), 1821–1832. https://doi.org/10.1016/S0252-9602(18)30849-X
38. M. Xiang, B. Zhang, V. Rădulescu, Superlinear Schrödinger-Kirchhoff type problems involving the fractional p-Laplacian and critical exponent, *Adv Nonlinear Anal.*, **9** (2020), 690–709. https://doi.org/10.1515/anona-2020-0021
39. M. Yang, Y. Ding, Existence of semiclassical states for a quasilinear Schrödinger equation with critical exponent in $\mathbb{R}^N$, *Annali di Matematica*, **192** (2013), 783–804. https://doi.org/10.1007/s10231-011-0246-6
40. H. Zhang, W. Liu, Existence of nontrivial solutions to perturbed p-Laplacian system in $\mathbb{R}^N$ involving critical nonlinearity, *Bound. Value Probl.*, **2012** (2012), 53. https://doi.org/10.1186/1687-2770-2012-53
41. B. Zhang, M. Squassina, X. Zhang, Fractional NLS equations with magnetic field, critical frequency and critical growth, *Manuscripta Math.*, **155** (2018), 115–140. https://doi.org/10.1007/s00229-017-0937-4
42. M. Zhen, B. Zhang, The Nehari manifold for fractional p-Laplacian system involving concave-conve nonlinearities and sign-changing weight functions, *Complex Var. Elliptic Equ.*, **66** (2021), 1731–1754. https://doi.org/10.1080/17476933.2020.1779237
43. W. Zhang, S. Yuan, L. Wen, Existence and concentration of ground states for fractional Choquard equation with indefinite potential, *Adv. Nonlinear Anal.*, **11** (2022), 1552–1578. https://doi.org/10.1515/anona-2022-0255
44. J. Zhang, W. Zhang, Semiclassical states for coupled nonlinear Schrödinger system with competing potentials, *J. Geom. Anal.*, **32** (2022), 114. https://doi.org/10.1007/s12220-022-00870-x
45. W. Zhang, J. Zhang, H. Mi, Ground states and multiple solutions for Hamiltonian elliptic system with gradient term, *Adv. Nonlinear Anal.*, **10** (2021), 331–352. https://doi.org/10.1515/anona-2020-0113
46. W. Zhang, J. Zhang, Multiplicity and concentration of positive solutions for fractional unbalanced double-phase problems, *J. Geom. Anal.*, **32** (2022), 235. https://doi.org/10.1007/s12220-022-00983-3