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Research article

A new strict decay rate for systems of longitudinal m-nonlinear viscoelastic wave equations

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Abstract: Recent years have been marked by a significant increase in interest in solving nonlinear equations that arise in various fields of natural science. This trend is associated with the creation of a new method of mathematical physics. The present study is devoted to the analysis of the propagation of m-nonlinear viscoelastic waves equations in an unbounded domain. The physical properties are determined by the equations of the linear theory of viscoelasticity. This article shows the main effect and interaction between the different weak and strong damping terms on the behavior of solutions. We found, under a novel condition on the kernel functions, an energy decay rate by using an appropriate energy estimates.

Keywords: damped wave equation; strong nonlinear system; global solution; exponential decay rate
Mathematics Subject Classification: 35B35, 35L05, 35L70

1. Introduction

Wave processes in viscoelastic and nonlinear viscoelastic equations that do not interact with a viscous fluid are considered in many references [1–5]. The present study is devoted to the analysis of the propagation of nonlinear longitudinal waves in $\mathbb{R}^n$ containing a viscous incompressible fluid inside. The physical properties are determined by the equations of the linear theory of viscoelasticity.

Let $x \in \mathbb{R}^n,\ t > 0,\ j = 1, 2, \ldots, m$. We consider the following system of m damped wave equations

$$\begin{cases} \left(|\partial_t u_j|^{\kappa-2}\partial_t u_j\right)_t + a\partial_t u_j & -\Theta(x)\Delta_x\left(u_j + \omega\partial_t u_j - \int_0^t \varpi_j(t-s)u_j(s)\,ds\right) \\ & = f_j(u_1, u_2, \ldots, u_m), \\ u_j(x,0) = u_{j0}(x), \\ \partial_t u_j(x,0) = u_{j1}(x), \end{cases} \tag{1.1}$$

where $a \in \mathbb{R}, \omega > 0, n \geq 3, \kappa \geq 2, u_j = u_j(x,t)$.

The functions $f_j(u_1, u_2, \ldots, u_m) \in (\mathbb{R}^m, \mathbb{R})$ are given by

$$f_j(u_1, u_2, \ldots, u_m) = (p+1)\Big[d\Big|\sum_{i=1}^m u_i\Big|^{(p-1)}\sum_{i=1}^m u_i + e|u_j|^{(p-3)/2}u_j|u_{j+1}|^{(p+1)/2}\Big],$$

and

$$f_m(u_1, u_2, \ldots, u_m) = (p+1)\Big[d\Big|\sum_{i=1}^m u_i\Big|^{(p-1)}\sum_{i=1}^m u_i + e|u_m|^{(p-3)/2}u_m|u_1|^{(p+1)/2}\Big],$$

with $d, e >,\ p > 3$.

One can find a function $\mathcal{F} \in C^1(\mathbb{R}^3, \mathbb{R})$ where

$$\sum_{j=1}^m u_j f_j(u_1, u_2, \ldots, u_m) = (p+1)\mathcal{F}(u_1, u_2, \ldots, u_m),\ \forall (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m, \tag{1.2}$$

satisfies

$$\mathcal{F}(u_1, u_2, \ldots, u_m) = \frac{1}{(p+1)}\Big|\sum_{j=1}^m u_j\Big|^{p+1} + \frac{2}{(p+1)}\Big|\sum_{j=1}^{m-1} u_j u_{j+1}\Big|^{(p+1)/2} + \frac{2}{(p+1)}|u_m u_1|^{(p+1)/2}. \tag{1.3}$$

For all $x \in \mathbb{R}^n$, the function $\Theta(x) > 0$ is a density and $(\Theta)^{-1} = 1/\Theta(x) \equiv \theta(x)$ such that

$$\theta \in L^\tau(\mathbb{R}^n) \quad \text{with} \quad \tau = \frac{2n}{2n - rn + 2r} \quad \text{for} \quad 2 \leq r \leq \frac{2n}{n-2}. \tag{1.4}$$

We define a new related spaces $\mathcal{H}$ as the closure of $C_0^\infty(\mathbb{R}^n)$, as in [6], we have

$$\mathcal{H} = \{v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \mid \nabla_x v \in L^2(\mathbb{R}^n)^n\},$$

with respect to the norm $\|v\|_{\mathcal{H}} = (v, v)_{\mathcal{H}}^{1/2}$ for

$$(v, w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla_x v \cdot \nabla_x w\, dx,$$

and $L^2_\theta(\mathbb{R}^n)$ as that to the norm $\|v\|_{L^2_\theta} = (v, v)_{L^2_\theta}^{1/2}$ for

$$(v, w)_{L^2_\theta} = \int_{\mathbb{R}^n} \theta v w\, dx.$$

For general $r \in [1, +\infty)$

$$\|v\|_{L^r_\theta} = \left(\int_{\mathbb{R}^n} \theta \, |v|^r \, dx \right)^{\frac{1}{r}},$$

is the norm of the weighted space $L^r_\theta(\mathbb{R}^n)$.

Non-stationary processes accompanied by the propagation of waves in various bodies are complex wave processes that are often encountered in many areas of technology, including aviation and rocketry.

A large number of materials used in construction and engineering have viscoelastic properties. Therefore, the problems of the theory of viscoelasticity have recently attracted special attention of many researchers and engineers in connection with the use of polymeric materials and plastics in various branches of production and the construction industry (please see [7–11]).

Although most of the advances in the theory of viscoelasticity are recent, the theory formulated for the linear isothermal case has been around for a long time. This is due to the contributions of authors such as Maxwell, Kelvin and Voigt. So Maxwell first introduced the law of deformation with respect to time in the form of a differential equation, which is still used for some materials today.

The fundamental foundations of the linear theory of viscoelasticity were formulated by L. Boltzmann. The basis of this theory is that the deformation at a given moment depends on all previous stresses. Boltzmann was also the first to give the equations of the three-dimensional theory of isotropic viscoelasticity.

It should be noted that the theory of viscoelasticity with slow deformation processes has received the main distribution, which has led to the development of the theory of creep and its applications in various fields of engineering and construction. At the same time, in many areas of technology, structures made of viscoelastic materials are subjected to impulsive actions, while wave processes occur in the deformable material.

In [12], the authors considered

$$\partial_{tt}u + a\partial_t u - \phi(x)\Delta_x \left(u + \omega\partial_t u - \int_0^t g(t-s)u(s)\,ds \right) = u|u|^{p-1}, \tag{1.5}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x), \\ \partial_t u(x,0) = u_1(x). \end{cases} \tag{1.6}$$

The existence of unique local and global solution in time was shown. The authors also proved a new decay rate of solution, please see [13–19].

In [20], a coupled system is considered as follows

$$\begin{cases} \partial_{tt}u - \Delta_x u + \int_0^t g(t-s)\Delta_x u(s)\,ds + \partial_t u = f_1(u,v) \\ \\ \partial_{tt}v - \Delta_x v + \int_0^t h(t-s)\Delta_x v(s)\,ds + \partial_t v = f_2(u,v). \end{cases} \tag{1.7}$$

Under appropriate conditions, the authors established a special decay result by multiplication techniques.

In this work, a system of n viscoelastic wave equations is studied. This system is the generalization of the one studied in [21], where $n = 2$, where the main goal was to study the existence, uniqueness and behavior of a solution for a coupled system of nonlinear viscoelastic wave equations with the presence of weak and strong damping terms. Based on the urgent need for applications and in order to give a wide scope to the applied sciences, we have extended the article [21] to become from two equations to a very large number n of equations and the very importance is to study the method of overlapping between the various terms, such as the sources, to become controlled and suitable for such a large non-linear system.

The paper brings new contributions to the prior literature mainly in what concerns new decay rate estimates of the energy.

We will use the Sobolev embedding and generalized Poincaré inequalities in the next Lemma.

Lemma 1.1. *[12] Let θ satisfy (1.4). For positive constants $C_\tau > 0$ and $C_P > 0$ depending only on θ and n, we have*

$$\|v\|_{\frac{2n}{n-2}} \leq C_\tau \|v\|_{\mathcal{H}},$$

and

$$\|v\|_{L^2_\theta} \leq C_P \|v\|_{\mathcal{H}},$$

for $v \in \mathcal{H}$.

Lemma 1.2. *[22] Let θ satisfy (1.4), then the estimates*

$$\|v\|_{L^r_\theta} \leq C_r \|v\|_{\mathcal{H}},$$

and

$$C_r = C_\tau \|\theta\|_\tau^{\frac{1}{r}},$$

hold for $v \in \mathcal{H}$. Here $\tau = 2n/(2n - rn + 2r)$ for $1 \leq r \leq 2n/(n-2)$.

The kernel functions $\varpi_j \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ is assumed to satisfy

$$1 - \overline{\varpi_j} = \rho_j > 0 \quad \text{for} \quad \overline{\varpi_j} = \int_0^{+\infty} \varpi_j(s)\, ds,\ \varpi_j'(t) \leq 0. \tag{1.8}$$

Let us note by

$$\mu(t) = \max_{t\geq 0} \Big\{\varpi_1(t), \varpi_2(t), \ldots, \varpi_m(t)\Big\}, \tag{1.9}$$

and

$$\mu_0(t) = \min_{t\geq 0} \Big\{ \int_0^t \varpi_1(s)ds, \int_0^t \varpi_2(s)ds, \ldots, \int_0^t \varpi_m(s)ds\Big\}. \tag{1.10}$$

There is a function $\chi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, such that the properties inspired from [12],

$$\varpi_j'(t) + \chi(\varpi_j(t)) \leq 0, \quad \chi(0) = 0, \quad \chi'(0) > 0 \quad \text{and} \quad \chi''(\xi) \geq 0,\ i = 1, 2, \ldots, m, \tag{1.11}$$

satisfied for any $\xi \geq 0$.

Holder and Young's inequalities give

$$\left\|u_i u_j\right\|^{(p+1)/2}_{L^{(p+1)/2}_\theta} \leq \left(\|u_i\|^2_{L^{(p+1)}_\theta} + \|u_j\|^2_{L^{(p+1)}_\theta}\right)^{(p+1)/2}$$

$$\leq \left(\rho_i\|u_i\|_{\mathcal{H}}^2 + \rho_j\|u_j\|_{\mathcal{H}}^2\right)^{(p+1)/2}. \tag{1.12}$$

Thanks to Minkowski's inequality to give

$$\begin{aligned}\left\|\sum_{j=1}^{m} u_j\right\|_{L_\theta^{(p+1)}}^{(p+1)} &\leq c\left(\sum_{j=1}^{m}\|u_j\|_{L_\theta^{(p+1)}}^2\right)^{(p+1)/2} \\ &\leq c\left(\sum_{j=1}^{m}\|u_j\|_{\mathcal{H}}^2\right)^{(p+1)/2}.\end{aligned}$$

Thus there exists a positive constant $\eta > 0$ such that

$$\begin{aligned}&\left\|\sum_{j=1}^{m} u_j\right\|_{L_\theta^{(p+1)}}^{(p+1)} + 2\left\|\sum_{j=1}^{m-1} u_j u_{j+1}\right\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} + 2\|u_m u_1\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} \\ &\leq \eta\left(\sum_{j=1}^{m}\rho_j\|u_j\|_{\mathcal{H}}^2\right)^{(p+1)/2}.\end{aligned} \tag{1.13}$$

We need to define positive constants λ_0 and $\mathcal{E}_0$ by

$$\lambda_0 \equiv \eta^{-1/(p-1)} \quad \text{and} \quad \mathcal{E}_0 = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\eta^{-2/(p-1)}. \tag{1.14}$$

Let us denote an eigenpair $\{(\lambda_i, e_i)\}_{i\in\mathbb{N}} \subset \mathbb{R}\times\mathcal{H}$ of

$$-\Theta(x)\Delta_x e_i = \lambda_i e_i \quad x \in \mathbb{R}^n,$$

for any $i \in \mathbb{N}$. Then

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots \uparrow +\infty,$$

holds and $\{e_i\}$ is a complete orthonormal system in $\mathcal{H}$.

Definition 1.1. *The vectors $(u_1, u_2, \ldots, u_m)$ is said a weak solution to (1.1) on $[0, T]$ if satisfies for $x \in \mathbb{R}^n$*

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(|\partial_t u_j|^{\kappa-2}\partial_t u_j\right)_t\varphi_j dx + a\int_{\mathbb{R}^n}\partial_t u_j\varphi_j dx \\ -&\int_{\mathbb{R}^n}\Theta(x)\Delta_x\left(u_j + \omega\partial_t u_j - \int_0^t \varpi_j(t-s)u_j(s)\,ds\right)\varphi_j dx \\ =&\int_{\mathbb{R}^n} f_j(u_1, u_2, \ldots, u_m)\varphi_j dx,\end{aligned} \tag{1.15}$$

for all test functions $\varphi_j \in \mathcal{H}$, $j = 1, 2, \ldots, m$ for almost all $t \in [0, T]$.

2. Main results

Theorem 2.1. *(Local existence result) Let*

$$1 < p \leq \frac{n+2}{n-2} \quad and \quad n \geq 3. \tag{2.1}$$

Let $(u_{10}, u_{20}, \ldots u_{m0}) \in \mathcal{H}^m$ *and* $(u_1, u_1, \ldots, u_m) \in [L^\kappa_\theta(\mathbb{R}^n)]^m$*. Under the assumptions (1.3), (1.4) and (1.8)–(1.11), suppose that*

$$a + \lambda_1 \omega > 0. \tag{2.2}$$

Then (1.1) has a unique local solution $(u_1, u_2, \ldots, u_m)$ *such that*

$$(u_1, u_2, \ldots, u_m) \in \mathcal{X}^m_T, \ \mathcal{X}_T \equiv C([0,T]; \mathcal{H}) \cap C^1([0,T]; L^\kappa_\theta(\mathbb{R}^n)),$$

for sufficiently small $T > 0$*.*

We show now that solution established in Theorem 2.1 is global in time without any additional conditions.

Let us introduce the potential energy $J : \mathcal{H}^m \to \mathbb{R}$ defined by

$$J(u_1, u_2, \ldots, u_m) = \sum_{j=1}^{m} \left(1 - \int_0^t \varpi_j(s)\, ds\right) \left\|u_j\right\|^2_{\mathcal{H}} + \left(\varpi_j \circ u_j\right). \tag{2.3}$$

The modified energy is defined by

$$\mathcal{E}(t) = \frac{\kappa - 1}{\kappa} \sum_{j=1}^{m} \left\|\partial_t u_j\right\|^\kappa_{L^\kappa_\theta} + \frac{1}{2} J(u_1, u_2, \ldots, u_m) - \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u_1, u_2, \ldots, u_m) dx, \tag{2.4}$$

here

$$\left(\varpi_j \circ w\right)(t) = \int_0^t \varpi_j(t-s) \left\|w(t) - w(s)\right\|^2_{\mathcal{H}}\, ds,$$

for any $w \in L^2(\mathbb{R}^n)$, $j = 1, 2, \ldots, m$.

Theorem 2.2. *(Global existence) Let (1.3), (1.4) and (1.8)–(1.11) hold. Under (2.1), (2.2) and for sufficiently small* $(u_{10}, u_{11}), (u_{20}, u_{21}), \ldots, (u_{m0}, u_{m1}) \in \mathcal{H} \times L^\kappa_\theta(\mathbb{R}^n)$*, problem (1.1) admits a unique global solution* $(u_1, u_2, \ldots, u_m)$ *such that*

$$(u_1, u_2, \ldots, u_m) \in \mathcal{X}^m, \ \mathcal{X} \equiv C([0, +\infty); \mathcal{H}) \cap C^1([0, +\infty); L^\kappa_\theta(\mathbb{R}^n)). \tag{2.5}$$

The nonclassical decay rate for solution is given in the next Theorem.

Theorem 2.3. *(Decay of solution) Let (1.3), (1.4) and (1.8)–(1.11) hold. Under conditions (2.1), (2.2) and*

$$\gamma = \eta \Big(\frac{2(p+1)}{p-1} \mathcal{E}(0)\Big)^{(p-1)/2} < 1, \tag{2.6}$$

there exists $t_0 > 0$ *depending only on* ϖ_j*,* a*,* ω*,* λ_1 *and* $H'(0)$ *such that*

$$0 \leq \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(-\int_{t_0}^t \frac{\mu(s)}{1 - \mu_0(t)}\right), \tag{2.7}$$

holds for all $t > t_0$*.*

Owing to the positivity of μ in (1.9), we have

$$0 \leq \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(-\int_{t_0}^{t} \mu(s)\,ds\right),$$

for a single wave equation. Condition (1.11) is imposed to make a different from [19, 23, 24], it leads $(\mu' + \nu\mu) \circ u$, here $\nu \in \mathbb{R}$.

Lemma 2.1. *For $(u_1, u_2, \ldots, u_m) \in \mathcal{X}_T^m$, the functional $\mathcal{E}(t)$ associated with problem (1.1) is a decreasing energy.*

Proof. For $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
&\mathcal{E}(t_2) - \mathcal{E}(t_1)\\
= &\int_{t_1}^{t_2} \frac{d}{dt} E(t)\,dt\\
= &-\sum_{j=1}^{m} \int_{t_1}^{t_2} \left(a \left\|\partial_t u_j\right\|_{L_\theta^2}^2 + \omega \left\|\partial_t u_j\right\|_{\mathcal{H}}^2 + \frac{1}{2}\varpi_j(t)\left\|u_j\right\|_{\mathcal{H}}^2 - \frac{1}{2}(\varpi_j' \circ u_j)\right) dt\\
\leq &\ 0,
\end{aligned}$$

owing to (1.8)–(1.11).

By (2.2), we get

$$(v, v)_* = \omega \int_{\mathbb{R}^n} |\nabla_x v|^2\,dx + a \int_{\mathbb{R}^n} \theta v^2\,dx \geq (\omega\lambda_1 + a) \int_{\mathbb{R}^n} \theta v^2\,dx \geq 0. \tag{2.8}$$

The following lemma yields.

Lemma 2.2. *Let θ satisfy (1.4). Under condition (2.2), we get*

$$\sqrt{\omega}\,\|v\|_{\mathcal{H}} \leq \|v\|_* \leq \sqrt{\omega + C_P^2}\,\|v\|_{\mathcal{H}},$$

for $v \in \mathcal{H}$.

3. Proofs of existence results

We will use some idea in [13, 19, 21]).

Proof. (Of Theorem 2.1.) Let $(u_{10}, u_{11}), (u_{20}, u_{21}), \ldots, (u_{m0}, u_{m1}) \in \mathcal{H} \times L_\theta^\kappa(\mathbb{R}^n)$. The presence of the nonlinear terms in the right hand side of our problem (1.1) gives us negative values of the energy. For this purpose, for any fixed $(u_1, u_2, \ldots, u_m) \in \mathcal{X}_T^m$, we can obtain first, a weak solution of the related system

$$\begin{cases}
\left(|\partial_t z_j|^{\kappa-2}\partial_t z_j\right)_t + a\partial_t z_j - \Theta(x)\Delta_x\left(z_j + \omega\partial_t z_j\right) + \Theta(x)\Delta_x \int_0^t \varpi_j(t-s) z_j(s)\,ds \\
\qquad = f_j(u_1, u_2, \ldots, u_m),\\
z_j(x, 0) = u_{j0}(x),\\
\partial_t z_j(x, 0) = u_{j1}(x).
\end{cases} \tag{3.1}$$

The Faedo-Galerkin's method consist to construct approximations of solutions $(z_{1n}, z_{2n}, \dots, z_{mn})$ for (3.1), then we obtain a prior estimates necessary to guarantee the convergence of approximations.

Let

$$W_{jn} = span\{e_{j1}, e_{j2}, ..., e_{jn}\}, j = 1, \dots, m.$$

Given initial data $u_{j0} \in \mathcal{H}$, $u_{j1} \in L^{\kappa}_{\theta}(\mathbb{R}^n)$, we define

$$z_{jn} = \sum_{i=1}^{n} g_{jin}(t) e_{ji}(x), \tag{3.2}$$

which satisfy the following approximate problem

$$\begin{aligned} &\left(\left(|\partial_t z_{jn}|^{\kappa-2}\partial_t z_{jn}\right)_t, e_{ji}\right) + \left(a\partial_t z_{jn}, e_{ji}\right) - \left(\Theta(x)\Delta_x \left(z_{jn} + \omega\partial_t z_{jn}\right), e_{ji}\right) \\ &= -\left(\Theta(x)\Delta_x \int_0^t \varpi_j(t-s) z_{jn}(s)\, ds, e_{ji}\right) + \left(f_j(u_1, u_2, \dots, u_m), e_{ji}\right), \end{aligned} \tag{3.3}$$

with initial conditions

$$z_{jn}(x, 0) = u^n_{j0}(x),\ \partial_t z_{jn}(x, 0) = u^n_{j1}(x), \tag{3.4}$$

which satisfies

$$\begin{aligned} &u^n_{j0} \to u_{j0},\ \ strongly\ \ in\ \ \mathcal{H}, \\ &u^{jn}_1 \to u_{j1},\ \ strongly\ \ in\ \ L^{\kappa}_{\theta}(\mathbb{R}^n). \end{aligned} \tag{3.5}$$

Taking $e_{ji} = g_{ji}$ in (3.3) yields the following Cauchy problem for a ordinary differential equation with unknown g^n_{ji}.

$$\begin{aligned} &\left(|g^n_{jit}(t)|^{\kappa-2} g^n_{jit}(t)\right)_t + a g^n_{ji}(t) + \lambda_i \left(g^n_{ji}(t) + \omega g^n_{jit}(t)\right) \\ &= \lambda_i \int_0^t \varpi_j(t-s) g^n_{ji}(s)\, ds + \left(f_j(u_1, u_2, \dots, u_m), g_{ji}\right). \end{aligned} \tag{3.6}$$

The problems (3.3) and (3.4) have a solutions $(g_{1in}, g_{2in}, \dots, g_{min})_{i=1,n} \in (H^3[0,T])^m$ and by using the embedding $H^m[0,T] \to C^m[0,T]$, we deduce that the solution $(g_{1in}, g_{2in}, \dots, g_{min})_{i=1,n} \in (C^2[0,T])^4$. In turn, this gives a unique $(z_{1n}, z_{2n}, \dots, z_{mn})$ defined by (3.2) and satisfying (3.3).

We return now to (1.1), and find

$$\top : (u_1, u_2, \dots, u_m) \mapsto (z_1, z_2, \dots, z_m)$$

from $\mathcal{X}^m_T$ to $\mathcal{X}^m_T$. We can now show that $\top$ is a contraction mapping in an appropriate subset of $\mathcal{X}^m_T$ for a small $T > 0$. Then $\top$ has a fixed point

$$\top(u_1, u_2, \dots, u_m) = (u_1, u_2, \dots, u_m),$$

which gives a unique solution in $\mathcal{X}^m_T$.

To show the global solution, we need to use

$$
\begin{aligned}
\mathcal{E}(t) &\geq \frac{1}{2}J(u_1,u_2,\ldots,u_m)-\int_{\mathbb{R}^n}\theta(x)\mathcal{F}(u_1,u_2,\ldots,u_m)dx\\
&\geq \frac{1}{2}J(u_1,u_2,\ldots,u_m)-\frac{1}{p+1}\left\|\sum_{j=1}^{m}u_j\right\|^{(p+1)}_{L_\theta^{(p+1)}}\\
&- \frac{2}{p+1}\Big(\Big\|\sum_{j=1}^{m-1}u_ju_{j+1}\Big\|^{(p+1)/2}_{L_\theta^{(p+1)/2}}+\|u_mu_1\|^{(p+1)/2}_{L_\theta^{(p+1)/2}}\Big)\\
&\geq \frac{1}{2}J(u_1,u_2,\ldots,u_m)-\frac{\eta}{p+1}\Big[\sum_{j=1}^{m}\rho_j\left\|u_j\right\|^2_{\mathcal{H}}\Big]^{(p+1)/2}\\
&\geq \frac{1}{2}J(u_1,u_2,\ldots,u_m)-\frac{\eta}{p+1}\Big(J(u_1,u_2,\ldots,u_m)\Big)^{(p+1)/2}\\
&= G(\beta),
\end{aligned}
\tag{3.7}
$$

here $\beta^2 = J(u_1,u_2,\ldots,u_m)$, for $t \in [0,T)$, where

$$G(\xi)=\frac{1}{2}\xi^2-\frac{\eta}{p+1}\xi^{(p+1)}.$$

Noting that $\mathcal{E}_0 = G(\lambda_0)$, given in (1.14). Then

$$
\begin{cases}
G(\xi)\geq 0, & in\ \ \xi\in[0,\lambda_0],\\
\\
G(\xi)<0, & in\ \ \xi>\lambda_0.
\end{cases}
\tag{3.8}
$$

Moreover, $\lim\limits_{\xi\to+\infty} G(\xi)\to-\infty$. Then

Lemma 3.1. *Let* $0 \leq \mathcal{E}(0) < \mathcal{E}_0$.
(i) *If* $\sum\limits_{j=1}^{m}\left\|u_{j0}\right\|^2_{\mathcal{H}} < \lambda_0^2$, *then*

$$J(u_1,u_2,\ldots,u_m)<\lambda_0^2,\ \forall t\in[0,T).$$

(ii) *If* $\sum\limits_{j=1}^{m}\left\|u_{j0}\right\|^2_{\mathcal{H}} > \lambda_0^2$, *then*

$$\sum_{j=1}^{m}\left\|u_j\right\|^2_{\mathcal{H}}>\lambda_1^2,\ \forall t\in[0,T),\lambda_1>\lambda_0.$$

Proof. Since $0 \leq \mathcal{E}(0) < \mathcal{E}_0 = G(\lambda_0)$, there exist ξ_1 and ξ_2 such that $G(\xi_1) = G(\xi_2) = \mathcal{E}(0)$ with $0 < \xi_1 < \lambda_0 < \xi_2$.
The case (i)**.** By (3.7), we have

$$G(J(u_{10},u_{20},\ldots u_{m0}))\leq\mathcal{E}(0)=G(\xi_1),$$

which implies that $J(u_{10}, u_{20}, \dots u_{m0}) \leq \xi_1^2$. Then we claim that $J(u_1, u_2, \dots, u_m) \leq \xi_1^2,\ \forall t \in [0, T)$. Moreover, there exists $t_0 \in (0, T)$ such that

$$\xi_1^2 < J(u_1(t_0), u_2(t_0), \dots, u_m(t_0)) < \xi_2^2.$$

Then

$$G(J(u_1(t_0), u_2(t_0), \dots, u_m(t_0)) > \mathcal{E}(0) \geq \mathcal{E}(t_0),$$

by Lemma 2.1, which contradicts (3.7). Hence we have

$$J(u_1, u_2, \dots, u_m) \leq \xi_1^2 < \lambda_0^2,\ \forall t \in [0, T).$$

The case (ii)**.** We can now show that $\sum_{j=1}^{m} \left\| u_{j0} \right\|_{\mathcal{H}}^2 \geq \xi_2^2$ and that $\sum_{j=1}^{m} \left\| u_j \right\|_{\mathcal{H}}^2 \geq \xi_2^2 > \lambda_0^2$ in the same way as (i).

Proof. (*Of Theorem 2.2.*) Let $(u_0, u_1), (u_{20}, u_{21}), \dots, (u_{m0}, u_{m1}) \in \mathcal{H} \times L_\theta^\kappa(\mathbb{R}^n)$ satisfy both $0 \leq \mathcal{E}(0) < \mathcal{E}_0$ and $\sum_{j=1}^{m} \left\| u_{j0} \right\|_{\mathcal{H}}^2 < \lambda_0^2$. By Lemmas 2.1 and 3.1, we have

$$\begin{aligned}
&\frac{2(\kappa-1)}{\kappa} \sum_{j=1}^{m} \left\| \partial_t u_j \right\|_{L_\theta^\kappa}^\kappa + \sum_{j=1}^{m} \rho_j \left\| u_j \right\|_{\mathcal{H}}^2 \\
&\leq \frac{2(\kappa-1)}{\kappa} \sum_{j=1}^{m} \left\| \partial_t u_j \right\|_{L_\theta^\kappa}^\kappa + \sum_{j=1}^{m} \Big[\Big(1 - \int_0^t \varpi_j(s)\, ds\Big) \left\| u_j \right\|_{\mathcal{H}}^2 + \big(\varpi_j \circ u_j\big) \Big] \\
&\leq 2\mathcal{E}(t) + \frac{2\eta}{p+1} \Big(\sum_{j=1}^{m} \rho_j \left\| u_j \right\|_{\mathcal{H}}^2 \Big)^{(p+1)/2} \\
&\leq 2\mathcal{E}(0) + \frac{2\eta}{p+1} \Big(J(u_1, u_2, \dots, u_m) \Big)^{(p+1)/2} \\
&\leq 2\mathcal{E}_0 + \frac{2\eta}{p+1} \lambda_0^{p+1} \\
&= \eta^{-2/(p-1)}.
\end{aligned} \tag{3.9}$$

The proof is completed.

4. Proof of decay results

Let

$$\begin{aligned}
\Lambda(u_1, u_2, \dots, u_m) &= \frac{1}{2} \sum_{j=1}^{m} \Big[\Big(1 - \int_0^t \varpi_j(s)\, ds\Big) \left\| u_j \right\|_{\mathcal{H}}^2 + \big(\varpi_j \circ u_j\big) \Big] \\
&\quad - \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u_1, u_2, \dots, u_m) dx,
\end{aligned}$$

$$\Pi(u_1, u_2, \dots, u_m) = \sum_{j=1}^{m} \Big[\Big(1 - \int_0^t \varpi_j(s)\, ds\Big) \left\| u_j \right\|_{\mathcal{H}}^2 + \big(\varpi_j \circ u_j\big) \Big]$$

$$- \quad (p+1)\int_{\mathbb{R}^n}\theta(x)\mathcal{F}(u_1,u_2,\ldots,u_m)dx.$$

Lemma 4.1. *Let $(u_1,u_2,\ldots,u_m)$ be the solution of problem (1.1). If*

$$\sum_{j=1}^{m}\left\|u_{j0}\right\|_{\mathcal{H}}^2-(p+1)\int_{\mathbb{R}^n}\theta(x)\mathcal{F}(u_1,u_2,\ldots,u_m)dx>0. \tag{4.1}$$

Then under condition (2.6), the functional $\Pi(u_1,u_2,\ldots,u_m)>0,\ \forall t>0$.

Proof. By (4.1) and continuity, there exists a time $t_1>0$ such that

$$\Pi(u_1,u_2,\ldots,u_m)\geq 0,\forall t<t_1.$$

Let

$$Y=\Big\{(u_1,u_2,\ldots,u_m)\mid \quad \Pi(u_1(t_0),u_2(t_0),\ldots,u_m(t_0))=0, \\ \Pi(u_1,u_2,\ldots,u_m)>0,\forall t\in[0,t_0)\Big\}. \tag{4.2}$$

Then, by (4.1), we have for all $(u_1,u_2,\ldots,u_m)\in Y$,

$$\begin{aligned}
&\Lambda(u_1,u_2,\ldots,u_m)\\
&=\frac{p-1}{2(p+1)}\sum_{j=1}^{m}\left(1-\int_0^t\varpi_j(s)\,ds\right)\left\|u_j\right\|_{\mathcal{H}}^2+\frac{p-1}{2(p+1)}\sum_{j=1}^{m}\left(\varpi_j\circ u_j\right)\\
&+\frac{1}{p+1}\Pi(u_1,u_2,\ldots,u_m)\\
&\geq\frac{p-1}{2(p+1)}\sum_{j=1}^{m}\left[\rho_j\left\|u_j\right\|_{\mathcal{H}}^2+\left(\varpi_j\circ u_j\right)\right].
\end{aligned}$$

Owing to (2.4), it follows for $(u_1,u_2,\ldots,u_m)\in Y$

$$\rho_j\left\|u_j\right\|_{\mathcal{H}}^2\leq\frac{2(p+1)}{p-1}\Lambda(u_1,u_2,\ldots,u_m)\leq\frac{2(p+1)}{p-1}\mathcal{E}(t)\leq\frac{2(p+1)}{p-1}\mathcal{E}(0). \tag{4.3}$$

By (1.13), (2.6) we have

$$\begin{aligned}
&(p+1)\int_{\mathbb{R}^n}\mathcal{F}(u_1(t_0),u_2(t_0),\ldots,u_m(t_0))\\
\leq\ &\eta\sum_{j=1}^{m}\left(\rho_j\left\|u_j(t_0)\right\|_{\mathcal{H}}^2\right)^{(p+1)/2}\\
\leq\ &\eta\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{(p-1)/2}\sum_{j=1}^{m}\rho_j\left\|u_j(t_0)\right\|_{\mathcal{H}}^2\\
\leq\ &\gamma\sum_{j=1}^{m}\rho_j\left\|u_j(t_0)\right\|_{\mathcal{H}}^2
\end{aligned}$$

$$
\begin{aligned}
&< \sum_{j=1}^{m}\Big(1-\int_0^{t_0}\varpi_j(s)ds\Big)\left\|u_j(t_0)\right\|_{\mathcal{H}}^2 \\
&< \sum_{j=1}^{m}\Big(1-\int_0^{t_0}\varpi_j(s)ds\Big)\left\|u_j(t_0)\right\|_{\mathcal{H}}^2 \\
&+ \sum_{j=1}^{m}(\varpi_j\circ u_j(t_0)), \qquad (4.4)
\end{aligned}
$$

hence $\Pi(u_1(t_0),u_2(t_0),\ldots,u_m(t_0)) > 0$ on Y, which contradicts the definition of Y since $\Pi(u_1(t_0),u_2(t_0),\ldots,u_m(t_0)) = 0$. Thus $\Pi(u_1,u_2,\ldots,u_m) > 0,\ \forall t > 0$.

We are now in position to prove the decay rate.

Proof. (*Of Theorem 2.3.*) By (1.13) and (4.3), we have for $t \geq 0$

$$
0 < \sum_{j=1}^{m}\rho_j\left\|u_j\right\|_{\mathcal{H}}^2 \leq \frac{2(p+1)}{p-1}\mathcal{E}(t). \qquad (4.5)
$$

Let

$$
I(t) = \frac{\mu(t)}{1-\mu_0(t)},
$$

where μ and μ_0 defined in (1.9) and (1.10).

Noting that $\lim\limits_{t\to+\infty}\mu(t) = 0$ by (1.8)–(1.10), we have

$$
\lim_{t\to+\infty} I(t) = 0,\ \ I(t) > 0,\ \forall t \geq 0.
$$

Then we take $t_0 > 0$ such that

$$
0 < \frac{2(\kappa-1)}{\kappa}I(t) < \min\left\{2\left(\omega\lambda_1+a\right),\chi'(0)\right\}, \qquad (4.6)
$$

with (1.11) for all $t > t_0$. Due to (2.4), we have

$$
\begin{aligned}
\mathcal{E}(t) &\leq \frac{(\kappa-1)}{\kappa}\sum_{j=1}^{m}\left\|\partial_t u_j\right\|_{L_\theta^\kappa}^\kappa + \frac{1}{2}\sum_{j=1}^{m}\left(\varpi_j\circ u_j\right) + \frac{1}{2}\sum_{j=1}^{m}\left(1-\int_0^t\varpi_j(s)\,ds\right)\left\|u_j\right\|_{\mathcal{H}}^2 \\
&\leq \frac{(\kappa-1)}{\kappa}\sum_{j=1}^{m}\left\|\partial_t u_j\right\|_{L_\theta^\kappa}^\kappa + \frac{1}{2}\sum_{j=1}^{m}\left(\varpi_j\circ u_j\right) + \frac{1}{2}(1-\mu_0(t))\sum_{j=1}^{m}\left\|u_j\right\|_{\mathcal{H}}^2.
\end{aligned}
$$

Then by definition of $I(t)$, we have

$$
I(t)\mathcal{E}(t) \leq \frac{(\kappa-1)}{\kappa}I(t)\sum_{j=1}^{m}\left\|\partial_t u_j\right\|_{L_\theta^\kappa}^\kappa + \frac{1}{2}\mu(t)\sum_{j=1}^{m}\left\|u_j\right\|_{\mathcal{H}}^2 + \frac{1}{2}I(t)\sum_{j=1}^{m}\left(\varpi_j\circ u_j\right), \qquad (4.7)
$$

and Lemma 2.1, we have for all $t_1, t_2 \geq 0$,

$$
\mathcal{E}(t_2) - \mathcal{E}(t_1)
$$

$$\leq -\int_{t_1}^{t_2}\left(a\sum_{j=1}^{m}\left\|\partial_t u_j\right\|_{L_\theta^2}^2+\omega\sum_{j=1}^{m}\left\|\partial_t u_j\right\|_{\mathcal{H}}^2+\frac{1}{2}\mu(t)\sum_{j=1}^{m}\left\|u_j\right\|_{\mathcal{H}}^2\right)dt$$
$$+\int_{t_1}^{t_2}\frac{1}{2}\sum_{j=1}^{m}(\varpi_j'\circ u_j)\,dt,$$

then

$$\mathcal{E}'(t) \leq -(\omega\lambda_1+a)\sum_{j=1}^{m}\left\|\partial_t u_j\right\|_{L_\theta^2}^2-\frac{1}{2}\mu(t)\sum_{j=1}^{m}\left\|u_j\right\|_{\mathcal{H}}^2+\frac{1}{2}\sum_{j=1}^{m}(\varpi_j'\circ u_j).$$

Then, by (4.6), $\forall t \geq t_0$, we get

$$\begin{aligned}
&\mathcal{E}'(t)+I(t)\mathcal{E}(t)\\
\leq\ & \left\{\frac{(\kappa-1)}{\kappa}I(t)-(\omega\lambda_1+a)\right\}\sum_{j=1}^{m}\left\|\partial_t u_j\right\|_{L_\theta^2}^2\\
+\ & \frac{1}{2}\sum_{j=1}^{m}(\varpi_j'\circ u_j)+\frac{1}{2}I(t)\sum_{j=1}^{m}(\varpi_j\circ u_j)\\
\leq\ & \frac{1}{2}\sum_{j=1}^{m}\int_0^t\left\{\varpi_j'(t-\tau)+I(t)\varpi_j(t-\tau)\right\}\left\|u_j(t)-u_j(\tau)\right\|_{\mathcal{H}}^2\,d\tau\\
\leq\ & \frac{1}{2}\sum_{j=1}^{m}\int_0^t\left\{\varpi_j'(\tau)+I(t)\varpi_j(\tau)\right\}\left\|u_j(t)-u_j(t-\tau)\right\|_{\mathcal{H}}^2\,d\tau\\
\leq\ & \frac{1}{2}\sum_{j=1}^{m}\int_0^t\left\{-\chi\big(\varpi_j(\tau)\big)+\chi'(0)\varpi_j(\tau)\right\}\left\|u_j(t)-u_j(t-\tau)\right\|_{\mathcal{H}}^2\,d\tau\\
\leq\ & 0,
\end{aligned}$$

by the convexity of χ and (1.11), we have

$$\chi(\xi)\geq\chi(0)+\chi'(0)\xi=\chi'(0)\xi.$$

Then

$$\mathcal{E}(t)\leq\mathcal{E}(t_0)\exp\left(-\int_{t_0}^{t}I(s)ds\right),$$

which completes the proof.

5. Conclusions

The analysis carried out showed that modeling of wave processes in $\mathbb{R}^n$, taking into account geometric and physical nonlinearities and dispersion, which is a consequence of thin-walledness, leads to the identification of effects that are impossible within the framework of linear theories.

The dissertation work developed the fundamentals of nonlinear wave dynamics in $\mathbb{R}^n$. Thus, a general theoretical approach to the study of wave processes in non-one-dimensional deformable systems has been developed.

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Conflict of interest

The authors declare there is no conflicts of interest.

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