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Research article

Extended DEA method for solving multi-objective transportation problem with Fermatean fuzzy sets

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Abstract: Data envelopment analysis (DEA) is a linear programming approach used to determine the relative efficiencies of multiple decision-making units (DMUs). A transportation problem (TP) is a special type of linear programming problem (LPP) which is used to minimize the total transportation cost or maximize the total transportation profit of transporting a product from multiple sources to multiple destinations. Because of the connection between the multi-objective TP (MOTP) and DEA, DEA-based techniques are more often used to handle practical TPs. The objective of this work is to investigate the TP with Fermatean fuzzy costs in the presence of numerous conflicting objectives. In particular, a Fermatean fuzzy DEA (FFDEA) method is proposed to solve the Fermatean fuzzy MOTP (FFMOTP). In this regard, every arc in FFMOTP is considered a DMU. Additionally, those objective functions that should be maximized will be used to define the outputs of DMUs, while those that should be minimized will be used to define the inputs of DMUs. As a consequence, two different Fermatean fuzzy effciency scores (FFESs) will be obtained for every arc by solving the FFDEA models. Therefore, unique FFESs will be obtained for every arc by finding the mean of these FFESs. Finally, the FFMOTP will be transformed into a single objective Fermatean fuzzy TP (FFTP) that can be solved by applying standard algorithms. A numerical example is illustrated to support the proposed method, and the results obtained by using the proposed method are compared to those of existing techniques. Moreover, the advantages of the proposed method are also discussed.

Keywords: multi-objective transportation problem; data envelopment analysis; Fermatean fuzzy arithmetic; triangular Fermatean fuzzy number
Mathematics Subject Classification: 90C32, 90C70

1. Introduction

Data envelopment analysis (DEA) is a relatively new "data-oriented" technique for assessing the performance of a group of two or more entities known as decision making units (DMUs) that convert multiple inputs into multiple outputs. DEA allows for the evaluation of multiple inputs and outputs at the same time without any assumptions on the distribution of data. Charnes et al. [1, 2] designed DEA as a non-parametric approach for assessing the performance of predefined DMUs.

The problems in real life are too complex, and this complexity includes uncertainty in the form of ambiguity, chance, or insufficient knowledge. Most of the parameters of the problem are defined using language statements. Therefore, treating the decision-maker's knowledge as fuzzy data will provide better results. In humanistic systems, fuzzy modelling is a mathematical way of expressing ambiguity and fuzziness. To handle the ambiguity and fuzziness of goods in real-world problems, Zadeh [3] developed the concept of the fuzzy set (FS). Applications of fuzzy set theory in matrix games can be seen in [4]. Meanwhile, the FS theory could not judge the nature of satisfaction and dissatisfaction with human judgments. To overcome this shortcoming, Atanassov [5] presented a theory of intuitionistic FSs (IFS) in 1986, which is an extension of the FS theory and is extremely effective in dealing with imprecise information in real-world applications. Yager [6, 7] developed the idea of a Pythagorean fuzzy set (PFS) in 2013, with the relaxing condition that the sum of the squares of the belongingness and non-membership degrees should not exceed 1. Due to the restriction in PFS, Senapati and Yager [8–10] introduced the theory of Fermatean fuzzy sets (FFS), a more generic model than PFS in which the sum of the cubes of membership and non-membership degrees should be less than or equal to 1. Further discussions and different applications related to Fermatean fuzzy sets are also observed in [11, 12].

Linear programming (LP) is a fundamental method that uses linear functions to represent complex connections and then discovers the optimal places. LP is used to find the optimal solution to a problem with given constraints. We transform a real-world problem into a mathematical model in LP. "Fuzzy LP" is concerned with the optimization of a variable function known as the "fuzzy objective function" subject to a system of fuzzy linear equations and/or inequalities known as "restrictions" or "fuzzy constraints." Bellman and Zadeh [13] introduced the idea of decision-making in a fuzzy environment. Zimmerman [14] proposed the concept of a fuzzy linear programming problem (LPP). Allahviranloo et al. [15] solved fully fuzzy LPPs. Akram et al. [16–20] proposed different methods for solving the fully Pythagorean fuzzy LPPs, and Mehmood et al. [21, 22] developed fully bipolar fuzzy LP models. Ahmad et al. [23] developed a novel method for assessing LPP in a bipolar single-valued neutrosophic environment.

A transportation problem (TP) is a special type of LPP in which goods are transported from multiple sources to multiple destinations, subject to the supply and demand of the sources and destinations, respectively. The basic idea of the TP is to minimize the total cost of transportation. Hitchcock [24] originally introduced the concept of transportation in 1941 to transport commodities from multiple sources to a number of destinations. Because of the relationship between the MOTP and DEA, DEA-based techniques are more suited for dealing with real-world TPs. The literature reviews of TPs and DEA is given in Table 1.

Table 1. Literature review of TPs and DEA.

Reference	Year	Significance Influence
Banker et al. [25]	1984	Developed some models in DEA
Ahn [26]	1988	Devised some statistical and DEA evaluations of relative efficiencies
Roll et al. [27]	1991	Controlling factor weights in DEA
Sengupta [28]	1992	Introduced a fuzzy technique in DEA
Kao and Liu [29]	2000	Measured fuzzy efficiency in DEA
Saati et al. [30]	2002	Efficiency analyzed and ranked DMUs using fuzzy data
Lertworasirikul et al. [31]	2003	Introduced possibility approach of fuzzy DEA
Zerafat et al. [32]	2003	Proposed an alternative approach to assignment problem using common set of weights in DEA
Cooper et al. [33]	2006	Provided introduction to DEA and its uses
Zhou et al. [34]	2008	A survey of DEA in energy and environmental studies
Guo [35]	2009	Applications of fuzzy DEA in locating problems
Lotfi et al. [36]	2009	Efficiency and effectiveness in multi-activity network DEA model
Lotfi et al. [37]	2010	Introduced relationship between multi-objective LP (MOLP) and DEA on CCR dual model
Mousavi-Avval et al. [38]	2011	Introduced an optimization approach for apple production using DEA
Amirteimoori [39]	2011	Developed an extended TP based on DEA
Amirteimoori [40]	2012	Devised an extended shortest path problem based on DEA
Nabavi-Pelesaraei [41]	2014	Introduced optimization of energy required and greenhouse gas emission in DEA
Zhu et al. [42]	2014	Applied a network DEA model to quantify the eco-efficiency of products
Azadi et al. [43]	2015	Developed a new fuzzy DEA model to evaluate efficiency in management context
Shirdel and Mortezaee [44]	2015	Proposed method for multi-criteria assignment problem using DEA
Azar et al. [45]	2016	Introduced new model to determine common set of weights in DEA
Mardania et al. [46]	2017	Presented a comprehensive review of DEA technique
Hatami-Marbini et al. [47]	2017	Measured fuzzy efficiency in DEA
Hatami-Marbini and Saati [48]	2018	Evaluated efficiency in two-stage DEA under fuzzy environment
Rizk-Allaha et al. [49]	2018	Developed MOTP under neutrosophic environment
Tavana et al. [50]	2018	Developed a hybrid DEA-MOLP model
Edalatpanah and Smarandache [51]	2019	Proposed DEA for simplified neutrosophic sets
Liu and Song [52]	2019	Group decision making based on DEA cross-efficiency using IFS
Edalatpanah [53]	2020	Developed DEA using triangular neutrosophic numbers
Bagheri et al. [54]	2020	Solved fully fuzzy MOTP using common set of weights in DEA
Soltani et al. [55]	2020	Developed a new two-stage DEA model in fuzzy environment
Sahoo [56]	2021	Studied Fermatean fuzzy TP based on new ranking function
Mondal et al. [57]	2021	Investigated intuitionistic fuzzy sustainable multi-objective multi-item multi-choice step fixed-charge solid TP
Ghosh et al. [58]	2021	Studied multi-objective fully intuitionistic fuzzy fixed-charge solid TP
Giri and Roy [59]	2022	Evaluated neutrosophic multi-objective green four-dimensional fixed-charge TP
Ghosh et al. [60]	2022	Studied carbon mechanism on sustainable multi-objective solid TP for waste management in Pythagorean hesitant fuzzy environment
Akram et al. [61]	2022	Obtained the solution of Fermatean fuzzy transportation problem

In the literature, numerous methods have been developed to solve the MOTP fuzzy environment and intuitionistic fuzzy environment. Bagheri et al. [62] solved the MOTP using the DEA technique in a fuzzy environment by considering cost coefficients as triangular fuzzy numbers. Since the FS theory could not judge the nature of satisfaction and dissatisfaction with human judgments, our objective is to extend the DEA technique to solve the MOTP in a Fermatean fuzzy environment, because the FFS can handle situations in which uncertainty and ambiguity include hesitation. Therefore, we investigate MOTP in a Fermatean fuzzy environment using triangular Fermatean fuzzy numbers (TFFNs). Our main contributions are as follows:

1) Formulating the model of DEA in a Fermatean fuzzy environment.
2) Solving FFMOTP using FFDEA method by transforming it into single objective FFTP and then solving single objective FFTP by converting it into a crisp one with the help of a ranking function.

3) Demonstrating the proposed method's validity with an example.
4) Comparing the results of the proposed method with those of existing techniques.
5) Providing the advantages of the proposed method over the existing techniques.

The rest of the paper is structured as follows: Section 2 gives some basic definitions and operations of triangular Fermatean fuzzy numbers (TFFNs). In Section 3, the mathematical models of FFMOTP and FFDEA are given. Section 4 presents the procedure for solving FFMOTP. A numerical example and a comparative analysis are given in Section 5. Section 6 concludes the study.

2. Preliminaries

Definition 2.1. [8] Let X be a universal set. A *Fermatean fuzzy set* (FFS) $\tilde{A}^F$ on X is an object of the form

$$\tilde{A}^F = \{\langle y, \mu_{\tilde{A}^F}(y), \nu_{\tilde{A}^F}(y)\rangle : y \in X\},$$

where $\mu_{\tilde{A}^F} : X \to [0,1]$, $\nu_{\tilde{A}^F} : X \to [0,1]$, and

$$0 \leq (\mu_{\tilde{A}^F}(y))^3 + (\nu_{\tilde{A}^F}(y))^3 \leq 1,$$

for all $y \in X$. The values $\mu_{\tilde{A}^F}(y)$ and $\nu_{\tilde{A}^F}(y)$ represent the membership and non-membership degrees of the element y in the set $\tilde{A}^F$, respectively. Further, for all $y \in X$,

$$\pi_{\tilde{A}^F}(y) = \sqrt[3]{1 - (\mu_{\tilde{A}^F}(y))^3 - (\nu_{\tilde{A}^F}(y))^3}$$

represents the degree of hesitation for the element y in $\tilde{A}^F$.

Definition 2.2. [9] A *Fermatean fuzzy number* (FFN) $\tilde{A}^F$ is a FFS defined on $\mathbb{R}$ with the following conditions:
1) normal, i.e., there exists $y \in \mathbb{R}$ such that

$$\mu_{\tilde{A}^F}(y) = 1 \quad (\text{so} \;\; \nu_{\tilde{A}^F}(y) = 0);$$

2) convex for the membership function (MF) ($\mu_{\tilde{A}^F}$), i.e.,

$$\mu_{\tilde{A}^F}(\delta x + (1-\delta)y) \geq \min\{\mu_{\tilde{A}^F}(x), \mu_{\tilde{A}^F}(y)\}, \forall x, y \in \mathbb{R}, \delta \in [0,1];$$

3) concave for the non-membership function (NMF) ($\nu_{\tilde{A}^F}$), i.e.,

$$\nu_{\tilde{A}^F}(\delta x + (1-\delta)y) \leq \max\{\nu_{\tilde{A}^F}(x), \nu_{\tilde{A}^F}(y)\}, \forall x, y \in \mathbb{R}, \delta \in [0,1].$$

We give here some basic definitions.

Definition 2.3. A *triangular Fermatean fuzzy number* (TFFN) $\tilde{A}^F = \{(u^l, u^m, u^r); p, q\}$ is a FFS with the MF ($\mu_{\tilde{A}^F}$) and NMF ($\nu_{\tilde{A}^F}$) given as

$$\mu_{\tilde{A}^F}(y) = \begin{cases} \dfrac{(y-u^l)p}{u^m - u^l}, & u^l \leq y < u^m, \\ p, & y = u^m, \\ \dfrac{(u^r - y)p}{u^r - u^m}, & u^m < y \leq u^r, \\ 0, & y < u^l \;\; or \;\; y > u^r, \end{cases}$$

$$\nu_{\tilde{A}^F}(y) = \begin{cases} \dfrac{[u^m - y + q(y - u^l)]}{u^m - u^l}, & u^l \leq y < u^m, \\ q, & y = u^m, \\ \dfrac{[y - u^m + q(u^r - y)]}{u^r - u^m}, & u^m < y \leq u^r, \\ 1, & y < u^l \; or \; y > u^r. \end{cases}$$

The values p and q represent the maximum value of MF ($\mu_{\tilde{A}^F}$) and minimum value of NMF ($\nu_{\tilde{A}^F}$), respectively, such that $p \in [0, 1]$, $q \in [0, 1,]$, and

$$0 \leq p^3 + q^3 \leq 1.$$

By taking $p = 1$ and $q = 0$ in Definition 2.3, the TFFN $\tilde{A}^F$ assumes the form $\tilde{A}^F = \{(u^l, u^m, u^r); (u^{l'}, u^m, u^{r'})\}$ whose MF ($\mu_{\tilde{A}^F}$) and NMF ($\nu_{\tilde{A}^F}$) can be represented by

$$\mu_{\tilde{A}^F}(y) = \begin{cases} \dfrac{y - u^l}{u^m - u^l}, & u^l \leq y < u^m, \\ 1, & y = u^m, \\ \dfrac{u^r - y}{u^r - u^m}, & u^m < y \leq u^r, \\ 0, & y < u^l \;\; or \;\; y > u^r, \end{cases}$$

$$\nu_{\tilde{A}^F}(y) = \begin{cases} \dfrac{u^m - y}{u^m - u^{l'}}, & u^{l'} \leq y < u^m, \\ 0, & y = u^m, \\ \dfrac{y - u^m}{u^{r'} - u^m}, & u^m < y \leq u^{r'}, \\ 1, & y < u^{l'} \;\; or \;\; y > u^{r'}, \end{cases}$$

where $u^{l'} \leq u^l \leq u^m \leq u^r \leq u^{r'}$. The graphical representation of TrFFN is given in Figure 1.

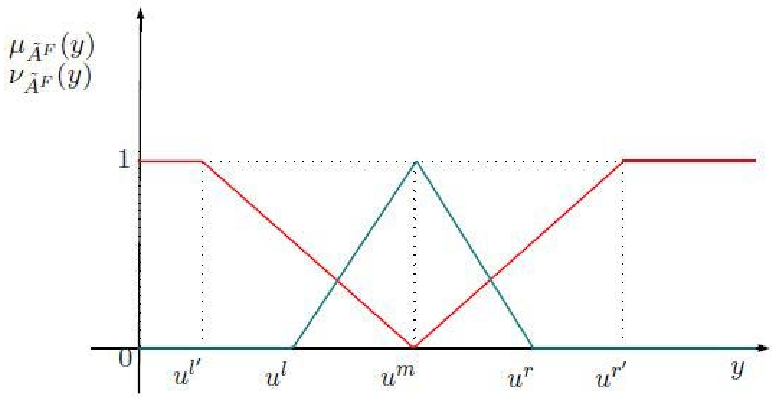

Figure 1. MF and NMF of TFFN.

Definition 2.4. A TFFN $\tilde{A}^F = \{(u^l, u^m, u^r); (u^{l'}, u^m, u^{r'})\}$ is regarded to be non-negative ($\tilde{A}^F \geq 0$) if $u^{l'} \geq 0$.

Definition 2.5. A TFFN $\tilde{A}^F = \{(u^l, u^m, u^r); (u^{l'}, u^m, u^{r'})\}$ is regarded to be non-positive ($\tilde{A}^F \leq 0$) if $u^{r'} \leq 0$.

Definition 2.6. A TFFN $\tilde{A}^F = \{(u^l, u^m, u^r); (u^{l'}, u^m, u^{r'})\}$ is regarded to be unrestricted if $u^{l'} \in \mathbb{R}$.

Definition 2.7. A TFFN $\tilde{A}^F = \tilde{0}$ if and only if $u^l = 0, u^m = 0, u^r = 0, u^{l'} = 0, u^{r'} = 0$.

Definition 2.8. Two TFFNs $\tilde{A}^F = \{(u^l, u^m, u^r); (u^{l'}, u^m, u^{r'})\}$, and $\tilde{B}^F = \{(v^l, v^m, v^r); (v^{l'}, v^m, v^{r'})\}$ are regarded to be equal if, and only if, $u^l = v^l$, $u^m = v^m$, $u^r = v^r$, $u^{l'} = v^{l'}$, $u^{r'} = v^{r'}$.

Definition 2.9. Let $\tilde{A}^F = \{(u^l, u^m, u^r); (u^{l'}, u^m, u^{r'})\}$, and $\tilde{B}^F = \{(v^l, v^m, v^r); (v^{l'}, v^m, v^{r'})\}$ be positive TFFNs. Then,

1) $\tilde{A}^F \oplus \tilde{B}^F = \{(u^l + v^l, u^m + v^m, u^r + v^r); (u^{l'} + v^{l'}, u^m + v^m, u^{r'} + v^{r'})\}$,
2) $\tilde{A}^F \ominus \tilde{B}^F = \{(u^l - v^r, u^m - v^m, u^r - v^l); (u^{l'} - v^{r'}, u^m - v^m, u^{r'} - v^{l'})\}$,
3) $\tilde{A}^F \otimes \tilde{B}^F = \{(u^l v^l, u^m v^m, u^r v^r); (u^{l'} v^{l'}, u^m v^m, u^{r'} v^{r'})\}$,
4) $\tilde{A}^F \oslash \tilde{B}^F = \{(u^l/v^l, u^m/v^m, u^r/v^r); (u^{l'}/v^{l'}, u^m/v^m, u^{r'}/v^{r'})\}$.

Definition 2.10. Let $\tilde{A}^F = \{(u^l, u^m, u^r); (u^{l'}, u^m, u^{r'})\}$ be a TFFN. Then, its ranking function is defined as

$$\Re(\tilde{A}^F) = \frac{(u^l + 4u^m + u^r) + (u^{l'} + 4u^m + u^{r'})}{12}.$$

Definition 2.11. Let $\tilde{A}^F = \{(u^l, u^m, u^r); (u^{l'}, u^m, u^{r'})\}$ and $\tilde{B}^F = \{(v^l, v^m, v^r); (v^{l'}, v^m, v^{r'})\}$ be two TFFNs. Then,

1) $\tilde{A}^F \leq \tilde{B}^F$ if $\Re(\tilde{A}^F) \leq \Re(\tilde{B}^F)$,
2) $\tilde{A}^F \geq \tilde{B}^F$ if $\Re(\tilde{A}^F) \geq \Re(\tilde{B}^F)$,
3) $\tilde{A}^F \approx \tilde{B}^F$ if $\Re(\tilde{A}^F) = \Re(\tilde{B}^F)$.

All indexes, parameters, and decision variables used in this manuscript are given in Table 2.

Table 2. Notation List for indices and parameters.

Symbol	Definition
j	Number of sources ($p = 1, 2, 3, \ldots, j$)
k	Number of destinations ($q = 1, 2, 3, \ldots, k$)
h	Total number of attributes ($a = 1, 2, 3, \ldots, h$)
$\tilde{c}^a_{pq}$	Unit Fermatean fuzzy cost of transportation from source p to destination q
x_{pq}	Fermatean fuzzy amount shipped from source p to destination q
s_p	Supply at p
d_q	Demand at q

3. Mathematical model of FFMOTP

The Fermatean fuzzy MOTP is a subclass of multi-objective LPP in which some of the parameters are expressed as FFNs. Assume that j sources contain various quantities of a product which must be delivered to k destinations. For each link (p, q) from source p to destination q, there are h Fermatean fuzzy attributes $\tilde{c}^a_{pq}$ $(a = 1, \ldots, h)$ for transportation. The goal is to find a feasible shipping plan from sources to destinations in order to optimize the objective functions. Assume s_p represents the supply of a product at the source p, and d_q represents the demand for a product at the destination q. Let x_{pq} represent the amount of product shipped from source p to destination q. Then, the FFMOTP having h Fermatean fuzzy objectives can be written as the following model:

$$\begin{aligned} \text{Optimize } Z_a &= \sum_{p=1}^{j}\sum_{q=1}^{k} \tilde{c}_{pq}^{a} \times x_{pq}, \quad a = 1,2,3,\dots,h, \\ \text{subject to } & \sum_{q=1}^{k} x_{pq} = s_p, \quad p = 1,2,3,\dots,j, \\ & \sum_{p=1}^{j} x_{pq} = d_q, \quad q = 1,2,3,\dots,k, \\ & x_{pq} \geq 0, \quad p = 1,2,3,\dots,j, \quad q = 1,2,3,\dots,k. \end{aligned} \tag{3.1}$$

However, the best possible solution of FFMOTP (3.1) is the ideal solution given as follows:

$$\begin{aligned} \tilde{f}_a = \text{Optimize } & \sum_{p=1}^{j}\sum_{q=1}^{k} \tilde{c}_{pq}^{a} \times x_{pq}, \quad a = 1,2,3,\dots,h, \\ \text{subject to } & \sum_{q=1}^{k} x_{pq} = s_p, \quad p = 1,2,3,\dots,j, \\ & \sum_{p=1}^{j} x_{pq} = d_q, \quad q = 1,2,3,\dots,k, \\ & x_{pq} \geq 0, \quad p = 1,2,3,\dots,j, \quad q = 1,2,3,\dots,k. \end{aligned} \tag{3.2}$$

Because the objective functions of the FFMOTP are incompatible, it is impossible to determine the ideal solution.

3.1. Mathematical model of FFDEA

Data envelopment analysis (DEA) is a mathematical approach for determining the relative efficiencies of decision making units (DMUs) with many inputs and outputs.

Suppose the efficiencies of n DMUs are to be evaluated. Each DMU_q ($q = 1,\dots,k$) produces s different Fermatean fuzzy outputs $\tilde{y}_q = (\tilde{y}_{1q},\dots,\tilde{y}_{sq})$, using m different Fermatean fuzzy inputs $\tilde{x}_p = (\tilde{x}_{1p},\dots,\tilde{x}_{mp})$. The model to evaluate the relative efficiency of DMU_r is as follows:

$$\begin{aligned} \text{Max } \tilde{\phi}_r &= \frac{\sum_{b=1}^{s} u_b \tilde{y}_{br}}{\sum_{p=1}^{j} v_p \tilde{x}_{pr}} \\ \text{subject to } \tilde{\phi}_q &= \frac{\sum_{b=1}^{s} u_b \tilde{y}_{bq}}{\sum_{p=1}^{j} v_p \tilde{x}_{pq}} \leq \tilde{1}, \quad q = 1,2,3,\dots,k, \\ & u_b, v_p \geq 0, \quad b = 1,2,3,\dots,s, \quad p = 1,2,3,\dots,j, \end{aligned} \tag{3.3}$$

where the first constraint in this model indicates that relative efficiency should be less than 1. The DMU with relative efficiency 1 would be the most efficient. Here, $\tilde{1} = \{(1, 1, 1); (1, 1, 1)\}$, and u_b ($b = 1, 2, 3, \ldots, s$) and v_p ($p = 1, 2, 3, \ldots, j$) are weights assigned to the outputs and inputs, respectively. An extended arithmetic approach based on Wang et al. [63] has been used in this work to solve the FFDEA model (3.3). Without loss of generality, all input and output data are assumed to be positive TFFNs to describe this approach briefly. Assume that the positive TFFNs $\tilde{x}_{pq} = \{(x^l_{pq}, x^m_{pq}, x^r_{pq}); (x^{l'}_{pq}, x^m_{pq}, x^{r'}_{pq})\}$ and $\tilde{y}_{bq} = \{(y^l_{bq}, y^m_{bq}, y^r_{bq}); (y^{l'}_{bq}, y^m_{bq}, y^{r'}_{bq})\}$ denotes the input and output data of DMU_q ($q = 1, \ldots, k$), respectively, for all $p = 1, 2, 3, \ldots, j$ and $b = 1, 2, 3, \ldots, s$. Then, using the Fermatean fuzzy arithmetic, the Fermatean fuzzy efficiency of DMU_t can be evaluated as follows:

$$
\begin{aligned}
\text{Max } \tilde{\phi}_t &\approx \{(\phi^l_t, \phi^m_t, \phi^r_t); (\phi^{l'}_t, \phi^m_t, \phi^{r'}_t)\} \\
&= \{(\frac{\sum_{b=1}^s u_b y^l_{bt}}{\sum_{p=1}^j v_p x^r_{pt}}, \frac{\sum_{b=1}^s u_b y^m_{bt}}{\sum_{p=1}^j v_p x^m_{pt}}, \frac{\sum_{b=1}^s u_b y^r_{bt}}{\sum_{p=1}^j v_p x^l_{pt}}); (\frac{\sum_{b=1}^s u_b y^{l'}_{bt}}{\sum_{p=1}^j v_p x^{r'}_{pt}}, \frac{\sum_{b=1}^s u_b y^m_{bt}}{\sum_{p=1}^j v_p x^m_{pt}}, \frac{\sum_{b=1}^s u_b y^{r'}_{bt}}{\sum_{p=1}^j v_p x^{l'}_{pt}})\}
\end{aligned}
$$

subject to

$$
\begin{aligned}
\tilde{\phi}_q &\approx \{(\phi^l_q, \phi^m_q, \phi^r_q); (\phi^{l'}_q, \phi^m_q, \phi^{r'}_q)\} \\
&= \{(\frac{\sum_{b=1}^s u_b y^l_{bq}}{\sum_{p=1}^j v_p x^r_{pq}}, \frac{\sum_{b=1}^s u_b y^m_{bq}}{\sum_{p=1}^j v_p x^m_{pq}}, \frac{\sum_{b=1}^s u_b y^r_{bq}}{\sum_{p=1}^j v_p x^l_{pq}}); (\frac{\sum_{b=1}^s u_b y^{l'}_{bq}}{\sum_{p=1}^j v_p x^{r'}_{pq}}, \frac{\sum_{b=1}^s u_b y^m_{bq}}{\sum_{p=1}^j v_p x^m_{pq}}, \frac{\sum_{b=1}^s u_b y^{r'}_{bq}}{\sum_{p=1}^j v_p x^{l'}_{pq}})\} \\
&\leq \{(1, 1, 1); (1, 1, 1)\}, \quad q = 1, \ldots, k, \\
& u_b, v_p \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad p = 1, 2, 3, \ldots, j.
\end{aligned} \tag{3.4}
$$

As $\phi^{r'}_q \leq 1$, $\phi^{l'}_q$, ϕ^l_q, ϕ^m_q and ϕ^r_q will also be less than or equal to one. To find the Fermatean fuzzy efficiency of DMU_t, model (3.4) is transformed into the following five LP models:

$$
\begin{aligned}
\text{Max } \phi^{l'}_t &= \frac{\sum_{b=1}^s u_b y^{l'}_{bt}}{\sum_{p=1}^j v_p x^{r'}_{pt}} \\
\text{subject to } \phi^{r'}_q &= \frac{\sum_{b=1}^s u_b y^{r'}_{bq}}{\sum_{p=1}^j v_p x^{l'}_{pq}} \leq 1, \quad q = 1, \ldots, k, \\
& u_b, v_p \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad p = 1, 2, 3, \ldots, j.
\end{aligned} \tag{3.5}
$$

Using the optimal weights of the above model, ϕ^l_t is computed as follows:

$$
\begin{aligned}
\text{Max } \phi^l_t &= \frac{\sum_{b=1}^s u_b y^l_{bt}}{\sum_{p=1}^j v_p x^r_{pt}} \\
\text{subject to } & \frac{\sum_{b=1}^s u_b y^{l'}_{bt}}{\sum_{p=1}^j v_p x^{r'}_{pt}} = \phi^{l'*}_t, \quad \phi^{r'}_q = \frac{\sum_{b=1}^s u_b y^{r'}_{bq}}{\sum_{p=1}^j v_p x^{l'}_{pq}} \leq 1, \quad q = 1, \ldots, k, \\
& u_b, v_p \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad p = 1, 2, 3, \ldots, j,
\end{aligned} \tag{3.6}
$$

where $\phi^{l'*}_t$ is the optimum value of the model (3.5). Then, using the optimal weights of the models (3.5) and (3.6), ϕ^m_t is determined as follows:

$$
\text{Max } \phi^m_t = \frac{\sum_{b=1}^s u_b y^m_{bt}}{\sum_{p=1}^j v_p x^m_{pt}}
$$

$$\text{subject to } \frac{\sum_{b=1}^{s} u_b y_{bt}^{l'}}{\sum_{p=1}^{j} v_p x_{pt}^{r'}} = \phi_t^{l'*}, \quad \frac{\sum_{b=1}^{s} u_b y_{bt}^{l}}{\sum_{p=1}^{j} v_p x_{pt}^{r}} = \phi_t^{l*},$$
$$\phi_q^{r'} = \frac{\sum_{b=1}^{s} u_b y_{bq}^{r'}}{\sum_{p=1}^{j} v_p x_{pq}^{l'}} \leq 1, \quad q = 1, \ldots, k,$$
$$u_b, v_p \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad p = 1, 2, 3, \ldots, j, \tag{3.7}$$

where $\phi_t^{l'*}$ and ϕ_t^{l*} are optimum values of the models (3.5) and (3.6), respectively. To determine ϕ_t^{r}, optimal weights of the models (3.5)–(3.7) are used as:

$$\text{Max } \phi_t^{r} = \frac{\sum_{b=1}^{s} u_b y_{bt}^{r}}{\sum_{p=1}^{j} v_p x_{pt}^{l}}$$
$$\text{subject to } \frac{\sum_{b=1}^{s} u_b y_{bt}^{l'}}{\sum_{p=1}^{j} v_p x_{pt}^{r'}} = \phi_t^{l'*}, \quad \frac{\sum_{b=1}^{s} u_b y_{bt}^{l}}{\sum_{p=1}^{j} v_p x_{pt}^{r}} = \phi_t^{l*},$$
$$\frac{\sum_{b=1}^{s} u_b y_{bt}^{m}}{\sum_{p=1}^{j} v_p x_{pt}^{m}} = \phi_t^{m*}, \quad \phi_q^{r'} = \frac{\sum_{b=1}^{s} u_b y_{bq}^{r'}}{\sum_{p=1}^{j} v_p x_{pq}^{l'}} \leq 1, \quad q = 1, \ldots, k,$$
$$u_b, v_p \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad p = 1, 2, 3, \ldots, j, \tag{3.8}$$

where $\phi_t^{l'*}$, ϕ_t^{l*} and ϕ_t^{m*} are optimum values of models (3.5)–(3.7), respectively. Now, using the optimal weights of the models (3.5)–(3.8), $\phi_t^{r'}$ is determined as follows:

$$\text{Max } \phi_t^{r'} = \frac{\sum_{b=1}^{s} u_b y_{bt}^{r'}}{\sum_{p=1}^{j} v_p x_{pt}^{l'}}$$
$$\text{subject to } \frac{\sum_{b=1}^{s} u_b y_{bt}^{l'}}{\sum_{p=1}^{j} v_p x_{pt}^{r'}} = \phi_t^{l'*}, \quad \frac{\sum_{b=1}^{s} u_b y_{bt}^{l}}{\sum_{p=1}^{j} v_p x_{pt}^{r}} = \phi_t^{l*},$$
$$\frac{\sum_{b=1}^{s} u_b y_{bt}^{m}}{\sum_{p=1}^{j} v_p x_{pt}^{m}} = \phi_t^{m*}, \quad \frac{\sum_{b=1}^{s} u_b y_{bt}^{r}}{\sum_{p=1}^{j} v_p x_{pt}^{l}} = \phi_t^{r*},$$
$$\phi_q^{r'} = \frac{\sum_{b=1}^{s} u_b y_{bq}^{r'}}{\sum_{p=1}^{j} v_p x_{pq}^{l'}} \leq 1, \quad q = 1, \ldots, k,$$
$$u_b, v_p \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad p = 1, 2, 3, \ldots, j, \tag{3.9}$$

where $\phi_t^{l'*}$, ϕ_t^{l*}, ϕ_t^{m*} and ϕ_t^{r*} are optimum values of models (3.5)–(3.8), respectively. As a result, $\phi_t^{l'*}$, ϕ_t^{l*}, ϕ_t^{m*}, ϕ_t^{r*} and $\phi_t^{r'*}$ are determined with the same set of weights.

Theorem 3.1. *The Fermatean fuzzy efficiency of DMU_t from models (3.5)–(3.9) yields a non-negative TFFN.*

Proof. Suppose $(u^*, v^*) = (u_1^*, \ldots, u_s^*, v_1^*, \ldots, v_j^*)$ is the optimum solution of model (3.9). Then, from the last constraints of the model, we have $(u^*, v^*) \geq 0$. Since, the Fermatean fuzzy input $\tilde{x}_{pq} = \{(x_{pq}^{l}, x_{pq}^{m}, x_{pq}^{r}); (x_{pq}^{l'}, x_{pq}^{m}, x_{pq}^{r'})\}$ and output $\tilde{y}_{bq} = \{(y_{bq}^{l}, y_{bq}^{m}, y_{bq}^{r}); (y_{bq}^{l'}, y_{bq}^{m}, y_{bq}^{r'})\}$ are non-negative TFFNs, we have

$$0 \leq x_{pq}^{l'} \leq x_{pq}^{l} \leq x_{pq}^{m} \leq x_{pq}^{r} \leq x_{pq}^{r'}, \quad p = 1, 2, 3, \ldots, j,$$

$$0 \le x_{bq}^{l'} \le x_{bq}^{l} \le x_{bq}^{m} \le x_{bq}^{r} \le x_{bq}^{r'}, \quad b = 1, 2, 3, \ldots, s.$$

Therefore,

$$0 \le \sum_{p=1}^{j} v_p^* x_{pq}^{l'} \le \sum_{p=1}^{j} v_p^* x_{pq}^{l} \le \sum_{p=1}^{j} v_p^* x_{pq}^{m} \le \sum_{p=1}^{j} v_p^* x_{pq}^{r} \le \sum_{p=1}^{j} v_p^* x_{pq}^{r'},$$
$$0 \le \sum_{b=1}^{s} u_b^* x_{bq}^{l'} \le \sum_{b=1}^{s} u_b^* x_{bq}^{l} \le \sum_{b=1}^{s} u_b^* x_{bq}^{m} \le \sum_{b=1}^{s} u_b^* x_{bq}^{r} \le \sum_{b=1}^{s} u_b^* x_{bq}^{r'}.$$

Consequently,

$$0 \le \frac{\sum_{b=1}^{s} u_b^* y_{bq}^{l'}}{\sum_{p=1}^{j} v_p^* x_{pq}^{r'}} \le \frac{\sum_{b=1}^{s} u_b^* y_{bq}^{l}}{\sum_{p=1}^{j} v_p^* x_{pq}^{r}} \le \frac{\sum_{b=1}^{s} u_b^* y_{bq}^{r}}{\sum_{p=1}^{j} v_p^* x_{pq}^{l}} \le \frac{\sum_{b=1}^{s} u_b^* y_{bq}^{r}}{\sum_{p=1}^{j} v_p^* x_{pq}^{l}} \le \frac{\sum_{b=1}^{s} u_b^* y_{bq}^{r'}}{\sum_{p=1}^{j} v_p^* x_{qp}^{l'}}, \; q = 1, 2, 3, \ldots, k.$$

This shows that $\{(\phi_t^{l*}, \phi_t^{m*}, \phi_t^{r*}); (\phi_t^{l'*}, \phi_t^{m*}, \phi_t^{r'*})\}$ preserves the form of a non-negative TFFN. □

The models (3.5)–(3.9) can be linearized into models as follows:

$$\begin{aligned} \text{Max } \phi_t^{l'} &= \sum_{b=1}^{s} u_b y_{bt}^{l'} \\ \text{subject to } & \sum_{p=1}^{j} v_p x_{pt}^{r'} = 1, \\ & \sum_{b=1}^{s} u_b y_{bq}^{r'} - \sum_{p=1}^{j} v_p x_{pq}^{l'} \le 0, \quad q = 1, \ldots, k, \\ & u_b, v_p \ge 0, \quad b = 1, 2, 3, \ldots, s, \quad p = 1, 2, 3, \ldots, j. \end{aligned} \tag{3.10}$$

$$\begin{aligned} \text{Max } \phi_t^{l} &= \sum_{b=1}^{s} u_b y_{bt}^{l} \\ \text{subject to } & \sum_{p=1}^{j} v_p x_{pt}^{r} = 1, \\ & \sum_{b=1}^{s} u_b y_{bt}^{l'} - \phi_t^{l'*} \sum_{p=1}^{j} v_p x_{pt}^{r'} \le 0, \\ & \sum_{b=1}^{s} u_b y_{bq}^{r'} - \sum_{p=1}^{j} v_p x_{pq}^{l'} \le 0, \quad q = 1, \ldots, k, \\ & u_b, v_p \ge 0, \quad b = 1, 2, 3, \ldots, s, \quad p = 1, 2, 3, \ldots, j. \end{aligned} \tag{3.11}$$

$$\text{Max } \phi_t^{m} = \sum_{b=1}^{s} u_b y_{bt}^{m}$$

$$
\begin{aligned}
\text{subject to } & \sum_{p=1}^{j} v_p x_{pt}^{m} = 1, \\
& \sum_{b=1}^{s} u_b y_{bt}^{l'} - \phi_t^{l'*} \sum_{p=1}^{j} v_p x_{pt}^{r'} \le 0, \\
& \sum_{b=1}^{s} u_b y_{bt}^{l} - \phi_t^{l*} \sum_{p=1}^{j} v_p x_{pt}^{r} \le 0, \\
& \sum_{b=1}^{s} u_b y_{bq}^{r'} - \sum_{p=1}^{j} v_p x_{pq}^{l'} \le 0, \quad q = 1, \ldots, k, \\
& u_b, v_p \ge 0, \quad b = 1, 2, 3, \ldots, s, \quad p = 1, 2, 3, \ldots, j.
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
\text{Max } \phi_t^{r} = & \sum_{b=1}^{s} u_b y_{bt}^{r} \\
\text{subject to } & \sum_{p=1}^{j} v_p x_{pt}^{l} = 1, \\
& \sum_{b=1}^{s} u_b y_{bt}^{l'} - \phi_t^{l'*} \sum_{p=1}^{j} v_p x_{pt}^{r'} \le 0, \\
& \sum_{b=1}^{s} u_b y_{bt}^{l} - \phi_t^{l*} \sum_{p=1}^{j} v_p x_{pt}^{r} \le 0, \\
& \sum_{b=1}^{s} u_b y_{bt}^{m} - \phi_t^{m*} \sum_{p=1}^{j} v_p x_{pt}^{m} \le 0, \\
& \sum_{b=1}^{s} u_b y_{bq}^{r'} - \sum_{p=1}^{j} v_p x_{pq}^{l'} \le 0, \quad q = 1, \ldots, k, \\
& u_b, v_p \ge 0, \quad b = 1, 2, 3, \ldots, s, \quad p = 1, 2, 3, \ldots, j.
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
\text{Max } \phi_t^{r'} = & \sum_{b=1}^{s} u_b y_{bt}^{r'} \\
\text{subject to } & \sum_{p=1}^{j} v_p x_{pt}^{l'} = 1, \\
& \sum_{b=1}^{s} u_b y_{bt}^{l'} - \phi_t^{l'*} \sum_{p=1}^{j} v_p x_{pt}^{r'} \le 0, \\
& \sum_{b=1}^{s} u_b y_{bt}^{l} - \phi_t^{l*} \sum_{p=1}^{j} v_p x_{pt}^{r} \le 0,
\end{aligned}
$$

$$\sum_{b=1}^{s} u_b y_{bt}^{m} - \phi_t^{m*} \sum_{p=1}^{j} v_p x_{pt}^{m} \leq 0,$$
$$\sum_{b=1}^{s} u_b y_{bt}^{r} - \phi_t^{r*} \sum_{p=1}^{j} v_p x_{pt}^{l} \leq 0,$$
$$\sum_{b=1}^{s} u_b y_{bq}^{r'} - \sum_{p=1}^{j} v_p x_{pq}^{l'} \leq 0, \quad q = 1, \ldots, k,$$
$$u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r. \tag{3.14}$$

4. Procedure to solve the FFMOTP

Consider the FFMOTP as given in model (3.1), with h Fermatean fuzzy attributes that need to be maximized and minimized. Every arc is related with h Fermatean fuzzy attributes. The arc attributes associated with the Fermatean fuzzy objectives that should be minimized are considered as Fermatean fuzzy input attributes, while the attributes associated with Fermatean fuzzy objectives that should be maximized are considered as Fermatean fuzzy output attributes. For every arc (p, q), two FFESs are calculated as criteria for the relative performance of the single objective transportation from p ($p = 1, \ldots, j$) to q ($q = 1, \ldots, k$). Finally, the average of these FFESs is calculated to get new FFESs of the arc. Then, the h Fermatean fuzzy attributes related to every arc are transformed into a single Fermatean fuzzy attribute, and the given FFMOTP is turned into a single objective Fermatean fuzzy transportation problem (FFTP).

To solve the model (3.1), the steps are as follows:

1) The relative performance of single objective transportation from source p to destination q, i.e., $\tilde{E}_{pq}^{1^*}$, is determined for every $q = 1, \ldots, k$. by using the source p as target and altering the destinations q. This is accomplished by solving the model:

$$\tilde{E}_{pq}^{(1*)} = \text{Max } \tilde{E}_{pq}^{(1)} = \frac{\sum_{b=1}^{s} u_b \tilde{y}_{pq}^{b}}{\sum_{d=1}^{r} v_d \tilde{x}_{pq}^{d}}$$
$$\text{subject to } \tilde{E}_{pf}^{(1)} = \frac{\sum_{b=1}^{s} u_b \tilde{y}_{pf}^{b}}{\sum_{d=1}^{r} v_d \tilde{x}_{pf}^{d}} \leq 1, \quad f = 1, \ldots, k,$$
$$u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r. \tag{4.1}$$

To determine the optimum value of model (4.1), i.e., $\tilde{E}_{pq}^{(1*)} = \{(E_{pq}^{(1*),l}, E_{pq}^{(1*),m}, E_{pq}^{(1*),r}); (E_{pq}^{(1*),l'}, E_{pq}^{(1*),m}, E_{pq}^{(1*),r'})\}$, each component of $\tilde{E}_{pq}^{(1*)}$ is calculated using the Fermatean fuzzy arithmetic approach discussed in the previous section. First, $E_{pq}^{(1*),l'}$ is determined as follows:

$$E_{pq}^{(1*),l'} = \text{Max } \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l'}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r'}}$$
$$\text{subject to } \frac{\sum_{b=1}^{s} u_b y_{pf}^{b,l'}}{\sum_{d=1}^{r} v_d x_{pf}^{d,r'}} \leq 1, \quad f = 1, \ldots, k,$$

$$u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r. \tag{4.2}$$

Then, using the optimal weights of model (4.2), $E_{pq}^{(1*),l}$ is computed as follows:

$$\begin{aligned}
E_{pq}^{(1*),l} = \text{Max}\ & \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r}} \\
\text{subject to}\ & \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l'}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r'}} = E_{pq}^{(1*),l'}, \quad \frac{\sum_{b=1}^{s} u_b y_{pf}^{b,l}}{\sum_{d=1}^{r} v_d x_{pf}^{d,r}} \leq 1, \quad f = 1, \ldots, k, \\
& u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r.
\end{aligned} \tag{4.3}$$

Using the optimal weights $E_{pq}^{(1*),l'}$ and $E_{pq}^{(1*),l}$ of the models (4.2) and (4.3), respectively, $E_{pq}^{(1*),m}$ is determined as

$$\begin{aligned}
E_{pq}^{(1*),m} = \text{Max}\ & \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,m}}{\sum_{d=1}^{r} v_d x_{pq}^{d,m}} \\
\text{subject to}\ & \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l'}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r'}} = E_{pq}^{(1*),l'}, \quad \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r}} = E_{pq}^{(1*),l}, \\
& \frac{\sum_{b=1}^{s} u_b y_{pf}^{b,m}}{\sum_{d=1}^{r} v_d x_{pf}^{d,m}} \leq 1, \quad f = 1, \ldots, k, \\
& u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r.
\end{aligned} \tag{4.4}$$

Using the optimal weights $E_{pq}^{(1*),l'}$, $E_{pq}^{(1*),l}$ and $E_{pq}^{(1*),r}$ of models (4.2)–(4.4), $E_{pq}^{(1*),r}$ is obtained as

$$\begin{aligned}
E_{pq}^{(1*),r} = \text{Max}\ & \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,r}}{\sum_{d=1}^{r} v_d x_{pq}^{d,l}} \\
\text{subject to}\ & \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l'}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r'}} = E_{pq}^{(1*),l'}, \quad \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r}} = E_{pq}^{(1*),l}, \\
& \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,m}}{\sum_{d=1}^{r} v_d x_{pq}^{d,m}} = E_{pq}^{(1*),m}, \quad \frac{\sum_{b=1}^{s} u_b y_{pf}^{b,r}}{\sum_{d=1}^{r} v_d x_{pf}^{d,l}} \leq 1, \quad f = 1, \ldots, k, \\
& u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r.
\end{aligned} \tag{4.5}$$

Finally, by using the optimal weights $E_{pq}^{(1*),l'}$, $E_{pq}^{(1*),l}$, $E_{pq}^{(1*),m}$ and $E_{pq}^{(1*),r}$ of models (4.2)–(4.5), $E_{pq}^{(1*),r'}$ is calculated as

$$\begin{aligned}
E_{pq}^{(1*),r'} = \text{Max}\ & \frac{\sum_{b=1}^{s} u_b \tilde{y}_{pq}^{b,r'}}{\sum_{d=1}^{r} v_d \tilde{x}_{pq}^{d,l'}} \\
\text{subject to}\ & \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l'}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r'}} = E_{pq}^{(1*),l'}, \quad \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r}} = E_{pq}^{(1*),l}, \\
& \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,m}}{\sum_{d=1}^{r} v_d x_{pq}^{d,m}} = E_{pq}^{(1*),m}, \quad \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,r}}{\sum_{d=1}^{r} v_d x_{pq}^{d,l}} = E_{pq}^{(1*),r},
\end{aligned}$$

$$\frac{\sum_{b=1}^{s} u_b y_{pf}^{b,r'}}{\sum_{d=1}^{r} v_d x_{pf}^{d,l'}} \leq 1, \quad f = 1, \ldots, k,$$
$$u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r. \tag{4.6}$$

The models (4.2)–(4.6) can be linearized into the following models:

$$\begin{aligned}
E_{pq}^{(1*),l'} = \text{Max } & \sum_{b=1}^{s} u_b y_{pq}^{b,l'} \\
\text{subject to } & \sum_{d=1}^{r} v_d x_{pq}^{d,r'} = 1, \\
& \sum_{b=1}^{s} u_b y_{pf}^{b,r'} - \sum_{d=1}^{r} v_d x_{pf}^{d,l'} \leq 0, \quad f = 1, \ldots, k, \\
& u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r.
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
E_{pq}^{(1*),l} = \text{Max } & \sum_{b=1}^{s} u_b y_{pq}^{b,l} \\
\text{subject to } & \sum_{d=1}^{r} v_d x_{pq}^{d,r} = 1, \\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l'} - E_{pq}^{(1*),l'} \sum_{d=1}^{r} v_d x_{pq}^{d,r'} = 0, \\
& \sum_{b=1}^{s} u_b y_{pf}^{b,r'} - \sum_{d=1}^{r} v_d x_{pf}^{d,l'} \leq 0, \quad f = 1, \ldots, k, \\
& u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r.
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
E_{pq}^{(1*),m} = \text{Max } & \sum_{b=1}^{s} u_b y_{pq}^{b,m} \\
\text{subject to } & \sum_{d=1}^{r} v_d x_{pq}^{d,m} = 1, \\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l'} - E_{pq}^{(1*),l'} \sum_{d=1}^{r} v_d x_{pq}^{d,r'} = 0, \\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l} - E_{pq}^{(1*),l} \sum_{d=1}^{r} v_d x_{pq}^{d,r} = 0, \\
& \sum_{b=1}^{s} u_b y_{pf}^{b,r'} - \sum_{d=1}^{r} v_d x_{pf}^{d,l'} \leq 0, \quad f = 1, \ldots, k, \\
& u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r.
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
E_{pq}^{(1*),r} = \text{Max } & \sum_{b=1}^{s} u_b y_{pq}^{b,r} \\
\text{subject to } & \sum_{d=1}^{r} v_d x_{pq}^{d,l} = 1, \\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l'} - E_{pq}^{(1*),l'} \sum_{d=1}^{r} v_d x_{pq}^{d,r'} = 0, \\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l} - E_{pq}^{(1*),l} \sum_{d=1}^{r} v_d x_{pq}^{d,r} = 0, \\
& \sum_{b=1}^{s} u_b y_{pq}^{b,m} - E_{pq}^{(1*),m} \sum_{d=1}^{r} v_d x_{pq}^{d,m} = 0, \\
& \sum_{b=1}^{s} u_b y_{pf}^{b,r'} - \sum_{d=1}^{r} v_d x_{pf}^{d,l'} \leq 0, \quad f = 1, \ldots, k, \\
& u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r.
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
E_{pq}^{(1*),r'} = \text{Max } & \sum_{b=1}^{s} u_b y_{pq}^{b,r'} \\
\text{subject to } & \sum_{d=1}^{r} v_d x_{pq}^{d,l'} = 1, \\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l'} - E_{pq}^{(1*),l'} \sum_{d=1}^{r} v_d x_{pq}^{d,r'} = 0, \\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l} - E_{pq}^{(1*),l} \sum_{d=1}^{r} v_d x_{pq}^{d,r} = 0, \\
& \sum_{b=1}^{s} u_b y_{pq}^{b,m} - E_{pq}^{(1*),m} \sum_{d=1}^{r} v_d x_{pq}^{d,m} = 0, \\
& \sum_{b=1}^{s} u_b y_{pq}^{b,r} - E_{pq}^{(1*),r} \sum_{d=1}^{r} v_d x_{pq}^{d,l} = 0, \\
& \sum_{b=1}^{s} u_b y_{pf}^{b,r'} - \sum_{d=1}^{r} v_d x_{pf}^{d,l'} \leq 0, \quad f = 1, \ldots, k, \\
& u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r.
\end{aligned} \tag{4.11}
$$

2) The relative performance of single objective transportation from source p to destination q, i.e., $\tilde{E}_{pq}^{2^*}$, is determined for every $p = 1, \ldots, j$ by using the destination q as target and altering the sources p. For this, we solve the model

$$
\tilde{E}_{pq}^{(2*)} = \text{Max } \tilde{E}_{pq}^{(2)} = \frac{\sum_{b=1}^{s} u_b \tilde{y}_{pq}^{b}}{\sum_{d=1}^{r} v_d \tilde{x}_{pq}^{d}}
$$

$$\text{subject to } \tilde{E}_{fq}^{(2)} = \frac{\sum_{b=1}^{s} u_b \tilde{y}_{fq}^{b}}{\sum_{d=1}^{r} v_p \tilde{x}_{fq}^{d}} \le 1, \quad f = 1, \ldots, j,$$
$$u_b, v_d \ge 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r. \tag{4.12}$$

To determine the optimum value of the model (4.12), i.e., $\tilde{E}_{pq}^{(2*)} = \{(E_{pq}^{(2*),l}, E_{pq}^{(2*),m}, E_{pq}^{(2*),r});$ $(E_{pq}^{(2*),l'}, E_{pq}^{(2*),m}, E_{pq}^{(2*),r'})\}$, each component of $\tilde{E}_{pq}^{(1*)}$ is determined using the Fermatean fuzzy arithmetic approach discussed in the previous section. First, $E_{pq}^{(2*),l'}$ is determined as follows:

$$E_{pq}^{(2*),l'} = \text{Max } \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l'}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r'}}$$
$$\text{subject to } \frac{\sum_{b=1}^{s} u_b \tilde{y}_{fq}^{b,r'}}{\sum_{d=1}^{r} v_d x_{fq}^{d,l'}} \le 1, \quad f = 1, \ldots, j,$$
$$u_b, v_d \ge 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r. \tag{4.13}$$

Then, using the optimal weights of the above model, $E_{pq}^{(2*),l}$ is computed as follows:

$$E_{pq}^{(2*),l} = \text{Max } \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r}}$$
$$\text{subject to } \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l'}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r'}} = E_{pq}^{(2*),l'}, \quad \frac{\sum_{b=1}^{s} u_b y_{fq}^{b,r'}}{\sum_{d=1}^{r} v_d x_{fq}^{d,l'}} \le 1, \quad f = 1, \ldots, j,$$
$$u_b, v_d \ge 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r. \tag{4.14}$$

Using the optimal weights of the models (4.13) and (4.14), $E_{pq}^{(2*),m}$ is determined as

$$E_{pq}^{(2*),m} = \text{Max } \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,m}}{\sum_{d=1}^{r} v_d x_{pq}^{d,m}}$$
$$\text{subject to } \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l'}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r'}} = E_{pq}^{(2*),l'}, \quad \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r}} = E_{pq}^{(2*),l},$$
$$\frac{\sum_{b=1}^{s} u_b y_{fq}^{b,r'}}{\sum_{d=1}^{r} v_d x_{fq}^{d,l'}} \le 1, \quad f = 1, \ldots, j,$$
$$u_b, v_d \ge 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r. \tag{4.15}$$

Using the optimal weights of the models (4.13)–(4.15), $E_{pq}^{(2*),r}$ is obtained as

$$E_{pq}^{(2*),r} = \text{Max } \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,r}}{\sum_{d=1}^{r} v_d x_{pq}^{d,l}}$$
$$\text{subject to } \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l'}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r'}} = E_{pq}^{(2*),l'}, \quad \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r}} = E_{pq}^{(2*),l},$$

$$\frac{\sum_{b=1}^{s} u_b y_{pq}^{b,m}}{\sum_{d=1}^{r} v_d x_{pq}^{d,m}} = E_{pq}^{(2*),m}, \quad \frac{\sum_{b=1}^{s} u_b y_{fq}^{b,r'}}{\sum_{d=1}^{r} v_d x_{fq}^{d,l'}} \leq 1, \quad f = 1, \dots, j,$$
$$u_b, v_d \geq 0, \quad b = 1, 2, 3, \dots, s, \quad d = 1, 2, 3, \dots, r. \tag{4.16}$$

Finally, by using the optimal weights of the models (4.13)–(4.16), $E_{pq}^{(1*),r'}$ is calculated as

$$\begin{aligned} E_{pq}^{(2*),r'} &= \text{Max } \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,r'}}{\sum_{d=1}^{r} v_d x_{pq}^{d,l'}} \\ \text{subject to } & \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l'}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r'}} = E_{pq}^{(2*),l'}, \quad \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,l}}{\sum_{d=1}^{r} v_d x_{pq}^{d,r}} = E_{pq}^{(2*),l}, \\ & \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,m}}{\sum_{d=1}^{r} v_d x_{pq}^{d,m}} = E_{pq}^{(2*),m}, \quad \frac{\sum_{b=1}^{s} u_b y_{pq}^{b,r}}{\sum_{d=1}^{r} v_d x_{pq}^{d,l}} = E_{pq}^{(2*),r}, \\ & \frac{\sum_{b=1}^{s} u_b y_{fq}^{b,r'}}{\sum_{d=1}^{r} v_d x_{fq}^{d,l'}} \leq 1, \quad f = 1, \dots, j, \\ & u_b, v_d \geq 0, \quad b = 1, 2, 3, \dots, s, \quad d = 1, 2, 3, \dots, r. \end{aligned} \tag{4.17}$$

The models (4.13)–(4.17) can be linearized into the following models:

$$\begin{aligned} E_{pq}^{(2*),l'} &= \text{Max } \sum_{b=1}^{s} u_b y_{pq}^{b,l'} \\ \text{subject to } & \sum_{d=1}^{r} v_d x_{pq}^{d,r'} = 1, \\ & \sum_{b=1}^{s} u_b y_{fq}^{b,l'} - \sum_{d=1}^{r} v_d x_{fq}^{d,r'} \leq 0, \quad f = 1, \dots, j, \\ & u_b, v_d \geq 0, \quad b = 1, 2, 3, \dots, s, \quad d = 1, 2, 3, \dots, r. \end{aligned} \tag{4.18}$$

$$\begin{aligned} E_{pq}^{(2*),l} &= \text{Max } \sum_{b=1}^{s} u_b y_{pq}^{b,l} \\ \text{subject to } & \sum_{d=1}^{r} v_d x_{pq}^{d,r} = 1, \\ & \sum_{b=1}^{s} u_b y_{pq}^{b,l'} - E_{pq}^{(1*),l'} \sum_{d=1}^{r} v_d x_{pq}^{d,r'} = 0, \\ & \sum_{b=1}^{s} u_b y_{fq}^{b,r'} - \sum_{d=1}^{r} v_d x_{fq}^{d,l'} \leq 0, \quad f = 1, \dots, j, \\ & u_b, v_d \geq 0, \quad b = 1, 2, 3, \dots, s, \quad d = 1, 2, 3, \dots, r. \end{aligned} \tag{4.19}$$

$$E_{pq}^{(2*),m} = \text{Max } \sum_{b=1}^{s} u_b y_{pq}^{b,m}$$

$$
\begin{aligned}
\text{subject to}\ & \sum_{d=1}^{r} v_d x_{pq}^{d,m} = 1,\\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l'} - E_{pq}^{(1*),l'} \sum_{d=1}^{r} v_d x_{pq}^{d,r'} = 0,\\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l} - E_{pq}^{(1*),l} \sum_{d=1}^{r} v_d x_{pq}^{d,r} = 0,\\
& \sum_{b=1}^{s} u_b y_{fq}^{b,r'} - \sum_{d=1}^{r} v_d x_{fq}^{d,l'} \leq 0,\quad f = 1,\ldots,j,\\
& u_b, v_d \geq 0,\quad b = 1,2,3,\ldots,s,\quad d = 1,2,3,\ldots,r.
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
E_{pq}^{(2*),r} = \text{Max}\ & \sum_{b=1}^{s} u_b y_{bq}^{b,r}\\
\text{subject to}\ & \sum_{d=1}^{r} v_d x_{pq}^{d,l} = 1,\\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l'} - E_{pq}^{(1*),l'} \sum_{d=1}^{r} v_d x_{pq}^{d,r'} = 0,\\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l} - E_{pq}^{(1*),l} \sum_{d=1}^{r} v_d x_{pq}^{d,r} = 0,\\
& \sum_{b=1}^{s} u_b y_{pq}^{b,m} - E_{pq}^{(1*),m} \sum_{d=1}^{r} v_d x_{pq}^{d,m} = 0,\\
& \sum_{b=1}^{s} u_b y_{fq}^{b,r'} - \sum_{d=1}^{r} v_d x_{fq}^{d,l'} \leq 0,\quad f = 1,\ldots,j,\\
& u_b, v_d \geq 0,\quad b = 1,2,3,\ldots,s,\quad d = 1,2,3,\ldots,r.
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
E_{pq}^{(2*),r'} = \text{Max}\ & \sum_{b=1}^{s} u_b y_{bq}^{b,r'}\\
\text{subject to}\ & \sum_{d=1}^{r} v_d x_{pq}^{d,l'} = 1,\\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l'} - E_{pq}^{(1*),l'} \sum_{d=1}^{r} v_d x_{pq}^{d,r'} = 0,\\
& \sum_{b=1}^{s} u_b y_{pq}^{b,l} - E_{pq}^{(1*),l} \sum_{d=1}^{r} v_d x_{pq}^{d,r} = 0,\\
& \sum_{b=1}^{s} u_b y_{pq}^{b,m} - E_{pq}^{(1*),m} \sum_{d=1}^{r} v_d x_{pq}^{d,m} = 0,
\end{aligned}
$$

$$\sum_{b=1}^{s} u_b y_{pq}^{b,r} - E_{pq}^{(1*),r} \sum_{d=1}^{r} v_d x_{pq}^{d,l} = 0,$$
$$\sum_{b=1}^{s} u_b y_{fq}^{b,r'} - \sum_{d=1}^{r} v_d \tilde{x}_{fq}^{d,l'} \leq 0, \quad f = 1, \ldots, j,$$
$$u_b, v_d \geq 0, \quad b = 1, 2, 3, \ldots, s, \quad d = 1, 2, 3, \ldots, r. \tag{4.22}$$

3) Now, the average of $\tilde{E}_{pq}^{1^*}$ and $\tilde{E}_{pq}^{2^*}$ is determined to get the new FFES $\tilde{E}_{pq}$ for every arc (p, q).

$$\tilde{E}_{pq} = \frac{\tilde{E}_{pq}^{1^*} + \tilde{E}_{pq}^{2^*}}{2}. \tag{4.23}$$

At the end, the h Fermatean fuzzy attributes have been turned into a single positive Fermatean fuzzy attribute $\tilde{E}_{pq}$, and the fully FFMOTP has been transformed into a single objective FFTP:

$$E^* = \max \sum_{p=1}^{j} \sum_{q=1}^{k} \tilde{E}_{pq} \times x_{pq}$$
$$\text{subject to} \quad \sum_{q=1}^{k} x_{pq} = s_p, \quad p = 1, \ldots, j,$$
$$\sum_{p=1}^{j} x_{pq} = d_q, \quad q = 1, \ldots, k,$$
$$x_{pq} \geq 0, \quad p = 1, \ldots, j, \quad q = 1, \ldots, k. \tag{4.24}$$

Finally, a transportation plan with the greatest Fermatean fuzzy efficiency will be obtained by solving the model (4.24).

It should be noticed that model (4.24) has no uncertainty with hesitation regarding the supply and demand of the product; rather, the only uncertainty with hesitation is related to the exact values of $\tilde{E}_{pq}$. These types of problems can be solved using a few practical techniques. One of these is the ranking function-based technique proposed by Mahmoodirad et al. [64]. To do this, it is sufficient to apply any random linear ranking function, substituting each Fermatean fuzzy number's rank with its corresponding Fermatean fuzzy number in the FFTP under discussion. In this manner, the FFTP is transformed into a crisp problem that can be quickly solved using the fundamental transportation methods. Their results are independent of the linear ranking function used. As a consequence, to obtain the crisp form of Fermatean fuzzy TP model (4.24), the ranking function $\Re(\tilde{A}^F) = \dfrac{(u^l + 4u^m + u^r) + (u^{l'} + 4u^m + u^{r'})}{12}$ is used, in which $\tilde{A}^F = \{(u^l, u^m, u^r); (u^{l'}, u^m, u^{r'})\}$ is a TFFN. In this way, the FFTP is converted into a crisp TP which can be easily solved by standard transportation techniques.

Theorem 4.1. *The optimum solution of the model (4.24) is an efficient solution of the model (3.1).*

Proof. It is important to note that the objective function of the model (4.24) is the model's (3.1) weighted sum objective function. Since the optimal solution of the weighted sum problem with positive weights is known, it will always be the most efficient solution of the multi-objective LPP under consideration [65]. Since the values of $\tilde{E}_{pq}$ in (4.23) are considered as the weights of weighted sum model (3.1), it is enough to prove that $\tilde{E}_{pq} \geq 0$. Using Eq (4.23), we show that $\tilde{E}_{pq}^{(1*)} = \{(\tilde{E}_{pq}^{(1*),l}, \tilde{E}_{pq}^{(1*),m}, \tilde{E}_{pq}^{(1*),r}); (\tilde{E}_{pq}^{(1*),l'}, \tilde{E}_{pq}^{(1*),m}, \tilde{E}_{pq}^{(1*),r'})\} > 0$, and $\tilde{E}_{pq}^{(2*)} = \{(\tilde{E}_{pq}^{(2*),l}, \tilde{E}_{pq}^{(2*),m}, \tilde{E}_{pq}^{(2*),r}); (\tilde{E}_{pq}^{(2*),l'}, \tilde{E}_{pq}^{(2*),m}, \tilde{E}_{pq}^{(2*),r'})\} > 0$. For this, we prove $\tilde{E}_{pq}^{(1*),l'} > 0$ and $\tilde{E}_{pq}^{(2*),l'} > 0$ by using the definition of a positive TFFN. As $\tilde{E}_{pq}^{(1*),l'}$ and $\tilde{E}_{pq}^{(2*),l'}$ are the optimal values of the classical input-oriented models of DEA, we have $0 < \tilde{E}_{pq}^{(1*),l'} \leq 1$ and $0 < \tilde{E}_{pq}^{(2*),l'} \leq 1$. □

In this section, we explore another popular approach for solving model (4.24) called the fuzzy programming approach. For this, consider $\tilde{E}_{pq} = \{(\tilde{E}_{pq}^{l}, \tilde{E}_{pq}^{m}, \tilde{E}_{pq}^{r}); (\tilde{E}_{pq}^{l'}, \tilde{E}_{pq}^{m}, \tilde{E}_{pq}^{r'})\}$, and the model is reduced to the multi-objective problem described below:

$$\begin{aligned}
\text{Min } Z_1 &= \sum_{p=1}^{j}\sum_{q=1}^{k}(\tilde{E}_{pq}^{l} - \tilde{E}_{pq}^{l'})x_{pq} \\
\text{Min } Z_2 &= \sum_{p=1}^{j}\sum_{q=1}^{k}(\tilde{E}_{pq}^{m} - \tilde{E}_{pq}^{l})x_{pq} \\
\text{Min } Z_3 &= \sum_{p=1}^{j}\sum_{q=1}^{k}\tilde{E}_{pq}^{m}x_{pq} \\
\text{Min } Z_4 &= \sum_{p=1}^{j}\sum_{q=1}^{k}(\tilde{E}_{pq}^{r} - \tilde{E}_{pq}^{m})x_{pq} \\
\text{Min } Z_5 &= \sum_{p=1}^{j}\sum_{q=1}^{k}(\tilde{E}_{pq}^{r'} - \tilde{E}_{pq}^{r})x_{pq}
\end{aligned}$$

$$\text{subject to constraints of model (4.24).} \tag{4.25}$$

To solve model (4.25), the positive ideal solution ($Z^{\oplus}$) and the negative ideal solution ($Z^{\ominus}$) are determined by solving the following LPPs:

$$Z_1^{\oplus} = \text{Min} \sum_{p=1}^{j}\sum_{q=1}^{k}(\tilde{E}_{pq}^{l} - \tilde{E}_{pq}^{l'})x_{pq} \qquad Z_1^{\ominus} = \text{Max} \sum_{p=1}^{j}\sum_{q=1}^{k}(\tilde{E}_{pq}^{l} - \tilde{E}_{pq}^{l'})x_{pq}$$

subject to constraints of model (4.24). subject to constraints of model (4.24).

$$Z_2^{\oplus} = \text{Min} \sum_{p=1}^{j}\sum_{q=1}^{k}(\tilde{E}_{pq}^{m} - \tilde{E}_{pq}^{l})x_{pq} \qquad Z_2^{\ominus} = \text{Max} \sum_{p=1}^{j}\sum_{q=1}^{k}(\tilde{E}_{pq}^{m} - \tilde{E}_{pq}^{l})x_{pq}$$

subject to constraints of model (4.24). subject to constraints of model (4.24).

$$Z_3^{\oplus} = \text{Max} \sum_{p=1}^{j}\sum_{q=1}^{k}\tilde{E}_{pq}^{m}x_{pq} \qquad Z_3^{\ominus} = \text{Min} \sum_{p=1}^{j}\sum_{q=1}^{k}\tilde{E}_{pq}^{m}x_{pq}$$

subject to constraints of model (4.24). subject to constraints of model (4.24).

$$Z_4^{\oplus} = \text{Max} \sum_{p=1}^{j}\sum_{q=1}^{k}(\tilde{E}_{pq}^{r} - \tilde{E}_{pq}^{m})x_{pq} \qquad Z_4^{\ominus} = \text{Min} \sum_{p=1}^{j}\sum_{q=1}^{k}(\tilde{E}_{pq}^{r} - \tilde{E}_{pq}^{m})x_{pq}$$

subject to constraints of model (4.24). subject to constraints of model (4.24).

$$Z_5^{\oplus} = \text{Max} \sum_{p=1}^{j} \sum_{q=1}^{k} (\tilde{E}_{pq}^{r'} - \tilde{E}_{pq}^{r}) x_{pq} \qquad Z_5^{\ominus} = \text{Min} \sum_{p=1}^{j} \sum_{q=1}^{k} (\tilde{E}_{pq}^{r'} - \tilde{E}_{pq}^{r}) x_{pq}$$

subject to constraints of model (4.24). subject to constraints of model (4.24). (4.26)

Then, the linear MFs of $\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3, \tilde{Z}_4$ and $\tilde{Z}_5$ are described as follows:

$$\mu_{\tilde{Z}_1} = (Z_1) \begin{cases} 1, & Z_1 < Z_1^{\oplus}, \\ \dfrac{Z_1^{\ominus} - Z_1}{Z_1^{\ominus} - Z_1^{\oplus}}, & Z_1^{\oplus} < Z_1 < Z_1^{\ominus}, \\ 0, & Z_1 > Z_1^{\ominus}, \end{cases} \tag{4.27}$$

$$\mu_{\tilde{Z}_2} = (Z_2) \begin{cases} 1, & Z_2 < Z_2^{\oplus}, \\ \dfrac{Z_2^{\ominus} - Z_2}{Z_2^{\ominus} - Z_2^{\oplus}}, & Z_2^{\oplus} < Z_1 < Z_2^{\ominus}, \\ 0, & Z_2 > Z_2^{\ominus}, \end{cases} \tag{4.28}$$

$$\mu_{\tilde{Z}_3} = (Z_3) \begin{cases} 1, & Z_3 > Z_3^{\oplus}, \\ \dfrac{Z_3 - Z_3^{\ominus}}{Z_3^{\oplus} - Z_3^{\ominus}}, & Z_3^{\ominus} < Z_3 < Z_3^{\oplus}, \\ 0, & Z_3 < Z_3^{\oplus}, \end{cases} \tag{4.29}$$

$$\mu_{\tilde{Z}_4} = (Z_4) \begin{cases} 1, & Z_4 > Z_4^{\oplus}, \\ \dfrac{Z_4 - Z_4^{\ominus}}{Z_4^{\oplus} - Z_4^{\ominus}}, & Z_4^{\ominus} < Z_4 < Z_4^{\oplus}, \\ 0, & Z_4 < Z_4^{\ominus}, \end{cases} \tag{4.30}$$

$$\mu_{\tilde{Z}_5}(Z_5) = \begin{cases} 1, & Z_5 > Z_5^{\oplus}, \\ \dfrac{Z_5 - Z_5^{\ominus}}{Z_5^{\oplus} - Z_5^{\ominus}}, & Z_5^{\ominus} < Z_5 < Z_5^{\oplus}, \\ 0, & Z_5 < Z_5^{\ominus}. \end{cases} \tag{4.31}$$

Finally, using the fuzzy programming approach, the following model is solved:

$$\begin{aligned} &\text{Max}\, \beta \\ &\text{subject to } \ \mu_{\tilde{Z}_p}(Z_p) \geq \beta, \quad p = 1, 2, 3, 4, 5, \\ &\qquad \text{constraints of model (4.24).} \end{aligned} \tag{4.32}$$

By inserting the MFs of (4.27)–(4.31) into the model (4.32), the following problem is obtained:

$$\begin{aligned} &\text{Max}\, \beta \\ &\text{subject to } \ Z_1 \leq Z_1^{\ominus} - (Z_1^{\ominus} - Z_1^{\oplus})\beta, \\ &\qquad Z_2 \leq Z_2^{\ominus} - (Z_2^{\ominus} - Z_2^{\oplus})\beta, \end{aligned}$$

$$Z_3 \geq Z_3^{\ominus} + (Z_3^{\oplus} - Z_3^{\ominus})\beta,$$
$$Z_4 \geq Z_4^{\ominus} + (Z_4^{\oplus} - Z_4^{\ominus})\beta,$$
$$Z_5 \geq Z_5^{\ominus} + (Z_5^{\oplus} - Z_5^{\ominus})\beta,$$
$$\text{constraints of model (4.24).} \tag{4.33}$$

5. Numerical example

Example 5.1. An automobile company has five assembly plants at five different towns A, B, C, D and E in a city. The company wants to deliver the cars to three markets in three towns J, K and L of another city to extend its business. The transportation cost, shipment value, and transportation profit are considered as TFFNs and are given in Table 3. The company wants to reduce transportation costs and maximize shipment value and transportation profit.

Table 3. Given data of example.

		J	K	L	s_p
	Trans. cost	{(680,685,692);(670)685,700}	{(361,370,380);(355,370,385)}	{(250,257,261);(245,257,270)}	
A	shipment value	{(265,271,279);(259,271,283)}	{(423,430,440);(415,430,450)}	{(416,423,435);(410,423,440)}	8
	Trans. profit	{(950,1000,1050);(900,1000,1090)}	{(740,760,790);(700,760,800)}	{(220,240,270);(200,240,310)}	
	Trans. cost	{(530,535,540);(525,535,545)}	{(320,327,335);(315,327,342)}	{(271,275,280);(265,275.283)}	
B	shipment value	{(285,292,300);(279,292,310)}	{(345,352,360);(335,352,370)}	{(657,665,670);(650,665,700)}	6
	Trans. profit	{(650,690,700);(630,690,740)}	{(500,565,590);(485,565,600)}	{(730,780,795);(700,780,815)}	
	Trans. cost	{(430,436,445);(420,436,450)}	{(308,315,322);(300,315,325)}	{(290,296,303);(285,296,310)}	
C	shipment value	{(720,780,800);(710,780,830)}	{(290,300,307);(281,300,315)}	{(739,780,795);(700,780,815)}	7
	Trans. profit	{(320,330,350);(300,330,365)}	{(1130,1190,1230);(1100,1190,1250)}	{(550,590,640);(510,590,670)}	
	Trans. cost	{(380,392,399);(375,392,408)}	{(394,400,410);(390,400,415)}	{(315,319,325);(310,319,329)}	
D	shipment value	{(631,650,670);(620,650,675)}	{(510,520,530);(505,520,540)}	{(435,450,465);(420,450,470)}	9
	Trans. profit	{(465,490,510);(430,490,550)}	{(2800,2900,3050);(2730,2900,3110)}	{(880,890,930);(850,890,960)}	
	Trans. cost	{(455,475,480);(450,475,490)}	{(451,460,469);(445,460,476)}	{(335,342,350);(330,342,360)}	
E	shipment value	{(475,485,490);(470,485,505)}	{(531,540,550);(525,540,560)}	{(590,630,650);(550,630,670)}	12
	Trans. profit	{(830,880,930);(800,880,950)}	{(475,490,520);(450,490,550)}	{(1550,1600,1650);(1500,1600,1700)}	
	d_q	15	14	13	

By treating the source p as target, the FFESs $\tilde{E}_{pq}^{(1*)}$ can be determined by using the model (4.1). To determine $\tilde{E}_{33}^{(1*)}$ the model is as follows:

$$\begin{aligned}
E_{33}^{1*} = \text{Max} \quad & \frac{[u_1\{(290,300,307);(281,300,315)\} + u_2\{(1130,1190,1230);(1100,1190,1250)\}]}{v_1\{(308,315,322);(300,315,325)\}} \\
\text{subject to} \quad & \frac{[u_1\{(720,780,800);(710,780,830)\} + u_2\{(320,330,350);(300,330,365)\}]}{v_1\{(430,436,445);(420,436,450)\}} \leq 1, \\
& \frac{[u_1\{(290,300,307);(281,300,315)\} + u_2\{(1130,1190,1230);(1100,1190,1250)\}]}{v_1\{(308,315,322);(300,315,325)\}} \leq 1, \\
& \frac{[u_1\{(739,780,795);(700,780,815)\} + u_2\{(550,590,640);(510,590,670)\}]}{v_1\{(290,296,303);(285,296,310)\}} \leq 1, \\
& u_1, u_2, v_1 \geq 0.
\end{aligned} \tag{5.1}$$

To solve the above model, five LP models (5.2)–(5.6) should be solved:

$$E_{33}^{(1*)l'} = \text{Max} \ \ 281u_1 + 1100u_2$$

$$
\begin{aligned}
\text{subject to}\ \ & 325v_1 = 1,\\
& 830u_1 + 365u_2 - 420v_1 \le 0,\\
& 315u_1 + 1250u_2 - 300v_1 \le 0,\\
& 815u_1 + 670u_2 - 285v_1 \le 0,\\
& u_1, u_2, v_1 \ge 0.
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
E_{33}^{(1*)l} = \text{Max}\ \ & 290u_1 + 1130u_2\\
\text{subject to}\ \ & 322v_1 = 1,\\
& 830u_1 + 365u_2 - 420v_1 \le 0,\\
& 315u_1 + 1250u_2 - 300v_1 \le 0,\\
& 815u_1 + 670u_2 - 285v_1 \le 0,\\
& 281u_1 + 1100u_2 - 325E_{33}^{(1*)l'}v_1 = 0,\\
& u_1, u_2, v_1 \ge 0.
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
E_{33}^{(1*)m} = \text{Max}\ \ & 300u_1 + 1190u_2\\
\text{subject to}\ \ & 315v_1 = 1,\\
& 830u_1 + 365u_2 - 420v_1 \le 0,\\
& 315u_1 + 1250u_2 - 300v_1 \le 0,\\
& 815u_1 + 670u_2 - 285v_1 \le 0,\\
& 281u_1 + 1100u_2 - 325E_{33}^{(1*)l'}v_1 = 0,\\
& 290u_1 + 1130u_2 - 322E_{33}^{(1*)l}v_1 = 0,\\
& u_1, u_2, v_1 \ge 0.
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
E_{33}^{(1*)r} = \text{Max}\ \ & 307u_1 + 1230u_2\\
\text{subject to}\ \ & 308v_1 = 1,\\
& 830u_1 + 365u_2 - 420v_1 \le 0,\\
& 315u_1 + 1250u_2 - 300v_1 \le 0,\\
& 815u_1 + 670u_2 - 285v_1 \le 0,\\
& 281u_1 + 1100u_2 - 325E_{33}^{(1*)l'}v_1 = 0,\\
& 290u_1 + 1130u_2 - 322E_{33}^{(1*)l}v_1 = 0,\\
& 300u_1 + 1190u_2 - 315E_{33}^{(1*)m}v_1 = 0,\\
& u_1, u_2, v_1 \ge 0.
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
E_{33}^{(1*)r'} = \text{Max}\ \ & 315u_1 + 1250u_2\\
\text{subject to}\ \ & 300v_1 = 1,
\end{aligned}
$$

$$
\begin{aligned}
&830u_1 + 365u_2 - 420v_1 \leq 0,\\
&315u_1 + 1250u_2 - 300v_1 \leq 0,\\
&815u_1 + 670u_2 - 285v_1 \leq 0,\\
&281u_1 + 1100u_2 - 325E_{33}^{(1*)l'}v_1 = 0,\\
&290u_1 + 1130u_2 - 322E_{33}^{(1*)l}v_1 = 0,\\
&300u_1 + 1190u_2 - 315E_{33}^{(1*)m}v_1 = 0,\\
&307u_1 + 1230u_2 - 308E_{33}^{(1*)r}v_1 = 0,\\
&u_1, u_2, v_1 \geq 0. \qquad (5.6)
\end{aligned}
$$

Similarly, the values of $\tilde{E}_{pq}^{(1*)}$ can be determined for the remaining arcs. The corresponding FFESs for other arcs are given in Table 4.

Table 4. values of $\tilde{E}_{pq}^{1*}$.

	J	K	L
A	{(0.61,0.65,0.69);(0.57,0.65,0.72)}	{(0.87,0.91,0.97);(0.83,0.91,1.00)}	{(0.89,0.91,0.97);(0.85,0.91,1.00)}
B	{(0.39,0.42,0.43);(0.38,0.42,0.46)}	{(0.49,0.56,0.60);(0.46,0.56,0.62)}	{(0.89,0.92,0.95);(0.87,0.92,1.00)}
C	{(0.57,0.63,0.65);(0.55,0.63,0.69)}	{(0.85,0.91,0.96);(0.81,0.91,1.00)}	{(0.85,0.92,0.96);(0.79,0.92,1.00)}
D	{(0.88,0.91,0.98);(0.84,0.91,1.00)}	{(0.89,0.93,0.97);(0.86,0.93,1.00)}	{(0.80,0.84,0.88);(0.76,0.84,0.90)}
E	{(0.49,0.50,0.53);(0.42,0.50,0.55)}	{(0.56,0.58,0.60);(0.54,0.58,0.62)}	{(0.86,0.91,0.96);(0.81,0.91,1.00)}

By treating the source q as target, the FFESs $\tilde{E}_{pq}^{(2*)}$ can be determined by using the model (4.12). To determine $\tilde{E}_{33}^{(2*)}$ the model is as follows:

$$
\begin{aligned}
E_{33}^{2*} = \text{Max} \;& \frac{[u_1\{(290, 300, 307); (281, 300, 315)\} + u_2\{(1130, 1190, 1230); (1100, 1190, 1250)\}]}{v_1\{(308, 315, 322); (300, 315, 325)\}}\\
\text{subject to} \;& \frac{[u_1\{(423, 430, 440); (415, 430, 450)\} + u_2\{(740, 760, 790); (700, 760, 800)\}]}{v_1\{(361, 370, 380); (355, 370, 385)\}} \leq 1,\\
& \frac{[u_1\{(345, 352, 360); (335, 352, 370)\} + u_2\{(500, 565, 590); (485, 565, 600)\}]}{v_1\{(320, 327, 335); (315, 327, 342)\}} \leq 1,\\
& \frac{[u_1\{(290, 300, 307); (281, 300, 315)\} + u_2\{(1130, 1190, 1230); (1100, 1190, 1250)\}]}{v_1\{(308, 315, 322); (300, 315, 325)\}} \leq 1,\\
& \frac{[u_1\{(510, 520, 530); (505, 520, 540)\} + u_2\{(2800, 2900, 3050); (2730, 2900, 3110)\}]}{v_1\{(394, 400, 410); (390, 400, 415)\}} \leq 1,\\
& \frac{[u_1\{(531, 540, 550); (525, 540, 560)\} + u_2\{(475, 490, 520); (450, 490, 550)\}]}{v_1\{(451, 460, 469); (445, 460, 476)\}} \leq 1,\\
& u_1, u_2, v_1 \geq 0. \qquad (5.7)
\end{aligned}
$$

To solve the above model, five LP models (5.8)–(5.12) should be solved:

$$
\begin{aligned}
E_{33}^{(2*)l'} = \text{Max} \;& 281u_1 + 1100u_2\\
\text{subject to} \;& 325v_1 = 1,\\
& 450u_1 + 800u_2 - 355v_1 \leq 0,
\end{aligned}
$$

$$
\begin{aligned}
& 370u_1 + 600u_2 - 315v_1 \le 0, \\
& 315u_1 + 1250u_2 - 300v_1 \le 0, \\
& 540u_1 + 3110u_2 - 390v_1 \le 0, \\
& 560u_1 + 550u_2 - 445v_1 \le 0, \\
& u_1, u_2, v_1 \ge 0.
\end{aligned} \tag{5.8}
$$

$$
\begin{aligned}
E_{33}^{(2*)l} = \text{Max}\ \ & 290u_1 + 1130u_2 \\
\text{subject to}\ \ & 322v_1 = 1, \\
& 450u_1 + 800u_2 - 355v_1 \le 0, \\
& 370u_1 + 600u_2 - 315v_1 \le 0, \\
& 315u_1 + 1250u_2 - 300v_1 \le 0, \\
& 540u_1 + 3110u_2 - 390v_1 \le 0, \\
& 560u_1 + 550u_2 - 445v_1 \le 0, \\
& 281u_1 + 1100u_2 - 325E_{33}^{(2*)l'}v_1 = 0, \\
& u_1, u_2, v_1 \ge 0.
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
E_{12}^{(2*)m} = \text{Max}\ \ & 430u_1 + 760u_2 \\
\text{subject to}\ \ & 370v_1 = 1, \\
& 450u_1 + 800u_2 - 355v_1 \le 0, \\
& 370u_1 + 600u_2 - 315v_1 \le 0, \\
& 315u_1 + 1250u_2 - 300v_1 \le 0, \\
& 540u_1 + 3110u_2 - 390v_1 \le 0, \\
& 560u_1 + 550u_2 - 445v_1 \le 0, \\
& 281u_1 + 1100u_2 - 325E_{33}^{(2*)l'}v_1 = 0, \\
& 290u_1 + 1130u_2 - 322E_{33}^{(2*)l}v_1 = 0, \\
& u_1, u_2, v_1 \ge 0.
\end{aligned} \tag{5.10}
$$

$$
\begin{aligned}
E_{12}^{(2*)r} = \text{Max}\ \ & 440u_1 + 790u_2 \\
\text{subject to}\ \ & 361v_1 = 1, \\
& 450u_1 + 800u_2 - 355v_1 \le 0, \\
& 370u_1 + 600u_2 - 315v_1 \le 0, \\
& 315u_1 + 1250u_2 - 300v_1 \le 0, \\
& 540u_1 + 3110u_2 - 390v_1 \le 0, \\
& 560u_1 + 550u_2 - 445v_1 \le 0, \\
& 281u_1 + 1100u_2 - 325E_{33}^{(2*)l'}v_1 = 0,
\end{aligned}
$$

$$290u_1 + 1130u_2 - 322E_{33}^{(2*)l}v_1 = 0,$$
$$300u_1 + 1190u_2 - 315E_{33}^{(2*)m}v_1 = 0,$$
$$u_1, u_2, v_1 \geq 0. \tag{5.11}$$

$$\begin{aligned} E_{12}^{(2*)r'} = \text{Max}\;\; & 450u_1 + 800u_2 \\ \text{subject to}\;\; & 355v_1 = 1, \\ & 450u_1 + 800u_2 - 355v_1 \leq 0, \\ & 370u_1 + 600u_2 - 315v_1 \leq 0, \\ & 315u_1 + 1250u_2 - 300v_1 \leq 0, \\ & 540u_1 + 3110u_2 - 390v_1 \leq 0, \\ & 560u_1 + 550u_2 - 445v_1 \leq 0, \\ & 281u_1 + 1100u_2 - 325E_{33}^{(2*)l'}v_1 = 0, \\ & 290u_1 + 1130u_2 - 322E_{33}^{(2*)l}v_1 = 0, \\ & 300u_1 + 1190u_2 - 315E_{33}^{(2*)m}v_1 = 0, \\ & 307u_1 + 1230u_2 - 308E_{33}^{(2*)r}v_1 = 0, \\ & u_1, u_2, v_1 \geq 0. \end{aligned} \tag{5.12}$$

Similarly, the values of $\tilde{E}_{pq}^{(2*)}$ can be determined for the remaining arcs. The corresponding FFESs for other arcs are given in Table 5.

Table 5. values of $\tilde{E}_{pq}^{2*}$.

	J	K	L
A	{(0.65,0.69,0.73);(0.61,0.69,0.77)}	{(0.80,0.84,0.88);(0.78,0.84,0.92)}	{(0.56,0.58,0.61);(0.53,0.58,0.63)}
B	{(0.57,0.61,0.63);(0.55,0.61,0.67)}	{(0.74,0.78,0.81);(0.71,0.78,0.85)}	{(0.88,0.92,0.94);(0.85,0.92,1.00)}
C	{(0.82,0.91,0.94);(0.80,0.91,1.00)}	{(0.65,0.69,0.72);(0.62,0.69,0.76)}	{(0.85,0.92,0.96);(0.79,0.92,1.00)}
D	{(0.86,0.91,0.97);(0.82,0.91,1.00)}	{(0.90,0.94,0.97);(0.88,0.94,1.00)}	{(0.60,0.63,0.66);(0.57,0.63,0.68)}
E	{(0.84,0.89,0.97);(0.80,0.89,1.00)}	{(0.82,0.85,0.88);(0.80,0.85,0.91)}	{(0.86,0.91,0.96);(0.81,0.91,1.00)}

Now, the average of FFESs $\tilde{E}_{pq}^{(1*)}$ and $\tilde{E}_{pq}^{(2*)}$ is determined to get new FFESs $\tilde{E}_{pq}$ which are given in Table 6.

Table 6. values of $\tilde{E}_{pq}$.

	J	K	L
A	{(0.63,0.67,0.71);(0.59,0.67,0.75)}	{(0.84,0.88,0.93);(0.81,0.88,0.96)}	{(0.73,0.75,0.79);(0.69,0.75,0.82)}
B	{(0.48,0.52,0.53);(0.47,0.52,0.57)}	{(0.62,0.67,0.71);(0.59,0.67,0.74)}	{(0.89,0.92,0.95);(0.86,0.92,1.00)}
C	{(0.70,0.77,0.80);(0.68,0.77,0.85)}	{(0.75,0.80,0.84);(0.72,0.80,0.88)}	{(0.85,0.92,0.96);(0.79,0.92,1.00)}
D	{(0.87,0.91,0.98);(0.83,0.91,1.00)}	{(0.90,0.94,0.97);(0.87,0.94,1.00)}	{(0.70,0.74,0.77);(0.67,0.74,0.79)}
E	{(0.67,0.70,0.75);(0.61,0.70,0.78)}	{(0.69,0.72,0.74);(0.67,0.72,0.77)}	{(0.86,0.91,0.96);(0.81,0.91,1.00)}

To find the solution of the model (3.1), the following single objective FFTP is solved:

$$\text{Max} \quad \sum_{p=1}^{5}\sum_{q=1}^{3}\tilde{E}_{pq}x_{pq}$$

$$\text{subject to} \quad \sum_{q=1}^{3} x_{1q} = 8, \quad \sum_{q=1}^{3} x_{2q} = 6, \quad \sum_{q=1}^{3} x_{3q} = 7, \quad \sum_{q=1}^{3} x_{4q} = 9,$$
$$\sum_{q=1}^{3} x_{5q} = 12, \quad \sum_{p=1}^{5} x_{p1} = 15, \quad \sum_{p=1}^{5} x_{p2} = 14, \quad \sum_{p=1}^{5} x_{p3} = 13,$$
$$x_{pq} \geq 0, \quad \text{for all} \quad p, q. \tag{5.13}$$

To solve the above model, we apply the ranking function $\Re(\tilde{A}^F) = \dfrac{(u^l + 4u^m + u^r) + (u^{l'} + 4u^m + u^{r'})}{12}$, where $\tilde{A}^F = \{(u^l, u^m, u^r); (u^{l'}, u^m, u^{r'})\}$ is a TFFN. Therefore, each FFES is replaced with its corresponding rank. The related results are given in Table 7.

Table 7. Crisp values of FFESs.

$\Re(\tilde{A}^F) = \dfrac{(u^l + 4u^m + u^r) + (u^{l'} + 4u^m + u^{r'})}{12}$		
0.67	0.88	0.75
0.52	0.67	0.92
0.77	0.80	0.91
0.91	0.94	0.74
0.70	0.72	0.91

$$\begin{aligned} \text{Max} \quad & 0.67x_{11} + 0.88x_{12} + 0.75x_{13} + 0.52x_{21} + 0.67x_{22} + 0.92x_{23} + 0.77x_{31} + 0.80x_{32} \\ & + 0.91x_{33} + 0.91x_{41} + 0.94x_{42} + 0.74x_{43} + 0.70x_{51} + 0.72x_{52} + 0.91x_{53} \end{aligned}$$
$$\text{subject to} \quad \sum_{q=1}^{3} x_{1q} = 8, \quad \sum_{q=1}^{3} x_{2q} = 6, \quad \sum_{q=1}^{3} x_{3q} = 7, \quad \sum_{q=1}^{3} x_{4q} = 9,$$
$$\sum_{q=1}^{3} x_{5q} = 12, \quad \sum_{p=1}^{5} x_{p1} = 15, \quad \sum_{p=1}^{5} x_{p2} = 14, \quad \sum_{p=1}^{5} x_{p3} = 13,$$
$$x_{pq} \geq 0, \quad \text{for all} \quad p, q. \tag{5.14}$$

Finally, at the end, by solving the model (5.14), a Fermatean fuzzy transportation plan with the maximum Fermatean fuzzy efficiency is determined as follows:

$$x_{12} = 8, \quad x_{23} = 6, \quad x_{31} = 7, \quad x_{41} = 3 \quad x_{42} = 6, \quad x_{51} = 5, \quad x_{53} = 7.$$

The Fermatean fuzzy values of objective functions of transportation cost, shipment value and transportation profit are $\{(15648, 16007, 16342); (15395, 16007, 16612)\}$, $\{(23824, 24795, 25330); (23280, 24795, 26090)\}$ and $\{(45735, 47540, 49570); (44070, 47540, 50805)\}$, respectively. The MFs and NMFs of transportation cost, shipment value and transportation profit are represented in Figures 2– 4, respectively.

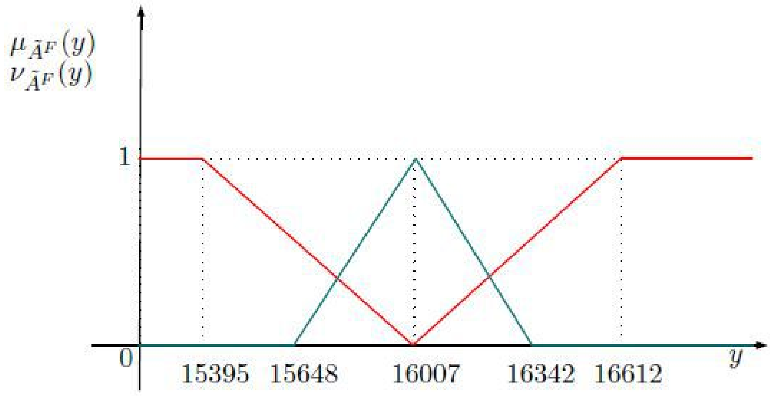

Figure 2. MF and NMF of transportation cost.

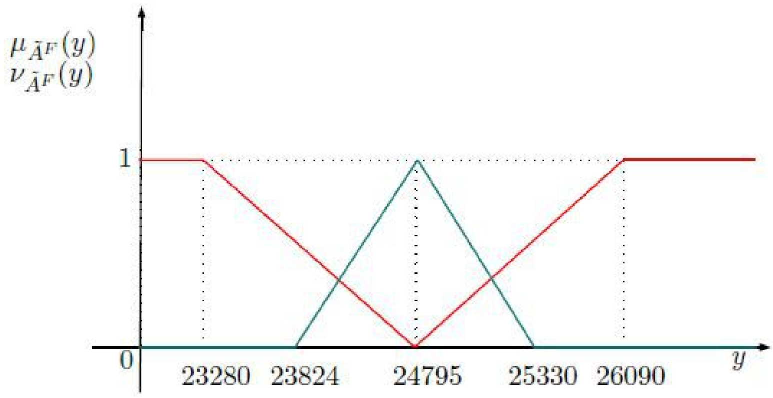

Figure 3. MF and NMF of shipping value.

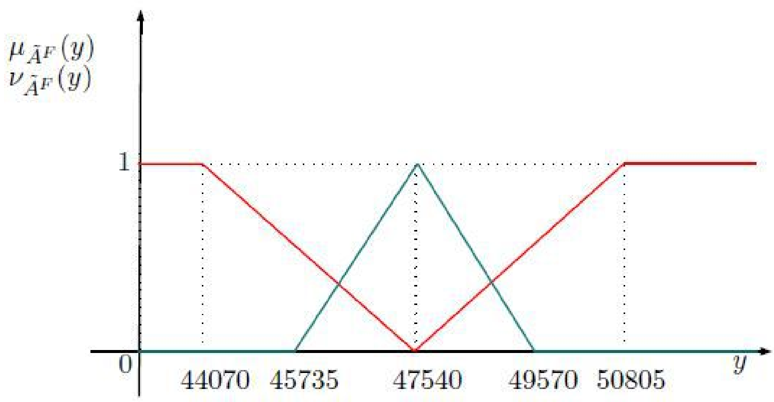

Figure 4. MF and NMF of transportation profit.

5.1. Comparative analysis

Now, if we solve the single objective FFTP (5.13) in Example 5.1 based on a fuzzy programming approach [65], we have the following.

$$Z_1 = \text{Min} 0.04x_{11} + 0.03x_{12} + 0.04x_{13} + 0.01x_{21} + 0.03x_{22} + 0.03x_{23} + 0.02x_{31} + 0.03x_{32}$$

$$
\begin{aligned}
& + 0.06x_{33} + 0.04x_{41} + 0.03x_{42} + 0.03x_{43} + 0.06x_{51} + 0.02x_{52} + 0.05x_{53} \\
Z_2 = \ & \text{Min} 0.04x_{11} + 0.04x_{12} + 0.02x_{13} + 0.04x_{21} + 0.05x_{22} + 0.03x_{23} + 0.07x_{31} + 0.05x_{32} \\
& + 0.07x_{33} + 0.04x_{41} + 0.04x_{42} + 0.04x_{43} + 0.03x_{51} + 0.03x_{52} + 0.05x_{53} \\
Z_3 = \ & \text{Max} 0.67x_{11} + 0.88x_{12} + 0.75x_{13} + 0.52x_{21} + 0.67x_{22} + 0.92x_{23} + 0.77x_{31} + 0.80x_{32} \\
& + 0.92x_{33} + 0.91x_{41} + 0.94x_{42} + 0.74x_{43} + 0.70x_{51} + 0.72x_{52} + 0.91x_{53} \qquad (5.15) \\
Z_4 = \ & \text{Max} 0.04x_{11} + 0.05x_{12} + 0.02x_{13} + 0.01x_{21} + 0.04x_{22} + 0.03x_{23} + 0.03x_{31} + 0.04x_{32} \\
& + 0.04x_{33} + 0.07x_{41} + 0.03x_{42} + 0.03x_{43} + 0.05x_{51} + 0.02x_{52} + 0.05x_{53} \\
Z_5 = \ & \text{Max} 0.04x_{11} + 0.03x_{12} + 0.03x_{13} + 0.04x_{21} + 0.03x_{22} + 0.05x_{23} + 0.05x_{31} + 0.04x_{32} \\
& + 0.04x_{33} + 0.02x_{41} + 0.03x_{42} + 0.02x_{43} + 0.03x_{51} + 0.03x_{52} + 0.04x_{53} \\
\text{subject to} \ & \text{constraints of model (5.13).}
\end{aligned}
$$

Then, the positive ideal solution ($Z^{\oplus}$) and negative ideal solution ($Z^{\ominus}$) are as follows:

$$
\begin{aligned}
Z_1^{\oplus} &= 1.01, \quad Z_1^{\ominus} = 1.92, \\
Z_2^{\oplus} &= 1.42, \quad Z_2^{\ominus} = 2.07, \\
Z_3^{\oplus} &= 36.19, \quad Z_3^{\ominus} = 29.55, \\
Z_4^{\oplus} &= 2.15, \quad Z_4^{\ominus} = 0.98, \\
Z_5^{\oplus} &= 1.67, \quad Z_5^{\ominus} = 1.24.
\end{aligned}
$$

Now, the following model is solved to obtain the solution of model (5.13):

$$
\begin{aligned}
\text{Max} \ & \beta \\
\text{subject to} \ & Z_1 + 0.91\beta \leq 1.92, \\
& Z_2 + 0.65\beta \leq 2.07, \\
& Z_3 - 6.64\beta \geq 29.55, \\
& Z_4 - 1.17\beta \geq 0.98, \\
& Z_5 - 0.43\beta \geq 1.24, \\
& \text{constraints of model (5.13).} \qquad (5.16)
\end{aligned}
$$

Therefore, the solution to the above model is $x_{11} = 3.85$, $x_{12} = 1.80$, $x_{13} = 2.35$, $x_{23} = 6$, $x_{31} = 3.51$, $x_{32} = 3.49$, $x_{41} = 7.63$, $x_{42} = 1.37$, $x_{52} = 7.35$, $x_{53} = 4.65$.

The objective values of shipping cost, shipping value and transportation profit are $\{(16379.14, 16699, 17009.81); (16108.06, 16699, 17290.29)\}$, $\{(22400.5, 23157.11, 23691.59); (21919.46, 23157.11, 24268.32)\}$ and $\{(33030.27, 34520.65, 35948.25); (31584.7, 34520.65, 37297.18)\}$. Note that by solving the single objective FFTP using the Mahmoodirad et al. [64] approach, the following solution is obtained: $x_{12} = 8$, $x_{23} = 6$, $x_{31} = 1$, $x_{32} = 6$, $x_{41} = 9$, $x_{51} = 5$, $x_{53} = 7$.

According to this solution, the Fermatean fuzzy shipping cost, Fermatean fuzzy shipment value and Fermatean fuzzy profit are {(14832,15233,15538);(14585,15233,15820)}, {(21970,22695,23212);(21396,22695,23810)} and {(36585,38240,39610);(35070,38240,40755)}, respectively. It is noted that the Fermatean fuzzy shipment value and profit obtained from the proposed approach are greater than those obtained from the Mahmoodriad et al. [64] approach. Therefore, the proposed approach is preferable by considering Fermatean fuzzy shipment value and profit.

Further, the suggested FFDEA technique for solving model (3.1) is now compared to an expanded version of the goal programming (GP) technique [54]. GP is a typical method for reducing a TP with several objective functions to a single objective function. The concept of GP is to reduce the distance between objective functions and an aspiration level vector either calculated by the decision maker or thet equals $\tilde{f} = (\tilde{f}_1^*, \tilde{f}_2^*, \ldots, \tilde{f}_h^*)$, where

$$\begin{aligned}
\tilde{f}_a^* =& Optimize \;\; \sum_{p=1}^{j}\sum_{q=1}^{k} \tilde{c}_{pq}^{a}\tilde{x}_{pq} \\
\text{subject to} \;\; & \sum_{q=1}^{k} \tilde{x}_{pq} = \tilde{s}_p, \;\; p = 1, \ldots, j, \\
& \sum_{p=1}^{j} \tilde{x}_{pq} = \tilde{d}_q, \;\; q = 1, \ldots, k, \\
& \tilde{x}_{pq} \geq \tilde{0}, \;\; p = 1, \ldots, j, \;\; q = 1, \ldots, k. \qquad (5.17)
\end{aligned}$$

Assume that, $\tilde{n}_d = \{(n_d^l, n_d^m, n_d^r); (n_d^{l'}, n_d^m, n_d^{r'})\}$ and $\tilde{p}_d = \{(p_d^l, p_d^m, p_d^r); (p_d^{l'}, p_d^m, p_d^{r'})\}$ are the under deviations and over deviations of the objectives $\tilde{f}_a = Optimize \sum_{p=1}^{j}\sum_{q=1}^{k} \tilde{c}_{pq}^{a}\tilde{x}_{pq}$ from their aspiration values $\tilde{f}_a^*$, respectively. Let $\tilde{f}_a = \sum_{p=1}^{j}\sum_{q=1}^{k} \tilde{c}_{pq}^{a}\tilde{x}_{pq}$ $(a = 1, \ldots, g)$ and $\tilde{f}_a = \sum_{p=1}^{j}\sum_{q=1}^{k} \tilde{c}_{pq}^{a}\tilde{x}_{pq}$ $(a = g+1, \ldots, g+s)$ be those objective functions of model (3.1) that should be minimized and maximized, respectively. In this way, the model (3.1) is transformed by GP into a minimization problem of the deviational parameters which minimizes the sum of deviational parameters as follows:

$$\begin{aligned}
\min \;\; & \sum_{a=1}^{g} \tilde{p}_a + \sum_{g+1}^{g+s} \tilde{n}_a \\
\text{subject to} \;\; & \sum_{p=1}^{j}\sum_{q=1}^{k} \tilde{c}_{pq}^{a}\tilde{x}_{pq} \leq \tilde{p}_a + \tilde{f}_a^*, \;\; a = 1, \ldots, g, \\
& \sum_{p=1}^{j}\sum_{q=1}^{k} \tilde{c}_{pq}^{a}\tilde{x}_{pq} + \tilde{n}_a \geq \tilde{f}_a^*, \;\; a = g+1, \ldots, g+s, \\
& \tilde{p}_a \geq \tilde{0}, \;\; a = 1, \ldots, g, \\
& \tilde{n}_a \geq \tilde{0}, \;\; a = g+1, \ldots, g+s, \\
& \sum_{q=1}^{k} \tilde{x}_{pq} = s_p, \;\; p = 1, \ldots, j, \\
& \sum_{p=1}^{j} \tilde{x}_{pq} = d_q, \;\; q = 1, \ldots, k, \\
& \tilde{x}_{pq} \geq \tilde{0}, \;\; p = 1, \ldots, j, \;\; q = 1, \ldots, k. \qquad (5.18)
\end{aligned}$$

For solving the model (5.18), assume

$$\begin{aligned}
\tilde{n}_a &= \{(n_a^l, n_a^m, n_a^r); (n_a^{l'}, n_a^m, n_a^{r'})\},\\
\tilde{p}_a &= \{(p_a^l, p_a^m, p_a^r); (p_a^{l'}, p_a^m, p_a^{r'})\},\\
\tilde{c}_{pq}^a &= \{(c_{pq}^{a,l}, c_{pq}^{a,m}, c_{pq}^{a,r}); (c_{pq}^{a,l'}, c_{pq}^{a,m}, c_{pq}^{a,r'})\} \text{ and}\\
\tilde{f}_a^* &= \{(f_a^{l^*}, f_a^{m^*}, f_a^{r^*}); (f_a^{l'^*}, f_a^{m^*}, f_a^{r'^*})\}.
\end{aligned}$$

In this way, model (5.18) is transformed into the following model using Definitions 2.8 and 2.9:

$$\begin{aligned}
\min \quad & \sum_{a=1}^{g}(p_a^{l'} + p_a^l + p_a^m + p_a^r + p_a^{r'}) + \sum_{g+1}^{g+s}(n_a^{l'} + n_a^l + n_a^m + n_a^r + n_a^{r'})\\
\text{subject to} \quad & \sum_{p=1}^{j}\sum_{q=1}^{k} c_{pq}^{a,l'} x_{pq}^{l'} \le p_a^{l'} + f_a^{l'^*}, \;\; a = 1, \dots, g,\\
& \sum_{p=1}^{j}\sum_{q=1}^{k} c_{pq}^{a,l} x_{pq}^{l} \le p_a^{l} + f_a^{l^*}, \;\; a = 1, \dots, g,\\
& \sum_{p=1}^{j}\sum_{q=1}^{k} c_{pq}^{a,m} x_{pq}^{m} \le p_a^{m} + f_a^{m^*}, \;\; a = 1, \dots, g,\\
& \sum_{p=1}^{j}\sum_{q=1}^{k} c_{pq}^{a,r} x_{pq}^{r} \le p_a^{r} + f_a^{r^*}, \;\; a = 1, \dots, g,\\
& \sum_{p=1}^{j}\sum_{q=1}^{k} c_{pq}^{a,r'} x_{pq}^{r'} \le p_a^{r'} + f_a^{r'^*}, \;\; a = 1, \dots, g,\\
& \sum_{p=1}^{j}\sum_{q=1}^{k} c_{ij}^{a,l'} x_{pq} + n_a^{l'} \ge f_a^{l'^*}, \;\; a = g+1, \dots, g+s,\\
& \sum_{p=1}^{j}\sum_{q=1}^{k} c_{pq}^{a,l} x_{pq} + n_a^{l} \ge f_a^{l^*}, \;\; a = g+1, \dots, g+s,\\
& \sum_{p=1}^{j}\sum_{q=1}^{k} c_{pq}^{a,m} x_{pq} + n_a^{m} \ge f_a^{m^*}, \;\; a = g+1, \dots, g+s,\\
& \sum_{p=1}^{j}\sum_{q=1}^{k} c_{pq}^{a,r} x_{pq} + n_a^{r} \ge f_a^{r^*}, \;\; a = g+1, \dots, g+s,\\
& \sum_{p=1}^{j}\sum_{q=1}^{k} c_{pq}^{a,r'} x_{pq} + n_a^{r'} \ge f_a^{r'^*}, \;\; a = g+1, \dots, g+s,\\
& p_a^{l'} \ge 0, \;\; a = 1, \dots, g,\\
& n_a^{l'} \ge 0, a = g+1, \dots, g+s,\\
& \text{constraints of model (4.24).}
\end{aligned} \tag{5.19}$$

The model (5.19) is a LPP that can be solved using the simplex method. For solving this example using the GP technique, the Fermatean fuzzy values of the objective functions of transportation cost, shipment value, and transportation profit are $\{(16445, 17205, 17665); (16055, 17205, 18135)\}$, $\{(21122, 21975, 22660); (20880, 21975, 23110)\}$, $\{(42355, 43205, 43925); (41846, 43205, 44435)\}$. Now, the solutions obtained by different approaches are given in Table 8.

Table 8. Comparison of solutions.

Approach	Transportation cost	Shipment value	Transportation profit
Proposed approach	{(15648, 16007, 16342); (15395, 16007, 16612)}	{(23824, 24795, 25330); (23280, 24795, 26090)}	{(45735, 47540, 49570); (44070, 47540, 50805)}
Fuzzy programming [65]	{(16379.14, 16699, 17009.81); (16108.06, 16699, 17290.29)}	{(22400.5, 23157.11, 23691.59); (21919.46, 23157.11, 24268.32)}	{(33030.27, 34520.65, 35948.25); (31584.7, 34520.65, 37297.18)}
Goal programming [54]	{(16445, 17205, 17665); (16055, 17205, 18135)}	{(21122, 21975, 22660); (20880, 21975, 23110)}	{(42355, 43205, 43925); (41846, 43205, 44435)}

As can be seen from Table 8, the Fermatean fuzzy transport cost determined using the proposed method is smaller than that obtained from the fuzzy programming and objective programming methods. Furthermore, the Fermatean fuzzy shipping value and Fermatean fuzzy profit determined using this method are larger than the values obtained from the fuzzy programming and objective programming methods. Therefore, the proposed method is more suitable for finding solutions for FFMOTP.

5.2. Advantages and limitations of the proposed problem

In general, the presented approach has many significant advantages over the fuzzy programming technique:

- The classical LP problem (4.33) used to solve FFTP (4.24) is not a transportation type problem. However, the LP problem resulting from the proposed method to solve FFTP (4.24) is a classical TP.
- The proposed approach provides an optimum solution for the FFTP (4.24) with integer values, but the fuzzy programming technique yields a Fermatean fuzzy optimal solution having non-integer values in the Fermatean fuzzy quantities of some goods to be moved from sources to endpoints that have no physical significance.
- The classical LP problem (4.33) used to solve FFTP (4.24) by applying the fuzzy programming technique has more variables and constraints as compared to the problem resulting from the proposed approach. Therefore, from a computational perspective, taking into account the number of variables and constraints, using the proposed method to solve FFTP (4.24) is highly effective in contrast to the fuzzy programming technique.
- Using the FFDEA method presented in this work, the FFMOTP (3.1) is transformed into a single objective FFTP without changing the structure of the TP. However, utilizing other techniques like goal programming and fuzzy approaches will increase the number of constraints for the problem by introducing new ones. As a result, if the problem seeks an integer optimal solution, the FFDEA method can be utilized by simply finding an optimal solution to the TP achieved by neglecting the integrality limitations, but the goal programming and fuzzy programming techniques are unable to obtain integer solutions without adding the integrality limitations.
- Our proposed method has some limitations also.
  (i) Our proposed method cannot be applied to unbalanced FFMOTP to obtain the Fermatean fuzzy optimal solution.

(ii) As long as the problem posed is nonlinear, our proposed method cannot be applied.
(iii) Our proposed method cannot be applied when all objective functions are minimized.

5.3. Managerial insights and results discussion

The proposed work can be broadly applied to numerous supply chain management and logistics operations. The proposed model can help organizations develop forward-looking network designs and consider sustainability impacts. The proposed method is very useful for dealing with indecisive uncertainty when the fuzzy variables are not sufficient to specify certain parameters in any logical process. Considering the uncertainty of transportation cost, shipment value, and transportation profit, DM will be able to operate the logistics system by transporting the right amount of product to the right demand centre at no additional cost. Therefore, the model constructs objective functions related to total transportation cost, shipment value, and transportation profit, which will help the organization stay in the global market by getting more transportation profit and shipment value and less transportation cost. Furthermore, the results obtained using our proposed method are inherently Fermatean fuzzy, i.e., they are normal forms of TFFNs. The results obtained using our proposed method are compared with existing methods with the help of a single real-world example. The transportation cost, shipment value, and transportation profit obtained using our proposed method are $\{(15648, 16007, 16342); (15395, 16007, 16612)\}$, $\{(23824, 24795, 25330); (23280, 24795, 26090)\}$ and $\{(45735, 47540, 49570); (44070, 47540, 50805)\}$. In addition, from the membership function of Fermatean fuzzy transportation cost $\{(15648, 16007, 16342); (15395, 16007, 16612)\}$, it is shown that degree of acceptance of transportation cost for DM increases when transportation cost increases from 15648 to 16007 and decreases when the transportation cost increases from 16007 to 16342, and the decision maker is completely satisfied when the transportation cost is 16007. However, the decision maker is fully unsatisfied, or the transportation cost is rejected when it lies beyond the interval $(15648, 16342)$. Furthermore, the non-membership function shows that the degree of rejection decreases when the transportation cost increases from 15395 to 16007, and it increases when the transportation cost increases from 16007 to 16612. When it exceeds $(16007, 16612)$, the transportation cost is completely rejected. Membership and non-membership functions for shipment value and transportation profit are represented in a similar fashion.

6. Conclusions

The TP is a form of LPP that is used to optimize resource allocation; it is a very important tool for managers and supply chain engineers to employ for cost optimization. The basic idea of TP is to determine the minimum total transportation cost for transporting a product from multiple sources to multiple destinations. We have formulated the MOTP in a Fermatean fuzzy environment. Next, we have developed an approach for solving the FFMOTP based on FFDEA, which is motivated by [62]. In this approach, every arc has been considered a DMU in the FFMOTP. Furthermore, those objective functions that should be maximized have been used to define the outputs of DMU, while those that should be minimized have been used to define the inputs of DMU. As a consequence, two different FFESs have been obtained for every arc by solving the FFDEA models. Then, by averaging these FFESs, unique FFESs for each arc have been determined. In this way, the FFMOTP has been transformed into a single objective FFTP. Finally, the FFTP has been converted into a classical TP by using the ranking function of the TFFN. A numerical example has been provided to illustrate the proposed method. The advantage of the proposed method is that it provides better results than the existing methods [64, 65]. In the future, we want to extend this work to the fractional TP and multi-objective fractional TP in the Fermatean fuzzy environment. Furthermore, we would like to point out that the proposed method cannot be used to determine the Fermatean fuzzy optimal solution of the MOTP when the parameters are generalized TFFNs. Therefore, further research to extend the proposed method to address these shortcomings is an interesting avenue for future research. We will report significant results from these ongoing projects in the near future.

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Conflict of interest

The authors declare no conflict of interest.

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