Mathematics

Research article

Properties of solutions for fractional-order linear system with differential equations

Shuo Wang[1]**, Juan Liu**[2] **and Xindong Zhang**[1,*]

[1] School of Mathematical Sciences, Xinjiang Normal University, Urumqi, Xinjiang 830017, China

[2] College of Big Data Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China

* **Correspondence:** Email: liaoyuan1126@163.com.

Abstract: In this paper, we study the analytical solutions of two-dimensional fractional-order linear system $\mathcal{D}_t^\alpha X(t) = AX(t)$ described by fractional differential equations, where $\mathcal{D}$ is the fractional derivative in the Caputo-Fabrizio sense and $A = (a_{ij})_{2\times2}$ is nonsingular coefficient matrix with $a_{ij} \in \mathbb{R}$. The analytical solutions of fractional-order linear system will be compared to the solution of classical linear system. Examples are provided to characterize the behavior of the solutions for fractional-order linear system.

1. Introduction

A fractional-order system means a system described by a fractional differential equation or fractional integral equation or by a system of such equations. A fractional differential equation is an equation which contains fractional derivative. There are many forms of fractional derivatives, such as Riemann-Liouville, Grünwald-Letnikov, Caputo and Caputo-Fabrizio fractional derivatives, one can refer to [1]. Fractional differential equations have been applied in various fields, and have been studied in the literature, for example, Lin and Xu studied the time-fractional diffusion equation using finite difference/spectral approximations in [2]; Zhuang et al. [3] considered the numerical methods for the variable-order fractional advection diffusion equation with a nonlinear source term; Properties of a new fractional derivative without singular kernel was studied by Losada and Nieto in [4]; Fardi and Khan considered the finite difference-spectral method for fractal mobile/immobiletransport model based on Caputo-Fabrizio derivative [5]; Solution of a fractional logistic ordinary differential equation

was obtained by Nieto in [6]. It has been found that the behavior of many physical systems can be properly described by using the fractional-order system theory. Generally speaking, the exact solutions of fractional differential equations are difficulty solved by some analytic methods. However, for some special fractional differential equations, we can still obtain their analytical solutions, such as fractional-order linear system involving Caputo-Fabrizio fractional derivative. Caputo-Fabrizio fractional derivative was proposed by Caputo and Fabrizio [7]. Algahtani [8] represented the model by Allen-Cahn with both derivatives (Atangana-Baleanu and Caputo-Fabrizio) in order to see their difference in a real world problem. Akman et al. [9] computed Caputo-Fabrizio derivatives of some known functions both theoretically and numerically. Baleanu et al. constructed a fractional order model of 2019 coronavirus disease transmission using Caputo-Fabrizio derivative in [10], and proposed a new fractional model for human liver involving Caputo-Fabrizio derivative with the exponential kernel in [11]. Aydogan et al. [12] investigated a new version of the mathematical model of Rabies disease by using the fractional Caputo-Fabrizio derivative. Additional researches on Caputo-Fabrizio fractional derivative can be found in [13–18], among others. There are many results about autonomous system, one can refer to [19–22]. Also, the fractional-order system has been studied by many scholars [23–25].

In this paper, we consider the Caputo-Fabrizio fractional-order real linear system. The main goal of this paper is to obtain the analytical solutions of fractional-order linear system, where the fractional derivative is in the Caputo-Fabrizio sense. The system is of the form

$$\mathcal{D}_t^\alpha X(t) = AX(t), \tag{1.1}$$

where $0 < \alpha < 1$, $X(t) = [x(t), y(t)]'$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. The fractional derivative $\mathcal{D}_t^\alpha x(t)$ is Caputo-Fabrizio fractional derivative of order α, defined as (one can see [4])

$$\mathcal{D}_t^\alpha x(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} x'(s) ds. \tag{1.2}$$

In [4], we know that $M(\alpha) = \frac{2}{2-\alpha}$. Thus, we have (one can see [7])

$$\mathcal{D}_t^\alpha x(t) = \frac{1}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} x'(s) ds. \tag{1.3}$$

For Caputo-Fabrizio fractional derivative, the corresponding fractional integral of a function $x(t)$ is

$$\mathcal{I}_t^\alpha x(t) = (1-\alpha)[x(t) - x_0] + \alpha \int_0^t x(s) ds, \tag{1.4}$$

where $x_0 = x(0)$ is the initial condition. By (1.4), if $x_0 \neq 0$ and $x(t) = x_0 e^{-\frac{\alpha}{1-\alpha}t}$, then we can obtain that $\mathcal{I}_t^\alpha x(t) = 0$. Hence the kernel of the operator $\mathcal{I}_t^\alpha$ is the one-dimensional subspace generated by $e^{-\frac{\alpha}{1-\alpha}t}$. This is quite different to the case of the usual fractional where $I_t^\alpha x(t) = 0$ implies that $x(t) = 0$. Thus, we have

$$\mathcal{I}_t^\alpha \mathcal{D}_t^\alpha x(t) = x(t) + c, \tag{1.5}$$

where c is an arbitrary constant. The behavior of classical fractional derivative (Rieman-Liouville or Caputo) is similar to the classical integer order derivative in the sense that it satisfies that the derivative

of the integral is the same function. However, for the new Caputo-Fabrizio derivative, we have

$$\mathcal{D}_t^\alpha \mathcal{I}_t^\alpha x(t) = x(t) - x_0 e^{-\frac{\alpha}{1-\alpha}t}. \tag{1.6}$$

Let $x(t)$ be the solution of (1.5), by (1.5) and (1.6), then $x(t)$ is the solution of (1.6) with a kernel $e^{-\frac{\alpha}{1-\alpha}t}$.

This paper is organized as follows. In Section 2, we solve the Caputo-Fabrizio fractional-order real linear system and obtain the analytical solutions which coincides with the solutions of classical linear system when $\alpha \to 1^-$. In Section 3, we plot the solutions of system (1.1) under given conditions at different values of α.

2. Solutions of fractional-order linear system

In this section, we consider fractional-order real linear system (1.1) under the assumption that $det(A) \neq 0$. According to the theoretical knowledge of linear algebra, if A is a nonsingular matrix, then there exists a nonsingular matrix T such that $B = T^{-1}AT$ and $X(t) = T\widetilde{X}(t)$, where B is a Jordan normal form and $\widetilde{X}(t) = [u(t), v(t)]'$. The corresponding characteristic polynomial of A is

$$P(\lambda) = det(A - \lambda I) = \lambda^2 - p\lambda + q \quad \text{with} \quad p = a_{11} + a_{22} \quad \text{and} \quad q = a_{11}a_{22} - a_{21}a_{22}, \tag{2.1}$$

which has zeros $\lambda = \frac{1}{2}(p - \sqrt{p^2 - 4q})$ and $\mu = \frac{1}{2}(p + \sqrt{p^2 - 4q})$. Thus, the system (1.1) will become

$$\mathcal{D}_t^\alpha \widetilde{X}(t) = T^{-1}AT\widetilde{X}(t) = B\widetilde{X}(t), \tag{2.2}$$

where B is one of the following normal forms:

$$\mathcal{L}(\lambda, \mu) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \mathcal{M}(\lambda) = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}, \quad \mathcal{R}(\eta, \gamma) = \begin{pmatrix} \eta & \gamma \\ -\gamma & \eta \end{pmatrix}.$$

Here λ, μ, η and γ are real numbers with $\mu \neq 0$ and $\gamma > 0$. If $\lambda = \mu \neq 0$, then it will become a special case of $\mathcal{L}(\lambda, \mu)$. The above three cases are denoted by $\mathcal{L}$ for left, $\mathcal{M}$ for middle and $\mathcal{R}$ for right, respectively. Therefore, we need to consider the case one by one.

If $A = \mathcal{L}(\lambda, \mu)$, then system (1.1) will become as follows

$$\begin{cases} \mathcal{D}_t^\alpha x(t) = \lambda x(t), \\ \\ \mathcal{D}_t^\alpha y(t) = \mu y(t). \end{cases} \tag{2.3}$$

Observe that for $\alpha = 1$, system (2.3) is recovered to the classical linear system as follows

$$\begin{cases} x'(t) = \lambda x(t), \\ \\ y'(t) = \mu y(t), \end{cases} \tag{2.4}$$

which has the solutions $x(t) = c_1 e^{\lambda t}$ and $y(t) = c_2 e^{\mu t}$.

The analytical solutions of system (2.3) can be obtained by the following theorem.

Theorem 2.1. *Let x_0 and y_0 be the initial conditions of system (2.3). Then the solutions of system (2.3) are $x(t) = c_1 e^{\frac{\lambda\alpha}{1-\lambda+\lambda\alpha}t}$ and $y(t) = c_2 e^{\frac{\mu\alpha}{1-\mu+\mu\alpha}t}$, where $c_1 = x_0$ and $c_2 = y_0$.*

Proof. If $x(t)$ and $y(t)$ are the solutions of (2.3), by (1.4) and (1.5), interating we get

$$
\begin{cases}
x(t) + c_1 = \lambda I_t^\alpha x(t) = \lambda(1 - \alpha)[x(t) - x(0)] + \lambda\alpha \int_0^t x(s)ds, \\
\\
y(t) + c_2 = \mu I_t^\alpha y(t) = \mu(1 - \alpha)[y(t) - y(0)] + \mu\alpha \int_0^t y(s)ds.
\end{cases}
\tag{2.5}
$$

Taking the first derivative both sides of (2.5), we get

$$
\begin{cases}
x'(t) = \lambda(1 - \alpha)x'(t) + \lambda\alpha x(t), \\
\\
y'(t) = \mu(1 - \alpha)y'(t) + \mu\alpha y(t).
\end{cases}
\tag{2.6}
$$

By (2.6), for $1 - \lambda + \lambda\alpha \neq 0$ and $1 - \mu + \mu\alpha \neq 0$, the analytical solutions of the system (2.3) with $0 < \alpha < 1$ are given by

$$
x(t) = c_1 e^{\frac{\lambda\alpha}{1-\lambda+\lambda\alpha}t} \text{ and } y(t) = c_2 e^{\frac{\mu\alpha}{1-\mu+\mu\alpha}t},
\tag{2.7}
$$

where $c_1 = x_0$ and $c_2 = y_0$ are real constants. This proves the theorem. □

If $A = \mathcal{M}(\lambda)$, then system (1.1) will become as follows

$$
\begin{cases}
\mathcal{D}_t^\alpha x(t) = \lambda x(t), \\
\\
\mathcal{D}_t^\alpha y(t) = x(t) + \lambda y(t).
\end{cases}
\tag{2.8}
$$

Observe that for $\alpha = 1$, system (2.8) is recovered to the classical linear system as follows

$$
\begin{cases}
x'(t) = \lambda x(t), \\
\\
y'(t) = x(t) + \lambda y(t),
\end{cases}
\tag{2.9}
$$

which has the solutions $x(t) = c_1 e^{\lambda t}$ and $y(t) = (c_1 t + c_2)e^{\lambda t}$.

The analytical solutions of system (2.8) can be obtained by the following theorem.

Theorem 2.2. *Let x_0 and y_0 be the initial conditions of system (2.8). Then the solutions of system (2.8) are $x(t) = c_1 e^{\frac{\lambda\alpha}{1-\lambda+\lambda\alpha}t}$ and $y(t) = \left(\frac{c_1\alpha}{(1-\lambda+\lambda\alpha)^2}t + c_2\right)e^{\frac{\lambda\alpha}{1-\lambda+\lambda\alpha}t}$, where $c_1 = x_0$ and $c_2 = y_0$.*

Proof. If $x(t)$ and $y(t)$ are the solutions of (2.8), by (1.4) and (1.5), interating we get

$$
\begin{cases}
x(t) + c_1 = \lambda(1 - \alpha)[x(t) - x_0] + \lambda\alpha \int_0^t x(s)ds, \\
\\
y(t) + c_2 = (1 - \alpha)[x(t) - x_0] + \alpha \int_0^t x(s)ds + \lambda(1 - \alpha)[y(t) - y(0)] + \lambda\alpha \int_0^t y(s)ds.
\end{cases}
\tag{2.10}
$$

Taking the first derivative both sides of (2.10), we get

$$
\begin{cases}
x'(t) = \lambda(1 - \alpha)x'(t) + \lambda\alpha x(t), \\
\\
y'(t) = (1 - \alpha)x'(t) + \alpha x(t) + \lambda(1 - \alpha)y'(t) + \lambda\alpha y(t).
\end{cases}
\tag{2.11}
$$

By (2.11), for $1 - \lambda + \lambda\alpha \neq 0$, the analytical solutions of the system (2.8) with $0 < \alpha < 1$ are given by

$$
x(t) = c_1 e^{\frac{\lambda\alpha}{1-\lambda+\lambda\alpha}t} \text{ and } y(t) = \left(\frac{c_1\alpha}{(1 - \lambda + \lambda\alpha)^2}t + c_2\right)e^{\frac{\lambda\alpha}{1-\lambda+\lambda\alpha}t},
\tag{2.12}
$$

where $c_1 = x_0$ and $c_2 = y_0$ are real constants. This proves the theorem. □

Similarly, if $A = \mathcal{R}(\eta, \gamma)$, then system (1.1) will become as follows

$$
\begin{cases}
\mathcal{D}_t^\alpha x(t) = \eta x(t) + \gamma y(t), \\
\mathcal{D}_t^\alpha y(t) = -\gamma x(t) + \eta y(t).
\end{cases} \tag{2.13}
$$

Observe that for $\alpha = 1$, system (2.13) is recovered to the classical linear system as follows

$$
\begin{cases}
x'(t) = \eta x(t) + \gamma y(t), \\
y'(t) = -\gamma x(t) + \eta y(t),
\end{cases} \tag{2.14}
$$

which has the solutions $x(t) = c_1 e^{\eta t} \cos(-\gamma t + c_2)$ and $y(t) = c_1 e^{\eta t} \sin(-\gamma t + c_2)$. Where c_1, c_2 satisfy that $x_0 = c_1 \cos c_2$ and $y_0 = c_1 \sin c_2$.

The analytical solutions of system (2.13) can be obtained by the following theorem.

Theorem 2.3. *Let x_0 and y_0 be the initial conditions of system (2.13). Then the solutions of system (2.13) are $x(t) = c_1 e^{\eta_1 t} \cos(-\gamma_1 t + c_2)$ and $y(t) = c_1 e^{\eta_1 t} \sin(-\gamma_1 t + c_2)$, where $\eta_1 = \frac{\eta\alpha - (\alpha - \alpha^2)(\eta^2 + \gamma^2)}{(1-\eta+\eta\alpha)^2 + (\gamma - \gamma\alpha)^2}$ and $\gamma_1 = \frac{\gamma\alpha}{(1-\eta+\eta\alpha)^2 + (\gamma - \gamma\alpha)^2}$, and c_1, c_2 satisfy that $x_0 = c_1 \cos c_2$ and $y_0 = c_1 \sin c_2$.*

Proof. If $x(t)$ and $y(t)$ are the solutions of (2.13), by (1.4) and (1.5), interating we get

$$
\begin{cases}
x(t) + c_1 = \eta(1-\alpha)[x(t) - x_0] + \eta\alpha \int_0^t x(s)ds + \gamma(1-\alpha)[y(t) - y_0] + \gamma\alpha \int_0^t y(s)ds, \\
y(t) + c_2 = -\gamma(1-\alpha)[x(t) - x_0] - \gamma\alpha \int_0^t x(s)ds + \eta(1-\alpha)[y(t) - y_0] + \eta\alpha \int_0^t y(s)ds.
\end{cases} \tag{2.15}
$$

Taking the first derivative both sides of (2.15), we get

$$
\begin{cases}
x'(t) = \eta(1-\alpha)x'(t) + \eta\alpha x(t) + \gamma(1-\alpha)y'(t) + \gamma\alpha y(t), \\
y'(t) = -\gamma(1-\alpha)x'(t) - \gamma\alpha x(t) + \eta(1-\alpha)y'(t) + \eta\alpha y(t).
\end{cases} \tag{2.16}
$$

By (2.16), we get

$$
\begin{cases}
x'(t) = \eta_1 x'(t) + \gamma_1 y(t), \\
y'(t) = -\gamma_1 x'(t) + \eta_1 y(t),
\end{cases} \tag{2.17}
$$

where $\eta_1 = \frac{\eta\alpha - (\alpha - \alpha^2)(\eta^2 + \gamma^2)}{(1-\eta+\eta\alpha)^2 + (\gamma - \gamma\alpha)^2}$ and $\gamma_1 = \frac{\gamma\alpha}{(1-\eta+\eta\alpha)^2 + (\gamma - \gamma\alpha)^2}$.

The system (2.17) can be solved explicitly by introducing polar coordinates $x = \rho \cos\theta$ and $y = \rho \sin\theta$, as follows. First note that

$$
x\frac{dx}{dt} + y\frac{dy}{dt} = \rho\frac{d\rho}{dt} \text{ and } x\frac{dy}{dt} - y\frac{dx}{dt} = \rho^2\frac{d\theta}{dt}.
$$

It follows that

$$
\rho'(t) = \eta_1 \rho \text{ and } \theta'(t) = -\gamma_1, \tag{2.18}
$$

thus, the solutions of system (2.18) may be expressed in the form

$$\rho(t) = c_1 e^{\eta_1 t} \text{ and } \theta(t) = -\gamma_1 t + c_2, \tag{2.19}$$

where $c_1 = \rho(0)$ and $c_2 = \theta(0)$.

By (2.19), the solutions of system (2.13) may be expressed in the form

$$x(t) = c_1 e^{\eta_1 t} \cos(-\gamma_1 t + c_2) \text{ and } y(t) = c_1 e^{\eta_1 t} \sin(-\gamma_1 t + c_2). \tag{2.20}$$

This proves the theorem. □

3. Computational results and discussion

In this section, we plot the solutions of system (1.1) under given conditions at different values of α. We consider the following three cases, that are $A = \mathcal{L}(\lambda, \mu)$, $A = \mathcal{M}(\lambda)$ and $A = \mathcal{R}(\eta, \gamma)$.
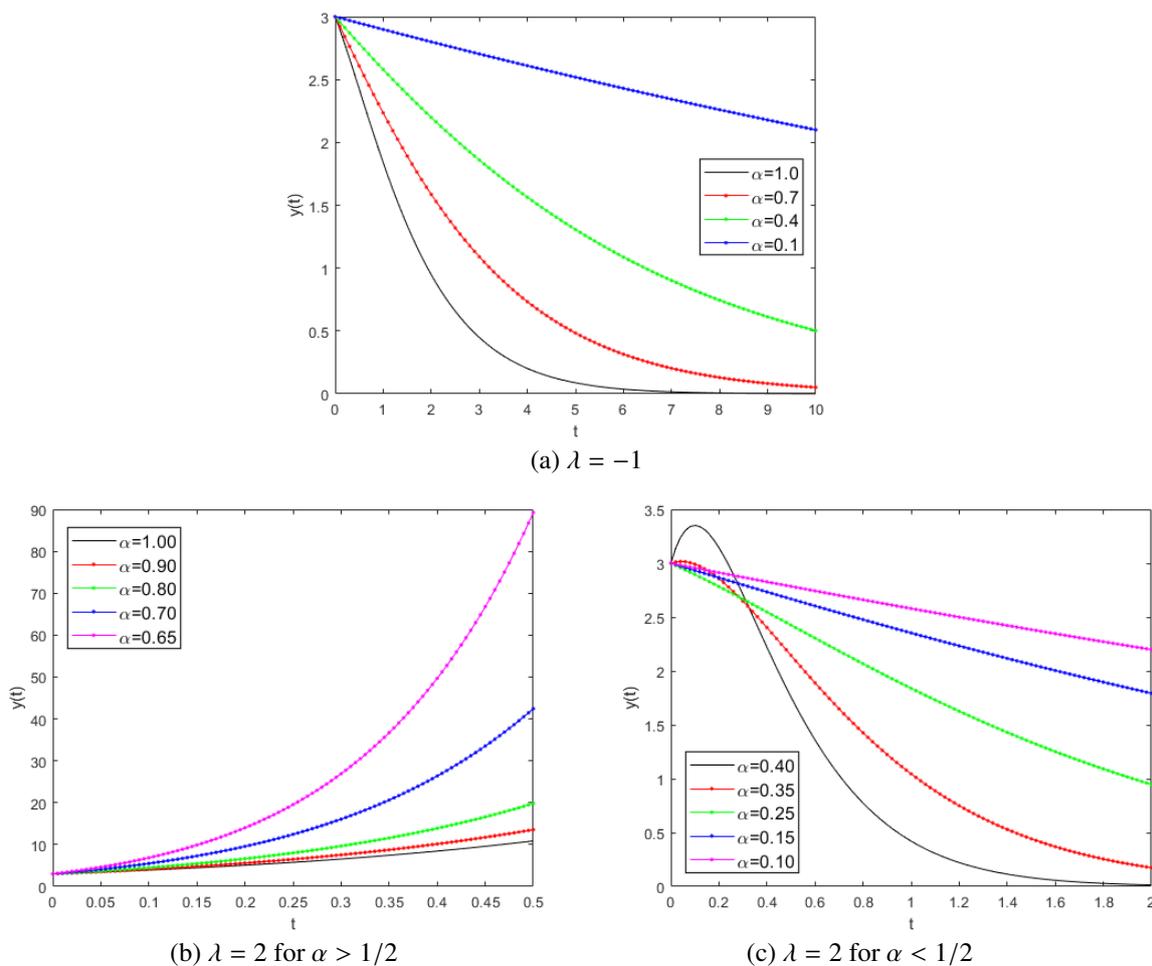
For $A = \mathcal{L}(\lambda, \mu)$. If $\lambda = -1$ or $\lambda = 2$, then the solutions $x(t)$ of system (2.3) for different α are shown in Figure 1 (similarly, the solutions of $y(t)$ can be obtained for given μ). For $\lambda = -1$, the results are shown in Figure 1 (a), which shows that $x(t) \to 0$ as $t \to +\infty$ and the speed of $x(t)$ approaching zero gradually becomes faster with the increase of α. For $\lambda = 2$, by Theorem 2.1, we know that $\alpha \neq 1/2$, and $\alpha = 1/2$ is the demarcation line, the results are shown in Figure 1 (b). If $\alpha > 1/2$, then $x(t)$ is increasing with the evolution of time and the speed increase of $x(t)$ becomes faster with the decrease of α. If $\alpha < 1/2$, then $x(t)$ is decreasing with the evolution of time and the speed decrease of $x(t)$ becomes faster with the increase of α.
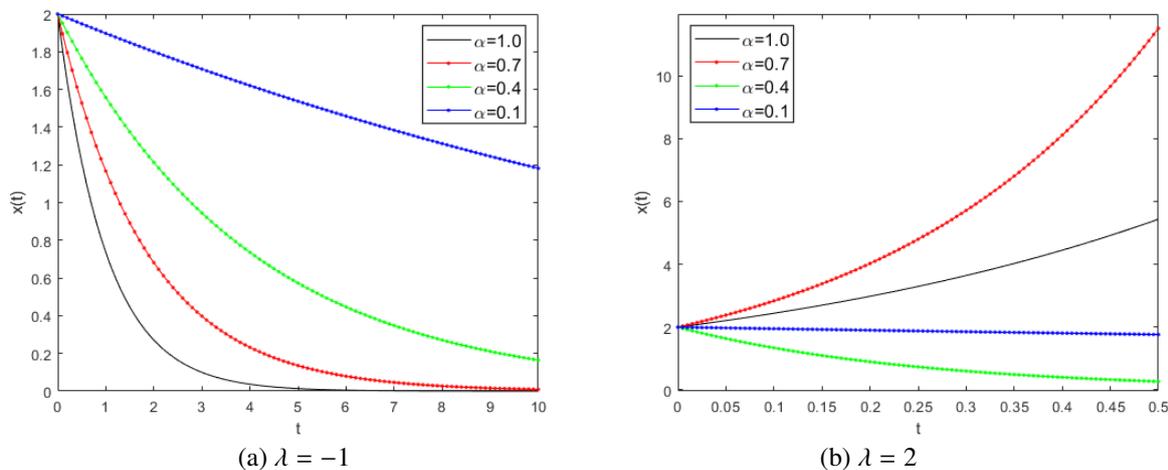
Figure 1. Solutions of system (2.3) for different values of α with $c_1 = 2$ and $c_2 = 3$.

(a) $\lambda = -1$ (b) $\lambda = 2$

For $A = \mathcal{M}(\lambda)$. If $\lambda = -1$ or $\lambda = 2$, then the solutions $y(t)$ of system (2.8) for different α are shown in Figure 2 ($x(t)$ can be obtained as in system (2.3)). For $\lambda = -1$, the results are shown in Figure 2 (a), which shows that $x(t) \to 0$ as $t \to +\infty$ and the speed of $x(t)$ approaching zero gradually becomes faster with the increase of α. For $\lambda = 2$, by Theorem 2.2, we know that $\alpha \neq 1/2$, and $\alpha = 1/2$ is the demarcation line. If $\alpha > 1/2$, then $x(t)$ is increasing with the evolution of time and the speed increase of $x(t)$ becomes faster with the decrease of α, the result is shown in Figure 2 (b). Figure 2 (c) shows the results for $\lambda = 2$ and $\alpha < 1/2$, on the whole, $x(t)$ is decreasing with the evolution of time and the

speed decrease of $x(t)$ becomes faster with the increase of α. However, a special phenomenon will occur in this case, that is $x(t)$ will increase firstly and then decrease when α reaches a critical value, in particular, this phenomenon is more obvious when $\alpha \to 0.5$.


(a) $\lambda = -1$

(b) $\lambda = 2$ for $\alpha > 1/2$

(c) $\lambda = 2$ for $\alpha < 1/2$

Figure 2. Solutions of system (2.8) for different values of α with $c_1 = 2$ and $c_2 = 3$.

For $A = \mathcal{R}(\eta, \gamma)$. If $\eta = 1/100$, $\gamma = \sqrt{199}/100$ and $c_1 = \rho(0) = 2$, $c_2 = \theta(0) = 3$, then the solutions $x(t)$ of system (2.13) are shown in Figure 3 (similarly, the Figure of $y(t)$ can be obtained). By Theorem 2.3, we know that $x(t) = 2\cos(-\sqrt{199}t/199 + 3)$ when $\alpha = 1/2$. If $\alpha > 1/2$, then $x(t)$ behaves an oscillating phenomenon and the amplitude becomes wide with the evolution of the time (Figure 3 (a)). If $\alpha < 1/2$, then $x(t)$ behaves an oscillating phenomenon and tends to zero with the evolution of the time. In particular, the speed of $x(t)$ approaching zero will firstly increase with the decrease of α. When α reaches a critical value, the speed of $x(t)$ approaching zero will slow down with the decrease of α (Figure 3 (b)).
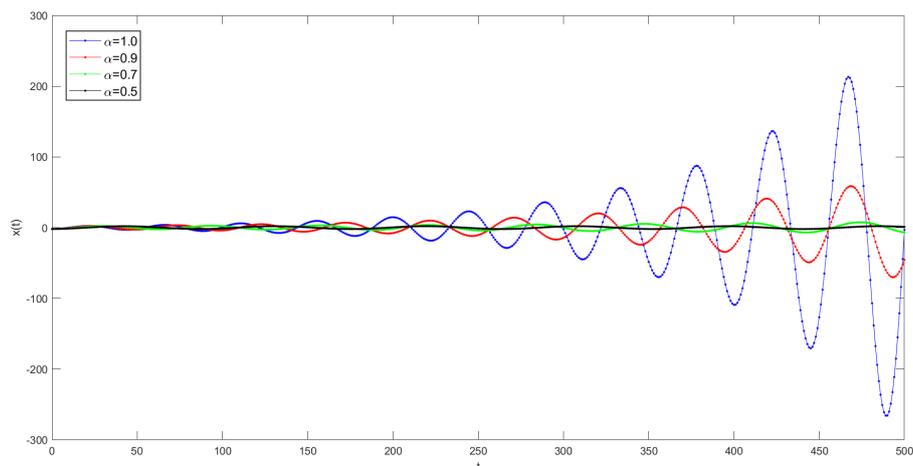
(a) $\eta = 1/100$ and $\gamma = \sqrt{199}/100$ for $\alpha \in \{0.5, 0.7, 0.9, 1.0\}$
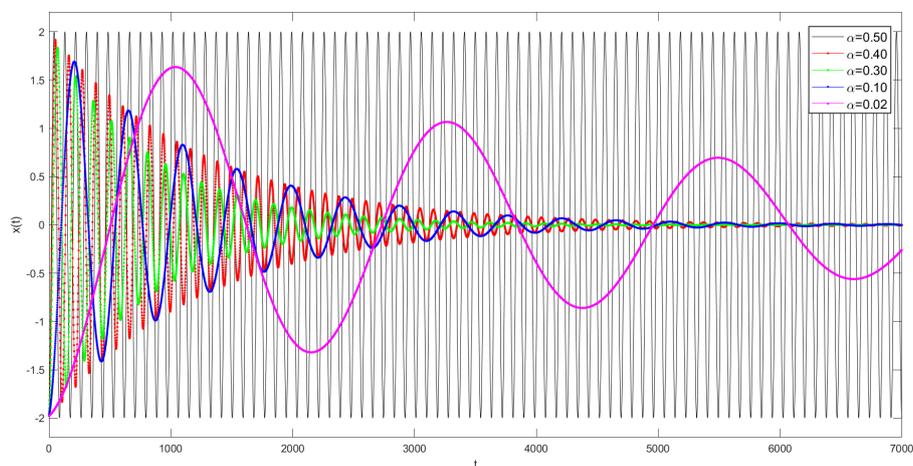
(b) $\eta = 1/100$ and $\gamma = \sqrt{199}/100$ for $\alpha \in \{0.02, 0.1, 0.3, 0.4, 0.5\}$

Figure 3. Solutions of system (2.13) for different values of α with $c_1 = 2$ and $c_2 = 3$.

4. Conclusions

In this paper, we study the analytical solutions for fractional-order linear system involving Caputo-Fabrizio fractional derivative. This problem is a natural generalization of the classical linear system. Compared with classical linear system, the solutions of fractional-order linear system have richer behaviors and abnormal phenomena for different α. From the examples, we can find that the results of this paper can be applied not only to fractional order linear system when $0 < \alpha < 1$, but also to classical linear systems as $\alpha = 1$, which means that the classical linear system is the limit form of fractional system. In the future, fractional-order nonlinear system can also be studied.

Acknowledgments

Conflict of interest

This work does not have any conflict of interest.

References

1. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.

2. Y. M. Lin, C. J. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, *J. Comput. Phys.*, **225** (2007), 1533–1552. https://doi.org/10.1016/j.jcp.2007.02.001

3. P. Zhuang, F. Liu, V. Anh, I. Turner, Numerical methods for the variable-order fractional advection diffusion equation with a nonlinear source term, *SIAM J. Numer. Anal.*, **47** (2009), 1760–1781. https://doi.org/10.1137/080730597

4. J. Losada, J. J. Nieto, Properties of a new fractional derivative without singular kernel, *Prog. Fract. Differ. Appl.*, **1** (2015), 87–92.

5. M. Fardi, Y. Khan, A novel finite difference-spectral method for fractal mobile/immobiletransport model based on Caputo-Fabrizio derivative, *Chaos Solitons Fract.*, **143** (2021), 110573. https://doi.org/10.1016/j.chaos.2020.110573

6. J. J. Nieto, Solution of a fractional logistic ordinary differential equation, *Appl. Math. Lett.*, **123** (2022), 107568. https://doi.org/10.1016/j.aml.2021.107568

7. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Prog. Fract. Differ. Appl.*, **1** (2015), 73–85.

8. O. J. J. Algahtani, Comparing the Atangana-Baleanu and Caputo-Fabrizio derivative with fractional order: Allen Cahn model, *Chaos Solitons Fract.*, **89** (2016), 552–559. https://doi.org/10.1016/j.chaos.2016.03.026

9. T. Akman, B. Yıldız, D. Baleanu, New discretization of Caputo-Fabrizio derivative, *Comput. Appl. Math.*, **37** (2018), 3307–3333. https://doi.org/10.1007/s40314-017-0514-1

10. D. Baleanu, H. Mohammadi, S. Rezapour, A fractional differential equation model for the COVID-19 transmission by using the Caputo-Fabrizio derivative, *Adv. Differ. Equ.*, **2020** (2020), 299. https://doi.org/10.1186/s13662-020-02762-2

11. D. Baleanu, A. Jajarmi, H. Mohammadi, S. Rezapour, A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative, *Chaos. Solitons Fract.*, **134** (2020), 109705. https://doi.org/10.1016/j.chaos.2020.109705

12. S. M. Aydogan, D. Baleanu, H. Mohammadi, S. Rezapour, On the mathematical model of Rabies by using the fractional Caputo-Fabrizio derivative, *Adv. Differ. Equ.*, **2020** (2020), 382. https://doi.org/10.1186/s13662-020-02798-4

13. M. Z. Xu, Y. J. Jian, Unsteady rotating electroosmotic flow with time-fractional Caputo-Fabrizio derivative, *Appl. Math. Lett.*, **100** (2020), 106015. https://doi.org/10.1016/j.aml.2019.106015

14. M. Caputo, M. Fabrizio, On the singular kernels for fractional derivatives. Some applications to partial differential equations, *Prog. Fract. Differ. Appl.*, **7** (2021), 79–82. https://doi.org/10.18576/pfda/070201

15. J. Losada, J. J. Nieto, Fractional integral associated to fractional derivatives with nonsingular kernels, *Prog. Fract. Differ. Appl.*, **7** (2021), 137–143.

16. F. Haq, I. Mahariq, T. Abdeljawad, N. Maliki, A new approach for the qualitative study of vector born disease using Caputo-Fabrizio derivative, *Numer. Methods Part. Differ. Equ.*, **37** (2021), 1809–1818. https://doi.org/10.1002/num.22728

17. N. Harrouche, S. Momani, S. Hasan, M. Al-Smadi, Computational algorithm for solving drug pharmacokinetic model under uncertainty with nonsingular kernel type Caputo-Fabrizio fractional derivative, *Alex. Eng. J.*, **60** (2021), 4347–4362. https://doi.org/10.1016/j.aej.2021.03.016

18. T. W. Zhang, Y. K. Li, Exponential Euler scheme of multi-delay Caputo-Fabrizio fractional-order differential equations, *Appl. Math. Lett.*, **124** (2022), 107709. https://doi.org/10.1016/j.aml.2021.107709

19. W. Walter, *Ordinary differential equations*, New York: Springer-verlag, 1998.

20. S. P. Bhaty, D. S. Bernstein, Finite-time stability of continuous autonomous systems, *SIAM J. Control Optim.*, **38** (2000), 751–766. https://doi.org/10.1137/S0363012997321358

21. R. P. Agarwal, D. O'Regan, *An introduction to ordinary differential equations*, Springer Science Business Media LLC, 2008.

22. J. C. Cortés, A. Navarro-Quiles, J. V. Romero, M. D. Roselló, Full solution of random autonomous first-order linear systems of difference equations. Application to construct random phase portrait for planar systems, *Appl. Math. Lett.*, **68** (2017), 150–156. https://doi.org/10.1016/j.aml.2016.12.015

23. W. M. Ahmad, W. M. Harb, On nonlinear control design for autonomous chaotic systems of integer and fractional orders, *Chaos Solitons Fract.*, **18** (2003), 693–701. https://doi.org/10.1016/S0960-0779(02)00644-6

24. C. G. Li, G. R. Chen, Chaos in the fractional order Chen system and its control, *Chaos Solitons Fract.*, **22** (2004), 549–554. https://doi.org/10.1016/j.chaos.2004.02.035

25. S. T. Kingni, V. T. Pham, S. Jafari, G. R. Kol, P. Woafo, Three-dimensional chaotic autonomous system with a circular equilibrium: Analysis, circuit implementation and its fractional-order form, *Circuits Syst. Signal Process.*, **35** (2016), 1933–1948. https://doi.org/10.1007/s00034-016-0259-x