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Research article

Blow-up criteria for the full compressible Navier-Stokes equations involving temperature in Vishik Spaces

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Abstract: In this paper, we consider the conditional regularity for the 3D incompressible Navier-Stokes equations in Vishik spaces. These results will be regarded an improvement of the results given by Huang-Li-Xin, (SIAM J. Math. Anal., 2011) and Jiu-Wang-Ye,(J. Evol. Equ., 2021).

Keywords: full compressible Navier-Stokes equations; strong solutions; blow-up criteria
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1. Introduction

We study the following system of Newton heat-conducting compressible fluid in three-dimensional space

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ \rho u_t + \rho u\cdot\nabla u + \nabla P(\rho,\theta) - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u = 0, \\ c_v[\rho\theta_t + \rho u\cdot\nabla\theta] + P\mathrm{div}\, u - \kappa\Delta\theta = \dfrac{\mu}{2}\left|\nabla u + (\nabla u)^{\mathrm{tr}}\right|^2 + \lambda(\mathrm{div}\, u)^2, \\ (\rho,u,\theta)|_{t=0} = (\rho_0,u_0,\theta_0). \end{cases} \tag{1.1}$$

Here, ρ, u, θ stand for the flow density, velocity and the absolute temperature, respectively. The scalar function P represents the pressure, the state equation of which is determined by

$$P = R\rho\theta, \qquad R > 0, \tag{1.2}$$

and κ is a positive constant and two constants μ and λ are the coefficients of viscosity satisfying the physical restrictions $\mu > 0,\ 2\mu + 3\lambda \geq 0$. The initial conditions satisfy

$$\rho(x,t) \to 0,\ u(x,t) \to 0,\ \theta(x,t) \to 0,\ \text{as } |x| \to \infty,\ \text{for } t \geq 0. \tag{1.3}$$

Let $\gamma > 0$. For all $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, we consider the following scaled functions:

$$\rho_\lambda = \rho(\lambda^2 t, \lambda x), \quad u_\lambda = \lambda u(\lambda^2 t, \lambda x), \quad \theta_\lambda = \lambda^2 \theta(\lambda^2 t, \lambda x). \tag{1.4}$$

There are huge literatures on the study of the existence of solutions to compressible Navier-Stokes equations, we only give a brief survey for blow-up criteria rather than the existence of solutions. When the initial data contain vacuums, after Xin's blow-up works [21, 22], the various result for blow up critria for strong solutions to the system (1.1) is investigated. In present paper, in particular, we focus on the Serrin type criteria (e.g. [6–9]) as

$$\limsup_{T \nearrow T^\star} \left(\|\mathrm{div}\, u\|_{L^1(0,T;L^\infty(\mathbb{R}^3))} + \|u\|_{L^p(0,T;L^q(\mathbb{R}^3))} \right) = \infty, \ \frac{2}{p} + \frac{3}{q} = 1, \ q > 3,$$

or

$$\limsup_{T \nearrow T^\star} \left(\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} + \|u\|_{L^p(0,T;L^q(\mathbb{R}^3))} \right) = \infty, \ \frac{2}{p} + \frac{3}{q} = 1, \ q > 3$$

and it is aimed to expand them into Vishik space motivated by the results of two recent papers Kanamaru [10] and Wu [20] (see also [2–5, 11–16, 18, 19] for other criteria containing Beale-Kato-Majda blow-up mechanism).

We remind the local well-posedness of strong solutions to the equations (1.1) (see [1]).

Theorem 1.1. Let $\lambda < 3\mu$. Suppose $u_0, \theta_0 \in (D^1 \cap D^2)(\mathbb{R}^3)$ and $\rho_0 \in (W^{1,q} \cap H^1 \cap L^1)(\mathbb{R}^3)$ for some $q \in (3, 6]$. If ρ_0 is nonnegative and the initial data satisfy the compatibility condition

$$\begin{aligned} -\mu\Delta u_0 - (\mu + \lambda)\nabla \mathrm{div}\, u_0 + \nabla P(\rho_0, \theta_0) &= \sqrt{\rho_0} g_1 \\ \Delta\theta_0 + \frac{\mu}{2}|\nabla u_0 + (\nabla u_0)^{\mathrm{tr}}|^2 + \lambda(\mathrm{div}\, u_0)^2 &= \sqrt{\rho_0} g_2 \end{aligned}$$

for vector fields $g_1, g_2 \in L^2(\mathbb{R}^3)$. Then there exist a time $T \in (0, \infty]$ and unique solution tp the equations (1.1)–(1.3), satisfying

$$(\rho, u, \theta) \in C([0, T); (L^1 \cap H^1 \cap W^{1,q}))(\mathbb{R}^3) \times C([0, T); (D^1 \cap D^2)(\mathbb{R}^3)) \times L^2([0, T); D^{2,q}(\mathbb{R}^3)),$$

$$(\rho_t, u_t, \theta_t) \in C([0, T); (L^2 \cap L^q)(\mathbb{R}^3)) \times L^2([0, T); D^1(\mathbb{R}^3)) \times L^2([0, T); D^1(\mathbb{R}^3)),$$
$$(\rho^{1/2} u_t, \rho^{1/2}\theta_t) \in L^\infty([0, T); L^2(\mathbb{R}^3)) \times L^\infty([0, T); L^2(\mathbb{R}^3)).$$

If the maximal existence time T^* is finite, then there holds

$$\limsup_{T \nearrow T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} + \|\theta\|_{L^{\frac{2q}{2q-3}}(0,T;\dot{V}^0_{q,\sigma,1}(\mathbb{R}^3))} \right) = \infty, \ q > \frac{3}{2}, \tag{1.5}$$

where $\sigma \in [1, \infty], \theta \in [1, \sigma]$.

Remark 1.1. In the light of the arguments in [7, 8], we observe that (1.5) be replaced by

$$\limsup_{T \nearrow T^*} \left(\|\mathrm{div}\, u\|_{L^1(0,T;L^\infty(\mathbb{R}^3))} + \|\theta\|_{L^p(0,T;L^q(\mathbb{R}^3))} \right) = \infty.$$

We note that the condition (1.5) is in scaling invariant norm in the sense of (1.4) for the temperature.

Remark 1.2. Without the restriction $\lambda < 3\mu$, in the case away from vacuum, through the argument in [9] and our proof, we obtain the similar results [9, Theorem 1.3] of what the authors in [9] says in Vishik space.

Next, we consider the full compressible Navier-Stokes equations without temperature.

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \nabla P(\rho) = 0, \\ (\rho, u)(x, 0) = (\rho_0, u_0)(x), \end{cases} \tag{1.6}$$

where ρ, u, and P are the density, velocity and pressure respectively. The equation of state is given by

$$P(\rho) = a\rho^\gamma, \quad (a > 0, \gamma > 1). \tag{1.7}$$

The constants μ and λ are the shear viscosity and the bulk viscosity coefficients respectively. They satisfy the following physical restrictions: $\mu > 0, \quad 3\lambda + 2\mu \geq 0$.

Through a similar scheme in Theorem 1.1, we also obtain the following result for the equations (1.6).

Theorem 1.2. Let (ρ, u) be a strong solution to the Cauchy problem (1.6)–(1.7) with the initial data (ρ_0, u_0) satisfy

$$0 \leq \rho_0 \in (L^1 \cap H^1 \cap W^{1,r})(\mathbb{R}^3), \quad u_0 \in (D^1 \cap D^2)(\mathbb{R}^3),$$

for some $r \in (3, \infty)$ and the compatibility condition:

$$-\mu\triangle u_0 - (\lambda+\mu)\nabla\operatorname{div} u_0 + \nabla P(\rho_0) = \rho_0^{1/2} g \quad \text{for some } g \in L^2(\mathbb{R}^3).$$

If $T^* < \infty$ is the maximal time of existence, then both

$$\lim_{T\to T^*}\left(\|\operatorname{div} u\|_{L^1(0,T;L^\infty(\mathbb{R}^3))} + \|u\|_{L^{\frac{2p}{p-3}}(0,T;\dot V^0_{q,\sigma,1}(\mathbb{R}^3))}\right) = \infty,$$

and

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} + \|u\|_{L^{\frac{2p}{p-3}}(0,T;\dot V^0_{q,\sigma,1}(\mathbb{R}^3))}\right) = \infty, \qquad 3 < p \leq \infty.$$

where $\sigma \in [1, \infty], \theta \in [1, \sigma]$.

2. Notations and some auxiliary lemmas

We follow the notation of [6] and [9]. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^3)$ represents the usual Lebesgue space. The classical Sobolev space $W^{k,p}(\mathbb{R}^3)$ is equipped with the norm $\|f\|_{W^{k,p}(\mathbb{R}^3)} = \sum_{\alpha=0}^{k} \|D^\alpha f\|_{L^p(\mathbb{R}^3)}$. A function f belongs to the homogeneous Sobolev spaces $D^{k,l}$ if $u \in L^1_{\text{loc}}(\mathbb{R}^3) : \|\nabla^k u\|_{L^l} < \infty$. $C > 0$ is an absolute constant which may be different from line to line unless otherwise stated in this paper. We also now introduce a Banach space $\dot V^s_{p,\sigma,\theta}(\mathbb{R}^3)$ which is larger than the homogeneous Besov space; see [10, 17].

Definition 2.1. *Let $s \in \mathbb{R}, p, \sigma \in [1, \infty], \theta \in [1, \sigma]$, the Vishik space $\dot{V}^s_{p,\sigma,\theta}$ is defined by*

$$\dot{V}^s_{p,\sigma,\theta}(\mathbb{R}^3) := \{f \in \mathcal{D}'(\mathbb{R}^3) : \|f\|_{\dot{V}^s_{p,\sigma,\theta}} < \infty\},$$

with the norm

$$\|f\|_{\dot{V}^s_{p,\sigma,\theta}(\mathbb{R}^3)} := \begin{cases} \sup_{N=1,2,\cdots} \dfrac{\left(\sum_{|j|<N} 2^{js\theta}\|\dot{\Delta}_j f\|^\theta_{L^p}\right)^{\frac{1}{\theta}}}{N^{\frac{1}{\theta}-\frac{1}{\sigma}}}, & \theta \neq \infty, \\ \|f\|_{\dot{B}^0_{p,\infty}(\mathbb{R}^N)}, \quad \theta = \infty. \end{cases}$$

Here $\mathcal{D}'(\mathbb{R}^3)$ is the dual space of $\mathcal{D}(\mathbb{R}^3) = \{f \in \mathcal{S}(\mathbb{R}^3); D^\alpha \hat{f}(0) = 0, \forall\, \alpha \in \mathbb{N}^3\}$. As mentioned in [20], we remind that the following continuous embeddings hold:

$$\dot{B}^s_{p,\sigma}(\mathbb{R}^3) = \dot{V}^s_{p,\sigma,\sigma}(\mathbb{R}^3) \subset \dot{V}^s_{p,\sigma,\theta_1}(\mathbb{R}^3) \subset \dot{V}^s_{p,\sigma,\theta_2}(\mathbb{R}^3) \subset \dot{V}^s_{p,\sigma,1}(\mathbb{R}^3)$$

for $s \in \mathbb{R}$, $p, \sigma \in [1, \infty]$ and $\theta_1, \theta_2 \in [1, \sigma]$ with $\theta_1 \geq \theta_2$.

In what follows, for simplicity, we write

$$L^p = L^p(\mathbb{R}^3),\ H^k = W^{k,2}(\mathbb{R}^3),\ D^k = D^{k,2}(\mathbb{R}^3),\ \dot{V}^s_{p,\sigma,\theta} := \dot{V}^s_{p,\sigma,\theta}(\mathbb{R}^3).$$

3. Proof of Theorem 1.1

We will prove Theorem 1.1 by a contradiction argument. Therefore, we assume that

$$\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\theta\|_{L^{\frac{2q}{2q-3}}(0,T;\dot{V}^0_{q,\sigma,1}(\mathbb{R}^3))} \leq C, \qquad \frac{2}{p} + \frac{3}{q} = 2,\ q > \frac{3}{2}. \tag{3.1}$$

Lemma 3.1. *Suppose that* (3.1) *is valid and $\lambda < 3\mu$, then there holds*

$$\sup_{0\leq t\leq T} \int_{\mathbb{R}^3} \Big[\frac{\mu}{2}|\nabla u|^2 + (\mu+\lambda)(div\, u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P div\, u + \frac{C}{2}\rho\theta^2 + \frac{C+1}{2\mu}\rho|u|^4\Big]$$
$$+ \int_0^T \Big[\kappa \int_{\mathbb{R}^3} |\nabla\theta|^2 + \frac{1}{2}\rho|\dot{u}|^2 + |u|^2\big|\nabla u\big|^2\Big] dt\ \leq C.$$

Proof. From Lemma 2.3 and Lemma 3.1 in [9], we know that

$$\frac{d}{dt}\int \Big[\frac{\mu}{2}|\nabla u|^2 + (\mu+\lambda)(\mathrm{div}\, u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\mathrm{div}\, u + \frac{C}{2}\rho\theta^2\Big]$$
$$+ \kappa\int |\nabla\theta|^2 + \frac{1}{2}\int \rho|\dot{u}|^2 \leq C\int \rho|\theta|^3 + C\int \rho|u|^2|\theta|^2 + C\int |u|^2|\nabla u|^2, \tag{3.2}$$

and

$$\frac{d}{dt}\int \rho|u|^4 + \int_{\mathbb{R}^3\cap\{|u|>0\}} |u|^2\big|\nabla u\big|^2 \leq C\int \rho|u|^2|\theta|^2. \tag{3.3}$$

Multiplying the inequality (3.3) by $(C + 1)$ and adding the result to the inequality (3.2), we have

$$\frac{d}{dt}\int \Big[\frac{\mu}{2}|\nabla u|^2 + (\mu+\lambda)(\mathrm{div}\, u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\mathrm{div}\, u + \frac{C}{2}\rho\theta^2 + \frac{C+1}{2\mu}\rho|u|^4\Big]$$

$$+\kappa\int|\nabla\theta|^2+\frac{1}{2}\int\rho|\dot{u}|^2+\int|u|^2\big|\nabla u\big|^2\le C\int\rho|\theta|^3+C\int\rho|u|^2|\theta|^2. \tag{3.4}$$

For the second term in the right hand side of (3.4), we note that

$$\int\rho|u|^2|\theta|^2=\int\rho^{\frac12}|\theta|\rho^{\frac12}|u|^2|\sum_{j<-N}\Delta_j\theta|+\int\rho^{\frac12}|\theta|\rho^{\frac12}|u|^2|\sum_{j=-N}^{j=N}\Delta_j\theta| \tag{3.5}$$

$$+\int\rho^{\frac12}|\theta|\rho^{\frac12}|u|^2|\sum_{j>-N}\Delta_j\theta|:=I+II+III.$$

Now, let's control each term sequentially by Hölder's inequality, interpolation inequality(for the term II below), Sobolev embedding theorem, Berstein's inequality and Young's inequalities:

(**The term (I)**):

$$I\le\|\sum_{j<-N}\Delta_j\theta\|_{L^\infty}\|\rho^{\frac12}\theta\|_{L^2}\|\rho^{\frac12}|u|^2\|_{L^2}\le C\sum_{j<-N}2^{\frac32 j}\|\theta\|_{L^2}\|\rho^{\frac12}\theta\|_{L^2}\|\rho^{\frac12}|u|^2\|_{L^2}$$

$$\le C2^{-3N^2}\|\theta\|_{L^2}^2\Big(\|\rho^{\frac12}\theta\|_{L^2}^2+\|\rho^{\frac12}|u|^2\|_{L^2}^2\Big).$$

(**The term (II)**):

$$\begin{aligned}
II&\le\sum_{j=-N}^{j=N}\|\Delta_j\theta\|_{L^q}\|\rho^{\frac12}\theta\|_{L^{\frac{2q}{q-1}}}\|\rho^{\frac12}|u|^2\|_{L^{\frac{2q}{q-1}}}\\
&\le\sum_{j=-N}^{j=N}\|\Delta_j\theta\|_{L^q}\|\rho^{\frac12}\theta\|_{L^2}^{1-\frac{3}{2q}}\|\rho^{\frac12}\theta\|_{L^6}^{\frac{3}{2q}}\|\rho^{\frac12}|u|^2\|_{L^2}^{1-\frac{3}{2q}}\|\rho^{\frac12}|u|^2\|_{L^6}^{\frac{3}{2q}}\\
&\le CN^{1-\frac1\sigma}\sup_{N=1,2,\cdots}\frac{\sum_{j=-N}^{j=N}\|\dot{\Delta}_ju\|_{L^q}}{N^{1-\frac1\sigma}}\|\rho^{\frac12}\theta\|_{L^2}^{1-\frac{3}{2q}}\|\rho^{\frac12}|u|^2\|_{L^2}^{1-\frac{3}{2q}}\|(\|\nabla\theta\|_{L^2}^{\frac3q}+\|\nabla|u|^2\|_{L^2}^{\frac3q})\\
&\le C\|\theta\|_{\dot{V}^0_{q,\sigma,1}}^{\frac{2q}{2q-3}}\|\rho^{\frac12}\theta\|_{L^2}\|\rho^{\frac12}|u|^2\|_{L^2}+\frac{1}{16}\|\nabla\theta\|_{L^2}^2+\frac{1}{16}\|\nabla|u|^2\|_{L^2}^2\\
&\le C\|\theta\|_{\dot{V}^0_{q,\sigma,1}}^{\frac{2q}{2q-3}}(\|\rho^{\frac12}\theta\|_{L^2}^2+\|\rho^{\frac12}|u|^2\|_{L^2}^2)+\frac{1}{16}\|\nabla\theta\|_{L^2}^2+\frac{1}{16}\|\nabla|u|^2\|_{L^2}^2.
\end{aligned} \tag{3.6}$$

(**The term (III)**):

$$III\le\sum_{j>N}\|\Delta_j\theta\|_{L^3}\|\rho^{\frac12}|u|^2\|_{L^6}\|\rho^{\frac12}\theta\|_{L^2}\le C\|\rho^{\frac12}|u|^2\|_{L^6}\sum_{j>N}2^{\frac12 j}\|\theta\|_{L^2}\|\rho^{\frac12}\theta\|_{L^2}$$

$$\le C2^{-\frac N2}\|\theta\|_{L^2}\|\nabla|u|^2\|_{L^2}\|\rho^{\frac12}\theta\|_{L^2}\le 2^{-N^2}\|\theta\|_{L^2}^2\|\rho^{\frac12}\theta\|_{L^2}^2+\frac{1}{32}\||u|\nabla|u|\|_{L^2}^2.$$

Summing up the estimates above, we have

$$\int\rho|u|^2|\theta|^2\le C2^{-3N^2}\|\theta\|_{L^2}^2\Big(\|\rho^{\frac12}\theta\|_{L^2}^2+\|\rho^{\frac12}|u|^2\|_{L^2}^2\Big)+C\|\theta\|_{\dot{V}^0_{q,\sigma,1}}^{\frac{2q}{2q-3}}(\|\rho^{\frac12}\theta\|_{L^2}^2+\|\rho^{\frac12}|u|^2\|_{L^2}^2)$$

$$+\frac{1}{16}\|\nabla\theta\|_{L^2}^2+\frac{1}{16}\|\nabla|u|^2\|_{L^2}^2+2^{-N^2}\|\theta\|_{L^2}^2\|\rho^{\frac12}\theta\|_{L^2}^2. \tag{3.7}$$

By similar above arguments, we get

$$\int \rho|\theta|^3 = \int \rho^{\frac{1}{2}}|\theta|\rho^{\frac{1}{2}}\theta|\theta| \le C2^{-3N^2}\|\theta\|_{L^2}^2\|\rho^{\frac{1}{2}}\theta\|_{L^2}^2 + C\|\theta\|_{\dot{V}^0_{q,\sigma,1}}^{\frac{2q}{2q-3}}\|\rho^{\frac{1}{2}}\theta\|_{L^2}^2 + \frac{1}{16}\|\nabla\theta\|_{L^2}^2 + \frac{1}{16}\|\nabla|u|^2\|_{L^2}^2 + 2^{-N^2}\|\theta\|_{L^2}^2\|\rho^{\frac{1}{2}}\theta\|_{L^2}^2. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.4), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int\Big[\frac{\mu}{2}|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\operatorname{div} u + \frac{C}{2}\rho\theta^2 + \frac{C+1}{2\mu}\rho|u|^4\Big] \\
&\quad + \frac{\kappa}{2}\int|\nabla\theta|^2 + \frac{1}{2}\int\rho|\dot{u}|^2 + \frac{1}{2}\int_{\mathbb{R}^3\cap\{|u|>0\}}|u|^2\big|\nabla u\big|^2 \\
&\le C2^{-3N^2}\|\theta\|_{L^2}^2\Big(\|\rho^{\frac{1}{2}}\theta\|_{L^2}^2 + \|\rho^{\frac{1}{2}}|u|^2\|_{L^2}^2\Big) + C\|\theta\|_{\dot{V}^0_{q,\sigma,1}}^{\frac{2q}{2q-3}}(\|\rho^{\frac{1}{2}}\theta\|_{L^2}^2 + \|\rho^{\frac{1}{2}}|u|^2\|_{L^2}^2) \\
&+ \frac{1}{16}\|\nabla\theta\|_{L^2}^2 + \frac{1}{16}\|\nabla|u|^2\|_{L^2}^2 + 2^{-N^2}\|\theta\|_{L^2}^2\|\rho^{\frac{1}{2}}\theta\|_{L^2}^2 \\
&\le C\Big(C2^{-3N^2+2^{-N^2}\|\theta\|_{L^2}^2}\|\theta\|_{L^2}^2 + \|\theta\|_{\dot{V}^0_{q,\sigma,1}}^{\frac{2q}{2q-3}}\Big)\Big(\int\Big[\frac{\mu}{2}|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\operatorname{div} u \\
&\qquad + \frac{C}{2}\rho\theta^2 + \frac{C+1}{2\mu}\rho|u|^4\Big]\Big) + \frac{1}{16}\|\nabla\theta\|_{L^2}^2 + \frac{1}{16}\|\nabla|u|^2\|_{L^2}^2,
\end{aligned} \tag{3.9}$$

where we used the fact that

$$\int\Big[(\mu+\lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\operatorname{div} u + \frac{C}{2}\rho\theta^2\Big] \ge \int\rho\theta^2,$$

for a sufficiently large constant $C > 0$. Now, choosing $N > 0$ sufficiently large such that $C2^{-N^2}\|\theta\|_{L^2}^2 \le \frac{1}{128}$, the estimate (3.9) becomes

$$\frac{d}{dt}\int\Big[\frac{\mu}{2}|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\operatorname{div} u + \frac{C}{2}\rho\theta^2 + \frac{C+1}{2\mu}\rho|u|^4\Big] \tag{3.10}$$

$$+\kappa\int|\nabla\theta|^2 + \frac{1}{2}\int\rho|\dot{u}|^2 + \int|u|^2\big|\nabla u\big|^2 \le C\int\rho|\theta|^3 + C\int\rho|u|^2|\theta|^2 \le CN\|u\|_{\dot{V}^0_{q,\sigma,1}}^{\frac{2p}{p-3}}\|\nabla u\|_{L^2}^2.$$

Then, Grönwall's inequality and (3.10) enables us to obtain that the desired results.

$$\sup_{0\le t\le T}\int_{\mathbb{R}^3}\Big[\frac{\mu}{2}|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\operatorname{div} u + \frac{C}{2}\rho\theta^2 + \frac{C+1}{2\mu}\rho|u|^4\Big]$$

$$+\int_0^T\Big[\kappa\int_{\mathbb{R}^3}|\nabla\theta|^2 + \frac{1}{2}\rho|\dot{u}|^2 + |u|^2\big|\nabla u\big|^2\Big]dt \le C.$$

□

Proof of Theorem 1.1. In the proof in Theorem 1.1 in [9], as long as Lemma 3.2 in [9] is only replaced by Lemma 3.1 in present paper, the proof is completed. □

4. Proof of Theorem 1.2

Let (ρ, u) be a strong solution to the problem (1.6)-(1.7) as described in Theorem 1.2. Then the standard energy estimate yields

$$\sup_{0\le t\le T}\left(\|\rho^{1/2}u(t)\|_{L^2}^2+\|\rho\|_{L^1}+\|\rho\|_{L^\gamma}^\gamma\right)+\int_0^T\|\nabla u\|_{L^2}^2dt\le C,\quad 0\le T<T^*. \tag{4.1}$$

We first prove Theorem 1.2 by a contradiction argument. Otherwise, there exists some constant $M_0>0$ such that

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)}+\|u\|_{L^{\frac{2p}{p-3}}(0,T;\dot{V}^0_{q,\sigma,1}(\mathbb{R}^3))}\right)\le M_0. \tag{4.2}$$

The first key estimate on ∇u will be given in the following lemma.

Lemma 4.1. *Under the condition (4.2), it holds that*

$$\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2+\int_0^T\int\rho u_t^2dxdt\le C,\quad 0\le T<T^*. \tag{4.3}$$

Proof. It follows from the momentum equations in (1.6) that

$$\triangle G=\mathrm{div}(\rho\dot{u}),\quad \mu\triangle\omega=\nabla\times(\rho\dot{u}),$$

where $\dot{v}:=v_t+u\cdot\nabla v,\quad G:=(2\mu+\lambda)\mathrm{div}u-P(\rho),\quad \omega:=\nabla\times u$ are the material derivative of f, the effective viscous flux G and the vorticity ω, respectively. In particular, for the effective viscous flux, it is well-known that

$$\|\nabla G\|_{L^p}\le\|\rho\dot{u}\|_{L^p},\ \forall p\in(1,+\infty),$$

and

$$\|\nabla G\|_{L^2}+\|\nabla\omega\|_{L^2}\le C(\|\rho u_t\|_{L^2}+\|\rho u\cdot\nabla u\|_{L^2}). \tag{4.4}$$

Multiplying the momentum equation $(1.6)_2$ by u_t and integrating the resulting equation over $\mathbb{R}^3$ gives

$$\frac{1}{2}\frac{d}{dt}\int\left(\mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}u)^2\right)dx+\int\rho u_t^2dx=\int P\mathrm{div}u_tdx-\int\rho u\cdot\nabla u\cdot u_tdx. \tag{4.5}$$

From $(1.6)_1$, we note that

$$P_t+\mathrm{div}(Pu)+(\gamma-1)P\mathrm{div}u=0.$$

For the first term in the right hand side of (4.5), one has

$$\int P\mathrm{div}u_tdx\le\frac{d}{dt}\int P\mathrm{div}udx+\frac{1}{8}\|\nabla G\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C. \tag{4.6}$$

Substituting (4.6) into (4.5), we have

$$\frac{d}{dt}\int\left(\frac{\mu}{2}|\nabla u|^2+\frac{\lambda+\mu}{2}(\mathrm{div}u)^2-P\mathrm{div}u\right)dx+\frac{1}{2}\int\rho u_t^2dx\le C\|\nabla u\|_{L^2}^2+\int\rho\left|u\cdot\nabla u\cdot u_t\right|dx+C.$$

For the second term in the right hand side of (4.5), we have

$$\int |\rho^{1/2}u\cdot\nabla u\cdot\rho^{1/2}u_t|\,dx \le \int |\rho^{1/2}\sum_{j<-N}\Delta_j u||\nabla u||\rho^{1/2}u_t|\,dx$$

$$+\int |\rho^{1/2}\sum_{j=-N}^{j=N}\Delta_j u||\nabla u||\rho^{1/2}u_t|\,dx+\int |\rho^{1/2}\sum_{j>-N}\Delta_j u||\nabla u||\rho^{1/2}u_t|\,dx := I+II+III.$$

In a similar way to (3.5), we let control each term sequentially.
(**The term (I)**):

$$I\le C2^{-3N^2}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+\frac{1}{32}\|\rho^{1/2}u_t\|_{L^2}^2.$$

(**The term (II)**):

$$II\le CN\|u\|_{\dot V^0_{p,\sigma,1}}^{\frac{2p}{p-3}}\|\nabla u\|_{L^2}^2+\frac{1}{32}\Big(\|\rho^{1/2}u_t\|_{L^2}+\|\nabla^2 u\|_{L^2}^2\Big).$$

(**The term (III)**):

$$III\le C2^{-N^2}\|u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2+\frac{1}{32}\|\rho^{1/2}u_t\|_{L^2}^2.$$

Summing up the estimates I–III, it is bounded by

$$C2^{-3N^2}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+CN\|u\|_{\dot V^0_{p,\sigma,1}}^{\frac{2p}{p-3}}\|\nabla u\|_{L^2}^2+C2^{-N^2}\|u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2 \tag{4.7}$$

$$+\frac{1}{16}\Big(\|\rho^{1/2}u_t\|_{L^2}+\|\nabla^2 u\|_{L^2}^2\Big).$$

On the other hand, due to (4.4), we note that

$$\|\nabla^2 u\|_{L^2(\mathbb{R}^3)}^2\le C(\|\sqrt{\rho}u_t\|_{L^2(\mathbb{R}^3)}^2+\|\rho u\cdot\nabla u\|_{L^2(\mathbb{R}^3)}^2) \tag{4.8}$$

$$\le C\|\sqrt{\rho}u_t\|_{L^2(\mathbb{R}^3)}^2+C2^{-3N^2}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+C2^{-N^2}\|u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2+CN\|u\|_{\dot V^0_{p,\sigma,1}}^{\frac{2p}{p-3}}\|\nabla u\|_{L^2}^2.$$

Collecting (4.7) and (4.8), we have

$$\frac{d}{dt}\int_{\emptyset}\Big(\frac{\mu}{2}|\nabla u|^2+\frac{\lambda+\mu}{2}(\mathrm{div}u)^2-P\mathrm{div}u\Big)\,dx+\frac{1}{4}\int \rho u_t^2\,dx+\int|\nabla G|^2\,dx \tag{4.9}$$

$$\le C\|\sqrt{\rho}u_t\|_{L^2(\mathbb{R}^3)}^2+C2^{-3N^2}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+C2^{-N^2}\|u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2+CN\|u\|_{\dot V^0_{p,\sigma,1}}^{\frac{2p}{p-3}}\|\nabla u\|_{L^2}^2.$$

Now, choosing $N>0$ sufficiently large such that $C2^{-N^2}\|u\|_{L^2}^2\le\frac{1}{128}$, (indeed, the constant $C>0$ is also depending on $\|\rho_0^{1/2}u_0\|_{L^2}^2$) the estimate (4.9) becomes

$$\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\frac{\mu}{2}|\nabla u|^2+\frac{\lambda+\mu}{2}(\mathrm{div}u)^2-P\mathrm{div}u+\rho|u|^2+\rho+\rho^\gamma\Big)\,dx \tag{4.10}$$

$$+\int_{\mathbb{R}^3}\Big(|\nabla G|^2+|\nabla u|^2+\frac{1}{4}\rho|u_t|^2\Big)\,dx\le CN\Big(\|u\|_{\dot V^0_{p,\sigma,1}}^{\frac{2p}{p-3}}+1\Big)\Big(\|\nabla u\|_{L^2}^2+1\Big),$$

which, together with (4.2) and Grönwall's inequality, gives (4.3). The proof of Lemma 4.1 is completed.

Proof of Theorem 1.2. In the proof in Theorem 1.1 in [6], as long as Lemma 3.1 in [6] is only replaced by Lemma 4.1 in our paper, the proof is completed. □

5. Appendix

For the convenience of the reader, we give the proof for (4.6), given in [6].

$$\begin{aligned}
&\int P\mathrm{div}u_t dx\\
&= \frac{d}{dt}\int P\mathrm{div}u dx - \int P_t \mathrm{div}u dx\\
&= \frac{d}{dt}\int P\mathrm{div}u dx + \int \mathrm{div}(Pu)\mathrm{div}u dx + (\gamma-1)\int P(\mathrm{div}u)^2 dx\\
&= \frac{d}{dt}\int P\mathrm{div}u dx - \int (Pu)\cdot\nabla\mathrm{div}u dx + (\gamma-1)\int P(\mathrm{div}u)^2 dx \qquad (5.1)\\
&= \frac{d}{dt}\int P\mathrm{div}u dx - \frac{1}{2\mu+\lambda}\int Pu\cdot\nabla G dx - \frac{1}{2(2\mu+\lambda)}\int P^2\mathrm{div}u dx\\
&\quad + (\gamma-1)\int P(\mathrm{div}u)^2 dx\\
&\le \frac{d}{dt}\int P\mathrm{div}u dx + \frac{1}{8}\|\nabla G\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + C,
\end{aligned}$$

6. Discussion

Our result is focused on the full compressible Navier-Stokes equationss. However, it is believed that our results can be expanded in various ways for the coupled equations or system. In this regard, we think of it as a future study and intend to produce more meaningful results.

7. Conclusions

The current paper results are Blow-up criteria for solutions in Vishik Space which is a weaker space to Besov space and Lebesgue space. It seems to be a meaningful result in this regard, and it is new.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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