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Research article

Fixed point approach to solve fractional differential equations in S^{JS}-metric spaces

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Abstract: This study aims to establish a new fixed point theorem in the framework of S^{JS}-metric spaces, recently introduced by Beg et al. We propose different principles of contraction using various techniques. The theorems obtained represent a new framework for other future work in the considered space. Also, we provide two applications of our results to linear system of equations and the following fractional differential equation

$$(\mathcal{P}):\left\{\begin{array}{rcl} D^{\lambda}x(t) & = & f(t,x(t)) = Fx(t) \text{ if } t\in I_0=(0,T] \\ x(0) & = & x(T)=r \end{array}\right\}.$$

These applications show the effectiveness of our approach as a powerful tool for solving several types of differential equations.

Keywords: fixed point; S^{JS}-metric spaces; system of linear equations; fractional differential equations
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1. Introduction

Fixed point theory has grown in importance to solve different problems derived from the theory of nonlinear differential equations, integral equations, engineering, and economics, by virtue of its extensive variety of applications. Banach [1] introduced the most well-known result in this field, which serves as a reference in this theory. Since then, several works have emerged by generalisation of Banach's Contraction Principle under different contractions in a various metric spaces. Indeed, new results were discovered by defining new metric spaces and presenting their topological properties. One of the most important spaces is the S-metric introduced by Sedghi [2]. The concept of S-metric space has caught the attention of researchers and has become a very attractive topic in fixed point theory. As a result of this growing interest, many other metric spaces were developed as a generalization of the latter [3–19] . Recently, Beg et al. [20, 21] introduced a new approach of S^{JS}-metric space. They studied the properties of S^{JS}-metric spaces and proved some fixed point theorems. Also, they presented the S^{JS} topological spaces induced by the S^{JS}-metric and obtained a classical result in conjunction with Cantor's intersection theorem in this context.

Motivated by the novelty of the S^{JS}-metric space, we decided to further study the fixed point results. Through this work, we establish new versions of some fixed point theorem on S^{JS}-metric spaces including various contractive conditions illustrated with some examples. We proved the existence and the uniqueness of fixed point using different techniques. Moreover, since the fixed point theory is a very important tool used to obtain solutions to different mathematical models, we propose in this work two applications of our results to the linear system of equations and to the fractional differential equation.

We start by briefly recalling some basic definitions and results for S^{JS}-metric spaces, due to Beg et al. [20], that will be needed in the sequel.

Definition 1.1. *[20] Consider a nonempty set Ω and a function $J : \Omega^3 \to [0, \infty)$. Let us define the set*

$$S(J, \Omega, \delta) = \{\{\delta_n\} \subset \Omega : \lim_{n\to\infty} J(\delta, \delta, \delta_n) = 0\}$$

for each $\delta \in \Omega$.

Definition 1.2. *[20] Let Ω be a nonempty set and $J : \Omega^3 \to [0, \infty)$ satisfy the following hypothesis:*

(i) $J(\delta, \xi, \nu) = 0$ implies $\delta = \xi = \nu$ for any $\delta, \xi, \nu \in \Omega$,
(ii) there exists some $b > 0$ such that for any $(\delta, \xi, \nu) \in \Omega^3$ and $\{\nu_n\} \in S(J, \Omega, \nu)$,

$$J(\delta, \xi, \nu) \leq b \limsup_{n\to\infty} (J(\delta, \delta, \nu_n) + J(\xi, \xi, \nu_n)).$$

Then the pair (Ω, J) is called an S^{JS}-metric space.
Moreover, if J also satisfies $J(\delta, \delta, \xi) = J(\xi, \xi, \delta)$ for all $\delta, \xi \in \Omega$, then we call it a symmetric S^{JS}-metric space.

Remark 1.3. *Note that $S(J, \Omega, \delta)$ in some cases can be empty. The following example presents a nonempty set of $S(J, \Omega, \delta)$.*

Example 1.4. *Let $\Omega = \mathbb{R}$ and, $J : \Omega^3 \to [0, \infty)$ be defined by $J(\delta, \xi, \nu) = |\delta-\xi|+|\xi-\nu|$ for all $\delta, \xi, \nu \in \mathbb{R}$. Let $\nu \in \mathbb{R}$ and the sequence (ν_n) be such that $\nu_n = \nu + \frac{1}{n}$.*
It is easy to see that $\lim_{n\to\infty} J(\nu, \nu, \nu+\frac{1}{n}) = 0$. Therefore, for every $\nu \in \mathbb{R}$ there exists a sequence $\nu_n = \nu+\frac{1}{n}$ such that $S(J, \Omega, \nu) \neq \emptyset$.

Definition 1.5. *Let (Ω, J) be an S^{JS}-metric space. A sequence $\{\delta_n\} \subset \Omega$ is said to be convergent to an element $\delta \in \Omega$ if $\{\delta_n\} \in S(J, \Omega, \delta)$.*

Example 1.6. *Let* $\Omega = \mathbb{R}$ *and* $J(\delta, \xi, \nu) = |\delta| + |\xi| + 2|\nu|$ *for all* $\delta, \xi, \nu \in \Omega$.
We have $J(\delta, \xi, \nu) = 0$ *imply that* $|\delta| + |\xi| + 2|\nu| = 0$ *which gives us* $|\delta| = |\xi| = |\nu| = 0$ *then the first condition of the Definition 1.2 is satisfied. Also, the symmetry of* J *is satisfied since we have* $J(\delta, \delta, \xi) = 2|\delta| + 2|\xi| = J(\xi, \xi, \delta)$. *Let* $\delta, \xi, \nu \in \Omega$ *and* ν_n *be a convergent sequence in* Ω *such that* $\lim_{n\to\infty} J(\nu, \nu, \nu_n) = 0$, *it leads to* $\limsup\limits_{n\to\infty} \nu_n = \nu$. *Then we have*

$$\begin{aligned}
J(\delta, \xi, \nu) &= |\delta| + |\xi| + 2|\nu| \\
&\leq 2(2|\delta| + 2|\xi| + 8|\nu|) \\
&= 2(2|\delta| + 2|\xi| + 2|\nu| + 6 \limsup_{n\to\infty} |\nu_n|) \\
&= 2(2|\delta| + 2\limsup_{n\to\infty} |\nu_n| + 2|\xi| + 2\limsup_{n\to\infty} |\nu_n| + 2|\nu| + 2\limsup_{n\to\infty} |\nu_n|) \\
&= 2\limsup_{n\to\infty} (2|\delta| + 2|\nu_n| + 2|\xi| + 2|\nu_n| + 2|\nu| + 2|\nu_n|) \\
&= 2\limsup_{n\to\infty} (J(\delta, \delta, \nu_n) + J(\xi, \xi, \nu_n) + J(\nu, \nu, \nu_n)).
\end{aligned}$$

Since $\lim\limits_{n\to\infty} J(\nu, \nu, \nu_n) = 0$, *then*

$$J(\delta, \xi, \nu) \leq 2\limsup_{n\to\infty} (J(\delta, \delta, \nu_n) + J(\xi, \xi, \nu_n)).$$

Then, all the assumptions of Definition 1.2 are satisfied. Hence, J *is a symmetric* S^{JS}*-metric space with* $b = 2$.

Proposition 1.7. *[20] In an* S^{JS}*-metric space* (Ω, J), *if* $\{\delta_n\}$ *converges to both* δ *and* ξ, *then* $\delta = \xi$.

Definition 1.8. *[20] Let* (Ω, J) *and* (Γ, J_1) *be two* S^{JS}*-metric spaces and* $\sigma : \Omega \to \Gamma$ *be a mapping. Then* σ *is called continuous at* $a_0 \in \Omega$ *if, for any* $\varepsilon > 0$, *there exists* $\tau > 0$ *such that, for any* $\delta \in \Omega$, $J_1(\sigma a_0, \sigma a_0, \sigma\delta) < \varepsilon$ *whenever* $J(a_0, a_0, \delta) < \tau$.

Definition 1.9. *[20] Let* (Ω, J) *be an* S^{JS}*-metric space. A sequence* $\delta_n \subset \Omega$ *is said to be Cauchy if* $\lim_{n,m\to\infty} J(\delta_n, \delta_n, \delta_m) = 0$.

Definition 1.10. *[20] An* S^{JS}*-metric space is said to be complete if every Cauchy sequence in* Ω *is convergent.*

Theorem 1.11. *[20] In an* S^{JS}*-metric space* (Ω, J) *if* σ *is continuous at* $a_0 \in \Omega$, *then for any sequence* $\delta_n \in S(J, \Omega, a_0)$ *implies* $\{\sigma\delta_n\} \in S(J, \Omega, \sigma a_0)$.

Definition 1.12. *Let* Ω *be any set and* $\sigma : \Omega \to \Omega$ *be a selfmap. For any given* $\delta \in \Omega$, *we define* $\sigma^n\delta$ *inductively by* $\sigma^0\delta = \delta$ *and* $\sigma^{n+1}\delta = \sigma(\sigma^n\delta)$. *For any* $\delta_0 \in \Omega$, *we define the sequence* $\{\delta_n\}_{n\geq 0} \subset \Omega$ *as follows*

$$\delta_n = \sigma\delta_{n-1} = \sigma^n\delta_0, \quad n = 1, 2, ... \tag{1.1}$$

Lemma 1.13. *[3] For every function* $\Phi : [0, +\infty) \longrightarrow [0, +\infty)$ *the following holds: if* Φ *is nondecreasing, then for each* $t > 0$, $\lim\limits_{n\to\infty} \Phi^n(t) = 0$ *implies that* $\Phi(t) < t$.

2. Fixed point theorems

Theorem 2.1. *Let* (Ω, J) *be an* S^{JS}*-complete symmetric metric space and* $\sigma : \Omega \to \Omega$ *be a continuous mapping satisfying*

$$J(\sigma\delta, \sigma\xi, \sigma\nu) \leq \phi(J(\delta, \xi, \nu)) \quad \text{for all } \delta, \xi, \nu \in \Omega,$$

where $\phi : [0, +\infty) \to [0, +\infty)$ *is an increasing function such that* $\lim_{n\to\infty} \phi^n(t) = 0$ *for each fixed* $t > 0$. *Then* σ *has a unique fixed point in* Ω.

Proof. Let $\delta_0 \in \Omega, \epsilon > 0$ and $\{\delta_n\}_{n\geq 0} \subset \Omega$ be the sequence defined in (1.1). Pick an arbitrary natural number k and consider the sequence $\{\delta_k\}_{k\geq 0}$ as defined in Definition 1.12. Now, consider the ball $B(\delta_k, \epsilon) := \{\xi \in \Omega : J(\delta_k, \delta_k, \xi) \leq \epsilon\}$. Note that $B(\delta_k, \epsilon) \neq \emptyset$ since $\delta_k \in B(\delta_k, \epsilon)$. We claim that σ maps the ball $B(\delta_k, \epsilon)$ onto itself. Indeed, Let $\nu \in B(\delta_k, \epsilon)$, then $J(\delta_k, \delta_k, \nu) \leq \epsilon$ which means that $\nu \in S(J, \Omega, \delta_k)$. Using the continuity of σ we get

$$\sigma\nu \in S(J, \Omega, \sigma\delta_k) = S(J, \Omega, \delta_{k+1}).$$

Therefore

$$\lim_{k\to\infty} J(\sigma\nu, \sigma\nu, \delta_{k+1}) = 0. \tag{2.1}$$

Now, assuming without loss of generality that $n = k + p$ for some constant $p \in \mathbb{N}$ we get

$$\begin{aligned} J(\delta_k, \delta_k, \sigma\nu) &\leq 2b \limsup_{n\to\infty} J(\delta_k, \delta_k, \delta_{n+1}) \\ &= 2b \limsup_{k\to\infty} J(\delta_k, \delta_k, \delta_{k+p+1}) \\ &\leq 2b \limsup_{k\to\infty} \phi^k(J(\delta_0, \delta_0, \delta_{p+1})). \end{aligned}$$

Using the property of ϕ, we obtain that $J(\delta_k, \delta_k, \sigma\nu) = 0 < \epsilon$ which affirms that $\sigma\nu \in B(\delta_k, \epsilon)$ and confirm our claim. Since $\delta_k \in B(\delta_k, \epsilon)$, we have $\sigma\delta_k \in B(\delta_k, \epsilon)$. By repeating this process, we get $\sigma^m\delta_k \in B(\delta_k, \epsilon)$ for all m. That is $\delta_l \in B(\delta_k, \epsilon)$ for all $l \geq k$. Therefore

$$J(\delta_m, \delta_m, \delta_l) < \epsilon \text{ for all } m, l > k.$$

Hence $\{\delta_k\}$ is a Cauchy sequence and owing to the completeness of Ω, there exists $u \in \Omega$ such that $\delta_k \to u$ as $k \to \infty$.
Moreover, $u = \lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \delta_{k+1} = \lim_{k\to\infty} \sigma\delta_k = \sigma u$. Thus, σ has u as a fixed point.
Let δ_1 and δ_2 be two fixed points of σ.

$$\begin{aligned} J(\delta_1, \delta_1, \delta_2) &= J(\sigma\delta_1, \sigma\delta_1, \sigma\delta_2) \\ &\leq \phi(J(\delta_1, \delta_1, \delta_2)). \end{aligned} \tag{2.2}$$

Using the property of the function ϕ and Lemma 1.13, we obtain from (2.2), $J(\delta_1, \delta_1, \delta_2) < J(\delta_1, \delta_1, \delta_2)$, then $J(\delta_1, \delta_1, \delta_2) = 0$ and $\delta_1 = \delta_2$, and σ has a unique fixed point in Ω . □

Theorem 2.2. *Let* (Ω, J) *be an* S^{JS}*-complete symmetric metric space and* σ *be a continuous self mapping on* Ω *satisfying*

$$J(\sigma\delta, \sigma\xi, \sigma\nu) \leq \alpha[J(\delta, \delta, \sigma\delta) + J(\xi, \xi, \sigma\xi) + J(\nu, \nu, \sigma\nu)], \tag{2.3}$$

$\forall \delta, \xi, \nu \in \Omega$, *where* $\alpha < \frac{1}{4}b$ *and* $\alpha \in [0, \frac{1}{3})$. *Then* σ *has a unique fixed point* $u \in \Omega$.

Proof. Consider $\delta_0 \in \Omega$ arbitrary and $\delta_n = \sigma^n \delta_0$. We denote by $J_n = J(\delta_n, \delta_n, \delta_{n+1})$. We assume that $J_n > 0$, for all $n \in \mathbb{N}$, otherwise, δ_n is a fixed point of σ for at least one $n \geq 0$. It follows from (2.3).

$$\begin{aligned}J_n = J(\delta_n, \delta_n, \delta_{n+1}) &= J(\sigma\delta_{n-1}, \sigma\delta_{n-1}, \sigma\delta_n)\\ &\leq \alpha[2J(\delta_{n-1}, \delta_{n-1}, \sigma\delta_{n-1}) + J(\delta_n, \delta_n, \sigma\delta_n)]\\ &= \alpha[2J(\delta_{n-1}, \delta_{n-1}, \delta_n) + J(\delta_n, \delta_n, \delta_{n+1})]\\ &= \alpha[2J_{n-1} + J_n].\end{aligned}$$

Therefore, $J_n \leq \frac{2\alpha}{1-\alpha} J_{n-1}$. Let $\gamma = \frac{2\alpha}{1-\alpha} < 1$ (since $\alpha < \frac{1}{3}$). By repeating this process we obtain $J_n \leq \gamma^n J_0$. Hence

$$\lim_{n\to\infty} J_n = 0. \tag{2.4}$$

Let us prove that $\{\delta_n\}$ is a Cauchy sequence. For all $n, m \in \mathbb{N}$, we have

$$\begin{aligned}J(\delta_n, \delta_n, \delta_m) &= J(\sigma\delta_{n-1}, \sigma\delta_{n-1}, \sigma\delta_{m-1})\\ &\leq \alpha[2J(\delta_{n-1}, \delta_{n-1}, \sigma\delta_{n-1}) + J(\delta_{m-1}, \delta_{m-1}, \sigma\delta_{m-1})]\\ &= \alpha[2J(\delta_{n-1}, \delta_{n-1}, \delta_n) + J(\delta_{m-1}, \delta_{m-1}, \delta_m)]\\ &= \alpha[2J_{n-1} + J_{m-1}].\end{aligned} \tag{2.5}$$

From (2.2) for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $J_{n-1} < \frac{\epsilon}{4}$ and $J_{m-1} < \frac{\epsilon}{2}$ for all $n, m > n_0$. Then using (2.4), we get $J(\delta_n, \delta_n, \delta_m) \leq \alpha\epsilon$. Since $\alpha < 1$, it follows that $J(\delta_n, \delta_n, \delta_m) \leq \epsilon$, $\forall n, m > n_0$. Therefore, $\{\delta_n\}$ is a Cauchy sequence in Ω and $\lim_{n,m\to\infty} J(\delta_n, \delta_n, \delta_m) = 0$. By completeness of Ω , there exists $u \in \Omega$ such that

$$\lim_{n\to\infty} J(\delta_n, \delta_n, u) = \lim_{n,m\to\infty} J(\delta_n, \delta_m, u) = J(u, u, u) = 0. \tag{2.6}$$

Now, we claim that $\sigma u = u$. From (2.6) we can conclude that $\{\delta_n\} \in S(J, \Omega, u)$. Using the triangle inequality, for any $n \in N$ we have

$$\begin{aligned}J(\sigma u, \sigma u, u) &\leq 2b \limsup_{n\to\infty} J(\sigma u, \sigma u, \delta_n)\\ &= 2b \limsup_{n\to\infty} J(\sigma u, \sigma u, \sigma\delta_{n-1})\\ &\leq 2b \limsup_{n\to\infty}[\alpha(2J(u, u, \sigma u) + J(\delta_{n-1}, \delta_{n-1}, \sigma\delta_{n-1}))]\\ &= 2b \limsup_{n\to\infty}[\alpha(2J(u, u, \sigma u) + J_{n-1})].\end{aligned} \tag{2.7}$$

From (2.4), (2.7) and the symmetry of the metric, we obtain:

$$\begin{aligned}J(\sigma u, \sigma u, u) &\leq 4b\alpha J(u, u, \sigma u)\\ &= 4b\alpha J(\sigma u, \sigma u, u).\end{aligned}$$

Then, $(1 - 4b\alpha)J(\sigma u, \sigma u, u) = 0$. Since $\alpha < \frac{1}{4b}$, then $J(\sigma u, \sigma u, u) = 0$, which gives $\sigma u = u$, and u is a fixed point of σ.

Let $v_1, v_2 \in \Omega$ be two fixed point of σ, $v_1 \neq v_2$ that is $\sigma v_1 = v_1$ and $\sigma v_2 = v_2$. From (2.3) we have

$$J(v_1, v_1, v_2) = J(\sigma v_1, \sigma v_1, \sigma v_2)$$

$$
\begin{aligned}
&\leq \alpha[2J(\nu_1,\nu_1,\sigma\nu_1)+J(\nu_2,\nu_2,\sigma\nu_2)]\\
&=2\alpha J(\nu_1,\nu_1,\nu_1)+\alpha J(\nu_2,\nu_2,\nu_2)\\
&=0.
\end{aligned}
$$

Therefore, $\nu_1 = \nu_2$. Hence, σ has a unique fixed point. □

Theorem 2.3. *Let (Ω, J) be an S^{JS} - complete symmetric metric space and $\sigma : \Omega \to \Omega$ be a continuous mapping satisfying*

$$J(\sigma\delta, \sigma\xi, \sigma\nu) \leq \lambda(\delta, \xi, \nu)J(\delta, \xi, \nu) \quad \forall \delta, \xi, \nu \in \Omega, \tag{2.8}$$

where $\lambda : \Omega^3 \to (0, 1)$ that satisfies the following condition; for every sequence $\{\delta_n\}_{n\geq 0}$ defined by $\delta_n = \sigma\delta_{n-1}$ we have $\lambda(\delta_n, \delta_n, \delta_m) \leq \lambda(\delta_{n-1}, \delta_{n-1}, \delta_{m-1})$. Then σ has a unique fixed point.

Proof. Consider the sequence $\{\delta_n = \sigma^n\delta_0\}$ defined in Definition (1.12). For all natural numbers n, m, we will prove that $\{\delta_n\}$ is a Cauchy sequence. Without loss of generality we suppose that $n < m$ and assume that these exists a constant $p \in N$ such that $m = n + p$. By using (2.8) we have:

$$
\begin{aligned}
J(\delta_n,\delta_n,\delta_m) &= J(\sigma\delta_{n-1},\sigma\delta_{n-1},\sigma\delta_{m-1})\\
&\leq \lambda(\delta_{n-1},\delta_{n-1},\delta_{m-1})J(\delta_{n-1},\delta_{n-1},\delta_{m-1})\\
&\leq \lambda^n(\delta_0,\delta_0,\delta_p)J(\delta_0,\delta_0,\delta_p).
\end{aligned}
$$

Taking the limit as $n \to \infty$ and since $0 < \lambda < 1$, we deduce that $\lim_{n,m\to\infty} J(\delta_n, \delta_n, \delta_m) = 0$, that is $\{\delta_n\}$ is a Cauchy sequence. Then, by the completeness of Ω, there exists $u \in \Omega$ such that

$$u = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \delta_{n-1}. \tag{2.9}$$

Let us prove that u is a fixed point of σ. From (2.9), we deduce that $u \in S(J, \Omega, \delta_n)$ and

$$\lim_{n\to\infty} J(u, u, \delta_n) = 0 \tag{2.10}$$

and

$$\lim_{n\to\infty} J(u, u, \delta_{n-1}) = 0. \tag{2.11}$$

Since, the sequence $\{\delta_n\}$ satisfying the condition (2.10) we can apply the triangle inequality as follows:

$$
\begin{aligned}
J(\sigma u,\sigma u,u) &\leq 2b\lim_{n\to\infty}\sup J(\sigma u,\sigma u,\delta_n)\\
&=2b\lim_{n\to\infty}\sup J(\sigma u,\sigma u,\sigma\delta_{n-1})\\
&\leq 2b\limsup_{n\to\infty}\lambda(u,u,\delta_{n-1})J(u,u,\delta_{n-1})\\
&\leq 2b\lambda(u,u,\delta_0)\lim_{n\to\infty}\sup J(u,u,\delta_{n-1}).
\end{aligned}
\tag{2.12}
$$

Taking into account (2.11) and (2.12) we deduce that u is a fixed point of σ.
Let, $\nu_1, \nu_2 \in \Omega$ be two fixed points of σ such that $\nu_1 \neq \nu_2$, that is, $\sigma\nu_1 = \nu_1$ and $\sigma\nu_2 = \nu_2$. Then,

$$
\begin{aligned}
J(\nu_1,\nu_1,\nu_2) &= J(\sigma\nu_1,\sigma\nu_1,\sigma\nu_2)\\
&\leq \lambda(\nu_1,\nu_1,\nu_2)J(\nu_1,\nu_1,\nu_2)\\
&< J(\nu_1,\nu_1,\nu_2).
\end{aligned}
$$

Then, $J(\nu_1, \nu_1, \nu_2) = 0$ which implies that $\nu_1 = \nu_2$. □

Example 2.4. *Let* $\Omega = [0,1]$ *and consider the* S^{JS} *- metric defined as*

$$J(\delta,\xi,\nu) = |\delta-\xi| + |\delta-\nu| \; for\ all\ \delta,\xi,\nu \in \Omega. \tag{2.13}$$

It is easy to verify that (Ω, J) *is a complete* S^{JS}*-metric space. Take* $\sigma(\delta) = \frac{\delta^2}{5}$*. Choose* $\lambda : \Omega^3 \to [0,1)$ *as* $\lambda(\delta,\xi,\nu) = \frac{\delta+\xi+\nu+1}{5}$*, then,* $\lambda \in A$*.*
Let $\delta,\xi,\nu,\in\Omega$*. Then*

$$\begin{aligned}
J(\sigma\delta,\sigma\xi,\sigma\nu) =& J(\frac{\delta^2}{5},\frac{\xi^2}{5},\frac{\nu^2}{5})\\
=&|\frac{\delta^2}{5}-\frac{\xi^2}{5}| - |\frac{\delta^2}{5}-\frac{\nu^2}{5}|\\
\leq&\frac{1}{5}[|\delta-\xi||\delta+\xi| + |\delta-\nu||\delta+\nu|]\\
\leq&\frac{1}{5}[|\delta-\xi||\delta+\xi+\nu+1| + |\delta-\nu||\delta+\xi+\nu+1|]\\
=&\frac{\delta+\xi+\nu+1}{5}[|\delta-\xi| + |\delta-\nu|]\\
=&\lambda(\delta,\xi,\nu)J(\delta,\xi,\nu).
\end{aligned}$$

Then, all the hypothesis of Theorem (2.3) are satisfied and based on this theorem the mapping σ *has a unique fixed point, which is* $u = 0$ *.*

Theorem 2.5. *Let* (Ω, J) *be a complete* S^{JS} *- symmetric metric space and* $\sigma : \Omega \to \Omega$ *be a continuous mapping such that :*

$$J(\sigma\delta,\sigma\xi,\sigma\nu) \leq \alpha J(\delta,\xi,\nu) + \beta(\delta,\sigma\delta,\sigma\delta) + \gamma J(\xi,\sigma\xi,\sigma\xi) + \delta J(\nu,\sigma\nu,\sigma\nu). \tag{2.14}$$

for all $\delta,\xi,\nu \in \Omega$ *where*

$$0 < \alpha+\beta < 1-\gamma-\delta \tag{2.15}$$
$$and\ 0 < \alpha < 1. \tag{2.16}$$

Then, there exists a unique fixed point of σ*.*

Proof. Let $\delta_0 \in \Omega$ be an arbitrary point of Ω and $\{\delta_n = \sigma^n\delta_0\}$ be a sequence in Ω. From (2.14) we have

$$\begin{aligned}
J(\delta_n,\delta_{n+1},\delta_{n+1}) =& J(\sigma\delta_{n-1},\sigma\delta_n,\sigma\delta_n)\\
\leq&\alpha J(\delta_{n-1},\delta_n,\delta_n) + \beta J(\delta_{n-1},\delta_n,\delta_n) + \gamma J(\delta_n,\delta_{n+1},\delta_{n+1}) + \delta J(\delta_n,\delta_{n+1},\delta_{n+1})\\
\leq&(\alpha+\beta)J(\delta_{n-1},\delta_n,\delta_n) + (\gamma+\delta)J(\delta_n,\delta_{n+1},\delta_{n+1}).
\end{aligned}$$

Then

$$J(\delta_n,\delta_{n+1},\delta_{n+1}) \leq \frac{\alpha+\beta}{1-\gamma-\delta}J(\delta_{n-1},\delta_n,\delta_n).$$

Taking $\lambda = \dfrac{\alpha+\beta}{1-\gamma-\delta}$, then from (2.15) we have $0 < \lambda < 1$. By induction we get $J(\delta_n, \delta_{n+1}, \delta_{n+1}) \leq \lambda^n J(\delta_0, \delta_1, \delta_1)$ which gives that

$$\lim_{n\to\infty} J(\delta_n, \delta_{n+1}, \delta_{n+1}) = 0. \tag{2.17}$$

We denote $J_n = J(\delta_n, \delta_{n+1}, \delta_{n+1})$. We will prove that $\{\delta_n\}$ is a Cauchy sequence in Ω.
For all $n, m \in N, n < m$ we assume w.l.o.g that there exists a fixed $p \in N$ such that $m = n + p$. we have

$$\begin{aligned}
J(\delta_n, \delta_n, \delta_m) &= J(\delta_n, \delta_n, \delta_{n+p}) = J(\sigma\delta_{n-1}, \sigma\delta_{n-1}, \sigma\delta_{n+p-1}) \\
&\leq \alpha J(\delta_{n-1}, \delta_{n-1}, \delta_{n+p-1}) + \beta J(\delta_{n-1}, \delta_n, \delta_n) + \gamma J(\delta_{n-1}, \delta_n, \delta_n) \\
&+ \delta J(\delta_{n+p-1}, \delta_{n+p}, \delta_{n+p}) \\
&= \alpha J(\delta_{n-1}, \delta_{n-1}, \delta_{n+p-1}) + (\beta+\gamma)J_{n-1} + \delta J_{n+p-1} \\
&\leq \alpha[\alpha J(\delta_{n-2}, \delta_{n-2}, \delta_{n+p-2}) + (\beta+\gamma)J_{n-2} + \delta J_{n+p-2}] + (\beta+\gamma)J_{n-1} \\
&+ \delta J_{n+P-1} \\
&= \alpha^2 J(\delta_{n-2}, \delta_{n-2}, \delta_{n+p-2}) + \alpha(\beta+\gamma)J_{n-2} + \alpha\delta J_{n+p-2} + (\beta+\gamma)J_{n-1} \\
&+ \delta J_{n+p-1} \\
&\vdots \\
&\leq \alpha^n J(\delta_0, \delta_0, \delta_p) + (\beta+\gamma)\sum_{k=1}^{n} \alpha^{k-1} J_{n-k} + \delta \sum_{k=1}^{n} \alpha^{k-1} J_{n+p-k}.
\end{aligned} \tag{2.18}$$

By taking the limit in (2.18) as $n \to \infty$ and using (2.16) and (2.17), we obtain

$$\lim_{n,m\to\infty} J(\delta_n, \delta_n, \delta_m) = 0.$$

Then, $\{\delta_n\}$ is a Cauchy sequence in Ω. By completeness, there exists $u \in \Omega$ such that $\delta_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} J(\delta_n, \delta_n, u) = \lim_{n,m\to\infty} J(\delta_m, \delta_m, u) = 0. \tag{2.19}$$

From (2.19), we deduce that $\{\delta_n\} \in S(J, \Omega, u)$. By using the triangle inequality we have

$$J(u, u, \sigma u) \leq 2b \limsup_{n\to\infty} J(u, u, \delta_n). \tag{2.20}$$

Therefore, from (2.19) and (2.20), we have $J(u, u, \sigma u) = 0$, then $u = \sigma u$.
Let $\nu_1, \nu_2 \in \Omega$ be two fixed points of σ, $\nu_1 \neq \nu_2$, that is, $\sigma\nu_1 = \nu_1$, $\sigma\nu_2 = \nu_2$.

$$\begin{aligned}
J(\nu_1, \nu_1, \nu_2) =& J(\sigma\nu_1, \sigma\nu_1, \sigma\nu_2) \\
\leq& \alpha J(\nu_1, \nu_1, \nu_2) + (\beta+\gamma)J(\nu_1, \sigma\nu_1, \sigma\nu_1) + \delta J(\nu_2, \sigma\nu_2, \sigma\nu_2) \\
=& \alpha J(\nu_1, \nu_1, \nu_2) + (\beta+\gamma)J(\nu_1, \nu_1, \nu_1) + \delta J(\nu_2, \nu_2, \nu_2).
\end{aligned}$$

Then, $(1-\alpha)J(\nu_1, \nu_1, \nu_2) \leq 0$. Using (2.16) we conclude that $J(\nu_1, \nu_1, \nu_2) = 0$. Then, $\nu_1 = \nu_2$. □

3. Applications

3.1. Linear system of equations

Consider the set $\Omega = \mathbb{R}^n$ where $\mathbb{R}$ is the set of real numbers and n a positive integer. Now, consider the symmetric S^{JS}-metric space (Ω, S^{JS}) defined by

$$J(\delta, \xi, \nu) = \max_{1\le i\le n} |\delta_i - \xi_i| + |\delta_i - \nu_i|$$

for all $\delta = (\delta_1, ..., \delta_n), \xi = (\xi_1, ..., \xi_n), \nu = (\nu_1, ..., \nu_n) \in \Omega$.

Theorem 3.1. *Consider the following system:*

$$\begin{cases} s_{11}\delta_1 + s_{12}\delta_2 + s_{13}\delta_3 + s_{1n}\delta_n = r_1, \\ s_{21}\delta_1 + s_{22}\delta_2 + s_{23}\delta_3 + s_{2n}\delta_n = r_2, \\ \vdots \\ s_{n1}\delta_1 + s_{n2}\delta_2 + s_{n3}\delta_3 + s_{nn}\delta_n = r_n. \end{cases}$$

if $\theta = \max_{1\le i\le n} \Big(\sum_{j=1, j\ne i}^{n} |s_{ij}| + |1 + s_{ii}| \Big) < 1$, *then the above linear system has a unique solution.*

Proof. Consider the map $\sigma : \Omega \to \Omega$ defined by $\sigma\delta = (B + I_n)\delta - r$ where

$$B = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix},$$

$\delta = (\delta_1, \delta_2, \cdots, \delta_n); \xi = (\xi_1, \xi_2, \cdots, \xi_n)$ and $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{R}^n$, I_n is the identity matrix for $n \times n$ matrices and $r = (r_1, r_2, \cdots, r_n) \in \mathbb{C}^n$. Let us prove that $J(\sigma\delta, \sigma\xi, \sigma\nu) \le \theta J(\delta, \xi, \nu)$, $\forall \delta, \xi, \nu \in \mathbb{R}^n$.
We denote by

$$\tilde{B} = B + I_n = (\tilde{b}_{ij}), \quad i, j = 1, ..., n,$$

with $\tilde{b}_{ij} = \begin{cases} s_{ij}, j \ne i \\ 1 + s_{ii}, j = i \end{cases}$ Hence,

$$\max_{1\le i\le n} \sum_{j=1}^{n} |\tilde{b}_{ij}| = \max_{1\le i\le n} \Big(\sum_{j=1, j\ne i}^{n} |s_{ij}| + |1 + s_{ii}| \Big) = \theta < 1.$$

On the other hand, for all $i = 1, ..., n$, we have

$$(\sigma\delta)_i - (\sigma\xi)_i = \sum_{j=1}^{n} \tilde{b}_{ij}(\delta_j - \xi_j), \tag{3.1}$$

$$(\sigma\delta)_i - (\sigma\nu)_i = \sum_{j=1}^{n} \tilde{b}_{ij}(\delta_j - \nu_j). \tag{3.2}$$

Therefore, using (3.1) and (3.2) we get

$$\begin{aligned}
J(\sigma\delta,\sigma\xi,\sigma\nu) &= \max_{1\leq i\leq n}\left(|(\sigma\delta)_i-(\sigma\xi)_i|+|(\sigma\delta)_i-(\sigma\nu)_i|\right)\\
&\leq \max_{1\leq i\leq n}\Big(\sum_{j=1}^{n}|\tilde{b}_{ij}||\delta_j-\xi_j|+\sum_{j=1}^{n}|\tilde{b}_{ij}||\delta_j-\nu_j|\Big)\\
&\leq \max_{1\leq i\leq n}\sum_{j=1}^{n}|\tilde{b}_{ij}|\max_{1\leq k\leq n}\left(|\delta_k-\xi_k|+|\delta_k-\nu_k|\right)\\
&= \theta J(\delta,\xi,\nu)=\Phi(J(\delta,\xi,\nu)),
\end{aligned}$$

where, $\Phi(t)=\theta t$, $\forall t\geq 0$. Note that, all the hypotheses of Theorem 2.1 are satisfied. Thus, σ has a unique fixed point. Therefore, the above linear system has a unique solution as desired. □

3.2. Fractional differential equation

In this section, we discuss the existence of a solution to the following problem:

$$(\mathcal{P}):\left\{\begin{array}{lcl} D^{\lambda}x(t) &=& f(t,x(t))=Fx(t) \text{ if } t\in I_0=(0,T]\\ x(0) &=& x(T)=r \end{array}\right\}$$

where $T>0$ and $f:I\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function, $I=[0,T]$ and $D^{\lambda}x$ denotes a Riemann-Liouville fractional derivative of x with $\lambda\in(0,1)$.
Let $C_{1-\lambda}(I,\mathbb{R})=\{f\in C((0,T],\mathbb{R}):t^{1-\lambda}f\in C(I,\mathbb{R})\}$. We define the following weighted norm

$$\|f\|^{*}=\max_{t\in[0,T]}t^{1-\lambda}|f(t)|.$$

Theorem 3.2. *Let $\lambda\in(0,1)$, $f\in C(I\times I,\mathbb{R})$ increasing and $0<\alpha<1$. In addition, we assume the following hypothesis:*

$$|f(u_1(t),v_1(t))-f(u_2(t),v_2(t))|\leq\frac{\Gamma(2\lambda)}{T^{2\lambda-1}}\alpha|v_1-v_2|.$$

Then the problem $(\mathcal{P})$ has a unique solution.

Proof. Problem $(\mathcal{P})$ is equivalent to the problem $\mathcal{M}x(t)=x(t)$ where

$$\mathcal{M}x(t)=rt^{\lambda-1}+\frac{1}{\Gamma(\lambda)}\int_0^t(t-s)^{\lambda-1}Fx(s)ds.$$

In fact, proving that the operator $\mathcal{M}$ has a fixed point is sufficient to say that problem $\mathcal{P}$ has a unique solution. To see this, assume that $\mathcal{M}x(t)=x(t)$ now apply D^{λ} to both side we get $D^{\lambda}x(t)=Fx(t)$. Therefore, we need to check that the hypothesis in Theorem 2.5 are satisfied where $\beta=\gamma=\delta=0$.
Indeed, let prove that $(A=C_{1-\lambda}(I,\mathbb{R}),J)$ is a complete S^{JS} metric space if we choose:

$$J(x,y,z)=\max_{[0,T]}t^{1-\lambda}\Big(|x(t)-y(t)|+|x(t)-z(t)|\Big),\ x,y,z\in C_{1-\lambda}(I,\mathbb{R}).$$

Let $x, y, z \in A$, If $J(x, y, z) = 0$, then $|x(t) - y(t)| + |x(t) - z(t)| =$ for all $t \in [0, T]$ which give us that $x = y = z$. On the other hand, let (z_n) be a convergent the sequence such that $\lim_{n\to\infty} J(z, z, z_n)$ which imply that $\lim_{n\to\infty} |z_n(t) - z(t)| = 0$, we have

$$\begin{aligned} J(x,y,z) &= \max_{[0,T]} t^{1-\lambda}\Big(|x(t) - y(t)| + |x(t) - z(t)|\Big) \\ &= \max_{[0,T]} t^{1-\lambda}\Big(|x(t) - z_n(t) + z_n(t) - y(t)| + |x(t) - z_n(t) + z_n(t) - z(t)|\Big) \\ &\leq \max_{[0,T]} t^{1-\lambda}\Big(2|x(t) - z_n(t)| + |z_n(t) - y(t)| + |z_n(t) - z(t)|\Big) \\ &\leq 2\lim_{n\to\infty} \sup \max_{[0,T]} t^{1-\lambda}\Big(|x(t) - z_n(t)| + |z_n(t) - y(t)|\Big) \\ &\leq 2\lim_{n\to\infty} \sup \max_{[0,T]} t^{1-\lambda}\Big(J(x,x,z_n) + J(y,y,z_n)\Big). \end{aligned}$$

Therefore, $(A = C_{1-\lambda}(I, \mathbb{R}), J)$ is an S^{JS} metric space.
The mapping $\mathcal{M}$ is increasing since f is increasing.
Now, we must prove that $\mathcal{M}$ is a contraction map. Let $x, y, z \in C_{1-\lambda}(J, \mathbb{R})$, $0 < \lambda < 1$.

$$\begin{aligned} J(\mathcal{M}x, \mathcal{M}y, \mathcal{M}z) &= \max_{[0,T]} t^{1-\lambda}\Big(|\mathcal{M}x(t) - \mathcal{M}y(t)| + |\mathcal{M}x(t) - \mathcal{M}z(t)|\Big) \\ &\leq \frac{1}{\Gamma(\lambda)} \max_{t\in[0,T]} t^{1-\lambda} \int_0^t (t-s)^{\lambda-1}\Big(|f(t,x(s)) - f(t,y(s))| \\ &+ |f(t,x(s)) - f(t,z(s))|\Big)ds. \end{aligned}$$

Subsequently, by the hypothesis of the theorem, we have:

$$\begin{aligned} J(\mathcal{M}x, \mathcal{M}y, \mathcal{M}z) &\leq \frac{1}{\Gamma(\lambda)} \max_{t\in[0,T]} t^{1-\lambda} \int_0^t (t-s)^{\lambda-1} \frac{\Gamma(2\lambda)}{T^{2\lambda-1}}\Big[\alpha|x(s) - y(s)| \\ &+ \alpha|x(s) - z(s)|\Big]ds \\ &\leq \frac{1}{\Gamma(\lambda)} \max_{t\in[0,T]} t^{1-\lambda} \int_0^t (t-s)^{\lambda-1} \frac{\Gamma(2\lambda)}{T^{2\lambda-1}}\Big[\alpha\|x - y\|^* s^{\lambda-1} \\ &+ \alpha\|x - z\|^* s^{\lambda-1}\Big]ds \\ &\leq \frac{1}{\Gamma(\lambda)} \max_{t\in[0,T]} t^{1-\lambda}\alpha(\|x - y\|^* + \|x - z\|^*)\frac{\Gamma(2\lambda)}{T^{2\lambda-1}} \int_0^t (t-s)^{\lambda-1} s^{\lambda-1} ds. \end{aligned}$$

From the Riemann-Liouville fractional integral we have

$$\int_0^t (t-s)^{\lambda-1} s^{\lambda-1} ds = \frac{\Gamma(\lambda)}{\Gamma(2\lambda)} t^{2\lambda-1}.$$

Therefore, we have

$$J(\mathcal{M}x, \mathcal{M}y, \mathcal{M}z) \leq \alpha J(x, y, z).$$

Thus, by Theorem 2.5, we deduce that $\mathcal{M}$ has a unique fixed point which leads us to conclude that equation $(\mathcal{P})$ has a unique solution as desired. □

4. Conclusions

Inspired by the novelty of the S^{JS}-metric spaces, we have proved in this article some new versions of fixed point results for different contraction mappings. In the proofs, in order to obtain the existence and the uniqueness of the fixed point we used the completeness of the S^{JS}-metric space and sometimes the symmetry of the metric. We have also presented two applications of our result to linear system of equations and fractional differential equations. These applications have shown the efficiency of our approach.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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