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Research article

Some new Riemann-Liouville fractional integral inequalities for interval-valued mappings

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Abstract: The notions of convex mappings and inequalities, which form a strong link and are key parts of classical analysis, have gotten a lot of attention recently. As a familiar extension of the classical one, interval-valued analysis is frequently used in the research of control theory, mathematical economy and so on. Motivated by the importance of convexity and inequality, our aim is to consider a new class of convex interval-valued mappings (I-$V\cdot Ms$) known as left and right (L-R) $\mathfrak{J}$-convex interval-valued mappings through pseudo-order relation ($\leq_p$) or partial order relation, because in interval space, both concepts coincide, so this order relation is defined in interval space. By using this concept, first we obtain Hermite-Hadamard (HH-) and Hermite-Hadamard-Fejér (HH-Fejér) type inequalities through pseudo-order relations via the Riemann-Liouville fractional integral operator. Moreover, we have shown that our results include a wide class of new and known inequalities for L-R $\mathfrak{J}$-convex- I-$V\cdot Ms$ and their variant forms as special cases. Under some mild restrictions, we have proved that the inclusion relation "⊆" is coincident to pseudo-order relation

"$\leq_p$" when the *I-V·M* is *L-R* $\mathfrak{J}$-convex or *L-R* $\mathfrak{J}$-concave. Results obtained in this paper can be viewed as an improvement and refinement of classical known results.

Keywords: *L-R* $\mathfrak{J}$-convex interval-valued mapping; interval Riemann-Liouville fractional integral operator; Hermite-Hadamard inequality; Hermite-Hadamard Fejér inequality
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1. Introduction

Convex analysis has played an important and fundamental part in the development of various fields of applied and pure science. In last few decades, much attention has been given in studying and distinguishing diverse directions of the classical idea of convexity. In the classical approach, a real-valued mapping $\Upsilon: K \to \mathbb{R}$ is called a convex mapping on K if

$$\Upsilon(\imath\varkappa + (1-\imath)\varpi) \leq \imath\Upsilon(\varkappa) + (1-\imath)\Upsilon(\varpi), \tag{1}$$

for all $\varkappa, \varpi \in K, \imath \in [0,1]$. If Υ is convex, then $-\Upsilon$ is concave. Recently, many extensions and generalizations of convex sets and convex mappings have been established, such as harmonic convexity [1], $\mathfrak{J}$-convexity [2], quasi convexity [3], Schur convexity [4,5], strong convexity [6,7], p-convexity [8], $(p,\mathfrak{J})$-convexity [9] and generalized convexity [10]. The main generalization of convex mappings is the discrete Jensen inequality [11], because it plays critical roles in probability theory, in optimization theory and in other fields of science. For more useful details, see [12–15] and the references therein.

The concept of convexity establishes a strong relationship with integral problems. Therefore, this field of research has attracted many authors to contribute their roles. Therefore, many inequalities have been introduced as applications of convex mappings. The representative results include the Gagliardo-Nirenberg-type inequality [16], Hardy-type inequality [17], Ostrowski-type inequality [18], Olsen-type inequality [19] and the most well-known inequality, namely, the Hermite-Hadamard inequality (HH-inequality, in short) [20]. The HH-inequality is an interesting outcome in convex analysis which is formulated for convex mappings. In [21], Fejér considered a major generalization of the HH-inequality, which is known as HH-Fejér inequality. This inequality basically depends upon the convex and symmetric mapping. With the assistance of the Fejér-inequality, many inequalities can be obtained through special symmetric mappings for convex mappings.

It is also a familiar fact that interval analysis [22] and fuzzy analysis [23] are considered to be two different fields of mathematics that provide tools to deal with data uncertainty. In general, interval analysis is typically used to deal with models whose data are composed of inaccuracies that may occur from certain kinds of measurements. On the other hand, without complete knowledge of the problem, fuzzy analysis can be used to deal with the models that were obtained. In addition, it is useful in the study of a wide range of issues in pure mathematics and applied sciences, such as operational analysis, computer science, management sciences, artificial intelligence, control engineering and decision making. Convex analysis has contributed significantly to the advancement of several sectors of practical and pure research. Convexity and non-convexity are also important concepts in optimization in the interval and fuzzy domains. Costa [24], Costa and Roman-Flores [25], Roman-Flores et al. [26] and Chalco-Cano et al. [27,28] have generalized several classical discrete and integral inequalities not only to the environment of the IVF and fuzzy-IVFs but also to more general set valued maps by Nikodem et al. [29] and Matkowski and Nikodem [30]. Zhang et al. [31] used a pseudo-order relation to establish a novel version of Jensen's inequalities for set-valued and fuzzy set-valued mappings, proving that these Jensen's inequalities are an extended form of Costa Jensen's inequalities [24]. With the use of fractional integral inequalities, a number of major fractional derivative and integral operators are systematically and successfully evaluated in the current environment [32–39]. Variants are well recognized to have several essential applications in all areas of mathematics as well as in other fields of natural science. Numerous types of variations, such as those bearing the names Jensen, Hermite–Hadamard, Hardy, Ostrowski, Minkowski and Opial et al., are noteworthy; they also have a substantial impact in important fields of research.

Moreover, recently, Khan et al. [40] introduced new versions of fuzzy interval-valued mappings (*F.I-V·Ms)*, known as $(\mathfrak{J}_1,\mathfrak{J}_2)$-convex *F.I-V·Ms*. As one step forward, Khan et al. introduced new classes of convex and generalized convex *F.I-V·Ms* and derived new HH type inequalities for $\mathfrak{J}$-convex *F.I-V·Ms* [41], $(\mathfrak{J}_1,\mathfrak{J}_2)$-preinvex *F.I-V·Ms* [42] and log-s-convex *F.I-V·Ms* in the second sense [43]. We refer readers to further analysis of the literature on the applications and properties of generalized convex mappings; see [44–59] and the references therein.

Inspired by the ongoing research work, the main aim of this paper is to introduce the class of *L-R* $\mathfrak{J}$-convex *I-V·Ms* and to establish inequalities of Jensen, Schur, H- and HH-Fejér type for *L-R* $\mathfrak{J}$-convex *I-V·Ms* by means of pseudo-order relations via Riemann-Liouville fractional integral operators. The main results of this paper also obtain some applications.

2. Preliminaries

Let $\mathfrak{R}_C$ be the set of all closed and bounded intervals of $\mathbb{R}$. Let $\mathfrak{R}_C^+$ and $\mathfrak{R}_C^-$ denote the set of all positive closed intervals and the set of all negative closed intervals of $\mathbb{R}$, correspondingly.
Remember the approaching notion, which Zhang et al. offered in [31].

Remark 2.1. [31] (i) The relation " $\leq_p$ " on $\mathfrak{R}_C$ which is expressed as

$$[\mathfrak{L}_*,\mathfrak{L}^*] \leq_p [\mathfrak{G}_*,\mathfrak{G}^*] \text{ if and only if } \mathfrak{L}_* \leq \mathfrak{G}_*,\ \mathfrak{L}^* \leq \mathfrak{G}^*,$$

for all $[\mathfrak{L}_*,\mathfrak{L}^*],\ [\mathfrak{G}_*,\mathfrak{G}^*] \in \mathfrak{R}_C$, is a pseudo-order relation.
(ii) Since "$\leq_p$" seems to be "left and right" on the real line, we refer to it as "left and right" (or "*L-R*" order, in short).

Theorem 2.2. [38] Assume that *I-V·M* $\Upsilon:[\mu,\omega] \subset \mathbb{R} \to \mathfrak{R}_C$ along with $\Upsilon(\varpi) = [\Upsilon_*(\varpi),\Upsilon^*(\varpi)]$, where $\varpi \in [\mu,\omega]$. We call $\Upsilon(\varpi)$ Lebesgue integrable if the mappings $\Upsilon_*(\varpi)$ and $\Upsilon^*(\varpi)$ both are integrable along with Lebesgue integrable over $[\mu,\omega]$. Furthermore, $\int_\mu^\omega \Upsilon(\varpi)d\varpi$ is defined as

$$\int_\mu^\omega \Upsilon(\varpi)d\varpi = \left[\int_\mu^\omega \Upsilon_*(\varpi)d\varpi, \int_\mu^\omega \Upsilon^*(\varpi)d\varpi\right]. \tag{2}$$

Definition 2.3. [37,39] Assume that *I-V·M* $\Upsilon:[\mu,\omega] \subset \mathbb{R} \to \mathfrak{R}_C$ along with $\Upsilon(\varpi) = [\Upsilon_*(\varpi),\Upsilon^*(\varpi)]$, where $\varpi \in [\mu,\omega]$, in which the mappings $\Upsilon_*(\varpi)$ and $\Upsilon^*(\varpi)$ both are Riemannian integrable over $[\mu,\omega]$. Furthermore, the interval-valued left-sided and right-sided fractional integral operators are defined as

$$\mathcal{J}^{\alpha}_{\mu^+}\, \Upsilon(\varpi) = \frac{1}{\Gamma(\alpha)}\int_{\mu}^{\varpi}(\varpi - \imath)^{\alpha-1}\Upsilon(\imath)\, d\imath, \quad (\varpi > \mu), \tag{3}$$

and

$$\mathcal{J}^{\alpha}_{\omega^-}\, \Upsilon(\varpi) = \frac{1}{\Gamma(\alpha)}\int_{\varpi}^{\omega}(\imath - \varpi)^{\alpha-1}\Upsilon(\imath)\, d\imath, \ (\varpi < \omega), \tag{4}$$

respectively, where $\Gamma(\varpi) = \int_0^{\infty} \imath^{\varpi-1}e^{-\imath}\, d\imath$ is the Euler gamma mapping.

Definition 2.3. [44] A real-valued mapping $\Upsilon: [\mu, \omega] \to \mathbb{R}^+$ is called a convex mapping if

$$\Upsilon(\imath\varkappa + (1-\imath)\varpi) \le \imath\Upsilon(\varkappa) + (1-\imath)\Upsilon(\varpi) \tag{5}$$

holds true for every $\varkappa, \varpi \in [\mu, \omega]$ together with $\imath \in [0,1]$. If (5) is reversed, then Υ is called concave.

Definition 2.4. [31]. The *I-V·M* $\Upsilon: [\mu, \omega] \to \mathfrak{R}_C^+$ is called *L-R* convex *I-V·M* on $[\mu, \omega]$ if

$$\Upsilon(\imath\varkappa + (1-\imath)\varpi) \le_p \imath\Upsilon(\varkappa) + (1-\imath)\Upsilon(\varpi) \tag{6}$$

holds true for every $\varkappa, \varpi \in [\mu, \omega]$ together with $\imath \in [0,1]$. If (6) is reversed, then Υ is called an *L-R* concave *I-V·M* on $[\mu, \omega]$. Υ is affine if and only if it is both an *L-R* convex and *L-R* concave *I-V·M*.

Remark 2.5. If $\Upsilon_*(\varpi) = \Upsilon^*(\varpi)$, then we obtain the inequality (5).

Definition 2.6. [47] Let $\mathfrak{J}_1, \mathfrak{J}_2: [0,1] \subseteq [\mu, \omega] \to \mathbb{R}^+$ such that $\mathfrak{J}_1, \mathfrak{J}_2 \not\equiv 0$. Then, *I-V·M* $\Upsilon: [\mu, \omega] \to \mathfrak{R}_C^+$ is said to be an *L-R* $(\mathfrak{J}_1, \mathfrak{J}_2)$-convex *I-V·M* on $[\mu, \omega]$ if

$$\Upsilon(\imath\varkappa + (1-\imath)\varpi) \le_p \mathfrak{J}_1(\imath)\,\mathfrak{J}_2(1-\imath)\,\Upsilon(\varkappa) + \mathfrak{J}_1(1-\imath)\,\mathfrak{J}_2(\imath)\,\Upsilon(\varpi) \tag{7}$$

holds true for every $\varkappa, \varpi \in [\mu, \omega]$ together with $\imath \in [0,1]$. If Υ is *L-R* $(\mathfrak{J}_1, \mathfrak{J}_2)$-concave on $[\mu, \omega]$, then inequality (7) is reversed.

Remark 2.7. [47] If $\mathfrak{J}_2(\imath) \equiv 1$, then an *L-R* $(\mathfrak{J}_1, \mathfrak{J}_2)$-preinvex *I-V·M* becomes an *L-R* $\mathfrak{J}$-convex *I-V·M*, that is,

$$\Upsilon(\imath\varkappa + (1-\imath)\varpi) \le_p \mathfrak{J}_1(\imath)\,\Upsilon(\varkappa) + \mathfrak{J}_1(1-\imath)\,\Upsilon(\varpi), \tag{8}$$

for every $\varkappa, \varpi \in [\mu, \omega], \imath \in [0,1]$.

If $\mathfrak{J}_1(\imath) = \imath, \mathfrak{J}_2(\imath) \equiv 1$, then from (7), we acquire (6).

If $\mathfrak{J}_1(\imath) \equiv 1 \equiv \mathfrak{J}_2(\imath)$, then from (7), we obtain the coming inequality:

$$\Upsilon(\imath\varkappa + (1-\imath)\varpi) \le_p \Upsilon(\varkappa) + \Upsilon(\varpi), \forall\, \varkappa, \varpi \in [\mu, \omega], \imath \in [0,1]. \tag{9}$$

Theorem 2.8. [47] Let $\mathfrak{J}: [0,1] \subseteq [\mu, \omega] \to \mathbb{R}^+$ be a non-negative real-valued mapping such that $\mathfrak{J} \not\equiv 0$ and let $\Upsilon: [\mu, \omega] \to \mathfrak{R}_C^+$ be an *I-V·M* such that

$$\Upsilon(\varpi) = [\Upsilon_*(\varpi), \Upsilon^*(\varpi)], \tag{10}$$

for all $\varpi \in [\mu, \omega]$. Then, Υ is an *L-R* $\mathfrak{J}$-convex *I-V·M* on $[\mu, \omega]$ if and only if $\Upsilon_*(\varpi)$ and $\Upsilon^*(\varpi)$ both are $\mathfrak{J}$-convex.

Example 2.9. We consider $\mathfrak{J}(\imath) = \imath$, for $\imath \in [0,1]$ and the *I-V·M* $\Upsilon: [0,4] \to \mathfrak{R}_C^+$ defined by $\Upsilon(\varpi) = \left[\varpi, 2e^{\varpi^2}\right]$. Since end point mappings $\Upsilon_*(\varpi)$, $\Upsilon^*(\varpi)$ are $\mathfrak{J}$-convex mappings. Hence, $\Upsilon(\varpi)$ is an *L-R* $\mathfrak{J}$-convex *I-V·M*.

Remark 2.10. [46] If Υ is a $\mathfrak{J}$-convex *I-V·M,* then Υ is an *L-R* $\mathfrak{J}$-concave *I-V·M* if and only if $\Upsilon_*(\varpi)$

is a $\mathfrak{I}$-affine mapping and $\Upsilon^*(\varpi)$ is a $\mathfrak{I}$-convex mapping such that

$$\Upsilon(\imath\varkappa+(1-\imath)\varpi) \supseteq \mathfrak{I}(\imath)\,\Upsilon(\varkappa)+\mathfrak{I}(1-\imath)\,\Upsilon(\varpi). \tag{11}$$

Similarly, if Υ is a $\mathfrak{I}$-concave *I-V·M,* then Υ is an *L-R* $\mathfrak{I}$-convex *I-V·M* if and only if $\Upsilon_*(\varpi)$ and $\Upsilon^*(\varpi)$ are $\mathfrak{I}$-concave and $\mathfrak{I}$-affine mappings, respectively, such that

$$\Upsilon(\imath\varkappa+(1-\imath)\varpi) \subseteq \mathfrak{I}(\imath)\,\Upsilon(\varkappa)+\mathfrak{I}(1-\imath)\,\Upsilon(\varpi). \tag{12}$$

3. Main results

In this section, we will propose some new Riemann-Liouville fractional integral HH type inequalities for L-R $\mathfrak{I}$-convex *I-V·Ms*.

Theorem 3.1. Let $\Upsilon:[\mu,\omega]\to\mathfrak{R}_C^+$ be an *L-R* $\mathfrak{I}$-convex *I-V·M* on $[\mu,\omega]$ such that $\Upsilon(\varpi)=[\Upsilon_*(\varpi),\Upsilon^*(\varpi)]$ for all $\varpi\in[\mu,\omega]$. Then, one has the successive coming relations:

$$\frac{1}{\alpha\,\mathfrak{I}\left(\frac{1}{2}\right)}\Upsilon\left(\frac{\mu+\omega}{2}\right) \leq_p \frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^\alpha\,\Upsilon(\omega)+\mathcal{J}_{\omega^-}^\alpha\,\Upsilon(\mu)\right]$$

$$\leq_p \left(\Upsilon(\mu)+\Upsilon(\omega)\right)\int_0^1 \imath^{\alpha-1}[\mathfrak{I}(\imath)-\mathfrak{I}(1-\imath)]d\imath. \tag{13}$$

If $\Upsilon(\varpi)$ is a concave *I-V·M*, then

$$\frac{1}{\alpha\,\mathfrak{I}\left(\frac{1}{2}\right)}\Upsilon\left(\frac{\mu+\omega}{2}\right) \geq_p \frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^\alpha\,\Upsilon(\omega)+\mathcal{J}_{\omega^-}^\alpha\,\Upsilon(\mu)\right]$$

$$\geq_p \left(\Upsilon(\mu)+\Upsilon(\omega)\right)\int_0^1 \imath^{\alpha-1}[\mathfrak{I}(\imath)-\mathfrak{I}(1-\imath)]d\imath\,. \tag{14}$$

Proof. Let $\Upsilon:[\mu,\omega]\to\mathfrak{R}_C^+$ be an *L-R* $\mathfrak{I}$-convex *I-V·M*. Then, by hypothesis, we have

$$\frac{1}{\mathfrak{I}\left(\frac{1}{2}\right)}\Upsilon\left(\frac{\mu+\omega}{2}\right) \leq_p \Upsilon(\imath\mu+(1-\imath)\omega)+\Upsilon\big((1-\imath)\mu+\imath\omega\big).$$

Therefore, we have

$$\frac{1}{\mathfrak{I}\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{\mu+\omega}{2}\right) \leq \Upsilon_*(\imath\mu+(1-\imath)\omega)+\Upsilon_*\big((1-\imath)\mu+\imath\omega\big),$$

$$\frac{1}{\mathfrak{I}\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{\mu+\omega}{2}\right) \leq \Upsilon^*(\imath\mu+(1-\imath)\omega)+\Upsilon^*\big((1-\imath)\mu+\imath\omega\big).$$

Multiplying both sides by $\imath^{\alpha-1}$ and integrating the obtained result with respect to $\imath$ over $(0,1)$, we have

$$\frac{1}{\mathfrak{I}\left(\frac{1}{2}\right)}\int_0^1 \imath^{\alpha-1}\Upsilon_*\left(\frac{\mu+\omega}{2}\right)d\imath$$

$$\leq \int_0^1 \imath^{\alpha-1}\Upsilon_*(\imath\mu+(1-\imath)\omega)d\imath+\int_0^1 \imath^{\alpha-1}\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)d\imath\,,$$

$$\frac{1}{\mathfrak{I}\left(\frac{1}{2}\right)}\int_0^1 \imath^{\alpha-1}\Upsilon^*\left(\frac{\mu+\omega}{2}\right)d\imath$$

$$\leq \int_0^1 \imath^{\alpha-1}\Upsilon^*(\imath\mu+(1-\imath)\omega)d\imath+\int_0^1 \imath^{\alpha-1}\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)d\imath\,.$$

Let $\varkappa = \imath\mu + (1-\imath)\omega$ and $\varpi = (1-\imath)\mu + \imath\omega$. Then, we have

$$\frac{1}{\alpha\,\Im\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{\mu+\omega}{2}\right) \le \frac{1}{(\omega-\mu)^{\alpha}}\int_{\mu}^{\omega}(\omega-\varkappa)^{\alpha-1}\Upsilon_*(\varkappa)\,d\varkappa$$
$$+\frac{1}{(\omega-\mu)^{\alpha}}\int_{\mu}^{\omega}(\varpi-\mu)^{\alpha-1}\Upsilon_*(\varpi)\,d\varpi$$
$$\frac{1}{\alpha\,\Im\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{\mu+\omega}{2}\right) \le \frac{1}{(\omega-\mu)^{\alpha}}\int_{\mu}^{\omega}(\omega-\varkappa)^{\alpha-1}\Upsilon^*(\varkappa)\,d\varkappa$$
$$+\frac{1}{(\omega-\mu)^{\alpha}}\int_{\mu}^{\omega}(\varpi-\mu)^{\alpha-1}\Upsilon^*(\varpi)\,d\varpi,$$
$$\le \frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon_*(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon_*(\mu)\right]$$
$$\le \frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon^*(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon^*(\mu)\right],$$

that is,

$$\frac{1}{\alpha\,\Im\left(\frac{1}{2}\right)}\left[\Upsilon_*\left(\frac{\mu+\omega}{2}\right), \Upsilon^*\left(\frac{\mu+\omega}{2}\right)\right] \le_p \frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon_*(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon_*(\mu)\right], \left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon^*(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon^*(\mu)\right]\right].$$

Thus,

$$\frac{1}{\alpha\,\Im\left(\frac{1}{2}\right)}\,\Upsilon\left(\frac{\mu+\omega}{2}\right) \le_p \frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon(\mu)\right]. \tag{15}$$

In a similar way as above, we have

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon(\mu)\right]$$
$$\le_p [\Upsilon(\mu) + \Upsilon(\omega)]\int_0^1 \imath^{\alpha-1}[\Im(\imath) - \Im(1-\imath)]d\imath. \tag{16}$$

Combining (15) and (16), we have

$$\frac{1}{\alpha\Im\left(\frac{1}{2}\right)}\Upsilon\left(\frac{\mu+\omega}{2}\right) \le_p \frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon(\mu)\right]$$
$$\le_p [\Upsilon(\mu) + \Upsilon(\omega)]\int_0^1 \imath^{\alpha-1}[\Im(\imath) - \Im(1-\imath)]d\imath,$$

that is, the required result.

Remark 3.2 From Theorem 3.1 we clearly see the following.

Let $\Upsilon_*(x)$ be a $\Im$-affine mapping and $\Upsilon^*(x)$ be a $\Im$-concave mapping. If $\Upsilon_*(x) \neq \Upsilon^*(x)$, then from Remark 2.10 and inequality (13), we acquire the coming inequality, which was achieved by Zhao et al. (see [48]):

$$\frac{1}{\alpha\Im\left(\frac{1}{2}\right)}\Upsilon\left(\frac{\mu+\omega}{2}\right) \supseteq \frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\mu^+}\,\Upsilon(\omega)+\mathcal{J}^{\alpha}_{\omega^-}\,\Upsilon(\mu)\right]$$

$$\supseteq [\Upsilon(\mu)+\Upsilon(\omega)]\int_0^1 \imath^{\alpha-1}[\Im(\imath)-\Im(1-\imath)]d\imath. \quad (17)$$

Let $\Upsilon_*(x)$ be a $\Im$-affine mapping and $\Upsilon^*(x)$ be a $\Im$-concave mapping with $\alpha = 1$. If $\Upsilon_*(x) \neq \Upsilon^*(x)$, then from Remark 2.10 and inequality (13), we acquire the coming inequality, which was achieved by Zhao et al. (see [36]):

$$\frac{1}{2\,\Im\left(\frac{1}{2}\right)}\,\Upsilon\left(\frac{\mu+\omega}{2}\right) \supseteq \frac{1}{\omega-\mu}\,(IR)\int_\mu^\omega \Upsilon(\varpi)d\varpi \supseteq [\Upsilon(\mu)+\,\Upsilon(\omega)]\int_0^1 \Im(\imath)\,d\imath. \quad (18)$$

If $\alpha = 1$, then (13) reduces to the inequality for an *L-R* $\Im$-convex *I-V·M* (see [50]):

$$\frac{1}{2\,\Im\left(\frac{1}{2}\right)}\,\Upsilon\left(\frac{\mu+\omega}{2}\right) \leq_p \frac{1}{\omega-\mu}\,(IR)\int_\mu^\omega \Upsilon(\varpi)d\varpi \leq_p [\Upsilon(\mu)+\,\Upsilon(\omega)]\int_0^1 \Im(\imath)\,d\imath. \quad (19)$$

If $\Im(\imath) = \imath$, then (13) reduces to the inequality for an *L-R* convex *I-V·M* (see [35]):

$$\Upsilon\left(\frac{\mu+\omega}{2}\right) \leq_p \frac{\Gamma(\alpha+1)}{2(\omega-\mu)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\mu^+}\,\Upsilon(\omega)+\mathcal{J}^{\alpha}_{\omega^-}\,\Upsilon(\mu)\right] \leq_p \frac{\Upsilon(\mu)+\Upsilon(\omega)}{2}. \quad (20)$$

Let $\alpha = 1$ and $\Im(\imath) = \imath$. Then, (13) reduces to the inequality for an *L-R* convex *I-V·M* given in [46]:

$$\Upsilon\left(\frac{\mu+\omega}{2}\right) \leq_p \frac{1}{\omega-\mu}\,(IR)\int_\mu^\omega \Upsilon(\varpi)d\varpi \leq_p \frac{\Upsilon(\mu)+\Upsilon(\omega)}{2}. \quad (21)$$

If $\Upsilon_*(\varpi) = \Upsilon^*(\varpi)$, then from (13) we get the following classical inequality:

$$\frac{1}{\alpha\,\Im\left(\frac{1}{2}\right)}\Upsilon\left(\frac{\mu+\omega}{2}\right) \leq \frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\mu^+}\,\Upsilon(\omega)+\mathcal{J}^{\alpha}_{\omega^-}\,\Upsilon(\mu)\right]$$

$$\leq [\Upsilon(\mu)+\Upsilon(\omega)]\int_0^1 \imath^{\alpha-1}[\Im(\imath)-\Im(1-\imath)]d\imath. \quad (22)$$

Let $\alpha = 1$ and $\Upsilon_*(\varpi) = \Upsilon^*(\varpi)$. Then, from (13) we obtain following classical inequality:

$$\frac{1}{2\,\Im\left(\frac{1}{2}\right)}\,\Upsilon\left(\frac{\mu+\omega}{2}\right) \leq \frac{1}{\omega-\mu}\,(R)\int_\mu^\omega \Upsilon(\varpi)d\varpi \leq [\Upsilon(\mu)+\,\Upsilon(\omega)]\int_0^1 \Im(\imath)\,d\imath. \quad (23)$$

Example 3.3. Let $\imath = \frac{1}{2}$, $\Im(\nu) = \nu$, for all $\nu \in [0,1]$and the *I-V·M* $\Upsilon\colon [\mu,\omega] = [2,3] \to \mathfrak{R}_C^+$, defined by

$$\Upsilon(\varpi)(\nu) = \begin{cases} \dfrac{\nu}{2-\varpi^{\frac{1}{2}}}, & \nu \in \left[0, 2-\varpi^{\frac{1}{2}}\right] \\ \dfrac{2\left(2-\varpi^{\frac{1}{2}}\right)-\nu}{2-\varpi^{\frac{1}{2}}}, & \nu \in (2-\varpi^{\frac{1}{2}}, 2\left(2-\varpi^{\frac{1}{2}}\right)] \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have $\Upsilon(\varpi) = [1,2]\left(2 - \varpi^{\frac{1}{2}}\right)$. Since end point mappings $\Upsilon_*(\varpi) = \left(2 - \varpi^{\frac{1}{2}}\right)$, $\Upsilon^*(\varpi) = 2\left(2 - \varpi^{\frac{1}{2}}\right)$ are $\mathfrak{J}$-convex mappings, $\Upsilon(\varpi)$ is an *L-R* $\mathfrak{J}$-convex *I-V·M*. We clearly see that $\Upsilon \in L([\mu,\omega], \mathfrak{R}_C^+)$, and

$$\frac{1}{\alpha\,\mathfrak{J}\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{\mu+\omega}{2}\right) = \Upsilon_*\left(\frac{5}{2}\right) = \frac{4-\sqrt{10}}{8}$$

$$\frac{1}{\alpha\,\mathfrak{J}\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{\mu+\omega}{2}\right) = \Upsilon^*\left(\frac{5}{2}\right) = \frac{4-\sqrt{10}}{4},$$

$$\frac{\Upsilon_*(\mu)+\Upsilon_*(\omega)}{2}\int_0^1 \nu^{\alpha-1}[\mathfrak{J}(\nu) - \mathfrak{J}(1-\nu)]d\nu = \left(4-\sqrt{2}-\sqrt{3}\right)$$

$$\frac{\Upsilon^*(\mu)+\Upsilon^*(\omega)}{2}\int_0^1 \nu^{\alpha-1}[\mathfrak{J}(\nu) - \mathfrak{J}(1-\nu)]d\nu = 2\left(4-\sqrt{2}-\sqrt{3}\right).$$

Note that

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon_*(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon_*(\mu)\right]$$

$$= \frac{\Gamma(\frac{1}{2})}{2}\frac{1}{\sqrt{\pi}}\int_2^3 (3-\varpi)^{\frac{-1}{2}}.\left(2-\varpi^{\frac{1}{2}}\right)d\varpi + \frac{\Gamma(\frac{1}{2})}{2}\frac{1}{\sqrt{\pi}}\int_2^3 (\varpi-2)^{\frac{-1}{2}}.\left(2-\varpi^{\frac{1}{2}}\right)d\varpi$$

$$= \frac{1}{2}\left[\frac{7393}{10{,}000} + \frac{9501}{10{,}000}\right]$$

$$= \frac{8447}{20{,}000}.$$

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon^*(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon^*(\mu)\right]$$

$$= \frac{\Gamma(\frac{1}{2})}{2}\frac{1}{\sqrt{\pi}}\int_2^3 (3-\varpi)^{\frac{-1}{2}}.2\left(2-\varpi^{\frac{1}{2}}\right)d\varpi + \frac{\Gamma(\frac{1}{2})}{2}\frac{1}{\sqrt{\pi}}\int_2^3 (\varpi-2)^{\frac{-1}{2}}.2\left(2-\varpi^{\frac{1}{2}}\right)d\varpi$$

$$= \left[\frac{7393}{10{,}000} + \frac{9501}{10{,}000}\right]$$

$$= \frac{8447}{10{,}000}.$$

Therefore,

$$\left[\frac{4-\sqrt{10}}{8}, \frac{4-\sqrt{10}}{4}\right] \leq_p \left[\frac{8447}{20{,}000}, \frac{8447}{10{,}000}\right] \leq_p \left[\left(4-\sqrt{2}-\sqrt{3}\right), 2\left(4-\sqrt{2}-\sqrt{3}\right)\right],$$

and Theorem 3.1 is illustrated.

We now derive some Riemann-Liouville fractional integral inequalities related to *HH* inequality for a product of two *L-R* $\mathfrak{J}$-convex *I-V·Ms*.

Theorem 3.4. Let $\Upsilon, \varphi : [\mu,\omega] \to \mathfrak{R}_C^+$ be *L-R* $\mathfrak{J}_1$-convex and *L-R* $\mathfrak{J}_2$-convex *I-V·Ms* on $[\mu,\omega]$, respectively, such that $\Upsilon(\varpi) = [\Upsilon_*(\varpi), \Upsilon^*(\varpi)]$ and $\varphi(\varpi) = [\varphi_*(\varpi), \varphi^*(\varpi)]$ for all $\varpi \in [\mu,\omega]$. Then, one has the successive coming relations:

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon(\omega)\times\varphi(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\Upsilon(\mu)\times\varphi(\mu)\right]$$

$$\leq_p \zeta(\mu,\omega)\int_0^1 \imath^{\alpha-1}[\mathfrak{J}_1(\imath)\mathfrak{J}_2(\imath)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)]d\imath$$

$$+\eta(\mu,\omega)\int_0^1 \imath^{\alpha-1}[\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(\imath)]d\imath.$$

where $\zeta(\mu,\omega)=\Upsilon(\mu)\times\varphi(\mu)+\varphi(\omega)\times\varphi(\omega)$, $\eta(\mu,\omega)=\Upsilon(\mu)\times\varphi(\omega)+\Upsilon(\omega)\times\varphi(\mu)$, and $\zeta(\mu,\omega)=[\zeta_*(\mu,\omega),\zeta^*(\mu,\omega)]$ and $\eta(\mu,\omega)=[\eta_*(\mu,\omega),\eta^*(\mu,\omega)]$.

Proof. Since Υ,φ both are *L-R* $\mathfrak{J}_1$-convex and *L-R* $\mathfrak{J}_2$-convex *I-V·Ms*, we have

$$\begin{aligned}\Upsilon_*(\imath\mu+(1-\imath)\omega)\leq\mathfrak{J}_1(\imath)\Upsilon_*(\mu)+\mathfrak{J}_1(1-\imath)\Upsilon_*(\omega)\\ \Upsilon^*(\imath\mu+(1-\imath)\omega)\leq\mathfrak{J}_1(\imath)\Upsilon^*(\mu)+\mathfrak{J}_1(1-\imath)\Upsilon^*(\omega)\end{aligned},$$

and

$$\begin{aligned}\varphi_*(\imath\mu+(1-\imath)\omega)\leq\mathfrak{J}_2(\imath)\varphi_*(\mu)+\mathfrak{J}_2(1-\imath)\varphi_*(\omega)\\ \varphi^*(\imath\mu+(1-\imath)\omega)\leq\mathfrak{J}_2(\imath)\varphi^*(\mu)+\mathfrak{J}_2(1-\imath)\varphi^*(\omega)\end{aligned}.$$

From the definition of an *L-R* $\mathfrak{J}$-convex *I-V·M*, we have

$$\begin{aligned}&\Upsilon_*(\imath\mu+(1-\imath)\omega)\times\varphi_*(\imath\mu+(1-\imath)\omega)\\ &\leq\mathfrak{J}_1(\imath)\mathfrak{J}_2(\imath)\Upsilon_*(\mu)\times\varphi_*(\mu)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)\Upsilon_*(\omega)\times\varphi_*(\omega)\\ &+\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)\Upsilon_*(\mu)\times\varphi_*(\omega)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(\imath)\Upsilon_*(\omega)\times\varphi_*(\mu)\\ &\Upsilon^*(\imath\mu+(1-\imath)\omega)\times\varphi^*(\imath\mu+(1-\imath)\omega)\\ &\leq\mathfrak{J}_1(\imath)\mathfrak{J}_2(\imath)\Upsilon^*(\mu)\times\varphi^*(\mu)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)\Upsilon^*(\omega)\times\varphi^*(\omega)\\ &+\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)\Upsilon^*(\mu)\times\varphi^*(\omega)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(\imath)\Upsilon^*(\omega)\times\varphi^*(\mu).\end{aligned}\tag{24}$$

Analogously, we have

$$\begin{aligned}&\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\varphi_*\big((1-\imath)\mu+\imath\omega\big)\\ &\leq\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)\Upsilon_*(\mu)\times\varphi_*(\mu)+\mathfrak{J}_1(\imath)\mathfrak{J}_2(\imath)\Upsilon_*(\omega)\times\varphi_*(\omega)\\ &+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(\imath)\Upsilon_*(\mu)\times\varphi_*(\omega)+\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)\Upsilon_*(\omega)\times\varphi_*(\mu)\\ &\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\times\varphi^*\big((1-\imath)\mu+\imath\omega\big)\\ &\leq\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)\Upsilon^*(\mu)\times\varphi^*(\mu)+\mathfrak{J}_1(\imath)\mathfrak{J}_2(\imath)\Upsilon^*(\omega)\times\varphi^*(\omega)\\ &+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(\imath)\Upsilon^*(\mu)\times\varphi^*(\omega)+\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)\Upsilon^*(\omega)\times\varphi^*(\mu).\end{aligned}\tag{25}$$

Adding (24) and (25), we have

$$\begin{aligned}&\Upsilon_*(\imath\mu+(1-\imath)\omega)\times\varphi_*(\imath\mu+(1-\imath)\omega)\\ &+\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\times\varphi_*\big((1-\imath)\mu+\imath\omega\big)\\ &\leq[\mathfrak{J}_1(\imath)\mathfrak{J}_2(\imath)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)][\Upsilon_*(\mu)\times\varphi_*(\mu)+\Upsilon_*(\omega)\times\varphi_*(\omega)]\\ &+[\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(\imath)][\Upsilon_*(\omega)\times\varphi_*(\mu)+\Upsilon_*(\mu)\times\varphi_*(\omega)]\\ &\Upsilon^*(\imath\mu+(1-\imath)\omega)\times\varphi^*(\imath\mu+(1-\imath)\omega)\\ &+\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\times\varphi^*\big((1-\imath)\mu+\imath\omega\big)\\ &\leq[\mathfrak{J}_1(\imath)\mathfrak{J}_2(\imath)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)][\Upsilon^*(\mu)\times\varphi^*(\mu)+\Upsilon^*(\omega)\times\varphi^*(\omega)]\\ &+[\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(\imath)][\Upsilon^*(\omega)\times\varphi^*(\mu)+\Upsilon^*(\mu)\times\varphi^*(\omega)].\end{aligned}\tag{26}$$

Taking the multiplication of (26) with $\imath^{\alpha-1}$ and integrating the obtained result with respect to $\imath$ over (0,1), we have

$$\int_0^1 \imath^{\alpha-1}\Upsilon_*(\imath\mu+(1-\imath)\omega)\times\varphi_*(\imath\mu+(1-\imath)\omega)$$
$$+\imath^{\alpha-1}\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\times\varphi_*\big((1-\imath)\mu+\imath\omega\big)d\imath$$
$$\leq \zeta_*(\mu,\omega)\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(\imath)+\Im_1(1-\imath)\Im_2(1-\imath)]\,d\imath$$
$$+\eta_*(\mu,\omega)\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(1-\imath)+\Im_1(1-\imath)\Im_2(\imath)]d\imath$$
$$\int_0^1 \imath^{\alpha-1}\Upsilon^*(\imath\mu+(1-\imath)\omega)\times\varphi^*(\imath\mu+(1-\imath)\omega)$$
$$+\imath^{\alpha-1}\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\times\varphi^*\big((1-\imath)\mu+\imath\omega\big)d\imath$$
$$\leq \zeta^*(\mu,\omega)\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(\imath)+\Im_1(1-\imath)\Im_2(1-\imath)]d\imath$$
$$+\eta^*(\mu,\omega)\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(1-\imath)+\Im_1(1-\imath)\Im_2(\imath)]d\imath.$$

It follows that

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon_*(\omega)\times\varphi_*(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon_*(\mu)\times\varphi_*(\mu)\right]$$
$$\leq \zeta_*(\mu,\omega)\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(\imath)+\Im_1(1-\imath)\Im_2(1-\imath)]d\imath$$
$$+\eta_*(\mu,\omega)\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(1-\imath)+\Im_1(1-\imath)\Im_2(\imath)]d\imath$$
$$\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon^*(\omega)\times\varphi^*(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon^*(\mu)\times\varphi^*(\mu)\right]$$
$$\leq\ \zeta^*\big((\mu,\omega)\big)\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(\imath)+\Im_1(1-\imath)\Im_2(1-\imath)]d\imath$$
$$+\eta^*(\mu,\omega)\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(1-\imath)+\Im_1(1-\imath)\Im_2(\imath)]d\imath.$$

It follows that

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon_*(\omega)\times\varphi_*(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon_*(\mu)\times\varphi_*(\mu),\ \mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon^*(\omega)\times\varphi^*(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon^*(\mu)\times\varphi^*(\mu)\right]$$
$$\leq_p\ [\zeta_*(\mu,\omega),\zeta^*(\mu,\omega)]\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(\imath)+\Im_1(1-\imath)\Im_2(1-\imath)]d\imath$$
$$+[\eta_*(\mu,\omega),\eta^*(\mu,\omega)]\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(1-\imath)+\Im_1(1-\imath)\Im_2(\imath)]d\imath,$$

that is,

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon(\omega)\times\varphi(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\Upsilon(\mu)\times\varphi(\mu)\right]$$
$$\leq_p \zeta(\mu,\omega)\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(\imath)+\Im_1(1-\imath)\Im_2(1-\imath)]d\imath$$
$$+\eta(\mu,\omega)\int_0^1 \imath^{\alpha-1}[\Im_1(\imath)\Im_2(1-\imath)+\Im_1(1-\imath)\Im_2(\imath)]d\imath.$$

Thus, the theorem has been established.

Theorem 3.5. Let $\Upsilon,\varphi:[\mu,\omega]\to\mathfrak{R}_C^+$ be two *L-R* $\Im_1$ -convex and *L-R* $\Im_2$ -convex *I-V·Ms*, respectively, such that $\Upsilon(\varpi)=[\Upsilon_*(\varpi),\Upsilon^*(\varpi)]$ and $\varphi(\varpi)=[\varphi_*(\varpi),\varphi^*(\varpi)]$ for all $\varpi\in[\mu,\omega]$. Then, one has the successive coming relations:

$$\frac{1}{\alpha\,\Im_1\left(\frac{1}{2}\right)\Im_2\left(\frac{1}{2}\right)}\,\Upsilon\left(\frac{\mu+\omega}{2}\right)\times\varphi\left(\frac{\mu+\omega}{2}\right)\leq_p\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon(\omega)\times\varphi(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon(\mu)\times\varphi(\mu)\right]$$
$$+\eta(\mu,\omega)\int_0^1[\imath^{\alpha-1}+(1-\imath)^{\alpha-1}]\Im_1(\imath)\Im_2(1-\imath)d\imath$$
$$+\zeta(\mu,\omega)\int_0^1[\imath^{\alpha-1}+(1-\imath)^{\alpha-1}]\Im_1(1-\imath)\Im_2(1-\imath)d\imath.$$

where $\zeta(\mu,\omega)=\Upsilon(\mu)\times\varphi(\mu)+\Upsilon(\omega)\times\varphi(\omega)$, $\eta(\mu,\omega)=\Upsilon(\mu)\times\varphi(\omega)+\Upsilon(\omega)\times\varphi(\mu)$, and $\zeta(\mu,\omega)=[\zeta_*(\mu,\omega),\zeta^*(\mu,\omega)]$ and $\eta(\mu,\omega)=[\eta_*(\mu,\omega),\eta^*(\mu,\omega)]$.

Proof. Consider that $\Upsilon,\varphi:[\mu,\omega]\to\mathfrak{R}_C^+$ are *L-R* $\mathfrak{J}_1$-convex and *L-R* $\mathfrak{J}_2$-convex *I-V·Ms*. Then, by hypothesis, we have

$$\Upsilon_*\left(\frac{\mu+\omega}{2}\right)\times\varphi_*\left(\frac{\mu+\omega}{2}\right)$$
$$\Upsilon^*\left(\frac{\mu+\omega}{2}\right)\times\varphi^*\left(\frac{\mu+\omega}{2}\right)$$

$$\leq\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\Upsilon_*(\imath\mu+(1-\imath)\omega)\times\varphi_*(\imath\mu+(1-\imath)\omega)\\+\Upsilon_*(\imath\mu+(1-\imath)\omega)\times\varphi_*\big((1-\imath)\mu+\imath\omega\big)\end{bmatrix}$$
$$+\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\times\varphi_*(\imath\mu+(1-\imath)\omega)\\+\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\times\varphi_*\big((1-\imath)\mu+\imath\omega\big)\end{bmatrix}$$
$$\leq\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\Upsilon^*(\imath\mu+(1-\imath)\omega)\times\varphi^*(\imath\mu+(1-\imath)\omega)\\+\Upsilon^*(\imath\mu+(1-\imath)\omega)\times\varphi^*\big((1-\imath)\mu+\imath\omega\big)\end{bmatrix}$$
$$+\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\times\varphi^*(\imath\mu+(1-\imath)\omega)\\+\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\times\varphi^*\big((1-\imath)\mu+\imath\omega\big)\end{bmatrix},$$

$$\leq\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\Upsilon_*(\imath\mu+(1-\imath)\omega)\times\varphi_*(\imath\mu+(1-\imath)\omega)\\+\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\times\varphi_*\big((1-\imath)\mu+\imath\omega\big)\end{bmatrix}$$
$$+\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\big(\mathfrak{J}_1(\imath)\Upsilon_*(\mu)+\mathfrak{J}_1(1-\imath)\Upsilon_*(\omega)\big)\\\times\big(\mathfrak{J}_2(1-\imath)\varphi_*(\mu)+\mathfrak{J}_2(\imath)\varphi_*(\omega)\big)\\+\big(\mathfrak{J}_1(1-\imath)\Upsilon_*(\mu)+\mathfrak{J}_1(\imath)\Upsilon_*(\omega)\big)\\\times\big(\mathfrak{J}_2(\imath)\varphi_*(\mu)+\mathfrak{J}_2(1-\imath)\varphi_*(\omega)\big)\end{bmatrix}$$
$$\leq\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\Upsilon^*(\imath\mu+(1-\imath)\omega)\times\varphi^*(\imath\mu+(1-\imath)\omega)\\+\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\times\varphi^*\big((1-\imath)\mu+\imath\omega\big)\end{bmatrix}$$
$$+\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\big(\mathfrak{J}_1(\imath)\Upsilon^*(\mu)+\mathfrak{J}_1(1-\imath)\Upsilon^*(\omega)\big)\\\times\big(\mathfrak{J}_2(1-\imath)\varphi^*(\mu)+\mathfrak{J}_2(\imath)\varphi^*(\omega)\big)\\+\big(\mathfrak{J}_1(1-\imath)\Upsilon^*(\mu)+\mathfrak{J}_1(\imath)\Upsilon^*(\omega)\big)\\\times\big(\mathfrak{J}_2(\imath)\varphi^*(\mu)+\mathfrak{J}_2(1-\imath)\varphi^*(\omega)\big)\end{bmatrix},$$

$$=\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\Upsilon_*(\imath\mu+(1-\imath)\omega)\times\varphi_*(\imath\mu+(1-\imath)\omega)\\+\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\times\varphi_*\big((1-\imath)\mu+\imath\omega\big)\end{bmatrix}$$
$$+\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\{\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(\imath)\}\eta_*\big((\mu,\omega)\big)\\+\{\mathfrak{J}_1(\imath)\mathfrak{J}_2(\imath)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)\}\zeta_*\big((\mu,\omega)\big)\end{bmatrix}$$
$$=\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\Upsilon^*(\imath\mu+(1-\imath)\omega)\times\varphi^*(\imath\mu+(1-\imath)\omega)\\+\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\times\varphi^*\big((1-\imath)\mu+\imath\omega\big)\end{bmatrix}$$
$$+\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix}\{\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(\imath)\}\eta^*(\mu,\omega)\\+\{\mathfrak{J}_1(\imath)\mathfrak{J}_2(\imath)+\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)\}\zeta^*(\mu,\omega)\end{bmatrix}. \tag{27}$$

Taking the multiplication of (27) with $\imath^{\alpha-1}$ and integrating over $(0,1)$, we get

$$\frac{1}{\alpha\,\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{\mu+\omega}{2}\right)\times\varphi_*\left(\frac{\mu+\omega}{2}\right)$$
$$\le\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon_*(\omega)\times\varphi_*(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon_*(\mu)\times\varphi_*(\mu)\right]$$
$$+\eta_*(\mu,\omega)\int_0^1[\imath^{\alpha-1}+(1-\imath)^{\alpha-1}]\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)d\imath$$
$$+\zeta_*(\mu,\omega)\int_0^1[\imath^{\alpha-1}+(1-\imath)^{\alpha-1}]\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)d\imath$$
$$\frac{1}{\alpha\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{\mu+\omega}{2}\right)\times\varphi^*\left(\frac{\mu+\omega}{2}\right)$$
$$\le\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon^*(\omega)\times\varphi^*(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon^*(\mu)\times\varphi^*(\mu)\right]$$
$$+\eta^*(\mu,\omega)\int_0^1[\imath^{\alpha-1}+(1-\imath)^{\alpha-1}]\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)d\imath$$
$$+\zeta^*(\mu,\omega)\int_0^1[\imath^{\alpha-1}+(1-\imath)^{\alpha-1}]\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)d\imath,$$

It follows that

$$\frac{1}{\alpha\,\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)}\,\Upsilon\left(\frac{\mu+\omega}{2}\right)\times\varphi\left(\frac{\mu+\omega}{2}\right)$$
$$\le_p\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon(\omega)\times\varphi(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon(\mu)\times\varphi(\mu)\right]$$
$$+\eta(\mu,\omega)\int_0^1[\imath^{\alpha-1}+(1-\imath)^{\alpha-1}]\mathfrak{J}_1(\imath)\mathfrak{J}_2(1-\imath)d\imath \qquad +\zeta(\mu,\omega)\int_0^1[\imath^{\alpha-1}+(1-\imath)^{\alpha-1}]\mathfrak{J}_1(1-\imath)\mathfrak{J}_2(1-\imath)d\imath$$

that is, the required result.

Next, we give some further refinements for interval fractional HH type inequalities in terms of the HH-Fejér inequality.

Theorem 3.6. Let $\Upsilon:[\mu,\omega]\to\mathfrak{R}_C^+$ be an *L-R* $\mathfrak{J}$-convex *I-V·M* with $\mu<\omega$, such that $\Upsilon(\varpi)=[\Upsilon_*(\varpi),\Upsilon^*(\varpi)]$ for all $\varpi\in[\mu,\omega]$. If $\mathfrak{B}:[\mu,\omega]\to\mathbb{R},\mathfrak{B}(\varpi)\ge 0$, symmetric with respect to $\frac{\mu+\omega}{2}$, then one has the successive coming relations:

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon\mathfrak{B}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon\mathfrak{B}(\mu)\right]\le_p\big(\Upsilon(\mu)+\Upsilon(\omega)\big)\int_0^1\imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)d\imath. \tag{28}$$

If Υ is a concave *I-V·M*, then inequality (28) is reversed.

Proof. Let Υ be an *L-R* $\mathfrak{J}$-convex *I-V·M* and $\imath^{\alpha-1}\mathfrak{B}(\imath\mu+(1-\imath)\omega)\ge 0$. Then, we have

$$\begin{aligned}&\imath^{\alpha-1}\Upsilon_*(\imath\mu+(1-\imath)\omega)\mathfrak{B}(\imath\mu+(1-\imath)\omega)\\&\le\imath^{\alpha-1}\big(\mathfrak{J}(\imath)\Upsilon_*(\mu)+\mathfrak{J}(1-\imath)\Upsilon_*(\omega)\big)\mathfrak{B}(\imath\mu+(1-\imath)\omega)\\&\imath^{\alpha-1}\Upsilon^*(\imath\mu+(1-\imath)\omega)\mathfrak{B}(\imath\mu+(1-\imath)\omega)\\&\le\imath^{\alpha-1}\big(\mathfrak{J}(\imath)\Upsilon^*(\mu)+\mathfrak{J}(1-\imath)\Upsilon^*(\omega)\big)\mathfrak{B}(\imath\mu+(1-\imath)\omega),\end{aligned}\tag{29}$$

and

$$\begin{aligned}
&\imath^{\alpha-1}\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\\
&\le \imath^{\alpha-1}\big(\mathfrak{J}(1-\imath)\Upsilon_*(\mu)+\mathfrak{J}(\imath)\Upsilon_*(\omega)\big)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\\
&\imath^{\alpha-1}\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\\
&\le \imath^{\alpha-1}\big(\mathfrak{J}(1-\imath)\Upsilon^*(\mu)+\mathfrak{J}(\imath)\Upsilon^*(\omega)\big)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big).
\end{aligned} \tag{30}$$

After adding (29) and (30) and integrating over $[0,1]$, we get

$$\begin{aligned}
&\int_0^1 \imath^{\alpha-1}\Upsilon_*(\imath\mu+(1-\imath)\omega)\mathfrak{V}(\imath\mu+(1-\imath)\omega)\,d\imath\\
&\quad+\int_0^1 \imath^{\alpha-1}\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath\\
&\le \int_0^1 \begin{bmatrix} \imath^{\alpha-1}\Upsilon_*(\mu)\big\{\mathfrak{J}(\imath)\mathfrak{V}(\imath\mu+(1-\imath)\omega)+\mathfrak{J}(1-\imath)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\big\}\\ +\imath^{\alpha-1}\Upsilon_*(\omega)\big\{\mathfrak{J}(1-\imath)\mathfrak{V}(\imath\mu+(1-\imath)\omega)+\mathfrak{J}(\imath)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\big\}\end{bmatrix} d\imath,\\
&= \Upsilon_*(\mu)\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{V}(\imath\mu+(1-\imath)\omega)\,d\imath\\
&\quad+\Upsilon_*(\omega)\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath,\\
&\int_0^1 \imath^{\alpha-1}\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath\\
&\quad+\int_0^1 \imath^{\alpha-1}\Upsilon^*(\imath\mu+(1-\imath)\omega)\mathfrak{V}(\imath\mu+(1-\imath)\omega)\,d\imath\\
&\le \int_0^1 \begin{bmatrix} \imath^{\alpha-1}\Upsilon^*(\mu)\big\{\mathfrak{J}(\imath)\mathfrak{V}(\imath\mu+(1-\imath)\omega)+\mathfrak{J}(1-\imath)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\big\}\\ +\imath^{\alpha-1}\Upsilon^*(\omega)\big\{\mathfrak{J}(1-\imath)\mathfrak{V}(\imath\mu+(1-\imath)\omega)+\mathfrak{J}(\imath)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\big\}\end{bmatrix} d\imath,\\
&= \Upsilon^*(\mu)\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{V}(\imath\mu+(1-\imath)\omega)\,d\imath\\
&\quad+\Upsilon^*(\omega)\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath.
\end{aligned} \tag{31}$$

Taking the right hand side of inequality (31), we have

$$\begin{aligned}
&\int_0^1 \imath^{\alpha-1}\Upsilon_*(\imath\mu+(1-\imath)\omega)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath\\
&\quad+\int_0^1 \imath^{\alpha-1}\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath\\
&= \frac{1}{(\omega-\mu)^\alpha}\int_\mu^\omega(\varpi-\mu)^{\alpha-1}\Upsilon_*(\mu+\omega-\varpi)\mathfrak{V}(\varpi)d\varpi\\
&\quad+\frac{1}{(\omega-\mu)^\alpha}\int_\mu^\omega(\varpi-\mu)^{\alpha-1}\Upsilon_*(\varpi)\mathfrak{V}(\varpi)d\varpi\\
&= \frac{1}{(\omega-\mu)^\alpha}\int_\mu^\omega(\omega-\varpi)^{\alpha-1}\Upsilon_*(\varpi)\mathfrak{V}(\mu+\omega-\varpi)d\varpi\\
&\quad+\frac{1}{(\omega-\mu)^\alpha}\int_\mu^\omega(\varpi-\mu)^{\alpha-1}\Upsilon_*(\varpi)\mathfrak{V}(\varpi)d\varpi\\
&= \frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\big[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon_*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon_*\mathfrak{V}(\mu)\big],\\
&\int_0^1 \imath^{\alpha-1}\Upsilon^*(\imath\mu+(1-\imath)\omega)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath\\
&\quad+\int_0^1 \imath^{\alpha-1}\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath\\
&= \frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\big[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon^*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon^*\mathfrak{V}(\mu)\big].
\end{aligned} \tag{32}$$

From (32), we have

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\, \Upsilon_*\mathfrak{B}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\, \Upsilon_*\mathfrak{B}(\mu)\right]$$
$$\leq \left(\Upsilon_*(\mu)+\Upsilon_*(\omega)\right)\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)$$
$$\frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\, \Upsilon^*\mathfrak{B}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\, \Upsilon^*\mathfrak{B}(\mu)\right]$$
$$\leq \left(\Upsilon^*(\mu)+\Upsilon^*(\omega)\right)\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big),$$

that is,

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\, \Upsilon_*\mathfrak{B}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\, \Upsilon_*\mathfrak{B}(\mu),\mathcal{J}_{\mu^+}^{\alpha}\, \Upsilon^*\mathfrak{B}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\, \Upsilon^*\mathfrak{B}(\mu)\right] \leq_p [\Upsilon_*(\mu)+\Upsilon_*(\omega),\Upsilon^*(\mu)+\Upsilon^*(\omega)]\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)d\imath.$$

Hence,

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\, \Upsilon\mathfrak{B}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\, \Upsilon\mathfrak{B}(\mu)\right] \leq_p \big(\Upsilon(\mu)+\Upsilon(\omega)\big)\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)d\imath.$$

Theorem 3.7. Let $\Upsilon:[\mu,\omega]\to\mathfrak{R}_C^+$ be an *L-R* $\mathfrak{J}$-convex *I-V·M* with $\mu<\omega$ such that $\Upsilon(\varpi)=[\Upsilon_*(\varpi),\Upsilon^*(\varpi)]$ for all $\varpi\in[\mu,\omega]$. Let $\Upsilon\in L([\mu,\omega],\mathfrak{R}_C^+)$ and $\mathfrak{B}:[\mu,\omega]\to\mathbb{R}, \mathfrak{B}(\varpi)\geq 0$, symmetric with respect to $\frac{\mu+\omega}{2}$. Then, one has the successive coming relations:

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\Upsilon\left(\frac{\mu+\omega}{2}\right)\left[\mathcal{J}_{\mu^+}^{\alpha}\, \mathfrak{B}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\, \mathfrak{B}(\mu)\right]\leq_p\left[\mathcal{J}_{\mu^+}^{\alpha}\, \Upsilon\mathfrak{B}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\, \Upsilon\mathfrak{B}(\mu)\right]. \tag{33}$$

If Υ is a concave *I-V·M*, then inequality (33) is reversed.

Proof. Since Υ is an *L-R* $\mathfrak{J}$-convex *I-V·M*, we have

$$\begin{aligned}\Upsilon_*\left(\frac{\mu+\omega}{2}\right)&\leq\mathfrak{J}\left(\frac{1}{2}\right)\Big(\Upsilon_*(\imath\mu+(1-\imath)\omega)+\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\Big)\\ \Upsilon^*\left(\frac{\mu+\omega}{2}\right)&\leq\mathfrak{J}\left(\frac{1}{2}\right)\Big(\Upsilon^*(\imath\mu+(1-\imath)\omega)+\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\Big).\end{aligned} \tag{34}$$

Since $\mathfrak{B}(\imath\mu+(1-\imath)\omega)=\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)$, by multiplying (34) by $\imath^{\alpha-1}\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)$ and integrating it with respect to $\imath$ over $[0,1]$, we obtain

$$\begin{aligned}&\Upsilon_*\left(\frac{\mu+\omega}{2}\right)\int_0^1 \imath^{\alpha-1}\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)d\imath\\ &\leq\mathfrak{J}\left(\frac{1}{2}\right)\begin{pmatrix}\int_0^1 \imath^{\alpha-1}\Upsilon_*(\imath\mu+(1-\imath)\omega)\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)d\imath\\ +\int_0^1 \imath^{\alpha-1}\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)d\imath\end{pmatrix},\\ &\Upsilon^*\left(\frac{\mu+\omega}{2}\right)\int_0^1 \imath^{\alpha-1}\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)d\imath\\ &\leq\mathfrak{J}\left(\frac{1}{2}\right)\begin{pmatrix}\int_0^1 \imath^{\alpha-1}\Upsilon^*(\imath\mu+(1-\imath)\omega)\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)d\imath\\ +\int_0^1 \imath^{\alpha-1}\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\mathfrak{B}\big((1-\imath)\mu+\imath\omega\big)d\imath\end{pmatrix}.\end{aligned} \tag{35}$$

Let $\varpi=(1-\imath)\mu+\imath\omega$. Then, from the right hand side of inequality (35), we have

$$\begin{aligned}
&\int_0^1 \imath^{\alpha-1}\Upsilon_*(\imath\mu+(1-\imath)\omega)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath \\
&+\int_0^1 \imath^{\alpha-1}\Upsilon_*\big((1-\imath)\mu+\imath\omega\big)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath \\
&=\frac{1}{(\omega-\mu)^\alpha}\int_\mu^\omega(\varpi-\mu)^{\alpha-1}\Upsilon_*(\mu+\omega-\varpi)\mathfrak{V}(\varpi)d\varpi \\
&\quad+\frac{1}{(\omega-\mu)^\alpha}\int_\mu^\omega(\varpi-\mu)^{\alpha-1}\Upsilon_*(\varpi)\mathfrak{V}(\varpi)d\varpi \\
&=\frac{1}{(\omega-\mu)^\alpha}\int_\mu^\omega(\varpi-\mu)^{\alpha-1}\Upsilon_*(\varpi)\mathfrak{V}(\mu-\omega-\varpi)d\varpi \\
&\quad+\frac{1}{(\omega-\mu)^\alpha}\int_\mu^\omega(\varpi-\mu)^{\alpha-1}\Upsilon_*(\varpi)\mathfrak{V}(\varpi)d\varpi \\
&=\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon_*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon_*\mathfrak{V}(\mu)\right], \\
&\int_0^1 \imath^{\alpha-1}\Upsilon^*(\imath\mu+(1-\imath)\omega)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath \\
&+\int_0^1 \imath^{\alpha-1}\Upsilon^*\big((1-\imath)\mu+\imath\omega\big)\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big)\,d\imath \\
&=\frac{\Gamma(\alpha)}{(\omega-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon^*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon^*\mathfrak{V}(\mu)\right].
\end{aligned} \tag{36}$$

Then, from (36), we have

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{\mu+\omega}{2}\right)\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\mathfrak{V}(\mu)\right]\le\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon_*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon_*\mathfrak{V}(\mu)\right]$$
$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{\mu+\omega}{2}\right)\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\mathfrak{V}(\mu)\right]\le\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon^*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon^*\mathfrak{V}(\mu)\right],$$

from which we have

$$\begin{aligned}
&\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\left[\Upsilon_*\left(\frac{\mu+\omega}{2}\right),\Upsilon^*\left(\frac{\mu+\omega}{2}\right)\right]\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\mathfrak{V}(\mu)\right] \\
&\quad\le_p\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon_*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon_*\mathfrak{V}(\mu),\ \mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon^*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon^*\mathfrak{V}(\mu)\right],
\end{aligned}$$

and it follows that

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\Upsilon\left(\frac{\mu+\omega}{2}\right)\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\mathfrak{V}(\mu)\right]\le_p\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon\mathfrak{V}(\mu)\right].$$

This completes the proof.

Remark 3.8. If $\mathfrak{V}(\varpi)=1$, then by combining (28) and (33), we get Theorem 3.1.
If $\mathfrak{J}(\imath)=\imath$, then by combining (25) and (30), we get the following inequality (see [35]):

$$\begin{aligned}
\Upsilon\left(\frac{\mu+\omega}{2}\right)\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\mathfrak{V}(\mu)\right]&\le_p\left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon\mathfrak{V}(\mu)\right] \\
&\le_p\frac{\Upsilon(\mu)+\Upsilon(\omega)}{2}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\,\mathfrak{V}(\mu)\right].
\end{aligned} \tag{37}$$

Let $\mathfrak{J}(\imath)=\imath$ and $\alpha=1$. Then, from (25) and (30), we obtain the following inequality for an *L*-*R* convex *I*-*V*·*M* (see [35]).

$$\Upsilon\left(\frac{\mu+\omega}{2}\right)\le_p\frac{1}{\int_\mu^\omega\mathfrak{V}(\varpi)d\varpi}\,(IR)\int_\mu^\omega\Upsilon(\varpi)\mathfrak{V}(\varpi)d\varpi\le_p\frac{\Upsilon(\mu)+\Upsilon(\omega)}{2}. \tag{38}$$

Let $\mathfrak{J}(\imath)=\imath$ and $\alpha=1=\mathfrak{V}(\varpi)$. Then, from (25) and (30), we obtain the following (see [35,50]):

$$\Upsilon\left(\frac{\mu+\omega}{2}\right) \leq_p (IR)\int_\mu^\omega \Upsilon(\varpi)d\varpi \leq_p \frac{\Upsilon(\mu)+\Upsilon(\omega)}{2}. \quad (39)$$

Let $\Upsilon_*(x)$ bea $\mathfrak{J}$-affine mapping and $\Upsilon^*(x)$ be a $\mathfrak{J}$-concave mapping. If $\Upsilon_*(x) \neq \Upsilon^*(x)$, then from Remark 2.10 and inequality (13), we acquire the fractional HH-Fejér inequality, which was achieved by Kalsoom et al. (see [49]).
Let $\Upsilon_*(x)$ be a $\mathfrak{J}$-affine mapping and $\Upsilon^*(x)$ be a $\mathfrak{J}$-concave mapping with $\mathfrak{J}(\imath) = \imath$ and $\alpha = 1$. If $\Upsilon_*(x) \neq \Upsilon^*(x)$, then from Remark 2.10, inequality (25) and (30), we acquire the coming inequality, which was achieved by Kalsoom et al. (see [49]):

$$\Upsilon\left(\frac{\mu+\omega}{2}\right)\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{V}(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\mathfrak{V}(\mu)\right] \supseteq \left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon\mathfrak{V}(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon\mathfrak{V}(\mu)\right]$$
$$\supseteq \frac{\Upsilon(\mu)+\Upsilon(\omega)}{2}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{V}(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\mathfrak{V}(\mu)\right]. \quad (40)$$

If $\Upsilon_*(\varpi) = \Upsilon^*(\varpi)$ and $\mathfrak{J}(\imath) = \imath$, then from (25) and (30), the following classical inequality can be obtained:

$$\Upsilon\left(\frac{\mu+\omega}{2}\right)\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{V}(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\mathfrak{V}(\mu)\right] \leq \left[\mathcal{J}_{\mu^+}^{\alpha}\,\Upsilon\mathfrak{V}(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\Upsilon\mathfrak{V}(\mu)\right]$$
$$\leq \frac{\Upsilon(\mu)+\Upsilon(\omega)}{2}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{V}(\omega) + \mathcal{J}_{\omega^-}^{\alpha}\,\mathfrak{V}(\mu)\right]. \quad (41)$$

If $\Upsilon_*(\varpi) = \Upsilon^*(\varpi)$, $\alpha = 1$, and $\mathfrak{J}(\imath) = \imath$, then from (25) and (30), we obtain the classical HH-Fejér inequality.
If $\Upsilon_*(\varpi) = \Upsilon^*(\varpi)$, $\mathfrak{V}(\varpi) = \alpha = 1$, and $\mathfrak{J}(\imath) = \imath$, then from (25) and (30), we get the classical HH-inequality.
Example 3.9. We consider the *I-V·M* $\Upsilon: [0,2] \to \mathfrak{R}_C^+$ defined by $\Upsilon(\varpi) = [1,2]\left(2-\sqrt{\varpi}\right)$. Since end point mappings $\Upsilon_*(\varpi), \Upsilon^*(\varpi)$ are *L-R* $\mathfrak{J}$-convex mappings, $\Upsilon(\varpi)$ is an *L-R* $\mathfrak{J}$-convex *I-V·M*. If

$$\mathfrak{V}(\varpi) = \begin{cases} \sqrt{\varpi}, & \sigma \in [0,1], \\ \sqrt{2-\varpi}, & \sigma \in (1,2], \end{cases}$$

then $\mathfrak{V}(2-\varpi) = \mathfrak{V}(\varpi) \geq 0$, for all $\varpi \in [0,2]$. Since $\Upsilon_*(\varpi) = \left(2-\sqrt{\varpi}\right)$ and $\Upsilon^*(\varpi) = 2\left(2-\sqrt{\varpi}\right)$. If $\mathfrak{J}(\imath) = \imath$ and $\alpha = \frac{1}{2}$, then we compute the following:

$$\begin{aligned} \frac{\Upsilon_*(\mu)+\Upsilon_*(\omega)}{2}\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big) &= \frac{\pi}{\sqrt{2}}\left(\frac{4-\sqrt{2}}{2}\right), \\ \frac{\Upsilon^*(\mu)+\Upsilon^*(\omega)}{2}\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big) &= \frac{\pi}{\sqrt{2}}\left(4-\sqrt{2}\right), \end{aligned} \quad (42)$$

$$\begin{aligned} \frac{\Upsilon_*(\mu)+\Upsilon_*(\omega)}{2}\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big) &= \frac{\pi}{\sqrt{2}}\left(\frac{4-\sqrt{2}}{2}\right), \\ \frac{\Upsilon^*(\mu)+\Upsilon^*(\omega)}{2}\int_0^1 \imath^{\alpha-1}[\mathfrak{J}(\imath)+\mathfrak{J}(1-\imath)]\mathfrak{V}\big((1-\imath)\mu+\imath\omega\big) &= \frac{\pi}{\sqrt{2}}\left(4-\sqrt{2}\right), \end{aligned}$$

$$\frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\ \Upsilon_*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\ \Upsilon_*\mathfrak{V}(\mu)\right]=\frac{1}{\sqrt{\pi}}\left(2\pi+\frac{4-8\sqrt{2}}{3}\right),$$
$$\frac{\Gamma(\alpha)}{(\omega-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\ \Upsilon^*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\ \Upsilon^*\mathfrak{V}(\mu)\right]=\frac{2}{\sqrt{\pi}}\left(2\pi+\frac{4-8\sqrt{2}}{3}\right). \tag{43}$$

From (42) and (43), we have

$$\frac{1}{\sqrt{\pi}}\left[\left(2\pi+\frac{4-8\sqrt{2}}{3}\right),2\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)\right]\leq_p\frac{\pi}{\sqrt{2}}\left[\frac{4-\sqrt{2}}{2},4-\sqrt{2}\right].$$

Hence, Theorem 3.6 is verified.
For Theorem 3.7, we have

$$\begin{aligned}&\mathcal{J}_{\mu^+}^{\alpha}\ \Upsilon_*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\ \Upsilon_*\mathfrak{V}(\mu)\\&=\frac{1}{\sqrt{\pi}}\int_0^2(2-\varpi)^{\frac{-1}{2}}\mathfrak{V}(\varpi)\left(2-\sqrt{\varpi}\right)d\varpi+\frac{1}{\sqrt{\pi}}\int_0^2(\varpi)^{\frac{-1}{2}}\ \mathfrak{V}(\varpi)\left(2-\sqrt{\varpi}\right)d\varpi\\&=\frac{1}{\sqrt{\pi}}\left(\pi+\frac{8-8\sqrt{2}}{3}\right)+\frac{1}{\sqrt{\pi}}\left(\pi-\frac{4}{3}\right)=\frac{1}{\sqrt{\pi}}\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)\\&\mathcal{J}_{\mu^+}^{\alpha}\ \Upsilon^*\mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\ \Upsilon^*\mathfrak{V}(\mu)\\&=\frac{2}{\sqrt{\pi}}\int_0^2(2-\varpi)^{\frac{-1}{2}}\mathfrak{V}(\varpi)\left(2-\sqrt{\varpi}\right)d\varpi+\frac{2}{\sqrt{\pi}}\int_0^2(\varpi)^{\frac{-1}{2}}\ \mathfrak{V}(\varpi)\left(2-\sqrt{\varpi}\right)d\varpi\\&=\frac{2}{\sqrt{\pi}}\left(\pi+\frac{8-8\sqrt{2}}{3}\right)+\frac{2}{\sqrt{\pi}}\left(\pi-\frac{4}{3}\right)=\frac{2}{\sqrt{\pi}}\left(2\pi+\frac{4-8\sqrt{2}}{3}\right).\end{aligned} \tag{44}$$

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{\mu+\omega}{2}\right)\left[\mathcal{J}_{\mu^+}^{\alpha}\ \mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\ \mathfrak{V}(\mu)\right]=\sqrt{\pi},$$
$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{\mu+\omega}{2}\right)\left[\mathcal{J}_{\mu^+}^{\alpha}\ \mathfrak{V}(\omega)+\mathcal{J}_{\omega^-}^{\alpha}\ \mathfrak{V}(\mu)\right]=2\sqrt{\pi}. \tag{45}$$

From (44) and (45), we have

$$\sqrt{\pi}[1,2]\leq_p\frac{1}{\sqrt{\pi}}\left[2\pi+\frac{4-8\sqrt{2}}{3},2\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)\right].$$

4. Conclusions and future plan

We introduced the class of left and right $\mathfrak{J}$-convex interval-valued mappings by means of pseudo-order relation and investigated some properties. Some novel inequalities for left and right $\mathfrak{J}$-convex interval-valued mappings were proved**.** The results of this study can be applied in optimization, uncertainty analysis and also different areas of applied and pure sciences. We intend to use various types of left and right convex interval-valued mappings to construct interval inequalities of interval-valued mappings by means of pseudo-order relations and the Riemann-Liouville fractional integral operator.

Conflicts of interest

The authors declare no conflict of interest.

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