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Research article

New classes of unified fractional integral inequalities

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Abstract: Many researchers in recent years have studied fractional integrals and derivatives. Some authors recently introduced generalized fractional integrals, the so-called unified fractional integrals. In this article, we establish certain new integral inequalities by employing the unified fractional integral operators. In fact, for a class of n ($n \in \mathbb{N}$), positive continuous and decreasing functions on $[v_1, v_2]$, certain new classes of integral inequalities are discussed. The inequalities established in this manuscript are more general forms of the classical inequalities given in the literature. The existing classical inequalities can be rectified by imposing the conditions stated in remarks. By imposing certain conditions on $\hbar$ and Λ available in the literature, many new forms of fractional integral inequalities can be produced.

Keywords: fractional integrals; fractional proportional integrals; fractional integral inequalities
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1. Introduction

Fractional calculus is presently primarily concerned with studying fractional-order integral and derivative functions over real and complex domains and their applications. Using arithmetic from classical analysis in the fractional analysis is critical in many cases for producing more realistic findings. Fractional order differential equations can handle a wide variety of mathematical models. Fractional mathematical models provide more comprehensive and accurate results than classical mathematical models because they are particular cases of fractional-order mathematical models. In classical analysis, integer orders are not a good model for nature. On the other hand, fractional computing allows us to look at any number of orders and come up with significantly more concrete objectives.

In [1–5], some researchers defined new fractional derivative operators by using exponential and Mittag-Leffler functions in the kernels. Such developments encourage further study into new concepts for combining fractional derivative and integral operators and obtaining fractional integral inequalities using these modified fractional derivatives and integral operators. In the theory of differential equations and applied mathematics, integral inequalities and their applications are crucial. Using the classical fractional integral, fractional derivative operators and their extensions, many different forms of classical integral inequalities and their modifications have been created [6–13].

Sarikaya and Budak investigated the (k, s)-Riemann-Liouville fractional integral and its applications in [14]. In [15], enlarged Hermite-Hadamard type inequalities are discovered using fractional integral operators. Agarwal et al. [16] used the k-fractional integrals operators to introduce Hermite-Hadamard type inequalities.

Using a family of n positive functions, Dahmani, in [17], presented certain classes of fractional integral inequalities. Using the (k, s)-fractional integral operators, the authors of [18] constructed fractional integral inequalities for a class of n ($n \in \mathbb{N}$), positive continuous and decreasing functions on $[a, b]$.

Using fractional conformable integrals, the authors [19–22] recently developed numerous forms of inequalities. In [29], Akin studied the boundedness and compactness of integral operators on time scales. Akin [30] established fractional maximal integrals to establish integral inequalities on time scales. New principles of non-linear integral inequalities are presented in time scales via diamond-α dynamic integrals and the nabla integral in [31]. Younus et al. [32] gave some new variants of Gronwall type inequalities on time scales. An interesting application of fractional integrals and differentials can be found in the works [33–35].

Definition 1.1. *[23] Let* $\Lambda : [0, \infty) \to [0, \infty)$ *be the function satisfying the hypothesis given below:*

$$\int_0^1 \frac{\Lambda(\upsilon)}{\upsilon} d\upsilon < \infty, \tag{1.1}$$

$$\frac{1}{K} \leq \frac{\Lambda(\hbar_1)}{\Lambda(\hbar_2)} \leq K, \quad \frac{1}{2} \leq \frac{\hbar_1}{\hbar_2} \leq 2, \tag{1.2}$$

$$\frac{\Lambda(\hbar_2)}{\hbar_2^2} \leq L \frac{\Lambda(\hbar_1)}{\hbar_1^2}, \quad \hbar_1 \leq \hbar_2, \tag{1.3}$$

$$|\frac{\Lambda(\hbar_2)}{\hbar_2^2} - \frac{\Lambda(\hbar_1)}{\hbar_1^2}| \leq M|\hbar_2 - \hbar_1| \frac{\Lambda(\hbar_2)}{\hbar_2^2}, \quad \frac{1}{2} \leq \frac{\hbar_1}{\hbar_2} \leq 2, \tag{1.4}$$

where $K, L, M > 0$ *and are independent of* $\hbar_1, \hbar_2 > 0$. *If* $\Lambda(\hbar_2)\hbar_2^\sigma$ *is increasing for some* $\sigma > 0$, *and* $\frac{\Lambda(\hbar_2)}{\hbar_2^\mu}$ *is decreasing for some* $\mu > 0$, *then* Λ *satisfies* (1.1)–(1.4).

Definition 1.2. *[25] Let the function* $\hbar$ *be differentiable and strictly increasing on* $[v_1, v_2]$ *and let the weighted function* $\omega(\theta) \neq 0$ *be defined on* $[v_1, v_2]$. *Let* $X_\omega^p(v_1, v_2)$, $1 \leq p \leq \infty$, *be the space of all Lebesgue measurable functions defined on* $[v_1, v_2]$ *for which* $\|\aleph\|_{X_\omega^p} < \infty$ *is*

$$\|\aleph\|_{X_\omega^p} = \left(\int_{v_1}^{v_2} | (\omega(\theta)\aleph(\theta) |)^p \hbar'(\theta) d\theta \right)^{\frac{1}{p}}, 1 \leq p < \infty,$$

and

$$\|\aleph|_{X_\omega^\infty} = ess \sup_{v_1 \le \theta \le v_2} | \omega(\theta)\aleph(\theta) | < \infty.$$

Note that $\aleph \in X_\omega^p(v_1, v_2) \leftrightarrow \omega(\theta)\aleph(\theta)(\hbar'(\theta))^{\frac{1}{p}} \in L_P(v_1, v_2)$ *for* $1 \le p < \infty$, *and* $\aleph \in X_\omega^\infty(v_1, v_2) \leftrightarrow \omega(\theta)\aleph(\theta) \in L_\infty(v_1, v_2)$.

Definition 1.3. *[24] The unified weighted left and right sided integral operators are respectively given below:*

$$\left({}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\aleph\right)(\lambda) = \omega^{-1}(\lambda) \int_{v_1}^{\lambda} \frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)} \omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad v_1 < \lambda, \tag{1.5}$$

and

$$\left({}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_2-}\aleph\right)(\lambda) = \omega^{-1}(\lambda) \int_{\lambda}^{v_2} \frac{\Lambda\left(\hbar(\upsilon) - \hbar(\lambda)\right)}{\hbar(\upsilon) - \hbar(\lambda)} \omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad v_2 > \lambda. \tag{1.6}$$

Remark 1.1. *Here, we discuss the following special cases of* (1.5) *and* (1.6) *by applying some specific conditions on* $\hbar$ *and* Λ.

(i) If we take $\Lambda(\hbar(\lambda)) = \hbar(\lambda)$, *then unified weighted integrals* (1.5) *and* (1.6) *will become*

$$\left({}^{\hbar}_{\omega}\mathcal{I}_{v_1+}\aleph\right)(\lambda) = \omega^{-1}(\lambda) \int_{v_1}^{\lambda} \omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad v_1 < \lambda,$$

and

$$\left({}^{\hbar}_{\omega}\mathcal{I}_{v_2-}\aleph\right)(\lambda) = \omega^{-1}(\lambda) \int_{\lambda}^{v_2} \omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad v_2 > \lambda,$$

respectively.

(ii) If we take $\hbar(\lambda) = \lambda$, *then the unified integrals* (1.5) *and* (1.6) *will become*

$$\left({}^{\hbar}_{\omega}\mathcal{I}_{v_1+}\aleph\right)(\lambda) = \omega^{-1}(\lambda) \int_{v_1}^{\lambda} \frac{\Lambda(\lambda - \upsilon)}{\lambda - \upsilon} \omega(\upsilon)\aleph(\upsilon)d\upsilon, \quad v_1 < \lambda, \tag{1.7}$$

and

$$\left({}^{\hbar}_{\omega}\mathcal{I}_{v_2-}\aleph\right)(\lambda) = \omega^{-1}(\lambda) \int_{\lambda}^{v_2} \frac{\Lambda(\upsilon - \lambda)}{\upsilon - \lambda} \omega(\upsilon)\aleph(\upsilon)d\upsilon, \quad v_2 > \lambda, \tag{1.8}$$

respectively.

(iii) If we take $\Lambda(\hbar(\lambda)) = \frac{\hbar(\lambda)^\zeta}{\Gamma(\zeta)}$, *then* (1.5) *and* (1.6) *will lead to the following generalized Riemann-Liouville fractional integrals, respectively, as defined by [25].*

$$\left({}^{\hbar}_{\omega}\mathcal{I}^{\zeta}_{v_1}\aleph\right)(\lambda) = \frac{\omega^{-1}(\lambda)}{\Gamma(\zeta)} \int_{v_1}^{\lambda} \left(\hbar(\lambda) - \hbar(\upsilon)\right)^{\zeta-1} \omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad v_1 < \lambda, \tag{1.9}$$

and

$$\left({}^{\hbar}_{\omega}\mathcal{I}^{\zeta}_{v_2-}\aleph\right)(\lambda) = \frac{\omega^{-1}(\lambda)}{\Gamma(\zeta)} \int_{\lambda}^{v_2} \left(\hbar(\upsilon) - \hbar(\lambda)\right)^{\zeta-1} \omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad v_2 > \lambda, \tag{1.10}$$

where $\zeta, \in \mathbb{C}$ *with* $\Re(\zeta) > 0$.

(iv) If we take $\hbar(\lambda) = \lambda$ *and* $\Lambda(\hbar(\lambda)) = \frac{\lambda^{\zeta}}{\Gamma(\zeta)}$, *then* (1.5) *and* (1.6) *reduce to the given weighted Riemann-Liouville fractional integrals, respectively:*

$$\left({}_{\omega}\mathcal{I}^{\zeta}_{\nu_1+}\aleph\right)(\lambda) = \frac{\omega^{-1}(\lambda)}{\Gamma(\zeta)}\int_{\nu_1}^{\lambda}(\lambda-\upsilon)^{\zeta-1}\omega(\upsilon)\aleph(\upsilon)d\upsilon, \quad \nu_1 < \lambda, \tag{1.11}$$

and

$$\left({}_{\omega}\mathcal{I}^{\zeta}_{\nu_2-}\aleph\right)(\lambda) = \frac{\omega^{-1}(\lambda)}{\Gamma(\zeta)}\int_{\lambda}^{\nu_2}(\upsilon-\lambda)^{\zeta-1}\omega(\upsilon)\aleph(\upsilon)d\upsilon, \quad \nu_2 > \lambda. \tag{1.12}$$

(v) If we take $\hbar(\lambda) = \ln\lambda$ *and* $\Lambda(\hbar(\lambda)) = \frac{(\ln\lambda)^{\zeta}}{\Gamma(\zeta)}$, *then* (1.5) *and* (1.6) *will lead to the weighted Hadamard integrals given below:*

$$\left({}_{\omega}\mathcal{I}^{\zeta}_{\nu_1+}\aleph\right)(\lambda) = \frac{\omega^{-1}(\lambda)}{\Gamma(\zeta)}\int_{\nu_1}^{\lambda}(\ln\lambda-\ln\upsilon)^{\zeta-1}\omega(\upsilon)\aleph(\upsilon)\frac{d\upsilon}{\upsilon}, \quad \nu_1 < \lambda, \tag{1.13}$$

and

$$\left({}_{\omega}\mathcal{I}^{\zeta}_{\nu_2-}\aleph\right)(\lambda) = \frac{\omega^{-1}(\lambda)}{\Gamma(\zeta)}\int_{\lambda}^{\nu_2}(\ln\upsilon-\ln\lambda)^{\zeta-1}\omega(\upsilon)\aleph(\upsilon)\frac{d\upsilon}{\upsilon}, \quad \nu_2 > \lambda. \tag{1.14}$$

(vi) If we take $\hbar(\lambda) = \lambda^{\eta}$ *and* $\Lambda(\hbar(\lambda)) = \frac{\lambda^{\eta}}{\eta}$, $\eta > 0$, *then* (1.5) *and* (1.6) *will become weighted Katugampola fractional integrals as follows:*

$$\left({}_{\omega}\mathcal{I}^{\zeta}_{\nu_1}\aleph\right)(\lambda) = \frac{\omega^{-1}(\lambda)}{\Gamma(\zeta)}\int_{\nu_1}^{\lambda}\left(\frac{\lambda^{\eta}-\upsilon^{\eta}}{\eta}\right)^{\zeta-1}\omega(\upsilon)\aleph(\upsilon)\frac{d\upsilon}{\upsilon^{1-\eta}}, \quad \nu_1 < \lambda, \tag{1.15}$$

and

$$\left({}_{\omega}\mathcal{I}^{\zeta}_{\nu_2}\aleph\right)(\lambda) = \frac{\omega^{-1}(\lambda)}{\Gamma(\zeta)}\int_{\lambda}^{\nu_2}\left(\frac{\upsilon^{\eta}-\lambda^{\eta}}{\eta}\right)^{\zeta-1}\omega(\upsilon)\aleph(\upsilon)\frac{d\upsilon}{\upsilon^{1-\eta}}, \quad \nu_2 > \lambda. \tag{1.16}$$

(vii) If we take $\hbar(\lambda) = \lambda$ *and* $\Lambda(\hbar(\lambda)) = \frac{\lambda}{\eta}\exp\left(-\frac{1-\eta}{\eta}\lambda\right)$, $\eta \in (0,1)$, *then* (1.5) *and* (1.6) *will reduce to the weighted fractional integrals given by*

$$\left({}_{\omega}\mathcal{I}^{\eta}_{\nu_1+}\aleph\right)(\lambda) = \frac{\omega^{-1}(\lambda)}{\eta}\int_{\nu_1}^{\lambda}\exp\left(-\frac{1-\eta}{\eta}(\lambda-\upsilon)\right)\omega(\upsilon)\aleph(\upsilon), \quad \nu_1 < \lambda, \tag{1.17}$$

and

$$\left({}_{\omega}\mathcal{I}^{\eta}_{\nu_2-}\aleph\right)(\lambda) = \frac{\omega^{-1}(\lambda)}{\eta}\int_{\lambda}^{\nu_2}\exp\left(-\frac{1-\eta}{\eta}(\upsilon-\lambda)\right)\omega(\upsilon)\aleph(\upsilon)d\upsilon, \quad \nu_2 > \lambda. \tag{1.18}$$

Remark 1.2. *(i) If we take* $\omega(\lambda) = 1$ *and* $\Lambda(\hbar(\lambda)) = \hbar(\lambda)$, *then* (1.5) *and* (1.6) *will become*

$$\left({}^{\hbar}\mathcal{I}_{\nu_1+}\aleph\right)(\lambda) = \int_{\nu_1}^{\lambda}\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad \nu_1 < \lambda,$$

and

$$\left({}^{\hbar}\mathcal{I}_{\nu_2-}\aleph\right)(\lambda) = \int_{\lambda}^{\nu_2}\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad \nu_2 > \lambda,$$

respectively.

(ii) If we take $\omega(\lambda) = 1$*, then* (1.5) *and* (1.6) *will lead to the unified integrals defined by [26] as follows:*

$$\left({}^{\hbar}\mathcal{I}^{\Lambda}_{\nu_1+}\aleph\right)(\lambda) = \int_{\nu_1}^{\lambda} \frac{\Lambda\left(\hbar(\lambda)-\hbar(\upsilon)\right)}{\hbar(\lambda)-\hbar(\upsilon)}\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad \nu_1 < \lambda, \tag{1.19}$$

and

$$\left({}^{\hbar}\mathcal{I}^{\Lambda}_{\nu_2-}\aleph\right)(\lambda) = \int_{\lambda}^{\nu_2} \frac{\Lambda\left(\hbar(\upsilon)-\hbar(\lambda)\right)}{\hbar(\upsilon)-\hbar(\lambda)}\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad \nu_2 > \lambda. \tag{1.20}$$

(iii) If we take $\omega(\lambda) = 1$ *and* $\hbar(\lambda) = \lambda$*, then* (1.5) *and* (1.6) *will reduce to the fractional integrals defined by [27] as follows:*

$$\left(\mathcal{I}^{\Lambda}_{\nu_1+}\aleph\right)(\lambda) = \int_{\nu_1}^{\lambda} \frac{\Lambda\left(\lambda-\upsilon\right)}{\lambda-\upsilon}\aleph(\upsilon)d\upsilon, \quad \nu_1 < \lambda, \tag{1.21}$$

and

$$\left({}^{\hbar}\mathcal{I}^{\Lambda}_{\nu_2-}\aleph\right)(\lambda) = \int_{\lambda}^{\nu_2} \frac{\Lambda\left(\upsilon-\lambda\right)}{\upsilon-\lambda}\aleph(\upsilon)d\upsilon, \quad \nu_2 > \lambda. \tag{1.22}$$

(iv) [28, 36] If we put $\omega(\lambda) = 1$ *and* $\Lambda(\hbar(\lambda)) = \frac{\hbar(\lambda)^{\zeta}}{\Gamma(\zeta)}$*, then* (1.5) *and* (1.6) *will become generalized Riemann-Liouville fractional integrals as follows:*

$$\left({}^{\hbar}\mathcal{I}^{\zeta}_{\nu_1}\aleph\right)(\lambda) = \frac{1}{\Gamma(\zeta)}\int_{\nu_1}^{\lambda} \left(\hbar(\lambda)-\hbar(\upsilon)\right)^{\zeta-1}\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad \nu_1 < \lambda, \tag{1.23}$$

and

$$\left({}^{\hbar}\mathcal{I}^{\zeta}_{\nu_2-}\aleph\right)(\lambda) = \frac{1}{\Gamma(\zeta)}\int_{\lambda}^{\nu_2} \left(\hbar(\upsilon)-\hbar(\lambda)\right)^{\zeta-1}\hbar'(\upsilon)\aleph(\upsilon)d\upsilon, \quad \nu_2 > \lambda, \tag{1.24}$$

where $\zeta, \in \mathbb{C}$ *with* $\Re(\zeta) > 0$.

(v) If we take $\omega(\lambda) = 1$, $\hbar(\lambda) = \lambda$ *and* $\Lambda(\hbar(\lambda)) = \frac{\lambda^{\zeta}}{\Gamma(\zeta)}$*, then* (1.5) *and* (1.6) *will reduce to the following Riemann-Liouville fractional integrals*

$$\left(\mathfrak{I}^{\zeta}_{\nu_1+}\aleph\right)(\lambda) = \frac{1}{\Gamma(\zeta)}\int_{\nu_1}^{\lambda} (\lambda-\upsilon)^{\zeta-1}\aleph(\upsilon)d\upsilon, \quad \nu_1 < \lambda, \tag{1.25}$$

and

$$\left(\mathfrak{I}^{\zeta}_{\nu_2-}\aleph\right)(\lambda) = \frac{1}{\Gamma(\zeta)}\int_{\lambda}^{\nu_2} (\upsilon-\lambda)^{\zeta-1}\aleph(\upsilon)d\upsilon, \quad \nu_2 > \lambda, \tag{1.26}$$

respectively (see, [28, 36]).

(vi) If we take $\omega(\lambda) = 1$, $\hbar(\lambda) = \ln\lambda$ *and* $\Lambda(\hbar(\lambda)) = \frac{(\ln\lambda)^{\zeta}}{\Gamma(\zeta)}$*, then* (1.5) *and* (1.6) *will respectively become the following Hadamard integrals [28, 36].*

$$\left(\mathcal{I}^{\zeta}_{\nu_1+}\aleph\right)(\lambda) = \frac{1}{\Gamma(\zeta)}\int_{\nu_1}^{\lambda} (\ln\lambda-\ln\upsilon)^{\zeta-1}\aleph(\upsilon)\frac{d\upsilon}{\upsilon}, \quad \nu_1 < \lambda,$$

and

$$\left(\mathcal{I}^{\zeta}_{\nu_2-}\aleph\right)(\lambda) = \frac{1}{\Gamma(\zeta)}\int_{\lambda}^{\nu_2} (\ln\upsilon-\ln\lambda)^{\zeta-1}\aleph(\upsilon)\frac{d\upsilon}{\upsilon}, \quad \nu_2 > \lambda.$$

(vii) *If we take* $\omega(\lambda) = 1$, $\hbar(\lambda) = \lambda^\eta$ *and* $\Lambda(\hbar(\lambda)) = \frac{\lambda^\eta}{\eta}$, $\eta > 0$, *then* (1.5) *and* (1.6) *will become Katugampola [37] integrals as follows:*

$$\left(\mathcal{I}^{\zeta}_{\nu_1}\aleph\right)(\lambda) = \frac{1}{\Gamma(\zeta)}\int_{\nu_1}^{\lambda}\left(\frac{\lambda^\eta - \upsilon^\eta}{\eta}\right)^{\zeta-1}\aleph(\upsilon)\frac{d\upsilon}{\upsilon^{1-\eta}}, \quad \nu_1 < \lambda,$$

and

$$\left(\mathcal{I}^{\zeta}_{\nu_2}\aleph\right)(\lambda) = \frac{1}{\Gamma(\zeta)}\int_{\lambda}^{\nu_2}\left(\frac{\upsilon^\eta - \lambda^\eta}{\eta}\right)^{\zeta-1}\aleph(\upsilon)\frac{d\upsilon}{\upsilon^{1-\eta}}, \quad \nu_2 > \lambda.$$

(viii) *If we take* $\omega(\lambda) = 1, \hbar(\lambda) = \lambda$ *and* $\Lambda(\hbar(\lambda)) = \frac{\lambda}{\eta}\exp\left(-\frac{1-\eta}{\eta}\lambda\right)$, $\eta \in (0, 1)$, *then* (1.5) *and* (1.6) *reduce to the integrals given below:*

$$\left(\mathcal{I}^{\eta}_{\nu_1+}\aleph\right)(\lambda) = \frac{1}{\eta}\int_{\nu_1}^{\lambda}\exp\left(-\frac{1-\eta}{\eta}(\lambda-\upsilon)\right)\aleph(\upsilon), \quad \nu_1 < \lambda,$$

and

$$\left(\mathcal{I}^{\eta}_{\nu_2-}\aleph\right)(\lambda) = \frac{1}{\eta}\int_{\lambda}^{\nu_2}\exp\left(-\frac{1-\eta}{\eta}(\upsilon-\lambda)\right)\aleph(\upsilon)d\upsilon, \quad \nu_2 > \lambda,$$

respectively. Similarly, (1.5) *and* (1.6) *will reduce to the integrals introduced by [22, 38–40].*

The main motivation of this paper is to establish certain new integral inequalities by employing the unified fractional integral operators. In fact, for a class of n ($n \in \mathbb{N}$) positive, continuous and decreasing functions on $[\nu_1, \nu_2]$, certain new classes of integral inequalities will be discussed. The inequalities obtained in this manuscript are more general forms of the classical inequalities given in the literature. The existing classical inequalities can be rectified by imposing the conditions stated in remarks. By imposing certain conditions on $\hbar$ and Λ available in the literature, many new forms of fractional integral inequalities can be produced. It is expected that the ideas and techniques of the paper will inspire interested readers.

2. Main results

In this section, we utilize the weighted integral (1.5) to obtain the refinement of some classical inequalities. Throughout the paper, we let the function $\hbar$ be an increasing and positive function on $[\nu_1, \nu_2]$ with a continuous derivative $\hbar'$ on (ν_1, ν_2). To do this, first we prove that the operators defined by (1.5) and (1.6) are bounded.

Theorem 2.1. *Let the functions* $\Lambda, \aleph : [\nu_1, \nu_2] \to \mathbb{R}$, $0 < \nu_1 < \nu_2$, *be positive and integrable functions. Let* $\hbar : [\nu_1, \nu_2] \to \mathbb{R}$ *be a positive, increasing function having a continuous derivative on* (ν_1, ν_2). *If* $\frac{\Lambda}{\lambda}$ *is increasing on* $[\nu_1, \nu_2]$, *then for* $\lambda \in [\nu_1, \nu_2]$, *we have*

$$\omega^{-1}(\lambda) \mid \int_{\nu_1}^{\lambda}\frac{\Lambda(\hbar(\lambda) - \hbar(\upsilon))}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon \mid\leq \omega^{-1}(\lambda)\Lambda(\hbar(\nu_2) - \hbar(\nu_1))\,\|\aleph\|_{X^p_\omega}$$

and

$$\omega^{-1}(\lambda) \mid \int_{\lambda}^{\nu_2}\frac{\Lambda(\hbar(\upsilon) - \hbar(\lambda))}{\hbar(\upsilon) - \hbar(\lambda)}\omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon \mid\leq \omega^{-1}(\lambda)\Lambda(\hbar(\nu_2) - \hbar(\nu_1))\,\|\aleph\|_{X^p_\omega}.$$

Furthermore, one can get

$$\omega^{-1}(\lambda)\mid\int_{v_1}^{\lambda}\frac{\Lambda(\hbar(\lambda)-\hbar(\upsilon))}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon\mid+\omega^{-1}(\lambda)\mid\int_{\lambda}^{v_2}\frac{\Lambda(\hbar(\upsilon)-\hbar(\lambda))}{\hbar(\upsilon)-\hbar(\lambda)}\omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon\mid$$

$$\leq 2\omega^{-1}(\lambda)\Lambda(\hbar(v_2)-\hbar(v_1))\,\|\aleph\|_{X_\omega^p}.$$

Proof. By the given hypothesis, $\hbar$ is increasing, and therefore for $\upsilon\in[v_1,\lambda)$, $\lambda\in[v_1,v_2]$, $\hbar(\lambda)-\hbar(\upsilon)\leq\hbar(\lambda)-\hbar(v_1)$. Also, since the function $\frac{\Lambda}{\lambda}$ is increasing, we have

$$\frac{\Lambda(\hbar(\lambda)-\hbar(\upsilon))}{\hbar(\lambda)-\hbar(\upsilon)}\leq\frac{\Lambda(\hbar(\lambda)-\hbar(v_1))}{\hbar(\lambda)-\hbar(v_1)}. \tag{2.1}$$

By the given hypothesis, $\aleph$ and ω are positive functions, and $\hbar$ is increasing and differentiable. Therefore, from (2.1), it follows that

$$\frac{\Lambda(\hbar(\lambda)-\hbar(\upsilon))}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)\leq\frac{\Lambda(\hbar(\lambda)-\hbar(v_1))}{\hbar(\lambda)-\hbar(v_1)}\omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon). \tag{2.2}$$

From this, the following inequality can be easily obtained

$$\omega^{-1}(\lambda)\mid\int_{v_1}^{\lambda}\frac{\Lambda(\hbar(\lambda)-\hbar(\upsilon))}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon\mid\leq\omega^{-1}(\lambda)\Lambda(\hbar(\lambda)-\hbar(v_1))\parallel\aleph\parallel_{X_\omega^p}. \tag{2.3}$$

Similarly, one can get

$$\omega^{-1}(\lambda)\mid\int_{\lambda}^{v_2}\frac{\Lambda(\hbar(\upsilon)-\hbar(\lambda))}{\hbar(\upsilon)-\hbar(\lambda)}\omega(\upsilon)\hbar'(\upsilon)\aleph(\upsilon)d\upsilon\mid\leq\omega^{-1}(\lambda)\Lambda(\hbar(v_2)-\hbar(v_1))\parallel\aleph\parallel_{X_\omega^p}. \tag{2.4}$$

Finally, by adding (2.3) and (2.4), we get the last inequality. □

Theorem 2.2. *Suppose that the function* $\aleph$ *is a positive, continuous and decreasing function on* $[v_1,v_2]$. *Let* $v_1<\lambda\leq v_2,\vartheta_1>0$, *and* $\sigma\geq\gamma>0$. *Then, for generalized integral operator* (1.5)*, we have*

$$\frac{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}[\aleph^{\sigma}(\lambda)]}{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}[\aleph^{\gamma}(\lambda)]}\geq\frac{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\aleph^{\sigma}(\lambda)\right]}{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\aleph^{\gamma}(\lambda)\right]}. \tag{2.5}$$

Proof. Since $\aleph$ is a positive, continuous and decreasing functions on the interval $[v_1,v_2]$, we have

$$\left((\rho-v_1)_1^{\vartheta}-(\upsilon-v_1)_1^{\vartheta}\right)(\aleph^{\sigma-\gamma}(\upsilon)-\aleph^{\sigma-\gamma}(\rho))\geq 0, \tag{2.6}$$

where $v_1<\lambda\leq v_2$, $\vartheta_1>0$, $\sigma\geq\gamma>0$, and $\upsilon,\rho\in[v_1,\lambda]$.

By (2.6), we have

$$(\rho-v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\upsilon)+(\upsilon-v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\rho)-(\rho-v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\rho)-(\upsilon-v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\upsilon)\geq 0. \tag{2.7}$$

Define a function

$$G(\lambda,\upsilon):=\frac{\Lambda(\hbar(\lambda)-\hbar(\upsilon))}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon). \tag{2.8}$$

We observe that the function $G(\lambda,\upsilon)$ remains positive for all $\upsilon \in (v_1,\lambda)$ and $v_1 < \upsilon < \lambda \le v_2$, as each term of the above function is positive in view of the conditions stated in Theorem 2.2. Therefore, multiplying (2.7) by

$$G(\lambda,\upsilon)\aleph^{\gamma}(\upsilon) = \frac{\Lambda\left(\hbar(\lambda)-\hbar(\upsilon)\right)}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\gamma}(\upsilon),$$

for $\upsilon \in (v_1,\lambda)$ and $v_1 < \lambda \le v_2$, we have

$$\begin{aligned}
&G(\lambda,\upsilon)\left[(\rho-v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\upsilon)+(\upsilon-v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\rho)-(\rho-v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\rho)-(\upsilon-v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\upsilon)\right]\aleph^{\gamma}(\upsilon)\\
&\quad=(\rho-v_1)_1^{\vartheta}\frac{\Lambda\left(\hbar(\lambda)-\hbar(\upsilon)\right)}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\gamma}(\upsilon)\aleph^{\sigma-\gamma}(\upsilon)\\
&\qquad+(\upsilon-v_1)_1^{\vartheta}\frac{\Lambda\left(\hbar(\lambda)-\hbar(\upsilon)\right)}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\gamma}(t)\aleph^{\sigma-\gamma}(\rho)\\
&\qquad-(\rho-v_1)_1^{\vartheta}\frac{\Lambda\left(\hbar(\lambda)-\hbar(\upsilon)\right)}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\gamma}(\upsilon)\aleph^{\sigma-\gamma}(\rho)\\
&\qquad-(\upsilon-v_1)_1^{\vartheta}\frac{\Lambda\left(\hbar(\lambda)-\hbar(\upsilon)\right)}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\gamma}(\upsilon)\aleph^{\sigma-\gamma}(\upsilon)\ge 0.
\end{aligned}\tag{2.9}$$

Integrating (2.9) with respect to υ over (v_1,λ), we have

$$\begin{aligned}
&(\rho-v_1)_1^{\vartheta}\int_{v_1}^{\lambda}\frac{\Lambda\left(\hbar(\lambda)-\hbar(\upsilon)\right)}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\sigma}(\upsilon)d\upsilon\\
&\quad+\aleph^{\sigma-\gamma}(\rho)\int_{v_1}^{\lambda}\frac{\Lambda\left(\hbar(\lambda)-\hbar(\upsilon)\right)}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)(\upsilon-v_1)_1^{\vartheta}\aleph^{\gamma}(\upsilon)d\upsilon\\
&\quad-(\rho-v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\rho)\int_{v_1}^{\lambda}\frac{\Lambda\left(\hbar(\lambda)-\hbar(\upsilon)\right)}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\gamma}(\upsilon)d\upsilon\\
&\quad-\int_{v_1}^{\lambda}\frac{\Lambda\left(\hbar(\lambda)-\hbar(\upsilon)\right)}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)(\upsilon-v_1)_1^{\vartheta}\aleph^{\sigma}(\upsilon)d\upsilon\ge 0.
\end{aligned}\tag{2.10}$$

Multiplying (2.10) by $\frac{1}{\omega(\lambda)}$, we get

$$\begin{aligned}
&(\rho-v_1)_1^{\vartheta}\,{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\aleph^{\sigma}(\lambda)\right]+\aleph^{\sigma-\gamma}(\rho)\,{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[(\lambda-v_1)_1^{\vartheta}\aleph^{\gamma}(\theta)\right]\\
&\quad-(\rho-v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\rho)\,{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\aleph^{\gamma}(\lambda)\right]-{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[(\lambda-v_1)_1^{\vartheta}\aleph^{\sigma}(\lambda)\right].
\end{aligned}\tag{2.11}$$

Multiplying (2.11) by

$$G(\lambda,\rho)\aleph^{\gamma}(\rho) = \frac{\Lambda\left(\hbar(\lambda)-\hbar(\rho)\right)}{\hbar(\lambda)-\hbar(\rho)}\omega(\rho)\hbar'(\rho)\aleph^{\gamma}(\rho)$$

for $\rho \in (v_1,\lambda)$ and $v_1 < \lambda \le v_2$, and integrating the resultant identity with respect to ρ over (v_1,λ), we get

$${}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\aleph^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[(\lambda-v_1)_1^{\vartheta}\aleph^{\gamma}(\lambda)\right]-{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[(\lambda-v_1)_1^{\vartheta}\aleph^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\aleph^{\gamma}(\lambda)\right]\ge 0.$$

It follows that

$${}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\aleph^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[(\lambda-v_1)_1^{\vartheta}\aleph^{\gamma}(\lambda)\right]\ge{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[(\lambda-v_1)_1^{\vartheta}\aleph^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\aleph^{\gamma}(\lambda)\right].$$

Dividing the above equation by ${}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[(\lambda-v_1)_1^{\vartheta}\aleph^{\gamma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\aleph^{\gamma}(\lambda)\right]$, we get the desired inequality (2.5). □

Remark 2.1. *If $\aleph$ is increasing on $[v_1, v_2]$, then the inequality in Theorem 2.2 will reverse.*

Remark 2.2. *If we take $\omega(\lambda) = 1$, $\hbar(\lambda) = \lambda$ and $\Lambda(\hbar(\lambda)) = \frac{\lambda^\zeta}{\Gamma(\zeta)}$, then Theorem 2.2 will reduce to Riemann-Liouville fractional integrals.*

Remark 2.3. *If we take $\omega(\lambda) = 1$, $\hbar(\lambda) = \lambda$, $\lambda = v_2$ and $\Lambda(\hbar(\lambda)) = \lambda$, then Theorem 2.2 will reduce to Theorem 3, proved earlier by Liu et al. [8].*

Example 2.1. *The special case of Theorem 2.2 is by taking $\omega(\lambda) = 1$, $\hbar(\lambda) = \lambda$, $\Lambda(\hbar(\lambda)) = \lambda$, $\sigma = 2$, $\gamma = \vartheta = 1$ and $\aleph(\upsilon) = 2 - \upsilon$, which is positive, continuous and decreasing on $[0, 1]$, and then we have*

$$\frac{\int_0^1 \aleph^2(\upsilon)d\upsilon}{\int_0^1 \aleph(\upsilon)d\upsilon} \geq \frac{\int_0^1 \upsilon\aleph^2(\upsilon)d\upsilon}{\int_0^1 \upsilon\aleph(\upsilon)d\upsilon}$$

i.e., 4.666 >1.375.

Theorem 2.3. *Suppose that the function $\aleph$ is a positive, continuous and decreasing function on $[v_1, v_2]$. Let $v_1 < \lambda \leq v_2, \vartheta_1 > 0$, and $\sigma \geq \gamma > 0$. Then, for generalized fractional integral (1.5), we have*

$$\frac{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\gamma(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\gamma(\lambda)\right]}{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^\gamma(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^\gamma(\lambda)\right]} \geq 1. \tag{2.12}$$

Proof. By multiplying both sides of (2.11) by

$$G(\lambda, \rho)\aleph^\gamma(\rho) = \frac{\Phi\left(\hbar(\lambda) - \hbar(\rho)\right)}{\hbar(\lambda) - \hbar(\rho)}\omega(\rho)\hbar'(\rho)\aleph^\gamma(\rho)$$

for $\rho \in (v_1, \lambda)$ and $v_1 < \lambda \leq v_2$, and integrating the resultant identity with respect to ρ over (v_1, λ), we get

$$\begin{aligned}&{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^\sigma(x)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\gamma(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\gamma(\lambda)\right]\\ &\quad - {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^\gamma(\lambda)\right] - {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^\gamma(\lambda)\right] \geq 0.\end{aligned}$$

It follows that

$$\begin{aligned}&{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^\sigma(x)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\gamma(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\gamma(\lambda)\right]\\ &\geq {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^\gamma(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^\gamma(\lambda)\right].\end{aligned} \tag{2.13}$$

Hence, dividing (2.13) by

$${}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^\gamma(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda - v_1)^{\vartheta}_1\aleph^\sigma(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^\gamma(\lambda)\right]$$

completes the proof. □

Remark 2.4. *Applying Theorem 2.3 for $\Lambda = \Phi$, we get Theorem 2.2.*

Theorem 2.4. *Suppose that the functions $\aleph$ and h_1 are positive and continuous on $[v_1, v_2]$ such that h_1 is increasing and $\aleph$ is decreasing on the interval $[v_1, v_2]$. Let $v_1 < \lambda \leq v_2, \vartheta_1 > 0$, and $\sigma \geq \gamma > 0$. Then, for generalized fractional integral* (1.5)*, we have*

$$\frac{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\sigma}(\lambda)\right]\,{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\gamma}(\lambda)\right]}{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\sigma}(\lambda)\right]\,{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right]} \geq 1. \tag{2.14}$$

Proof. By the hypothesis given in Theorem 2.4, we can write

$$\left(h_1^{\vartheta}(\rho) - h_1^{\vartheta}(\upsilon)\right)\left(\aleph^{\sigma-\gamma}(\upsilon) - \aleph^{\sigma-\gamma}(\rho)\right) \geq 0, \tag{2.15}$$

where $v_1 < \lambda \leq v_2, \vartheta_1 > 0, \sigma \geq \gamma > 0$, and $\upsilon, \rho \in [v_1, \lambda]$.

From (2.15), we have

$$h_1^{\vartheta}(\rho)\aleph^{\sigma-\gamma}(\upsilon) + h_1^{\vartheta}(\upsilon)\aleph^{\sigma-\gamma}(\rho) - h_1^{\vartheta}(\rho)\aleph^{\sigma-\gamma}(\rho) - h_1^{\vartheta}(\upsilon)\aleph^{\sigma-\gamma}(\upsilon) \geq 0. \tag{2.16}$$

Multiplying both sides of (2.16) by

$$G(\lambda, \upsilon)\aleph^{\gamma}(\upsilon) = \frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\gamma}(\upsilon)$$

for $\upsilon \in (v_1, \lambda)$ and $v_1 < \lambda \leq v_2$, we have

$$\begin{aligned}
&G(\lambda, \upsilon)\aleph^{\gamma}(\upsilon)\left[h_1^{\vartheta}(\rho)\aleph^{\sigma-\gamma}(\upsilon) + h_1^{\vartheta}(\upsilon)\aleph^{\sigma-\gamma}(\rho) - h_1^{\vartheta}(\rho)\aleph^{\sigma-\gamma}(\rho) - h_1^{\vartheta}(\upsilon)\aleph^{\sigma-\gamma}(\upsilon)\right] \\
&= h_1^{\vartheta}(\rho)\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\sigma}(\upsilon) \\
&\quad + h_1^{\vartheta}(\upsilon)\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\sigma-\gamma}(\rho)\aleph^{\sigma}(\upsilon) \\
&\quad - h_1^{\vartheta}(\rho)\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\sigma-\gamma}(\rho)\aleph^{\sigma}(\upsilon) \\
&\quad - h_1^{\vartheta}(\upsilon)_1^{\vartheta}\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\sigma}(\upsilon) \geq 0.
\end{aligned} \tag{2.17}$$

Integrating (2.17) with respect to υ over (v_1, λ), we have

$$\begin{aligned}
&h_1^{\vartheta}(\rho)\int_a^x \frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\sigma}(\upsilon)d\upsilon \\
&+ \aleph_q^{\sigma-\gamma_q}(\rho)\int_{v_1}^{\lambda} \frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)h_1^{\vartheta}(\upsilon)\aleph^{\gamma}(\upsilon)d\upsilon \\
&- h_1^{\vartheta}(\rho)\aleph^{\sigma-\gamma}(\rho)\int_{v_1}^{\lambda} \frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\aleph^{\gamma}(\upsilon)d\upsilon \\
&- \int_{v_1}^{\lambda} \frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)h_1^{\vartheta}(\upsilon)\aleph^{\sigma}(\upsilon)d\upsilon \geq 0.
\end{aligned} \tag{2.18}$$

Multiplying (2.18) by $\frac{1}{\omega(\lambda)}$ and in view of (1.5), we can write

$$h_1^{\vartheta}(\rho)\,{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\sigma}(\lambda)\right] + \aleph^{\sigma-\gamma}(\rho)\,{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\gamma}(\lambda)\right]$$

$$- h_1^{\vartheta}(\rho)\aleph^{\sigma-\gamma}(\rho)\, {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right] - {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\gamma}(\lambda)\right] \geq 0. \tag{2.19}$$

Again, multiplying (2.19) by

$$G(\lambda,\rho)\aleph^{\gamma}(\rho) = \frac{\Lambda\left(\hbar(\lambda)-\hbar(\rho)\right)}{\hbar(\lambda)-\hbar(\rho)}\omega(\rho)\hbar'(\rho)\aleph^{\gamma}(\rho)$$

for $\rho \in (v_1, \lambda)$ and $v_1 < \lambda \leq v_2$, and integrating the resultant identity with respect to ρ over (v_1, λ), we get

$${}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\gamma}(\lambda)\right] - {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right] \geq 0.$$

This can be written as

$${}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\gamma}(\lambda)\right] \, geq \, {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right],$$

which completes the desired inequality (2.14) of Theorem 2.4. □

Theorem 2.5. *Suppose that the functions* $\aleph$ *and* h_1 *are positive and continuous on* $[v_1, v_2]$ *such that* h_1 *is increasing and* $\aleph$ *is decreasing on the interval* $[v_1, v_2]$*. Let* $v_1 < x \leq v_2, \vartheta_1 > 0,$ *and* $\sigma \geq \gamma > 0.$ *Then, for generalized integral* (1.5)*, we have*

$$\frac{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\gamma}(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\gamma}(\lambda)\right]}{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\gamma}\lambda)\right]} \geq 1. \tag{2.20}$$

Proof. Multiplying (2.19) by

$$G(\lambda,\rho)\aleph^{\gamma}(\rho) = \frac{\Phi\left(\hbar(\lambda)-\hbar(\rho)\right)}{\hbar(\lambda)-\hbar(\rho)}\omega(\rho)\hbar'(\rho)\aleph^{\gamma}(\rho)$$

for $\rho \in (v_1, \lambda)$ and $v_1 < \lambda \leq v_2$, and integrating the resultant identity with respect to ρ over (v_1, λ), we get

$$\begin{aligned} &{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\gamma}(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\gamma}(\lambda)\right] \\ &\quad - {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right] - {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right] \geq 0. \end{aligned}$$

It follows that

$$\begin{aligned} &{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\gamma}(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\gamma}(\lambda)\right] \\ &\geq {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right]. \end{aligned}$$

Dividing both sides by

$${}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right] + {}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[h_1^{\vartheta}(\lambda)\aleph^{\sigma}(\lambda)\right] {}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right]$$

gives the desired inequality (2.20). □

Remark 2.5. *Taking* $\Lambda = \Phi$ *in Theorem 2.5, we get Theorem 2.4.*

3. Unified integral inequalities for a class of decreasing positive functions

In this section, we utilize the left unified integral operator (1.5) to establish some inequalities for a class of decreasing positive functions.

Theorem 3.1. *Suppose that the functions* $(\aleph_l)_{l=1,2,3,\dots,n}$ *be n positive, continuous and decreasing functions on* $[v_1, v_2]$. *Let* $v_1 < \lambda \leq v_2, \vartheta_1 > 0$, *and* $\sigma \geq \gamma_q > 0$ *for any fixed* $q \in \{1, 2, 3, \dots, n\}$. *Then, for generalized fractional integral operator* (1.5)*, we have*

$$\frac{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]}{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]} \geq \frac{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)_1^{\vartheta}\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]}{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)_1^{\vartheta}\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]}. \tag{3.1}$$

Proof. Since $(\aleph_l)_{l=1,2,3,\dots,n}$ are n positive, continuous and decreasing functions on the interval $[v_1, v_2]$, we have

$$\left((\rho - v_1)_1^{\vartheta} - (\upsilon - v_1)_1^{\vartheta}\right)\left(\aleph_q^{\sigma-\gamma_q}(\upsilon) - \aleph_q^{\sigma-\gamma_q}(\rho)\right) \geq 0 \tag{3.2}$$

for any fixed $q \in \{1, 2, 3, \dots, n\}, v_1 < \lambda \leq v_2, \vartheta_1 > 0, \sigma \geq \gamma_q > 0$ and $\upsilon, \rho \in [v_1, \lambda]$. By (3.2), we have

$$(\rho - v_1)_1^{\vartheta}\aleph_q^{\sigma-\gamma_q}(\upsilon) + (\upsilon - v_1)_1^{\vartheta}\aleph_q^{\sigma-\gamma_q}(\rho) \geq (\rho - v_1)_1^{\vartheta}\aleph_q^{\sigma-\gamma_q}(\rho) + (\upsilon - v_1)_1^{\vartheta}\aleph_q^{\sigma-\gamma_q}(\upsilon). \tag{3.3}$$

Multiplying both sides of (3.3) by

$$G(\lambda, \upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon) = \frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(t)$$

for $\rho \in (v_1, \lambda)$ and $v_1 < \lambda \leq v_2$, we have

$$\begin{aligned}
G(\lambda, \upsilon)&\Big[(\rho - v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\upsilon) + (\upsilon - a)_1^{\vartheta}\aleph^{\sigma-\gamma}(\rho) - (\rho - v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\rho) \\
&-(\upsilon - v_1)_1^{\vartheta}\aleph^{\sigma-\gamma}(\upsilon)\Big]\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon) \\
&= (\rho - v_1)_1^{\vartheta}\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(t)\aleph_q^{\sigma-\gamma_q}(\upsilon) \\
&\quad + (\upsilon - v_1)_1^{\vartheta}\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\rho) \\
&\geq (\rho - v_1)_1^{\vartheta}\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\rho) \\
&\quad + (\upsilon - v_1)_1^{\vartheta}\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\upsilon).
\end{aligned} \tag{3.4}$$

Integrating (3.4) with respect to υ over (v_1, λ), we have

$$(\rho - v_1)_1^{\vartheta}\int_{v_1}^{\lambda}\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(t)\aleph_q^{\sigma-\gamma_q}(\upsilon)d\upsilon$$

$$
\begin{aligned}
&+\aleph_q^{\sigma-\gamma_q}(\rho)\int_{v_1}^{\lambda}\frac{\Lambda(\hbar(\lambda)-\hbar(\upsilon))}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)d\upsilon\\
&\geq(\rho-v_1)_1^{\vartheta}\aleph_q^{\sigma-\gamma_q}(\rho)\int_{v_1}^{\lambda}\frac{\Lambda(\hbar(\lambda)-\hbar(\upsilon))}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)d\upsilon\\
&+\int_{v_1}^{\lambda}\frac{\Lambda(\hbar(\lambda)-\hbar(\upsilon))}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)(\upsilon-v_1)_1^{\vartheta}\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\upsilon)d\upsilon.
\end{aligned}
$$

In view of (1.5), it follows that

$$
\begin{aligned}
&(\rho-v_1)_1^{\vartheta}\,{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]+\aleph_q^{\sigma-\gamma_q}(\rho)\,{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\geq(\rho-v_1)_1^{\vartheta}\aleph_q^{\sigma-\gamma_q}(\rho)\,{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]-{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right].
\end{aligned}
\tag{3.5}
$$

Again, multiplying both sides of (3.5) by

$$
G(\lambda,\rho)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\rho)=\frac{\Lambda(\hbar(\lambda)-\hbar(\rho))}{\hbar(\lambda)-\hbar(\rho)}\omega(\rho)\hbar'(\rho)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\rho)
$$

for $\rho\in(v_1,\lambda)$ and $v_1<\lambda\leq v_2$, and integrating the resultant identity with respect to ρ over (v_1,λ), we get

$$
\begin{aligned}
&{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\geq{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right],
\end{aligned}
$$

which gives the required inequality (3.1). □

Remark 3.1. *If we consider that $(\aleph_l)_{l=1,2,3,\ldots,n}$ are increasing functions on $[v_1,v_2]$, then the inequality in Theorem 3.1 will reverse .*

Remark 3.2. *If we take $\omega(\lambda)=1$, $\hbar(\lambda)=\lambda$ and $\Lambda(\hbar(\lambda))=\frac{\lambda^{\zeta}}{\Gamma(\zeta)}$, then Theorem 3.1 will reduce to the result proved by Dahmani [17].*

Remark 3.3. *If we take $\omega(\lambda)=1$, $\hbar(\lambda)=\lambda$, $\lambda=v_2$, $n=1$ and $\Lambda(\hbar(\lambda))=\lambda$, then Theorem 3.1 will reduce to Theorem 3, proved earlier by Liu et al. [8].*

Theorem 3.2. *Suppose that the functions $(\aleph_l)_{l=1,2,3,\ldots,n}$ are n positive, continuous and decreasing functions on $[v_1,v_2]$. Let $v_1<\lambda\leq v_2,\vartheta_1>0$, and $\sigma\geq\gamma_q>0$ for any fixed $q\in\{1,2,3,\ldots,n\}$. Then, for generalized fractional integral* (1.5)*, we have*

$$
\frac{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]+{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]}{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]+{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]}\geq 1. \tag{3.6}
$$

Proof. Multiplying both sides of (3.5) by

$$G(\lambda,\rho)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\rho)=\frac{\Phi\left(\hbar(\lambda)-\hbar(\rho)\right)}{\hbar(\lambda)-\hbar(\rho)}\omega(\rho)\hbar'(\rho)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\rho)$$

for $\rho\in(v_1,\lambda)$ and $v_1<\lambda\leq v_2$, and integrating the resultant identity with respect to ρ over (v_1,λ), we get

$$\begin{aligned}
&{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\quad+{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\geq{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\quad+{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]. \qquad (3.7)
\end{aligned}$$

It follows that

$$\begin{aligned}
&{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\quad+{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\geq{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\quad+{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]. \qquad (3.8)
\end{aligned}$$

Hence, dividing (3.8) by

$$\begin{aligned}
&{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\quad+{}^{\hbar}_{\omega}\mathcal{I}^{\Phi}_{v_1+}\left[(\lambda-v_1)_1^{\vartheta}\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]
\end{aligned}$$

completes the proof. □

Remark 3.4. *Applying Theorem 3.2 for $\Lambda=\Phi$, we get Theorem 3.1.*

Theorem 3.3. *Suppose that the functions* $(\aleph_l)_{l=1,2,3,\dots,n}$ *and* h_1 *are positive and continuous on* $[v_1, v_2]$ *such that* h_1 *is increasing and* $(\aleph_l)_{l=1,2,3,\dots,n}$ *are decreasing on* $[v_1, v_2]$. *Let* $v_1 < \lambda \leq v_2, \vartheta_1 > 0$, *and* $\sigma \geq \gamma_q > 0$ *for any fixed* $q \in \{1, 2, 3, \dots, n\}$. *Then, for generalized fractional integral* (1.5)*, we have*

$$\frac{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]}{{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[h_1^{\vartheta}(\lambda)\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}^{\hbar}_{\omega}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]} \geq 1. \tag{3.9}$$

Proof. Under the hypothesis given in Theorem 3.3, we can write

$$\left(h_1^{\vartheta}(\rho) - h_1^{\vartheta}(\upsilon)\right)\left(\aleph_q^{\sigma-\gamma_q}(\upsilon) - \aleph_q^{\sigma-\gamma_q}(\rho)\right) \geq 0 \tag{3.10}$$

for any fixed $q \in \{1, 2, 3, \dots, n\}, v_1 < \lambda \leq v_2, \vartheta_1 > 0, \sigma \geq \gamma_q > 0$ and $\upsilon, \rho \in [v_1, \lambda]$.

From (3.10), we can write

$$h_1^{\vartheta}(\rho)\aleph_q^{\sigma-\gamma_q}(\upsilon) + h_1^{\vartheta}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\rho) - h_1^{\vartheta}(\rho)\aleph_q^{\sigma-\gamma_q}(\rho) - h_1^{\vartheta}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\upsilon) \geq 0. \tag{3.11}$$

Multiplying (3.11) by

$$G(\lambda,\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon) = \frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)$$

for $\upsilon \in (v_1, \lambda)$ and $v_1 < \lambda \leq v_2$, we have

$$\begin{aligned}
&h_1^{\vartheta}(\rho)\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\upsilon)\\
&+ h_1^{\vartheta}(\upsilon)\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\rho)\\
&- h_1^{\vartheta}(\rho)\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\rho)\\
&- h_1^{\vartheta}(\upsilon)_1^{\vartheta}\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\upsilon) \geq 0.
\end{aligned} \tag{3.12}$$

Integrating (3.12) with respect to υ over (v_1, λ), we have

$$\begin{aligned}
&h_1^{\vartheta}(\rho)\int_{v_1}^{\lambda}\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\upsilon)d\upsilon\\
&+ \aleph_q^{\sigma-\gamma_q}(\rho)\int_{v_1}^{\lambda}\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)h_1^{\vartheta}(\upsilon)_1^{\vartheta}\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)d\upsilon\\
&- h_1^{\vartheta}(\rho)\aleph_q^{\sigma-\gamma_q}(\rho)\int_{v_1}^{\lambda}\frac{\Lambda\left(\hbar(\lambda) - \hbar(\upsilon)\right)}{\hbar(\lambda) - \hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)d\upsilon\\
&- \int_{v_1}^{\lambda}h_1^{\vartheta}(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\upsilon)\aleph_q^{\sigma-\gamma_q}(\upsilon)d\upsilon \geq 0.
\end{aligned} \tag{3.13}$$

In view of (1.5), we can write from (3.13)

$$h_1^{\vartheta}(\rho)\,{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]+\aleph_q^{\sigma-\gamma_q}(\rho)\,{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[h_1^{\vartheta}(\lambda)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]$$
$$-h_1^{\vartheta}(\rho)\aleph_q^{\sigma-\gamma_q}(\rho)\,{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]-{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[h_1^{\vartheta}(\lambda)\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]\geq 0. \tag{3.14}$$

Again, multiplying (3.14) by

$$G(\lambda,\rho)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\rho)=\frac{\Lambda\left(\hbar(\lambda)-\hbar(\upsilon)\right)}{\hbar(\lambda)-\hbar(\upsilon)}\omega(\upsilon)\hbar'(\upsilon)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\rho)$$

for $\rho\in(v_1,\lambda)$ and $v_1<\lambda\leq v_2$, and integrating the resultant identity with respect to ρ over (a,x), we get

$${}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[h_1^{\vartheta}(\lambda)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]$$
$$-{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[h_1^{\vartheta}(\lambda)\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\geq 0,$$

which gives the required inequality (3.9). □

Theorem 3.4. *Suppose that the functions $(\aleph_l)_{l=1,2,3,\ldots,n}$ and h_1 are positive and continuous on $[v_1,v_2]$ such that h_1 is increasing and $(\aleph_l)_{l=1,2,3,\ldots,n}$ are decreasing on the interval $[v_1,v_2]$. Let $v_1<\lambda\leq v_2,\vartheta_1>0$, and $\sigma\geq\gamma_q>0$ for any fixed $q\in\{1,2,3,\ldots,n\}$. Then, for generalized fractional integral* (1.5)*, we have*

$$\frac{{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[h_1^{\vartheta}(\lambda)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]+{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[h_1^{\vartheta}(\lambda)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]}{{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[h_1^{\vartheta}(\lambda)\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]+{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[h_1^{\vartheta}(\lambda)\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]}\geq 1. \tag{3.15}$$

Proof. Multiplying (3.14) by

$$G(\lambda,\rho)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\rho)=\frac{\Phi\left(\hbar(\lambda)-\hbar(\rho)\right)}{\hbar(\lambda)-\hbar(\rho)}\omega(\rho)\hbar'(\rho)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\rho)$$

for $\rho\in(v_1,\lambda)$ and $v_1<\lambda\leq v_2$, and integrating the resultant identity with respect to ρ over (v_1,λ), we get

$${}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[h_1^{\vartheta}(\lambda)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]$$
$$+{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[h_1^{\vartheta}(\lambda)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]$$

$$- {}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[h_1^{\vartheta}(\lambda)\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]$$
$$- {}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[h_1^{\vartheta}(\lambda)\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\geq 0.$$

This can be written as

$$\begin{aligned}
&{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[h_1^{\vartheta}(\lambda)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\quad + {}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[h_1^{\vartheta}(\lambda)\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\geq {}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[h_1^{\vartheta}(\lambda)\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\quad + {}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[h_1^{\vartheta}(\lambda)\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right].
\end{aligned}$$

Dividing both sides by

$$\begin{aligned}
&{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[h_1^{\vartheta}(\lambda)\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]\\
&\quad + {}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Phi}\left[h_1^{\vartheta}(\lambda)\prod_{l\neq q}^{n}\aleph_l^{\gamma_l}\aleph_q^{\sigma}(\lambda)\right]{}_{\omega}^{\hbar}\mathcal{I}_{v_1+}^{\Lambda}\left[\prod_{l=1}^{n}\aleph_l^{\gamma_l}(\lambda)\right]
\end{aligned}$$

gives the desired inequality (3.15). □

Remark 3.5. *Applying Theorem 3.4 for $\Lambda = \Phi$, we get Theorem 3.3. Similarly, we can establish the inequalities for the right generalized proportional fractional integral defined by* (1.6).

4. Special cases

By using our main results, we get the following certain new and well-known inequalities in terms of well-known fractional integral operators:

Corollary 4.1. *Suppose that the function $\aleph$ is a positive, continuous and decreasing function on* $[v_1, v_2]$. *Let $v_1 < \lambda \leq v_2, \vartheta_1 > 0$, and $\sigma \geq \gamma > 0$. Then, for generalized integral operator* (1.7) *(see, for example, [27]), we have*

$$\frac{\mathcal{I}_{v_1+}^{\Lambda}\left[\aleph^{\sigma}(\lambda)\right]}{\mathcal{I}_{v_1+}^{\Lambda}\left[\aleph^{\gamma}(\lambda)\right]}\geq\frac{\mathcal{I}_{v_1+}^{\Lambda}\left[(\lambda-v_1)_1^{\vartheta}\aleph^{\sigma}(\lambda)\right]}{\mathcal{I}_{v_1+}^{\Lambda}\left[(\lambda-v_1)_1^{\vartheta}\aleph^{\gamma}(\lambda)\right]}.$$

Proof. Taking $\omega = 1, \hbar(\lambda) = \lambda$ and $\Lambda(\hbar(\lambda)) = \lambda$ in Theorem 2.2, we get the desired result. □

Corollary 4.2. *Suppose that the functions* $(\aleph_l)_{l=1,2,3,\dots,n}$ *are n positive, continuous and decreasing on* $[v_1, v_2]$*. Let* $v_1 < \lambda \leq v_2, \vartheta_1 > 0$*, and* $\sigma \geq \gamma_q > 0$ *for any fixed* $q \in \{1, 2, 3, \dots, n\}$*. Then, for generalized fractional integral operator* (1.7)*, we have*

$$\frac{\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l\neq q}^{n} \aleph_l^{\gamma_l} \aleph_q^{\sigma}(\lambda)\right]}{\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l=1}^{n} \aleph_l^{\gamma_l}(\lambda)\right]} \geq \frac{\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)_1^{\vartheta} \prod_{l\neq q}^{n} \aleph_l^{\gamma_l} \aleph_q^{\sigma}(\lambda)\right]}{\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)_1^{\vartheta} \prod_{l=1}^{n} \aleph_l^{\gamma_l}(\lambda)\right]}.$$

Proof. Taking $\omega = 1$, $\hbar(\lambda) = \lambda$ and $\Lambda(\hbar(\lambda)) = \lambda$ in Theorem 3.1, one can get the desired result. □

Similarly, by taking $\omega = 1$ in Theorems 2.2 and 3.1, one can get results for the fractional integral defined by Farid [26] as follows:

Corollary 4.3. *Suppose that the function* $\aleph$ *is a positive, continuous and decreasing function on* $[v_1, v_2]$*. Let* $v_1 < \lambda \leq v_2, \vartheta_1 > 0$*, and* $\sigma \geq \gamma > 0$*. Then, for generalized integral operator* (1.19)*, we have*

$$\frac{{}^{\hbar}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\sigma}(\lambda)\right]}{{}^{\hbar}\mathcal{I}^{\Lambda}_{v_1+}\left[\aleph^{\gamma}(\lambda)\right]} \geq \frac{{}^{\hbar}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)_1^{\vartheta} \aleph^{\sigma}(\lambda)\right]}{{}^{\hbar}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)_1^{\vartheta} \aleph^{\gamma}(\lambda)\right]}.$$

Corollary 4.4. *Suppose that the functions* $(\aleph_l)_{l=1,2,3,\dots,n}$ *are n positive, continuous and decreasing on* $[v_1, v_2]$*. Let* $v_1 < \lambda \leq v_2, \vartheta_1 > 0$*, and* $\sigma \geq \gamma_q > 0$ *for any fixed* $q \in \{1, 2, 3, \dots, n\}$*. Then, for generalized fractional integral operator* (1.19)*, we have*

$$\frac{{}^{\hbar}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l\neq q}^{n} \aleph_l^{\gamma_l} \aleph_q^{\sigma}(\lambda)\right]}{{}^{\hbar}\mathcal{I}^{\Lambda}_{v_1+}\left[\prod_{l=1}^{n} \aleph_l^{\gamma_l}(\lambda)\right]} \geq \frac{{}^{\hbar}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)_1^{\vartheta} \prod_{l\neq q}^{n} \aleph_l^{\gamma_l} \aleph_q^{\sigma}(\lambda)\right]}{{}^{\hbar}\mathcal{I}^{\Lambda}_{v_1+}\left[(\lambda - v_1)_1^{\vartheta} \prod_{l=1}^{n} \aleph_l^{\gamma_l}(\lambda)\right]},$$

where $\hbar$ *is a differentiable, increasing and continuous function on* $[v_1, v_2]$.

5. Conclusions

In this present investigation, we generalized many classical inequalities discussed in the literature via unified fractional integrals and proved that the operators defined in (1.5) and (1.6) are bounded. We developed certain new classes of unified fractional integral inequalities for a class of n ($n \in \mathbb{N}$) positive, continuous and decreasing functions on $[v_1, v_2]$. Certain special cases of the main result are discussed in Section 4. By applying specific conditions on $\hbar$ and Λ as given in the literature, we can produce certain new classes of inequalities as discussed in Remark 1.1. We hope that our ideas and techniques of this paper will inspire interested readers working in this field.

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Conflict of interest

The authors declare no conflict of interest.

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