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Research article

Characterizations of Fock-type spaces of eigenfunctions on $\mathbb{R}^n$

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Abstract: In this paper, we prove a norm equivalence for an exponential type weighted integral of an eigenfunction and its derivative on $\mathbb{R}^n$. As applications, we characterize Fock-type spaces of eigenfunctions on $\mathbb{R}^n$ in terms of Lipschitz type conditions and double integral conditions. These obtained results are extensions of the corresponding ones in classcial Fock space.

Keywords: eigenfunction; Fock space; Lipschitz condition; double integral condition
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1. Introduction

For $n \geq 2$, let $\mathbb{R}^n$ denote the n-dimensional real vector space. For two column vectors $x, y \in \mathbb{R}^n$, we use $\langle x, y\rangle$ to denote the inner product of x and y. The ball in $\mathbb{R}^n$ with center a and radius r is denoted by $\mathbb{B}(a, r)$. In particular, we write $\mathbb{B} = \mathbb{B}(0, 1)$ and $\mathbb{B}_r = \mathbb{B}(0, r)$. Let dv be the volume measure on $\mathbb{R}^n$ and $d\sigma$ the normalized surface measure on the unit sphere $\mathbb{S} = \partial\mathbb{B}$.

Given $\alpha > 0, m \in \mathbb{N}$ and $t \in \mathbb{R}$, the *t-weighted (α, m)-Gaussian measure $dG_{\alpha,m,t}$* on $\mathbb{R}^n$ is given by

$$dG_{\alpha,m,t}(x) = C_{\alpha,m,t}e^{-\alpha|x|^m}\frac{dv(x)}{(1 + |x|)^t},$$

where $C_{\alpha,m,t}$ is the positive constant to be the normalized volume measure. In particular, if $m = 2, t = 0$, $dG_{\alpha,2,0}$ is the classical Gaussian measure on $\mathbb{R}^n$ (cf. [1]).

For $\lambda \geq 0$, we denote by $H_\lambda(\mathbb{R}^n)$ the set of all eigenfunctions of the Laplacian with eigenvalue λ on $\mathbb{R}^n$, i.e.,

$$H_\lambda(\mathbb{R}^n) = \{f \in C^2 : \Delta f = \lambda f\},$$

where Δ is the ordinary Laplace operator on $\mathbb{R}^n$. Obviously, if $\lambda = 0$, $H_0(\mathbb{R}^n)$ is the set of all harmonic functions on $\mathbb{R}^n$.

Let $0 < p < \infty$, $s > -1$ and f be a holomorphic function on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. The famous Hardy-Littlewood theorem asserts that

$$\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^s dA(z) \approx |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p+s} dA(z), \tag{1.1}$$

where dA is the normalized area measure on $\mathbb{C}$ so that $A(\mathbb{D}) = 1$ (cf. [2]).

It is known that the integral estimate (1.1) plays an important role in the theory of holomorphic functions. For the generalizations and applications of (1.1) to the spaces of holomorphic functions, harmonic functions, and solutions to certain PDEs, see [3–10] and the references therein.

Let $\mathbb{C}^n$ be the n-dimensional complex vector space. In recent years a special class of holomorphic function spaces, the so-called *holomorphic Fock space* $\mathbb{F}(\mathbb{C}^n)$, has attracted much attention. See [10–16] for a summary of recent research on $\mathbb{F}(\mathbb{C}^n)$. For $0 < p < \infty$ and $\alpha > 0$, recall that an entire function f on $\mathbb{C}^n$ is said to belong to the Fock space $\mathbb{F}(\mathbb{C}^n)$ if

$$\|f\|_{p,\alpha}^p = \int_{\mathbb{C}^n} \left| f(z) e^{-\alpha|z|^2} \right|^p dv(z) < \infty.$$

In [12], Hu considered an analog of (1.1) in the setting of $\mathbb{F}(\mathbb{C}^n)$ and proved that

$$\|f\|_{p,\alpha}^p \approx |f(0)|^p + \int_{\mathbb{C}^n} \left| \nabla f(z)(1 + |z|)^{-1} e^{-\alpha|z|^2} \right|^p dv(z). \tag{1.2}$$

As a consequence of (1.2), he obtained the boundedness and compactness of Cesàro operators from one Fock space to another. For the further generalizations of (1.2) to holomorphic Fock spaces with some general differential weights, see [11, 13–15]. By applying these results, Cho et al. characterized Fock-type spaces in terms of Lipschitz type conditions and double integral conditions (cf. [13, 14]).

Since the eigenfunctions can be viewed as extensions of holomorphic functions on the complex vector space, it is interesting to establish analogous of the equivalence of norms (1.1) and (1.2) in the setting of $H_\lambda(\mathbb{R}^n)$. In [8], Stoll extended (1.1) to the setting of $H_\lambda(\mathbb{B})$ ([8, Theorem 5.1]). Furthermore, by using this result, he established some harmonic majorants criteria for eigenfunctions with finite Dirichlet integrals on a bounded domain Ω of $\mathbb{R}^n$ ([8, Theorem 5.2]). Motivated by the results in [11–14], we consider a similar norm equivalence (1.2) in the setting of $H_\lambda(\mathbb{R}^n)$ in this note.

For $1 < p < \infty$ and $\alpha > 0$, the *Fock-type space* $F^p_{\alpha,m,t}(\mathbb{R}^n)$ consists of all $f \in H_\lambda(\mathbb{R}^n)$ such that

$$\|f\|^p_{F^p_{\alpha,m,t}} = \int_{\mathbb{R}^n} \left| f(x) e^{-\alpha|x|^m} \right|^p \frac{dv(x)}{(1 + |x|)^t} < \infty.$$

Especially, when $m = 2$, $t = \lambda = 0$, $F^p_{\alpha,2,0}(\mathbb{R}^n)$ becomes the harmonic Fock space (cf. [17]).

Theorem 1.1. *Let* $1 < p < \infty$, $\alpha > 0, m \in \mathbb{N}$, $t \in \mathbb{R}$. *Then*

$$\|f\|^p_{F^p_{\alpha,m,t}} \approx |f(0)|^p + \int_{\mathbb{R}^n} \left| \frac{\nabla f(x) e^{-\alpha|x|^m}}{1 + |x|^{m-1}} \right|^p \frac{dv(x)}{(1 + |x|)^t}, \tag{1.3}$$

for all $f \in H_\lambda(\mathbb{R}^n)$.

As an application of Theorem 1.1, we obtain a Lipschitz type characterization for the Fock-type space $F^p_{\alpha,m,t}(\mathbb{R}^n)$.

Theorem 1.2. *Let $1 < p < \infty$, $\alpha > 0, m \in \mathbb{N}$, $q \geq 0$, $t \in \mathbb{R}$ and $f \in H_\lambda(\mathbb{R}^n)$. Then the following two statements are equivalent on $\mathbb{R}^n$:*
(*a*) $f \in F^p_{\alpha,m,t}(\mathbb{R}^n)$;
(*b*) *There exists a positive continuous function $g \in L^p(dG_{\alpha p,m,t-pq(m-1)})$ such that*

$$\frac{|f(x) - f(y)|}{|x - y|} \leq (1 + |x|^{m-1} + |y|^{m-1})^{1+q}(g(x) + g(y))$$

for all $x, y \in \mathbb{R}^n$ with $x \neq y$.

For $m \in \mathbb{N}$, $s \in \mathbb{R}$, $r > 0$ and $f \in H_0(\mathbb{R}^n)$ (i.e. f is harmonic), we define

$$Lf(x, y) = f(x) - f(y)$$

and

$$L^s_r f(x, y) = [Lf(x, y)]e^{s|x|^m}\chi_{E_r(x)}(y),$$

where $\chi_{E_r(x)}$ denotes the characteristic function of Euclidean ball $E_r(x)$ (see its definition in Section 2).

In our final result, we discuss the double integral characterization for harmonic Fock-type spaces.

Theorem 1.3. *Let $1 < p < \infty$, $\alpha > 0, m \in \mathbb{N}$, $t, s \in \mathbb{R}$, $q \geq 0$ and $f \in H_0(\mathbb{R}^n)$. Then the following statements are equivalent on $\mathbb{R}^n$:*
(*a*) $f \in F^p_{\alpha,m,t}(\mathbb{R}^n)$;
(*b*) $Lf \in L^p(dG_{\alpha p,m,t} \times dG_{\alpha p,m,t})$;
(*c*) $L^s_r f \in L^p(dG_{\beta p,m,\gamma} \times dG_{\beta p,m,\gamma})$, *where* $\beta = \frac{s+\alpha}{2}, \gamma = \frac{t-n(m-1)}{2}$.

Lipschitz type characterization for Bergman spaces with standard weights on the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ in terms of the Euclidean, hyperbolic, and pseudo-hyperbolic metrics was original established by Wulan and Zhu ([9, Theorem 1.1]). As an application, double integral characterizations for weighted Bergman spaces in the unit ball in $\mathbb{C}^n$ were proved in [18, 19]. For the further generalizations of these results to harmonic Bergman space and holomorphic Fock space, we refer to [3, 4, 6, 13, 14].

The rest of this paper is organized as follows. In Section 2, some necessary terminology and notation will be introduced. In Section 3, we shall prove Theorem 1.1. The proof of Theorem 1.2 will be presented in Section 4 by applying Theorem 1.1. The final Section 5 is devoted to the proof of Theorems 1.3. Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. For nonnegative quantities X and Y, $X \lesssim Y$ means that X is dominated by Y times some inessential positive constant. We write $X \approx Y$ if $Y \lesssim X \lesssim Y$.

2. Preliminaries

In this section, we introduce notations and collect some preliminaries results that involve eigenfunctions on $\mathbb{R}^n$.

For $0 < p < \infty$, $\lambda \geq 0$ and $f \in H_\lambda(\mathbb{R}^n)$, the p-th *integral mean* of f on $r\mathbb{S}$ is defined as

$$M_p(f, r) = \Big(\int_{\mathbb{S}} |f(r\xi)|^p d\sigma(\xi)\Big)^{\frac{1}{p}}, \quad 0 < r < \infty.$$

Lemma 2.1. *Let $1 \le p < \infty$, $\lambda \ge 0$ and $f \in H_\lambda(\mathbb{B})$. Then both $M_p^p(f, r)$ and $M_p^p(\nabla f, r)$ are increasing with $0 < r < 1$.*

Proof. We first prove the monotonicity of $M_p^p(f, r)$. Let $f \in H_\lambda(\mathbb{B})$ and $\mathcal{Z}_f$ be the zero set of f on $\mathbb{B}$. Then

$$\Delta|f|^p = p(p-1)|f|^{p-2}|\nabla f|^2 + p\lambda|f|^p \ge 0,$$

which implies that $|f|^p$ is subharmonic on $\mathbb{B} \setminus \mathcal{Z}_f$. Note that at each point of $\mathcal{Z}_f$ the mean value inequality trivially holds, and thus $|f|^p$ is subharmonic on $\mathbb{B}$. It follows from Green's theorem, we know that $M_p^p(f, r)$ is increasing with $0 < r < 1$.

Now we come to prove the monotonicity of $M_p^p(\nabla f, r)$. In view of the definition of $H_\lambda(\mathbb{B})$, it is easy to see that if $f \in H_\lambda(\mathbb{B})$, then $f \in C^\infty$. This gives

$$\Delta\partial_i f = \partial_i \Delta f = \lambda \partial_i f, \quad i \in \{1, 2, ..., n\},$$

which implies that the partial derivative $\partial_i f$ also belongs to $H_\lambda(\mathbb{B})$. By a discussion similar to the above, the monotonicity of $M_p^p(\nabla f, r)$ follows.

For $m \in \mathbb{N}$, $r > 0$ and $a \in \mathbb{R}^n$, the Euclidean ball $E_r(a)$ in $\mathbb{R}^n$ is defined as

$$E_r(a) = \Big\{x \in \mathbb{R}^n : \ |x - a| < \frac{r}{1 + |a|^{m-1}}\Big\}.$$

Lemma 2.2. *Let $m \in \mathbb{N}$, $a \in \mathbb{R}^n$ and $r > 0$. Then for any $x \in E_r(a)$,*

$$e^{|x|^m} \approx e^{|a|^m} \quad \text{and} \quad 1 + |x|^{m-1} \approx 1 + |a|^{m-1}.$$

Lemma 2.3. *Let $1 < p < \infty$, $0 < \alpha < \infty$, $k \in \mathbb{R}, m \in \mathbb{N}$ and f be a locally integrable function on $[0, \infty)$. Then there exists a constant C such that*

$$\int_0^\infty \Big| \int_0^r f(t)dt \Big|^p (1 + r)^k e^{-\alpha r^m} dr \le C \int_0^\infty |f(r)|^p (1 + r)^{k-(m-1)p} e^{-\alpha r^m} dr.$$

Proof. Let $\phi(r) = (1 + r)^k e^{-\alpha r^m}$, $\varphi(r) = (1 + r)^{k-(m-1)p} e^{-\alpha r^m}$ and p' be the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. By simple computations, we have

$$\lim_{r\to\infty} \frac{\int_r^\infty \phi(t)dt}{r^{k-m+1} e^{-\alpha r^m}} = \frac{1}{\alpha m}$$

and

$$\lim_{r\to\infty} \frac{\int_0^r \varphi(t)^{1-p'} dt}{r^{-\frac{k-(m-1)p}{p-1} - m + 1} e^{\frac{\alpha}{p-1} r^m}} = \frac{p-1}{\alpha m}.$$

This gives that

$$\begin{aligned}
& \lim_{r\to\infty} \Big(\int_r^\infty \phi(r)dr\Big)^{\frac{1}{p}} \Big(\int_0^r \varphi(r)^{1-p'} dr\Big)^{\frac{1}{p'}} \\
= \ & \Big(\frac{1}{\alpha m}\Big)^{\frac{1}{p}} \Big(\frac{p-1}{\alpha m}\Big)^{\frac{1}{p'}} \lim_{r\to\infty} (r^{k-m+1} e^{-\alpha r^m})^{\frac{1}{p}} (r^{-\frac{k-(m-1)p}{p-1} - m + 1} e^{\frac{\alpha}{p-1} r^m})^{\frac{1}{p'}} \\
= \ & (\frac{1}{\alpha m})^{\frac{1}{p}} (\frac{p-1}{\alpha m})^{\frac{1}{p'}} \in (0, \infty).
\end{aligned}$$

Since $\int_0^\infty \phi(r)dr < \infty$ and $\varphi(r)^{1-p'} \in C[0, R]$ for $R > 0$, it concludes that

$$\sup_{r\in[0,\infty)} \Big(\int_r^\infty \phi(r)dr\Big)^{\frac{1}{p}} \Big(\int_0^r \varphi(r)^{1-p'}dr\Big)^{\frac{1}{p'}} < \infty.$$

Applying Riemann-Liouville integral theorem in [20], the assertion of this lemma follows.

We end this section with some inequalities concerning eigenfunctions in $H_\lambda(\mathbb{R}^n)$ which are useful for our investigations (cf. [8]).

Lemma 2.4. *Let* $1 \le p < \infty$*,* $r > 0$ *and* $f \in H_\lambda(\mathbb{R}^n)$*. Then there exists some positive constant C such that*

(*i*) $|f(x)|^p \le \frac{C}{r^n} \int_{\mathbb{B}(x,r)} |f(y)|^p dv(y)$;

(*ii*) $|\nabla f(x)|^p \le \frac{C}{r^n} \int_{\mathbb{B}(x,r)} |\nabla f(y)|^p dv(y)$;

(*iii*) $|\nabla f(x)|^p \le \frac{C}{r^{n+p}} \int_{\mathbb{B}(x,r)} |f(y)|^p dv(y)$.

3. Proof of Theorem 1.1

In this section, we divide the proof of Theorem 1.1 into the following two parts.

Proposition 3.1. *Let* $1 < p < \infty$*,* $\alpha > 0, m \in \mathbb{N}$*,* $t \in \mathbb{R}$*. Then*

$$\int_{\mathbb{R}^n} \Big|\frac{\nabla f(x)e^{-\alpha|x|^m}}{1 + |x|^{m-1}}\Big|^p \frac{dv(x)}{(1 + |x|)^t} \lesssim \int_{\mathbb{R}^n} |f(x)|^p e^{-\alpha p|x|^m} \frac{dv(x)}{(1 + |x|)^t} \tag{3.1}$$

for all $f \in H_\lambda(\mathbb{R}^n)$*.*

Proof. By the subharmonicity of $|f(x)|^p$ and Lemma 2.4, we have

$$|\nabla f(x)|^p \lesssim \frac{(1 + |x|^{m-1})^{n+p}}{\omega_n r^{n+p}} \int_{E_r(x)} |f(y)|^p dv(y), \tag{3.2}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. It follows Lemma 2.2, (3.2) can be rewritten as

$$\Big|\frac{\nabla f(x)e^{-\alpha|x|^m}}{1 + |x|^{m-1}}\Big|^p \lesssim \frac{(1 + |x|^{m-1})^n}{\omega_n r^{n+p}} \int_{E_r(x)} |f(y)|^p e^{-\alpha p|y|^m} dv(y).$$

Combing this with Fubini's theorem, we obtain that

$$\begin{aligned} &\int_{\mathbb{R}^n} \Big|\frac{\nabla f(x)e^{-\alpha|x|^m}}{1 + |x|^{m-1}}\Big|^p \frac{dv(x)}{(1 + |x|)^t} \\ \lesssim\ &\int_{\mathbb{R}^n} \frac{(1 + |x|^{m-1})^n}{(1 + |x|)^t} \int_{E_r(x)} |f(y)|^p e^{-\alpha p|y|^m} dv(y)dv(x) \end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{R}^n} |f(y)|^p e^{-\alpha p|y|^m} dv(y) \int_{E_r(y)} \frac{(1+|x|^{m-1})^n}{(1+|x|)^t} dv(x) \\
&\lesssim \int_{\mathbb{R}^n} |f(y)|^p e^{-\alpha p|y|^m} \frac{dv(y)}{(1+|y|)^t}.
\end{aligned}
$$

This proves the result.

Proposition 3.2. *Let* $1 < p < \infty$*,* $\alpha > 0, m \in \mathbb{N}$*,* $t \in \mathbb{R}$*. Then*

$$
\int_{\mathbb{R}^n} |f(x) - f(0)|^p e^{-\alpha p|x|^m} \frac{dv(x)}{(1+|x|)^t} \lesssim \int_{\mathbb{R}^n} \Big| \frac{\nabla f(x) e^{-\alpha|x|^m}}{1+|x|^{m-1}} \Big|^p \frac{dv(x)}{(1+|x|)^t} \tag{3.3}
$$

for all $f \in H_\lambda(\mathbb{R}^n)$*.*

Proof. To simplify our notation, set $\partial_\rho f(\rho\zeta) = \frac{\partial f(\rho\zeta)}{\partial\rho}$, where $\rho > 0$ and $\zeta \in \mathbb{S}$. By the fundamental theorem of calculus,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |f(x) - f(0)|^p e^{-\alpha p|x|^m} \frac{dv(x)}{(1+|x|)^t} \\
&\lesssim \int_0^\infty \int_{\mathbb{S}} n r^{n-1} |f(r\zeta) - f(0)|^p e^{-\alpha p r^m} \frac{d\sigma(\zeta) dr}{(1+r)^t} \\
&\lesssim \int_0^\infty \int_{\mathbb{S}} n r^{n-1} \Big| \Big(\int_0^r \partial_\rho f(\rho\zeta) d\rho \Big) e^{-\alpha r^m} \Big|^p \frac{d\sigma(\zeta) dr}{(1+r)^t} \\
&\lesssim \int_0^\infty \int_{\mathbb{S}} n r^{n-1} \Big| \Big(\int_0^r |\nabla f(\rho\zeta)| d\rho \Big) \Big|^p e^{-\alpha p r^m} \frac{d\sigma(\zeta) dr}{(1+r)^t} \\
&\lesssim \int_{\mathbb{S}} \int_0^\infty r^{n-1} \Big| \nabla f(r\zeta) \Big|^p e^{-\alpha p r^m} \frac{dr d\sigma(\zeta)}{(1+r)^{t+p(m-1)}},
\end{aligned}
$$

where the last inequality follows from Lemma 2.3.

Hence, by the monotonicity of $M_p^p(\nabla f, r)$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |f(x) - f(0)|^p e^{-\alpha p|x|^m} \frac{dv(x)}{(1+|x|)^t} \\
&\lesssim \int_0^\infty M_p^p(\nabla f, r) e^{-\alpha p r^m} \frac{((\frac{2}{3})^{n-1} + r^{n-1}) dr}{(1+r)^{t+p(m-1)}} \\
&\lesssim \Big\{ \int_0^{\frac{2}{3}} + \int_{\frac{2}{3}}^\infty \Big\} M_p^p(\nabla f, r) e^{-\alpha p r^m} \frac{((\frac{2}{3})^{n-1} + r^{n-1}) dr}{(1+r)^{t+p(m-1)}} \\
&\lesssim \Big\{ M_p^p(\nabla f, \frac{2}{3}) + \int_{\frac{2}{3}}^\infty r^{n-1} M_p^p(\nabla f, r) e^{-\alpha p r^m} \frac{dr}{(1+r)^{t+p(m-1)}} \Big\} \\
&\lesssim \Big\{ \int_{\frac{2}{3}}^1 + \int_{\frac{2}{3}}^\infty \Big\} r^{n-1} M_p^p(\nabla f, r) e^{-\alpha p r^m} \frac{dr}{(1+r)^{t+p(m-1)}} \\
&\lesssim \int_{\mathbb{R}^n} \Big| \frac{\nabla f(x) e^{-\alpha|x|^m}}{1+|x|^{m-1}} \Big|^p \frac{dv(x)}{(1+|x|)^t},
\end{aligned}
$$

as required. The proof of this proposition is finished.

Proof of Theorem 1.1. Gathering Propositions 3.1 and 3.2, (1.3) follows.

4. Lipschitz type characterization

In this section, we discuss the Lipschitz type characterization for the space $F^p_{\alpha,m,t}(\mathbb{R}^n)$ by applying Theorem 1.1.

For $x \in \mathbb{R}^n$, $r > 0$ and $m \in \mathbb{N}$, set

$$\Omega_r(x) = \{y \in \mathbb{R}^n : |x-y|(1+|x|^{m-1}+|y|^{m-1}) < r\}.$$

Obviously, we have $\Omega_r(x) \subset E_r(x)$.

Proof of Theorem 1.2. We first prove $(b) \Rightarrow (a)$. Assume that (b) holds. Fixing x and letting y approach x in the direction of each real coordinate axis, we get

$$|\partial_i f(x)| \lesssim (1+|x|^{m-1})^{1+q} g(x)$$

for each $i \in \{1,2,...,n\}$. Thus, we have

$$\frac{|\nabla f(x)|}{1+|x|^{m-1}} \lesssim (1+|x|^{m-1})^q g(x), \quad x \in \mathbb{B}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^p e^{-\alpha p|x|^m}}{(1+|x|^{m-1})^p} \frac{dv(x)}{(1+|x|)^t} &\lesssim \int_{\mathbb{R}^n} (1+|x|^{m-1})^{pq} |g(x)|^p e^{-\alpha p|x|^m} \frac{dv(x)}{(1+|x|)^t} \\ &\lesssim \int_{\mathbb{R}^n} |g(x)|^p e^{-\alpha p|x|^m} \frac{dv(x)}{(1+|x|)^{t-pq(m-1)}}. \end{aligned}$$

It follows from the assumption $g \in L^p(dG_{\alpha p,m,t-pq(m-1)})$ that

$$\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^p e^{-\alpha p|x|^m}}{(1+|x|^{m-1})^p} \frac{dv(x)}{(1+|x|)^t} < \infty.$$

Hence $f \in F^p_{\alpha,m,t}(\mathbb{R}^n)$ by Theorem 1.1.

For the converse, we assume $f \in F^p_{\alpha,m,t}(\mathbb{R}^n)$. Fix $r > 0$ and consider any two points $x, y \in \mathbb{R}^n$ with $y \in \Omega_r(x)$. Since $sy + (1-s)x \in E_r(x)$ for $0 \le s \le 1$, it is given that

$$\begin{aligned} |f(x) - f(y)| &= \Big| \int_0^1 \frac{df}{ds}(sy + (1-s)x) ds \Big| \\ &\le \sqrt{n}|x-y| \int_0^1 |\nabla f(sy + (1-s)x)| ds \\ &\lesssim |x-y| \sup\{|\nabla f(\xi)| : \xi \in E_r(x)\}. \end{aligned}$$

Note that for each $\xi \in E_r(x)$,

$$1 + |\xi|^{m-1} \approx 1 + |x|^{m-1} \approx 1 + |x|^{m-1} + |y|^{m-1},$$

and thus

$$|f(x) - f(y)| \le |x-y|(1+|x|^{m-1}+|y|^{m-1})^{1+q} h(x),$$

where

$$h(x) = C(r) \sup_{\xi \in E_r(x)} \frac{|\nabla f(\xi)|}{(1+|\xi|^{m-1})^{1+q}}.$$

If $y \notin \Omega_r(x)$, that is,

$$|x-y|(1+|x|^{m-1}+|y|^{m-1}) \geq r,$$

then the triangle inequality implies

$$\begin{aligned} & |f(x)-f(y)| \\ \leq\ & \frac{|x-y|(1+|x|^{m-1}+|y|^{m-1})}{r}(|f(x)|+|f(y)|) \\ \lesssim\ & \frac{|x-y|(1+|x|^{m-1}+|y|^{m-1})^{1+q}}{r}\Big(\frac{|f(x)|}{(1+|x|^{m-1})^q}+\frac{|f(y)|}{(1+|y|^{m-1})^q}\Big). \end{aligned}$$

By letting $g(x) = h(x) + \frac{|f(x)|}{r(1+|x|^{m-1})^q}$, we obtain

$$|f(x)-f(y)| \leq |x-y|(1+|x|^{m-1}+|y|^{m-1})^q(g(x)+g(y))$$

for all $x, y \in \mathbb{R}^n$. It is clear that $\frac{|f(x)|}{r(1+|x|^{m-1})^q} \in L^p(dG_{\alpha p,m,t-pq(m-1)})$ from the assumption $f \in F^p_{\alpha,m,t}(\mathbb{R}^n)$ and thus g is the desired function provided that $h \in L^p(dG_{\alpha p,m,t-pq(m-1)})$.

Now, we claim that $h \in L^p(dG_{\alpha p,m,t-pq(m-1)})$. From the definition of $E_r(x)$, it is easy for us to find $r_1 > r$ such that $E_r(\xi) \subset E_{r_1}(x)$ for each $\xi \in E_r(x)$. By Lemmas 2.2 and 2.4, we deduces that

$$\begin{aligned} \frac{|\nabla f(\xi)|^p}{(1+|\xi|^{m-1})^{p(1+q)}} &\leq (1+|\xi|^{m-1})^{n-pq}\int_{E_r(\xi)}|f(y)|^p dv(y) \\ &\lesssim (1+|x|^{m-1})^{n-pq}\int_{E_{r_1}(x)}|f(y)|^p dv(y). \end{aligned}$$

Taking the supremum over all $\xi \in E_r(x)$ leads to

$$|h(x)|^p \lesssim (1+|x|^{m-1})^{n-pq}\int_{E_{r_1}(x)}|f(y)|^p dv(y).$$

Integrating both sides of the above inequality against the measure $dG_{\alpha p,m,t-pq(m-1)}$ and applying Fubini's theorem, we have

$$\begin{aligned} & \int_{\mathbb{R}^n}|h(x)|^p dG_{\alpha p,m,t-pq(m-1)} \\ =\ & \int_{\mathbb{R}^n}\left|h(x)e^{-\alpha|x|^m}\right|^p\frac{dv(x)}{(1+|x|)^{t-pq(m-1)}} \\ \lesssim\ & \int_{\mathbb{R}^n}\frac{(1+|x|^{m-1})^{n-pq}e^{-\alpha p|x|^m}}{(1+|x|)^{t-pq(m-1)}}\int_{E_{r_1}(x)}|f(y)|^p dv(y)dv(x) \\ \lesssim\ & \int_{\mathbb{R}^n}|f(y)|^p dv(y)\int_{E_{r_1}(y)}\frac{(1+|x|^{m-1})^{n-pq}e^{-\alpha p|x|^m}}{(1+|x|)^{t-pq(m-1)}}dv(x). \end{aligned}$$

It follows from Lemma 2.2 again that

$$\int_{\mathbb{R}^n} |h(x)|^p dG_{\alpha p,m,t-pq(m-1)} \lesssim \int_{\mathbb{R}^n} |f(y)|^p e^{-\alpha p|y|^m} \frac{dv(y)}{(1+|y|)^t},$$

which is what we need.

The proof of Theorem 1.2 is complete.

From the proof of Theorem 1.2, the following local version of Theorem 1.2 can be easily derived for arbitrary $q \in \mathbb{R}$.

Theorem 4.1. *Let* $1 < p < \infty$, $\alpha > 0, m \in \mathbb{N}$, $t, q \in \mathbb{R}$ *and* $f \in H_\lambda(\mathbb{R}^n)$. *Then the following two statements are equivalent on* $\mathbb{R}^n$*:*
(a) $f \in F^p_{\alpha,m,t}(\mathbb{R}^n)$;
(b) *There exists a positive continuous function* $g \in L^p(dG_{\alpha p,m,t-pq(m-1)})$ *such that*

$$\frac{|f(x) - f(y)|}{|x - y|} \leq (1 + |x|^{m-1} + |y|^{m-1})^{1+q}\big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{R}^n$ *with* $y \in \Omega_r(x)$ *and* $x \neq y$.

5. Double integral characterization

In this section, we shall prove Theorem 1.3.

Theorem 5.1. *Let* $1 < p < \infty$, $\alpha > 0, m \in \mathbb{N}$, $t \in \mathbb{R}$ *and* $f \in H_0(\mathbb{R}^n)$. *Then the following two statements are equivalent on* $\mathbb{R}^n$*:*
(a) $f \in F^p_{\alpha,m,t}(\mathbb{R}^n)$;
(b) $Lf \in L^p(dG_{\alpha p,m,t} \times dG_{\alpha p,m,t})$.

Proof. Let $f \in H_0(\mathbb{R}^n)$. We first assume that (a) holds. Then

$$\begin{aligned}
&||Lf||^p_{L^p(dG_{\alpha p,m,t} \times dG_{\alpha p,m,t})} \\
= & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x) - f(y)|^p dG_{\alpha p,m,t}(x) dG_{\alpha p,m,t}(y) \\
\lesssim & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (|f(x)|^p + |f(y)|^p) dG_{\alpha p,m,t}(x) dG_{\alpha p,m,t}(y) \\
\lesssim & \int_{\mathbb{R}^n} |f(x)|^p dG_{\alpha p,m,t}(x)
\end{aligned}$$

and thus (b) holds.

Conversely, assume (b) holds. Fixing $x \in \mathbb{B}$ and replacing f by $f - f(x)$, it follows from Lemma 2.4, we have

$$\begin{aligned}
|f(x) - f(0)|^p &\lesssim \int_{E_r(0)} |f(x) - f(y)|^p dv(y) \\
&\lesssim \int_{E_r(0)} |f(x) - f(y)|^p dG_{\alpha p,m,t}(y)
\end{aligned}$$

$$\lesssim \int_{\mathbb{R}^n} |f(x) - f(y)|^p dG_{\alpha p,m,t}(y).$$

Integrating both sides of the above against the measure $dG_{\alpha p,m,t}(x)$ gives

$$\int_{\mathbb{R}^n} |f(x) - f(0)|^p dG_{\alpha p,m,t}(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x) - f(y)|^p dG_{\alpha p,m,t}(y) dG_{\alpha p,m,t}(x),$$

from which we see that $f \in F^p_{\alpha,m,t}(\mathbb{R}^n)$. The proof of this theorem is finished.

Now, we come to characterize $F^p_{\alpha,m,t}(\mathbb{R}^n)$ in terms of double integral of $L^s_r f$ as follows.

Theorem 5.2. *Let $1 < p < \infty$, $\alpha > 0, m \in \mathbb{N}$, $t, s \in \mathbb{R}$ and $f \in H_0(\mathbb{R}^n)$. Then the following two statements are equivalent on $\mathbb{R}^n$:*
(a) $f \in F^p_{\alpha,m,t}(\mathbb{R}^n)$;
(b) $L^s_r f \in L^p(dG_{\beta p,m,\gamma} \times dG_{\beta p,m,\gamma})$, where $\beta = \frac{s+\alpha}{2}, \gamma = \frac{t-n(m-1)}{2}$.

Proof. Let us first assume that (a) holds. Then

$$\begin{aligned}
&||L^s_r f||^p_{L^p(dG_{\beta p,m,\gamma}\times dG_{\beta p,m,\gamma})} \\
= &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x) - f(y)|^p e^{sp|x|^m} \chi_{E_r(x)}(y) dG_{\beta p,m,\gamma}(x) dG_{\beta p,m,\gamma}(y) \\
\lesssim &\int_{\mathbb{R}^n}\int_{E_r(x)} (|f(x)|^p + |f(y)|^p) e^{sp|x|^m} dG_{\beta p,m,\gamma}(y) dG_{\beta p,m,\gamma}(x) \\
\lesssim &\int_{\mathbb{R}^n}\int_{E_r(x)} |f(x)|^p e^{sp|x|^m} dG_{\beta p,m,\gamma}(y) dG_{\beta p,m,\gamma}(x) \\
&+ \int_{\mathbb{R}^n}\int_{E_r(x)} |f(y)|^p e^{sp|y|^m} dG_{\beta p,m,\gamma}(y) dG_{\beta p,m,\gamma}(x).
\end{aligned}$$

By applying Lemma 2.2 and Fubini's theorem, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{E_r(x)} |f(x)|^p e^{sp|x|^m} dG_{\beta p,m,\gamma}(y) dG_{\beta p,m,\gamma}(x) \\
\lesssim &\int_{\mathbb{R}^n} |f(x)|^p e^{sp|x|^m} dG_{\beta p,m,\gamma}(x) \int_{E_r(x)} e^{-\beta p|y|^m} \frac{dv(y)}{(1 + |y|)^\gamma} \\
\lesssim &\int_{\mathbb{R}^n} |f(x)|^p dG_{\alpha p,m,t}(x)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{E_r(x)} |f(y)|^p e^{sp|y|^m} dG_{\beta p,m,\gamma}(y) dG_{\beta p,m,\gamma}(x) \\
\lesssim &\int_{\mathbb{R}^n} |f(y)|^p e^{sp|y|^m} dG_{\beta p,m,\gamma}(y) \int_{E_r(y)} dG_{\beta p,m,\gamma}(x) \\
\lesssim &\int_{\mathbb{R}^n} |f(y)|^p dG_{\alpha p,m,t}(y).
\end{aligned}$$

Therefore

$$\|L_r^s\|^p_{L^p(G_{\alpha p,m,t}\times G_{\alpha p,m,t})} \lesssim \int_{\mathbb{R}^n} |f(x)|^p dG_{\alpha p,m,t}(x).$$

Conversely, we assume (b) holds. Fixing $x \in \mathbb{B}$ and $f \in H_0(\mathbb{R}^n)$, let

$$g_x(y) = [f(y) - f(x)]e^{s|x|^m}.$$

Then it is easy to check that $g_x(y) \in H_0(\mathbb{R}^n)$ and $\nabla g_x(x) = \nabla f(x)e^{s|x|^m}$. Applying Lemmas 2.2 and 2.4, we obtain

$$\Big(\frac{|\nabla f(x)|e^{s|x|^m}}{1+|x|^{m-1}}\Big)^p \leq (1+|x|^{m-1})^n \int_{E_r(x)} |f(y)-f(x)|^p e^{sp|x|^m} dv(y).$$

By integrating both sides of the above against the measure $dG_{(s+\alpha)p,m,t}(x)$ and Lemma 2.2 again, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big|\frac{\nabla f(x)e^{-\alpha|x|^m}}{1+|x|^{m-1}}\Big|^p \frac{dv(x)}{(1+|x|)^t} \\
\leq\ & \int_{\mathbb{R}^n} (1+|x|^{m-1})^n \int_{E_r(x)} |f(y)-f(x)|^p e^{sp|x|^m} dv(y) dG_{(s+\alpha)p,m,t}(x) \\
\leq\ & \int_{\mathbb{R}^n}\int_{E_r(x)} |f(y)-f(x)|^p e^{sp|x|^m} dv(y) dG_{(s+\alpha)p,m,t-n(m-1)}(x) \\
\leq\ & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|^p e^{sp|x|^m} \chi_{E_r(x)}(y) dG_{\beta p,m,\gamma}(x) dG_{\beta p,m,\gamma}(y).
\end{aligned}$$

Hence, by Theorem 1.1, we obtain

$$\int_{\mathbb{R}^n} |f(x)|^p dG_{\alpha p,m,t}(x) \lesssim \|L_r^s f\|^p_{L^p(dG_{\beta p,m,\gamma}\times dG_{\beta p,m,\gamma})}.$$

The proof of this theorem is complete.

6. Conclusions

We obtain a norm equivalence for an exponential type weighted integral of an eigenfunction and its derivative on $\mathbb{R}^n$. By using this result, we characterize Fock-type spaces of eigenfunctions on $\mathbb{R}^n$ in terms of Lipschitz type conditions and double integral conditions. All of these results are extensions of the corresponding ones in classcial Fock space.

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Conflict of interest

The authors declare that there is no conflicts of interest regarding the publication of this article.

References

1. W. Urbina, On singular integrals with respect to the Gaussian measure, *Ann. Scuola Norm. Super. Pisa Cl. Sci.*, **17** (1990), 531–567.
2. H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, New York: Springer-Verlag, 2000.
3. B. R. Choe, K. Nam, Double integral characterizations of harmonic Bergman spaces, *J. Math. Anal. Appl.*, **379** (2011), 889–909. https://doi.org/10.1016/j.jmaa.2011.02.024
4. X. Fu, J. Qiao, Bergman spaces, Bloch spaces and integral means of p-harmonic functions, *Bull. Korean Math. Soc.*, **58** (2021), 481–495. https://doi.org/10.4134/BKMS.B200367
5. J. Kinnunen, M. Kotilainen, V. Latvala, Hardy-Littlewood type gradient estimates for quasiminimizers, *Boll. Unione Mat. Ital.*, **3** (2010), 125–136.
6. K. Nam, Lipschitz type characterizations of harmonic Bergman spaces, *Bull. Korean Math. Soc.*, **50** (2013), 1277–1288. https://doi.org/10.4134/BKMS.2013.50.4.1277
7. M. Pavlović, J. Peláez, An equivalence for weighted integrals of an analytic function and its derivative, *Math. Nachr.*, **281** (2008), 1612–1623. https://doi.org/10.1002/mana.200510701
8. M. Stoll, Harmonic majorants for eigenfunctions of the Laplacian with finite Dirichlet integrals, *J. Math. Anal. Appl.*, **274** (2002), 788–811. https://doi.org/10.1016/S0022-247X(02)00364-5
9. H. Wulan, K. Zhu, Lipschitz type characterizations for Bergman spaces, *Can. Math. Bull.*, **52** (2009), 613–626. https://doi.org/10.4153/CMB-2009-060-6
10. K. Zhu, *Analysis on Fock spaces*, Graduate Texts in Mathematics, New York: Springer, 2012.
11. J. M. Ha, H. R. Cho, H. W. Lee, A norm equivalence for the mixed norm of Fock type, *Complex Var. Elliptic Equ.*, **61** (2016), 1644–1655. https://doi.org/10.1080/17476933.2016.1197916
12. Z. Hu, Equivalent norms on Fock spaces with some application to extended Cesàro operators, *Proc. Amer. Math. Soc.*, **141** (2013), 2829–2840. https://doi.org/10.1090/S0002-9939-2013-11550-9
13. B. R. Choe, K. Nam, New characterizations for the weighted fock spaces, *Complex Anal. Oper. Theory.*, **13** (2019), 2671–2686. https://doi.org/10.1007/s11785-018-0850-1
14. J. M. Ha, K. Nam, Characterizations for the Fock-type spaces, *Bull. Korean Math. Soc.*, **56** (2019), 745–756. http://dx.doi.org/10.4134/BKMS.b180540
15. H. R. Cho, K. Zhu, Fock-Sobolev spaces and their Carleson measures, *J. Funct. Anal.*, **263** (2012), 2483–2506. https://doi.org/10.1016/j.jfa.2012.08.003
16. N. Constantin, A Volterra-type integration operator on Fock spaces, *Proc. Amer. Math. Soc.*, **140** (2012), 4247–4257. http://dx.doi.org/10.1090/S0002-9939-2012-11541-2
17. M. Engliš, Berezin transform on the harmonic Fock space, *J. Math. Anal. Appl.*, **367** (2010), 75–97. https://doi.org/10.1016/j.jmaa.2009.12.028

18. S. Li, H. Wulan, R. Zhao, K. Zhu, A characterization of Bergman spaces on the unit ball of $\mathbb{C}^n$, *Glasgow Math. J.*, **51** (2009), 315–330. https://doi.org/10.1017/S0017089509004996

19. S. Li, H. Wulan, K. Zhu, A characterization of Bergman spaces on the unit ball of $\mathbb{C}^n$ II, *Can. Math. Bull.*, **55** (2012), 146–152. https://doi.org/10.4153/CMB-2011-047-6

20. F. J. Martin-Reyes, E. Sawyer, Weighted inequalities for Riemann-Liouville fractional integrals of order one and greater, *Proc. Amer. Math. Soc.*, **106** (1989), 727–733. https://doi.org/10.1090/S0002-9939-1989-0965246-8

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