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Research article

On the solutions of certain fractional kinetic matrix equations involving Hadamard fractional integrals

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Abstract: Currently, matrix fractional differential equations have several applications in diverse fields, including mathematical analysis, control systems, economics, optimization theory, physics, astrophysics and engineering. In this line of research, we introduce generalized fractional kinetic equations including extended k-Hurwitz-Lerch zeta-matrix functions. By applying the Hadamard fractional integral properties and via the Mellin integral transform, we present the solution of fractional kinetic matrix equations involving families of Hurwitz-Lerch zeta matrix functions. In addition, we consider a number of specific instances of our key results.

Keywords: Hadamard fractional integral; fractional kinetic equations; Hurwitz-Lerch zeta-matrix functions
Mathematics Subject Classification: 33B15, 33C05, 33C20, 34A05

1. Introduction

In mathematical analysis, mathematical physics, mathematical modelings and engineering processes, fractional calculus is able to work requisitely in solving certain boundary value problems or certain integral equations. Although there exist in the literature many definitions for fractional integral operators, the Riemann-Liouville and Caputo are the most common for fractional integrals. Among the many fractional integrals operators is the Hadamard fractional integral, the definition of which goes back to the works of Hadamard in 1892 [1]. Recently, many studies on the Hadamard fractional integral and its applications in various fields have been achieved, including those by Butzer et al. [2, 3], Pooseh et al. [4], Farid and Habibullah [5], Azam et al. [6], Abbas et al. [7], Boutiara et al. [8] and Ahmed et al. [9].

On the contrary, the mainstream and, perhaps, the most effective approach to the field of differential equations is fractional calculus approach, which has been recently discussed in fundamental works (for instance, see [10–12]). It is known that fractional differential equations are generalizations of differential equations in an arbitrary non-integer-order setting. Among the many fractional differential equations are the fractional kinetic equations (FKEs). The kinetic (reaction-type) equations have prime importance as a mathematical tool widely used to describe several astrophysical and physical phenomena [13]. In [14], the authors considered the FKE involving the Riemann-Liouville fractional integral. Recently, using various integral transforms, FKEs comprising a large array of special functions have been extensively applied to elucidate and solve many significant problems of physical phenomenons (see, e.g., [15–20]).

Nowadays, owing to the significance of the earlier work on FKEs and other important fractional differential equations, one should note that many researchers became interested in analyzing the scalar classic cases of the differential equations in a matrix setting. The use of matrix fractional differential equations (MFDEs) has been applied in several fields such as those related to statistics, physical phenomena, simulating reduction problems, communication systems and allied sciences; for instance, see [21–26] and the references cited therein.

In consideration of the aforementioned works, the current study was designed to highlight establishing an extensive form of the fractional kinetic matrix equation (FKME) involving families of the Hurwitz-Lerch zeta matrix functions by using the technique of the Hadamard fractional integral operator via the Mellin integral transform. In addition to these, solutions of the FKMEs under special conditions of the families of the Hurwitz-Lerch zeta matrix functions have been reported. It is also worth noting that this work is primarily analytical and designed to develop new properties using the proposed algorithm, which are needed for future applications.

2. Preliminaries

In this section, we recall some definitions and terminologies that will be used to prove the main results. Throughout our present work, let $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$, and $\mathbb{C}$ be the sets of positive integers, positive real numbers, non-positive integers, and complex numbers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. In addition, let $\mathbb{C}^{m\times m}$ be the vector space of all of the square matrices of order $m \in \mathbb{N}$ with entries contained in $\mathbb{C}$. Further, let I and $\mathtt{0}$ denote the identity and zero matrices corresponding to a square matrix of any order, respectively.

If $\mathrm{T} \in \mathbb{C}^{m\times m}$, the spectrum $\sigma(\mathrm{T})$ is the set of all eigenvalues of T for which we denote

$$\mu(\mathrm{T}) = \max\{\mathrm{Re}(\xi) : \xi \in \sigma(\mathrm{T})\} \quad \text{and} \quad \widetilde{\mu}(\mathrm{T}) = \min\{\mathrm{Re}(\xi) : \xi \in \sigma(\mathrm{T})\}, \tag{2.1}$$

where $\mu(\mathrm{T})$ refers to the spectral abscissa of T and for which $\widetilde{\mu}(\mathrm{T}) = -\mu(-\mathrm{T})$. A matrix T is said to be a positive stable if and only if $\widetilde{\mu}(\mathrm{T}) > 0$.

Definition 2.1. *If* T *is a positive stable matrix in* $\mathbb{C}^{m\times m}$ *and* $k \in \mathbb{R}^+$, *then the k- gamma matrix function* $\Gamma_k(\mathrm{T})$ *is well-defined as follows (cf. [27]):*

$$\Gamma_k(\mathrm{T}) = \int_0^\infty v^{\mathrm{T}-I}\, e^{-\frac{v^k}{k}}\, dv, \quad v^{\mathrm{T}-I} := \exp\left((\mathrm{T} - I)\ln v\right). \tag{2.2}$$

If T *is a matrix in* $\mathbb{C}^{m\times m}$ *such that* $\mathrm{T}+\ell kI$ *is an invertible matrix for every* $\ell\in\mathbb{N}_0$ *and* $k\in\mathbb{R}^+$*, then* $\Gamma_k(\mathrm{T})$ *is invertible, its inverse is* $\Gamma_k^{-1}(\mathrm{T})$*, and the k-Pochhammer matrix symbol is defined by*

$$(\mathrm{T})_{\ell,k}=\mathrm{T}(\mathrm{T}+kI)\cdots(T+(\ell-1)kI)=\Gamma_k(\mathrm{T}+\ell kI)\,\Gamma_k^{-1}(\mathrm{T})\quad(\ell\in\mathbb{N}_0,k\in\mathbb{R}^+).\tag{2.3}$$

Definition 2.2. *[24, 28] (Fractional matrix power)*

For a nonsingular matrix T *in* $\mathbb{C}^{m\times m}$ *we define* T^{ν} *a for an arbitrary real number* ν *by* $\mathrm{T}^{\nu}=\exp(\nu\log\mathrm{T})$*, where the logarithm is the principal matrix logarithm.*

In general, it is not true that $(\mathrm{T}^{\nu})^{\mu}=(\mathrm{T}^{\mu})^{\nu}$ *for real* ν *and* μ*, although for symmetric positive definite matrices this identity does hold because the eigenvalues are real and positive.*

If $X=\mathrm{T}^{\nu}$*, does it follow that* $\mathrm{T}=X^{\frac{1}{\nu}}$*? Clearly, the answer is no in general because, for example* $X=\mathrm{T}^2$*, does not imply* $\mathrm{T}=X^{1/2}$*. Using the matrix unwinding function it can be shown that* $(\mathrm{T}^{\nu})^{\frac{1}{\nu}}=\mathrm{T}$ *for* $\nu\in[-1,1]$*. Hence the function* $G(\mathrm{T})=\mathrm{T}^{\frac{1}{\nu}}$ *is the inverse function of* $F(\mathrm{T})=\mathrm{T}^{\nu}$ *for* $\nu\in[-1,1]$*.*

Definition 2.3. *Let* T *be a positive stable matrix in* $\mathbb{C}^{m\times m}$*. The generalized Riemann* ζ *matrix function [29] is defined by*

$$\zeta(T,\beta)=\sum_{n=0}^{\infty}(\beta+n)^{-T},\quad\beta\in\mathbb{C}.\tag{2.4}$$

In particular, by setting $\beta=1$*, (2.4) turns into a matrix analogue of the Riemann* ζ *function.*

Definition 2.4. *Let* T *be a positive stable matrix in* $\mathbb{C}^{m\times m}$*. Then, an extension of the k-gamma function of a matrix argument (2.2) is defined in [27] as follows:*

$$\Gamma_k^{\vartheta}(T)=\int_0^{\infty}v^{T-I}\,e^{\left(-\frac{v^k}{k}-\frac{\vartheta^k}{kv^k}\right)}\,dv\qquad(\vartheta\in\mathbb{R}_0^+,\ \ k\in\mathbb{R}^+).\tag{2.5}$$

Definition 2.5. *[30] Let* T*,* D*,* E *and* F *be positive stable matrices in* $\mathbb{C}^{m\times m}$*, such that* $T+\ell I$ *and* $F+\ell I$ *are invertible for all* $\ell\in\mathbb{N}_0$*,* $\sigma\in\mathbb{R}_0^+$*,* $k\in\mathbb{R}^+$*, and* $\alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-$*. Then, for* $|w|<1$*, the generalized extended k-Hurwitz-Lerch* $\zeta-$ *matrix function is defined by:*

$${}_2\Theta_1^{T;k,\alpha;\sigma}\left[\begin{array}{c}D,E\\F\end{array};w\right]=\sum_{n=0}^{\infty}(n+\alpha)^{-T}\,(D;\sigma)_{n,k}\,(E)_{n,k}\,[(F)_{n,k}]^{-1}\,\frac{w^n}{n!},\tag{2.6}$$

where $(D;\sigma)_{n,k}$ *is the generalized k-Pochhammer matrix symbols, which are defined as*

$$(D;\sigma)_{n,k}=\begin{cases}\Gamma_k^{\sigma}(D+nI)\,\Gamma_k^{-1}(D), & (\widetilde{\mu}(D)>0,\sigma,k\in\mathbb{R}^+,n\in\mathbb{N})\\ (D)_{n,k}, & (\sigma=0,\ k\in\mathbb{R}^+,n\in\mathbb{N})\\ I, & (n=0,\ \sigma=0,\ k=1)\end{cases}\tag{2.7}$$

Remark 2.1. *Some particular cases of (2.6) are in the following representations:*

i- *For $k = 1$ in (2.6), we get the matrix version of the result in [31] as follows*

$$ {}_2\Theta_1^{T;\alpha;\sigma}\left[\begin{array}{c} D, E \\ F \end{array} ; w\right] = \sum_{n=0}^{\infty} (n+\alpha)^{-T}\ (D;\sigma)_n\ (E)_n\ [(F)_n)]^{-1}\frac{w^n}{n!}, \tag{2.8} $$

where $T, D, E, F \in \mathbb{C}^{m\times m}, \alpha \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $\sigma \in \mathbb{R}_0^+$ when $|w| < 1$ and $\widetilde{\mu}(T + F - D - E) > 1$ when $|w| = 1$.

ii- *If we set $\sigma = 0$ in (2.6), it reduces to the following k-analogue of the generalized Hurwitz-Lerch $\zeta-$ matrix function:*

$$ {}_2\Theta_1^{T;k,\alpha}\left[\begin{array}{c} D, E \\ F \end{array} ; w\right] = \sum_{n=0}^{\infty} (n+\alpha)^{-T}\ (D)_{n,k}\ (E)_{n,k}\ [(F)_{n,k})]^{-1}\ \frac{w^n}{n!}, \tag{2.9} $$

where $T, D, E, F \in \mathbb{C}^{m\times m}, \alpha \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $k \in \mathbb{R}^+$ when $|w| < 1$ and $\widetilde{\mu}(T + F - D - E) > 1$ when $|w| = 1$.

iii- *When $k = 1$ and $\sigma = 0$ we obtain the matrix version of the definition in [32] as follows*

$$ {}_2\Theta_1^{T;\alpha}\left[\begin{array}{c} D, E \\ F \end{array} ; w\right] = \sum_{n=0}^{\infty} (n+\alpha)^{-T}\ (D)_n\ (E)_n\ [(F)_n)]^{-1}\ \frac{w^n}{n!}, \tag{2.10} $$

where $T, D, E, F \in \mathbb{C}^{m\times m}$ and $\alpha \in \mathbb{C}\setminus\mathbb{Z}_0^-$ when $|w| < 1$ and $\widetilde{\mu}(T + F - D - E) > 1$ when $|w| = 1$.

iv- *Further, when $k = 1$ and $E = F$ of (2.6) can be easily seen to yield the following matrix version of the result in [33]*

$$ {}_1\Theta_0^{T;\alpha;\sigma}(D; w) = \sum_{n=0}^{\infty} (n+\alpha)^{-T}\ (D;\sigma)_n\ \frac{w^n}{n!}, \tag{2.11} $$

where $T,\ D \in \mathbb{C}^{m\times m},\ \sigma,\ \alpha \in \mathbb{C}\setminus\mathbb{Z}_0^-,$ when $|w| < 1$ and $\widetilde{\mu}(T - D) > 1$ when $|w| = 1$.

v- *Further, when $\sigma = 0$ in (2.11), it is easily seen to yield the following k- Hurwitz-Lerch $\zeta-$ matrix function which is a generalization of the result in [34]*

$$ {}_1\Theta_0^{T;k;\alpha}(D; w) = \sum_{n=0}^{\infty} (n+\alpha)^{-T}\ (D)_{n,k}\ \frac{w^n}{n!}, \tag{2.12} $$

where $T,\ D \in \mathbb{C}^{m\times m}$ and $\alpha \in \mathbb{C}\setminus\mathbb{Z}_0^-,$ when $|w| < 1$ and $\widetilde{\mu}(T - D) > 1$ when $|w| = 1$.

vi- *We can easily retrieve the classical Hurwitz-Lerch $\zeta-$ function defined in [31] from (2.12), when $T = \mu \in \mathbb{C}^{1\times 1}$ and $D = 1 \in \mathbb{C}^{1\times 1}$.*

Definition 2.6. *The Mellin transform of a suitable integrable function $\mathfrak{G}(t)$ is defined [35], as usual, by*

$$ G(\delta) = \mathcal{M}\Big\{\mathfrak{G}(t) : t \to \delta\Big\} = \int_0^{\infty} t^{\delta-1}\ \mathfrak{G}(t)\ dt \qquad (\delta \in \mathbb{C}), \tag{2.13} $$

provided that the improper integral in (2.13) *exists. And the inverse Mellin transform is*

$$\mathfrak{G}(t) = \mathcal{M}^{-1}\Big\{G(\delta) : \delta \to t\Big\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-\delta}\, G(\delta)\, d\delta \qquad (c = \mathrm{Re}\,\delta). \tag{2.14}$$

Further, the Mellin convolution of two functions $\theta(t)$ and $\phi(t)$ is defined as

$$(\theta * \phi)(t) = \int_0^t \theta\left(\frac{t}{x}\right)\phi(x)\frac{dx}{x}. \tag{2.15}$$

Lemma 2.1. *[30] For a matrix $R \in \mathbb{C}^{m\times m}$, $\sigma \in \mathbb{R}_0^+$, and $k, \delta \in \mathbb{R}^+$; then, we have*

$$\mathcal{M}\Big\{\Gamma_k^{\sigma}(R) : \delta\Big\} = \Gamma_k(\delta I)\,\Gamma_k(R + \delta I) \qquad (\widetilde{\mu}(R + \delta I) > 0 \text{ when } k = 1), \tag{2.16}$$

where $\Gamma_k^{\sigma}(R)$ is the extended k-gamma of the matrix argument defined in (2.5).

Theorem 2.1. *[30] Under the conditions of the hypothesis in Definition 2.5, the Mellin transform of the extended k-Hurwitz-Lerch $\zeta-$ matrix function is given by*

$$\begin{aligned}&\mathcal{M}\left\{ {}_2\Theta_1^{T;k,\alpha;\sigma}\left[\begin{array}{c} D, E \\ F \end{array} ; w\right] : \sigma \to \delta\right\}\\ &= \Gamma_k(\delta)\,(D)_{\delta,k}\ {}_2\Theta_1^{T;k,\alpha;\sigma}\left[\begin{array}{c} D + \delta I, E \\ F \end{array} ; w\right],\end{aligned} \tag{2.17}$$

where $\Re(\delta) > 0$ and $\widetilde{\mu}(D + \delta I) > 0$ when $\sigma = 0$ and $k = 1$.

Definition 2.7. *[35] Let* $\mathrm{Re}(\gamma) > 0$. *The left-sided and the right-sided Hadamard fractional integrals of order $\gamma \in \mathbb{C}$ are defined, respectively as*

$$\left({}_H I_+^{\gamma} f\right)(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \left(\log\frac{t}{\tau}\right)^{\gamma-1}\frac{f(\tau)}{\tau}d\tau,\ \ t > 0,$$

and

$$\left({}_H I_-^{\gamma} f\right)(t) = \frac{1}{\Gamma(\gamma)}\int_t^{\infty} \left(\log\frac{\tau}{t}\right)^{\gamma-1}\frac{f(\tau)}{\tau}d\tau,\ \ t > 0.$$

Lemma 2.2. *[35] If* $\mathrm{Re}(\gamma) > 0$*, $\tau \in \mathbb{C}$, and the Mellin transform $\mathcal{M}(f)(\tau)$ exists for a function f, then the following holds true:*

$$\mathcal{M}\left({}_H I_+^{\gamma} f\right)(\tau) = (-\tau)^{-\gamma}\left(\mathcal{M}f\right)(\tau),\ \ \mathrm{Re}(\tau) < 0,$$

and

$$\mathcal{M}\left({}_H I_-^{\gamma} f\right)(\tau) = (\tau)^{-\gamma}\left(\mathcal{M}f\right)(\tau),\ \ \mathrm{Re}(\tau) > 0.$$

Theorem 2.2. *[36] For $t \in [0, \xi]$*

$$\mathcal{M}[f(t)](\tau) = \mathcal{F}(\tau) = \int_0^{\xi} \xi^{-\tau} t^{\tau-1} f(t)dt$$

and

$$f(t) = \mathcal{M}^{-1}[\mathcal{F}(\tau)](t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{t^{-\tau}}{\tau}\mathcal{F}(\tau)d\tau.$$

3. Main results

In this section we are going to study FKEs involving Hadamard fractional integrals associated with a generalized extended k-Hurwitz-Lerch $\zeta-$ matrix functions.

Theorem 3.1. *Let T_μ, D_μ, E_μ, F_μ and C be positive stable matrices in $\mathbb{C}^{m\times m}$ such that $T_\mu + \ell I$ and $F_\mu + \ell I$ are invertible for all $\mu \in \mathbb{N}$, $\ell \in \mathbb{N}_0$ $\delta, \sigma \in \mathbb{R}_0^+$, $d, k, \xi \in \mathbb{R}^+$ and $\alpha_\mu \in \mathbb{C}\setminus\mathbb{Z}_0^-$. Then, for $\gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $t \in [0,\xi]$ and ${}_2\Theta_1^{T_\mu;k_\mu,\alpha_\mu;\sigma}$ is generalized from (2.6); the generalized FKME*

$$N(t)I - N_0 t^{\delta-1}\prod_{\mu=1}^{n} {}_2\Theta_1^{T_\mu;k_\mu,\alpha_\mu;\sigma}\left[\begin{array}{c} D_\mu, E_\mu \\ F_\mu \end{array}; d^\gamma t^\gamma\right] = -C^\gamma {}_H I_t^\gamma N(t) \tag{3.1}$$

is solvable. The solution to (3.1) *is given by*

$$\begin{aligned} N(t)I =& N_0\, \xi^{\delta-1}\log(t)\prod_{\mu=1}^{n}\sum_{s=0}^{\infty}(s+\alpha_\mu)^{-T_\mu}(D_\mu;\sigma)_{n,k_\mu}(E_\mu)_{n,k_\mu}[(F_\mu)_{n,k_\mu}]^{-1} \\ &\times\left(\frac{d^{\gamma s}\xi^{\gamma s}}{s!}\right)^{\mu}\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty}\left[-\left(\log t^{C}\right)^{\gamma}\right]^{r}\left[\log t^{-(\gamma\mu s+\delta-1)}\right]^{\ell}\Gamma[1-(\gamma r+\ell+2)]. \end{aligned}\tag{3.2}$$

Proof. According to Lemma 2.2, if $\mathcal{N}(z)$ is the Mellin transform of $N(t)$ we have

$$\mathcal{M}\left[{}_H I_t^\gamma N(t)\right](z) = z^{-\gamma}\mathcal{N}(z).$$

Applying the Mellin transform to (3.1), gives

$$\begin{aligned} \mathcal{N}(z)[I+z^{-\gamma}C^\gamma] =& N_0\prod_{\mu=1}^{n}\sum_{s=0}^{\infty}\left(s+\alpha_\mu\right)^{-T_\mu}(D_\mu;\sigma)_{s,k_\mu}(E_\mu)_{s,k_\mu}\left[(F_\mu)_{s,k_\mu}\right]^{-1} \\ &\times\left(\frac{d^{\gamma s}}{s!}\right)^{\mu}\mathcal{M}\left[t^{\gamma\mu s+\delta-1}\right](z). \end{aligned}$$

But, for $t \in [0,\xi]$

$$\mathcal{M}\left[t^{\gamma\mu s+\delta-1}\right](z) = \frac{\xi^{\gamma\mu s+\delta-1}}{z+\gamma\mu s+\delta-1},\ z \in \mathbb{C}.$$

Hence,

$$\begin{aligned} \mathcal{N}(z)I =& N_0\prod_{\mu=1}^{n}\sum_{s=0}^{\infty}\left(s+\alpha_\mu\right)^{-T_\mu}(D_\mu;\sigma)_{s,k_\mu}(E_\mu)_{s,k_\mu}\left[(F_\mu)_{s,k_\mu}\right]^{-1} \\ &\times\left(\frac{d^{\gamma s}}{s!}\right)^{\mu}\xi^{\gamma\mu s+\delta-1}\sum_{r=0}^{\infty}(-1)^r C^{\gamma r}\frac{z^{-\gamma r}}{z+\gamma\mu s+\delta-1}. \end{aligned}\tag{3.3}$$

Since

$$\begin{aligned} \mathcal{M}^{-1}\left[\frac{z^{-\gamma r}}{z+\gamma\mu s+\delta-1}\right](t) &= \int_0^\infty \frac{t^{-z}}{z}\frac{z^{-\gamma r}}{z+\gamma\mu s+\delta-1}dz \\ &= \sum_{\ell=0}^{\infty}[-(\gamma\mu s+\delta-1)]^{\ell}\int_0^\infty t^{-z}z^{-(\gamma r+\ell+2)}dz \\ &= \sum_{\ell=0}^{\infty}[-(\gamma\mu s+\delta-1)]^{\ell}[\log t]^{\gamma r+\ell+1}\Gamma[1-(\gamma r+\ell+2)], \end{aligned}$$

taking the inverse Mellin transform on both sides of (3.3), yields

$$\begin{aligned}\mathcal{N}(z)I = N_0 \prod_{\mu=1}^{n} \sum_{s=0}^{\infty} \left(s+\alpha_\mu\right)^{-T_\mu} (D_\mu;\sigma)_{s,k_\mu}(E_\mu)_{s,k_\mu}\left[(F_\mu)_{s,k_\mu}\right]^{-1} \\ \times \left(\frac{d^{\gamma s}}{s!}\right)^{\mu} \xi^{\gamma\mu s+\delta-1} \sum_{r=0}^{\infty}(-1)^r C^{\gamma r} \sum_{\ell=0}^{\infty}[-(\gamma\mu s+\delta-1)]^{\ell} \\ \times [\log t]^{\gamma r+\ell+1}\Gamma[1-(\gamma r+\ell+2)],\end{aligned}$$

which is the targeted result of (3.2). □

Continuing the same process, we obtain the following corollaries.

Corollary 3.1. *Let T_μ, D_μ, E_μ, F_μ and C be positive stable matrices in $\mathbb{C}^{m\times m}$ such that $T_\mu+\ell I$ and $F_\mu+\ell I$ are invertible for all $\mu\in\mathbb{N}$, $\ell\in\mathbb{N}_0$, $\sigma\in\mathbb{R}_0^+$, $d,k,\xi\in\mathbb{R}^+$ and $\alpha_\mu\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then, for $\gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $t\in[0,\xi]$ the generalized FKE*

$$N(t)I - N_0 \prod_{\mu=1}^{n} {}_2\Theta_1^{T_\mu;k_\mu,\alpha_\mu;\sigma}\left[\begin{matrix} D_\mu, E_\mu \\ F_\mu \end{matrix} ;d^\gamma t^\gamma\right] = -C^\gamma {}_H I_t^\gamma N(t) \tag{3.4}$$

is solvable. The solution to (3.4) *is given by*

$$\begin{aligned}N(t)I = N_0\ \log(t) \prod_{\mu=1}^{n}\sum_{s=0}^{\infty}(s+\alpha_\mu)^{-T_\mu}(D_\mu;\sigma)_{n,k_\mu}(E_\mu)_{n,k_\mu}[(F_\mu)_{n,k_\mu}]^{-1} \\ \times\left(\frac{d^{\gamma s}\xi^{\gamma s}}{s!}\right)^{\mu}\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty}\left[-\left(\log t^{C}\right)^{\gamma}\right]^{r}\left[\log t^{-(\gamma\mu s)}\right]^{\ell}\Gamma[1-(\gamma r+\ell+2)].\end{aligned} \tag{3.5}$$

Corollary 3.2. *Let T, D, E, F and C be positive stable matrices in $\mathbb{C}^{m\times m}$ such that $T+\ell I$ and $F+\ell I$ are invertible for all $\ell\in\mathbb{N}_0$, $\delta,\sigma\in\mathbb{R}_0^+$, $d,k,\xi\in\mathbb{R}^+$ and $\alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then, for $\gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $t\in[0,\xi]$ the generalized FKE*

$$N(t)I - N_0 t^{\delta-1}{}_2\Theta_1^{T;k,\alpha;\sigma}\left[\begin{matrix} D, E \\ F \end{matrix} ;d^\gamma t^\gamma\right] = -C^\gamma {}_H I_t^\gamma N(t) \tag{3.6}$$

is solvable, and ${}_2\Theta_1^{T;k,\alpha;\sigma}$ is as defined in (2.6). The solution to (3.7) *is given by*

$$\begin{aligned}N(t)I = N_0\xi^{\delta-1}\log(t)\sum_{s=0}^{\infty}(s+\alpha)^{-T}(D;\sigma)_{n,k}(E)_{n,k}[(F)_{n,k}]^{-1} \\ \times\left(\frac{d^{\gamma s}\xi^{\gamma s}}{s!}\right)^{\mu}\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty}\left[-\left(\log t^{C}\right)^{\gamma}\right]^{r}\left[\log t^{-(\gamma\mu s+\delta-1)}\right]^{\ell}\Gamma[1-(\gamma r+\ell+2)].\end{aligned} \tag{3.7}$$

Corollary 3.3. *Let T, D, E, F and C be positive stable matrices in $\mathbb{C}^{m\times m}$ such that $T+\ell I$ and $F+\ell I$ are invertible for all $\ell\in\mathbb{N}_0$, $\sigma\in\mathbb{R}_0^+$, $d,k,\xi\in\mathbb{R}^+$, and $\alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then, for $\gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $t\in[0,\xi]$ the generalized FKE*

$$N(t)I - N_0\ {}_2\Theta_1^{T;k,\alpha;\sigma}\left[\begin{matrix} D, E \\ F \end{matrix} ;d^\gamma t^\gamma\right] = -C^\gamma {}_H I_t^\gamma N(t) \tag{3.8}$$

is solvable, and ${}_2\Theta_1^{T;k,\alpha;\sigma}$ *as defined (2.6). The solution to* (3.8) *is given by*

$$\begin{aligned} N(t)I = {}& N_0 \ \log(t) \sum_{s=0}^{\infty}(s+\alpha)^{-T}(D;\sigma)_{n,k}(E)_{n,k}[(F)_{n,k}]^{-1} \\ & \times\left(\frac{d^{\gamma s}\xi^{\gamma s}}{s!}\right)^{\mu}\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty}\left[-\left(\log t^{C}\right)^{\gamma}\right]^{r}\left[\log t^{-(\gamma\mu s)}\right]^{\ell}\Gamma[1-(\gamma r+\ell+2)]. \end{aligned} \tag{3.9}$$

Theorem 3.2. *Let* T, D, E, F *and* C *be positive stable matrices in* $\mathbb{C}^{m\times m}$ *such that* $T+\ell I$ *and* $F+\ell I$ *are invertible for all* $\ell\in\mathbb{N}_0$, $\sigma\in\mathbb{R}_0^+$, $k,\xi\in\mathbb{R}^+$ *and* $\alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-$. *Then, for* $\gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *and* ${}_2\Theta_1^{T;k,\alpha;\sigma}$ *as defined in (2.6) the generalized FKE*

$$N(\sigma)I - N_0\,{}_2\Theta_1^{T;k,\alpha;\sigma}\left[\begin{array}{c} D,E \\ F \end{array};w\right] = -C^{\gamma}{}_H I_{\sigma}^{\gamma}N(\sigma) \tag{3.10}$$

is solvable and its solution is given by

$$\mathcal{N}(z)I = N_0\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r C^{\gamma r}(n+\alpha)^{-T}\left\{(D;\sigma)_{n,k} * (\log\sigma)^{\gamma r-1}\right\}(E)_{n,k}\left[(F)_{n,k}\right]^{-1}\frac{w^n}{n!}, \tag{3.11}$$

where $*$ *refers to (2.15).*

Proof. Applying the Mellin transform to (3.10) gives

$$\begin{aligned} \mathcal{N}(z)\left[I + C^{\gamma}z^{-\gamma}\right] = {}& N_0\sum_{n=0}^{\infty}(n+\alpha)^{-T}\Gamma_k^{-1}(D)\mathcal{M}\Big[\Gamma_k^{\sigma}(D+nI)\Big] \\ & \times (E)_{n,k}\left[(F)_{n,k}\right]^{-1}\frac{w^n}{n!}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \mathcal{N}(z)I = {}& N_0\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r C^{\gamma r}(n+\alpha)^{-T}\Gamma_k^{-1}(D)\Big\{z^{-\gamma r}\mathcal{M}\Big[\Gamma_k^{\sigma}(D+nI)\Big]\Big\}\times \\ & \times (E)_{n,k}\left[(F)_{n,k}\right]^{-1}\frac{w^n}{n!}. \end{aligned}$$

Applying the Mellin convolution theorem (2.15), we get

$$\begin{aligned} \mathcal{N}(z)I = {}& N_0\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r C^{\gamma r}(n+\alpha)^{-T}\Gamma_k^{-1}(D)\Big\{\Gamma_k^{\sigma}(D+nI) * (\log\sigma)^{\gamma r-1}\Big\} \\ & \times (E)_{n,k}\left[(F)_{n,k}\right]^{-1}\frac{w^n}{n!}, \end{aligned}$$

which is the targeted result of (3.11). □

Using the same argument, we obtain the following corollaries.

Corollary 3.4. *Let T, D, E, F and C be positive stable matrices in $\mathbb{C}^{m\times m}$ such that $T+\ell I$ and $F+\ell I$ are invertible for all $\ell\in\mathbb{N}_0$, $\sigma\in\mathbb{R}_0^+$ and $\alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then, for $\gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and ${}_2\Theta_1^{T;\alpha;\sigma}$ as defined by (2.8) the generalized FKE*

$$N(\sigma)I - N_0\, {}_2\Theta_1^{T;\alpha;\sigma}\left[\begin{array}{c} D, E \\ F \end{array}; w\right] = -C^{\gamma}\, {}_H I_{\sigma}^{\gamma} N(\sigma) \tag{3.12}$$

is solvable and its solution is given by

$$\mathcal{N}(z)I = N_0 \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r C^{\gamma r}(n+\alpha)^{-T}\left\{(D;\sigma)_n * (\log\sigma)^{\gamma r-1}\right\}(E)_n\left[(F)_n\right]^{-1}\frac{w^n}{n!}, \tag{3.13}$$

where $*$ refers to (2.15).

Corollary 3.5. *Let T, D and C be positive stable matrices in $\mathbb{C}^{m\times m}$ such that $T+\ell I$ is invertible for all $\ell\in\mathbb{N}_0$, $\sigma\in\mathbb{R}_0^+$, $\xi\in\mathbb{R}^+$ and $\alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then, for $\gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and ${}_1\Theta_0^{T;\alpha;\sigma}$ as defined by (2.11) the generalized FKE*

$$N(\sigma)I - N_0\, {}_1\Theta_0^{T;\alpha;\sigma}[D; w] = -C^{\gamma}\, {}_H I_{\sigma}^{\gamma} N(\sigma) \tag{3.14}$$

is solvable and its solution is given by

$$\mathcal{N}(z)I = N_0 \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r C^{\gamma r}(n+\alpha)^{-T}\left\{(D;\sigma)_n * (\log\sigma)^{\gamma r-1}\right\}\frac{w^n}{n!}. \tag{3.15}$$

where $*$ refers to (2.15).

Remark 3.1. *Similarly, and using special cases in Remark 2.1, we can indicate other results similar to Theorem 3.1 and Theorem 3.2.*

4. Conclusions

Considering the efficiency and high significance of FKEs in various fields of applied science and engineering, and as motivated by recent studies [19, 20, 30], we employed the Hadamard fractional integral operator via the Mellin integral transform to discuss the generalization of some FKEs including families of Hurwitz-Lerch zeta matrix functions. Solutions to certain FKMEs involving families of Hurwitz-Lerch zeta matrix functions have also been established. Further, our main findings under suitable matrix parametric constraints, yielded numerous known and new results through the use of zeta matrix functions that may prove to be very useful for applications in various fields of physics, engineering and technology.

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Conflicts of interest

This work does not have any conflict of interest.

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