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Research article

New class of convex interval-valued functions and Riemann Liouville fractional integral inequalities

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Abstract: The appreciation of inequalities in convexity is critical for fractional calculus and its application in a variety of fields. In this paper, we provide a unique analysis based on Hermite-Hadamard inequalities in the context of newly defined class of convexity which is known as left and right harmonically h-convex IVF (left and right $\mathcal{H}$-h-convex IVF), as well as associated integral and fractional inequalities, are addressed by the suggested technique. Because of its intriguing character in the numerical sciences, there is a strong link between fractional operators and convexity. There have also been several exceptional circumstances studied, and numerous well-known Hermite-Hadamard inequalities have been derived for left and right $\mathcal{H}$-h-convex IVF. Moreover, some applications are also presented in terms of special cases which are discussed in this study. The plan's outcomes demonstrate that the approach may be implemented immediately and is computationally simple and precise. We believe, our findings, generalize certain well-known new and classical harmonically convexity discoveries from the literature.

Keywords: harmonic convex set; left and right harmonically h-convex interval-valued functions; Riemann Liouville Fractional Hermite-Hadamard type inequalities
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1. Introduction

The study of fractional order integral and derivative functions over real and complex domains, as well as its applications, is now the focus of fractional calculus. In many issues, using arithmetic from classical analysis in fractional analysis is crucial for generating more realistic results. Differential equations of fractional order are capable of handling a wide range of mathematical models. Because fractional mathematical models are special instances of fractional order mathematical models, they have more broad and accurate conclusions than classical mathematical models. Integer orders aren't an effective model for nature in classical analysis. On the other side, fractional computing allows us to look at any number of orders and come up with far more quantifiable goals.

Wang et al. [1] looked into fractional integral identities for a differentiable mapping involving Riemann–Liouville fractional integrals and Hadamard fractional integrals, and came up with some inequalities using standard convex, r-convex, m-convex, s-convex, (s, m)-convex, (β, m)-convex, Işcan [10] also used fractional integrals for preinvex functions to get various Hermite–Hadamard type inequalities. See [2–7] for other applications of the Hermite-Hadamard inequality.

Moore [8] used interval arithmetic, interval-valued functions, and integrals of interval-valued functions to establish arbitrarily sharp upper and lower limits on accurate solutions to various problems in practical mathematics. Moore [8] shown that if a real-valued function $\Upsilon(\varpi)$ meets an ordinary Lipschitz condition in Y, $|\Upsilon(\varpi) - \Upsilon(y)| \leq L|\varpi - y|$, for $\varpi, y \in Y$, then the united extension of is a Lipschitz interval extension in Y. Hilger [9] proposed a time scales theory that may be used to combine the study of discrete and continuous dynamical systems. The widespread use of dynamic equations and integral inequalities on time scales in domains as diverse as electrical engineering, quantum physics, heat transfer, neural networks, combinatorics, and population dynamics [9] has highlighted the need for this theory. Young's inequality, Hölder's inequality, Minkoswki's inequality, Jensen's inequality, Steffensen's inequality, Hermite–Hadamard inequality and Opial type inequality were all explored by Agarwal et al. [9]. Srivastava et al. [10] discovered some generic time scale weighted Opial type inequalities in 2010. Srivastava et al. [11] also proposed several time-based expansions and generalizations of Maroni's inequality. Under some favorable conditions, Wei et al. [12] created a local fractional integral counterpart of Anderson's inequality on fractal space and also demonstrated that the fractal space local fractional integral inequality is a novel extension of Anderson's inequality Tunç et al. [13] also constructed an identity for local fractional integrals and derived numerous modifications of the well-known Steffensen's inequality for fractional integrals. [10,14] and the references therein might be consulted for further information.

Bhurjee and Panda [15] identified the parametric form of an interval-valued function and devised a technique to investigate the existence of a generic interval optimization issue solution. Using the notion of the generalized Hukuhara difference, Lupulescu [16] developed differentiability and integrability for interval-valued functions on time scales. Cano et al. [17] developed a novel form of Ostrowski inequality for gH-differentiable interval-valued functions in 2015, and achieved an extension of the class of real functions that are not always differentiable. For gH-differentiable interval-valued functions, Cano et al. [17] found error limitations to quadrature rules. In addition, Roy and Panda [18] developed the idea of the -monotonic property of interval-valued functions in the higher dimension and used extended Hukuhara differentiability to obtain various conclusions.

The findings of Iscan [19] and Noor et al. [20] have largely influenced our research. The concept of harmonically h-convexity for interval-valued functions is introduced first. Then, given the introduced class of functions, we show certain new Hermite-Hadamard type inequalities. The conclusions from [19,20] have interval-valued analogues in our inequalities. We refer to [21–37] and the references therein for further information on real-valued and interval-valued functions.

Furthermore, Khan et al. introduced the different classes of convex functions like (h_1, h_2)-preinvex fuzzy IVFs [38], log-s-convex fuzzy IVFs in the second sense [39], harmonically convex fuzzy IVFs [40], (h_1, h_2)-convex fuzzy IVFs [41], generalized p-convex fuzzy IVFs [42] and introduced Hermite-Hadamard type inequalities of these functions. For more information, see [43–63] and the references are therein.

The following is a breakdown of the paper's structure. Following the preliminaries in Section 2, Section 3 introduces the left and right harmonically h-convexity idea for interval-valued functions and proves new Hermite-Hadamard fractional integral type inequalities. Section 4 concludes with conclusions and further study.

2. Preliminaries

First, we offer some background information on interval-valued functions, the theory of convexity, interval-valued integration, and interval-valued fractional integration, which will be utilized throughout the article.

We offer some fundamental arithmetic regarding interval analysis in this paragraph, which will be quite useful throughout the article.

$$\mathcal{Z} = [\mathcal{Z}_*, \mathcal{Z}^*],\ \mathcal{Q} = [\mathcal{Q}_*, \mathcal{Q}^*]\ (\mathcal{Z}_* \le \varkappa \le \mathcal{Z}^* \text{ and } \mathcal{Q}_* \le z \le \mathcal{Q}^*, \varkappa, z \in \mathbb{R})$$

$$\mathcal{Z} + \mathcal{Q} = [\mathcal{Z}_*, \mathcal{Z}^*] + [\mathcal{Q}_*, \mathcal{Q}^*] = [\mathcal{Z}_* + \mathcal{Q}_*, \mathcal{Z}^* + \mathcal{Q}^*], \tag{1}$$

$$\mathcal{Z} - \mathcal{Q} = [\mathcal{Z}_*, \mathcal{Z}^*] - [\mathcal{Q}_*, \mathcal{Q}^*] = [\mathcal{Z}_* - \mathcal{Q}_*, \mathcal{Z}^* - \mathcal{Q}^*], \tag{2}$$

$$\mathcal{Z} \times \mathcal{Q} = [\mathcal{Z}_*, \mathcal{Z}^*] \times [\mathcal{Q}_*, \mathcal{Q}^*] = [\min\mathcal{X}, \max\mathcal{X}] \tag{3}$$

$\min\mathcal{X} = \min\{\mathcal{Z}_*\mathcal{Q}_*, \mathcal{Z}^*\mathcal{Q}_*, \mathcal{Z}_*\mathcal{Q}^*, \mathcal{Z}^*\mathcal{Q}^*\}$, $\max\mathcal{X} = \max\{\mathcal{Z}_*\mathcal{Q}_*, \mathcal{Z}^*\mathcal{Q}_*, \mathcal{Z}_*\mathcal{Q}^*, \mathcal{Z}^*\mathcal{Q}^*\}$

$$\nu.[\mathcal{Z}_*, \mathcal{Z}^*] = \begin{cases} [\nu\mathcal{Z}_*, \nu\mathcal{Z}^*] & \text{if } \nu > 0, \\ \{0\} & \text{if } \nu = 0, \\ [\nu\mathcal{Z}^*, \nu\mathcal{Z}_*] & \text{if } \nu < 0. \end{cases} \tag{4}$$

Let $\mathcal{K}_C$, $\mathcal{K}_C^+$, $\mathcal{K}_C^-$ be the set of all closed intervals of $\mathbb{R}$, the set of all closed positive intervals of $\mathbb{R}$ and the set of all closed negative intervals of $\mathbb{R}$. Then, $\mathcal{K}_C$, $\mathcal{K}_C^+$, and $\mathcal{K}_C^-$ are defined as

$$\mathcal{K}_C = \{[\mathcal{Z}_*, \mathcal{Z}^*]: \mathcal{Z}_*, \mathcal{Z}^* \in \mathbb{R} \text{ and } \mathcal{Z}_* \le \mathcal{Z}^*\},$$

$$\mathcal{K}_C^+ = \{[\mathcal{Z}_*, \mathcal{Z}^*]: \mathcal{Z}_*, \mathcal{Z}^* \in \mathcal{K}_C \text{ and } \mathcal{Z}_* > 0\},$$

$$\mathcal{K}_C^- = \{[\mathcal{Z}_*, \mathcal{Z}^*]: \mathcal{Z}_*, \mathcal{Z}^* \in \mathcal{K}_C \text{ and} \mathcal{Z}^* < 0\}.$$

For $[\mathcal{Z}_*, \mathcal{Z}^*]$, $[Q_*, Q^*] \in \mathcal{K}_C$, the inclusion " $\subseteq$ " is defined by $[\mathcal{Z}_*, \mathcal{Z}^*] \subseteq [Q_*, Q^*]$, if and only if, $Q_* \leq \mathcal{Z}_*$, $\mathcal{Z}^* \leq Q^*$.

Remark 2.1. [36] The relation " $\leq_p$ " defined on $\mathcal{K}_C$ by

$$[Q_*, Q^*] \leq_p [\mathcal{Z}_*, \mathcal{Z}^*] \text{ if and only if } Q_* \leq \mathcal{Z}_*,\ Q^* \leq \mathcal{Z}^*, \tag{5}$$

for all $[Q_*, Q^*]$, $[\mathcal{Z}_*, \mathcal{Z}^*] \in \mathcal{K}_C$, it is an pseudo-order relation.

Theorem 2.2. [9] If $\Upsilon: [\mu, \upsilon] \subset \mathbb{R} \to \mathcal{K}_C$ is an *IV-F* on such that $\Upsilon(\omega) = [\Upsilon_*(\omega), \Upsilon^*(\omega)]$, then Υ is Riemann integrable over $[\mu, \upsilon]$ if and only if, $\Upsilon_*(\omega)$ and $\Upsilon^*(\omega)$ are both Riemann integrable over $[\mu, \upsilon]$ such that

$$(IR)\int_\mu^\upsilon \Upsilon(\omega)d\omega = \left[(R)\int_\mu^\upsilon \Upsilon_*(\omega)d\omega, (R)\int_\mu^\upsilon \Upsilon^*(\omega)d\omega\right], \tag{6}$$

where, $\Upsilon_*, \Upsilon^*: [\mu, \upsilon] \to \mathbb{R}$.

The following interval Riemann-Liouville fractional integral operators were introduced by Budak et al. [26] and Lupulescu [37]:

Definition 2.3. [26,37] Let $\beta > 0$ and $L([\mu, \upsilon], \mathcal{K}_C)$ be the collection of all Lebesgue measurable IVFs on$[\mu, \upsilon]$. Then the interval left and right Riemann-Liouville fractional integral of Υ with order $\beta > 0$ are defined by

$$\mathcal{J}_{\mu^+}^{\beta} \Upsilon(\varpi) = \frac{1}{\Gamma(\beta)}\int_\mu^\varpi (\varpi - \theta)^{\beta-1}\Upsilon(\theta)\, d\theta, \quad (\varpi > \mu), \tag{7}$$

and

$$\mathcal{J}_{\upsilon^-}^{\beta} \Upsilon(\varpi) = \frac{1}{\Gamma(\beta)}\int_\varpi^\upsilon (\theta - \varpi)^{\beta-1}\Upsilon(\theta)\, d\theta, \quad (\varpi < \upsilon), \tag{8}$$

respectively, where $\Gamma(\varpi) = \int_0^\infty \theta^{\varpi-1}e^{-\theta}\, d\theta$ is the Euler gamma function. The interval left and right Riemann-Liouville fractional integral ϖ based on left and right end point functions can be defined, that is

$$\begin{aligned}\mathcal{J}_{\mu^+}^{\beta} \Upsilon(\varpi) &= \frac{1}{\Gamma(\beta)}\int_\mu^\varpi (\varpi - \theta)^{\beta-1}\Upsilon(\theta)\, d\theta \\ &= \frac{1}{\Gamma(\beta)}\int_\mu^\varpi (\varpi - \theta)^{\beta-1}[\Upsilon_*(\theta), \Upsilon^*(\theta)]\, d\theta, (\varpi > \mu),\end{aligned} \tag{9}$$

where

$$\mathcal{J}_{\mu^+}^{\beta} \Upsilon_*(\varpi) = \frac{1}{\Gamma(\beta)}\int_\mu^\varpi (\varpi - \theta)^{\beta-1}\Upsilon_*(\theta)\, d\theta, \ (\varpi > \mu), \tag{10}$$

and

$$\mathcal{J}_{\mu^+}^{\beta} \Upsilon^*(\varpi) = \frac{1}{\Gamma(\beta)}\int_\mu^\varpi (\varpi - \theta)^{\beta-1}\Upsilon^*(\theta)\, d\theta, \ (\varpi > \mu), \tag{11}$$

Similarly, the left and right end point functions can be used to define the right Riemann-Liouville fractional integral Υ of ϖ.

Definition 2.4. [19] A set $K = [\mu, \upsilon] \subset \mathbb{R}^+ = (0, \infty)$ is said to be harmonically convex set, if, for all $\varpi, z \in K, \theta \in [0,1]$, we have

$$\frac{\varpi z}{\theta\varpi+(1-\theta)z} \in K. \tag{12}$$

Definition 2.5. [19] The $\Upsilon: [\mu, \upsilon] \to \mathbb{R}^+$ is called harmonically convex function on $[\mu, \upsilon]$ if

$$\Upsilon\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right) \leq (1-\theta)\Upsilon(\varpi) + \theta\Upsilon(z), \tag{13}$$

for all $\varpi, z \in [\mu, \upsilon], \theta \in [0,1]$. If (13) is reversed then, Υ is called harmonically concave IVF on $[\mu, \upsilon]$.
Definition 2.6. [20] The positive real-valued function $\Upsilon: [\mu, \upsilon] \to \mathbb{R}^+$ is called $\mathcal{H}$−h-convex function on $[\mu, \upsilon]$ if

$$\Upsilon\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right) \leq h(1-\theta)\Upsilon(\varpi) + h(\theta)\Upsilon(z), \tag{14}$$

for all $\varpi, z \in [\mu, \upsilon], \theta \in [0,1]$, and $h: [0,1] \subseteq [\mu, \upsilon] \to \mathbb{R}^+$ such that $h \not\equiv 0$. If (14) is reversed then, Υ is called $\mathcal{H}$−h-concave function on $[\mu, \upsilon]$. The set of all $\mathcal{H}$−h-convex ($\mathcal{H}$−h-concave) functions is denoted by

$$HSX([\mu, \upsilon], \mathbb{R}^+, h) \qquad \big(HSV([\mu, \upsilon], \mathbb{R}^+, h)\big).$$

Definition 2.7. [35] The IVF $\Upsilon: [\mu, \upsilon] \to \mathcal{K}_C^+$ is called left and right h-convex IVF on $[\mu, \upsilon]$ if

$$\Upsilon(\,(1-\theta)\varpi + \theta z) \leq_p h(1-\theta)\Upsilon(\varpi) + h(\theta)\Upsilon(z), \tag{15}$$

for all $\varpi, z \in [\mu, \upsilon], \theta \in [0,1]$, and $h: [0,1] \subseteq [\mu, \upsilon] \to \mathbb{R}^+$ such that $h \not\equiv 0$. If (15) is reversed then, Υ is called left and right h-concave IVF on $[\mu, \upsilon]$. The set of all left and right h-convex (left and right h-concave) IVF is denoted by

$$LRSX([\mu, \upsilon], \mathbb{R}^+, h) \qquad \big(LRSV([\mu, \upsilon], \mathbb{R}^+, h)\big).$$

Definition 2.8. The IVF $\Upsilon: [\mu, \upsilon] \to \mathcal{K}_C^+$ is called left and right harmonically convex IVF (left and right $\mathcal{H}$-convex IVF) on $[\mu, \upsilon]$ if

$$\Upsilon\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right) \leq_p (1-\theta)\Upsilon(\varpi) + \theta\Upsilon(z), \tag{16}$$

for all $\varpi, z \in [\mu, \upsilon], \theta \in [0,1]$. If (16) is reversed then, Υ is called left and right harmonically concave IVF (left and right $\mathcal{H}$-concave IVF) on $[\mu, \upsilon]$.
Definition 2.9. The IVF $\Upsilon: [\mu, \upsilon] \to \mathcal{K}_C^+$ is called left and right harmonically h-convex (left and right $\mathcal{H}$−h-convex IVF) on $[\mu, \upsilon]$ if

$$\Upsilon\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right) \leq_p h(1-\theta)\Upsilon(\varpi) + h(\theta)\Upsilon(z), \tag{17}$$

for all $\varpi, z \in [\mu, \upsilon], \theta \in [0,1]$, for all $\varpi \in [\mu, \upsilon]$ and $h: [0,1] \subseteq [\mu, \upsilon] \to \mathbb{R}^+$ such that $h \not\equiv 0$. If (17) is reversed then, Υ is called left and right $\mathcal{H}$−h-concave IVF on $[\mu, \upsilon]$. The set of all left and right $\mathcal{H}$−h-convex (left and right $\mathcal{H}$−h-concave) IVF is denoted by

$$LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h) \qquad \big(LRHSV([\mu,\upsilon],\mathcal{K}_C^+,h)\big).$$

Theorem 2.10. Let $[\mu,\upsilon]$ be harmonically convex set, and let $\Upsilon:[\mu,\upsilon]\to\mathcal{K}_C^+$ be an IVF such that

$$\Upsilon(\varpi)=[\Upsilon_*(\varpi),\Upsilon^*(\varpi)],\forall\,\varpi\in[\mu,\upsilon]. \tag{18}$$

for all $\varpi\in[\mu,\upsilon]$. Then, $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h)$, if and only if, $\Upsilon_*(\varpi)$, $\Upsilon^*(\varpi)\in HSX([\mu,\upsilon],\mathbb{R}^+,h)$.

Proof. Assume that $\Upsilon_*(\varpi)$, $\Upsilon^*(\varpi)\in HSX([\mu,\upsilon],\mathbb{R}^+,h)$. Then, from (15), we have

$$\Upsilon_*\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)\le h(1-\theta)\Upsilon_*(\varpi)+h(\theta)\Upsilon_*(z),\ \forall\,\varpi,z\in[\mu,\upsilon],\theta\in[0,1],$$

And

$$\Upsilon^*\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)\le h(1-\theta)\Upsilon^*(\varpi)+h(\theta)\Upsilon^*(z),\forall\,\varpi,z\in[\mu,\upsilon],\theta\in[0,1].$$

Then by (18), (4) and (5), we obtain

$$\Upsilon\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)=[\Upsilon_*(\theta\varpi+(1-\theta)z),\Upsilon^*(\theta\varpi+(1-\theta)z)],$$

$$\le_p h(1-\theta)[\Upsilon_*(\varpi),\Upsilon^*(\varpi)]+h(\theta)[\Upsilon_*(z),\Upsilon^*(z)],$$

that is

$$\Upsilon\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)\le_p h(1-\theta)\Upsilon(\varpi)+h(\theta)\Upsilon(z),\ \forall\,\varpi,z\in[\mu,\upsilon],\theta\in[0,1].$$

Hence, $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h)$.

Conversely, let $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h)$. Then for all $\varpi,z\in[\mu,\upsilon]$, $\theta\in[0,1]$, we have

$$\Upsilon\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)\le_p h(1-\theta)\Upsilon(\varpi)+h(\theta)\Upsilon(z),$$

Therefore, from (18), left side of above inequality, we have

$$\Upsilon\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)=\left[\Upsilon_*\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right),\Upsilon^*\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)\right].$$

Again, from (18), we obtain

$$h(1-\theta)\Upsilon(\varpi)+h(\theta)\Upsilon(z)=h(1-\theta)[\Upsilon_*(\varpi),\Upsilon^*(\varpi)]+h(\theta)[\Upsilon_*(z),\Upsilon^*(z)],$$

for all $\varpi,z\in[\mu,\upsilon]$, $\theta\in[0,1]$. Then by $\mathcal{H}-h$-convexity of Υ, we have for all $\varpi,z\in[\mu,\upsilon]$, $\theta\in[0,1]$ such that

$$\Upsilon_*\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)\le h(1-\theta)\Upsilon_*(\varpi)+h(\theta)\Upsilon_*(z),$$

and

$$\Upsilon^*\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right) \leq h(1-\theta)\Upsilon^*(\varpi)+h(\theta)\Upsilon^*(z),$$

Hence, $\Upsilon_*(\varpi)$, $\Upsilon^*(\varpi) \in HSX([\mu,\upsilon],\mathbb{R}^+,h)$.

Remark 2.11. On fixing $\Upsilon_*(\varpi)=\Upsilon^*(\varpi)$, then from Definition 2.9, we obtain Definition 2.6. On fixing $h(\theta)=\theta$, then from Definition 2.9, we obtain Definition 2.8.

Example 2.12. We consider the IVFs $\Upsilon:[0,2]\to\mathcal{K}_C^+$ defined by,

$$\Upsilon(\varpi)=[1,2]\sqrt{\varpi}. \tag{19}$$

Since $\Upsilon_*(\varpi)$, $\Upsilon^*(\varpi)\in HSX([\mu,\upsilon],\mathbb{R}^+,h)$ with $h(\theta)=\theta$. Hence, $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h)$.

We shall develop a relationship between h-convex IVF and $\mathcal{H}-h$-convex IVF in the next finding.

Theorem 2.13. Let $\Upsilon:[\mu,\upsilon]\to\mathcal{K}_C^+$ be a IVF such that $\Upsilon(\varpi)=[\Upsilon_*(\varpi),\Upsilon^*(\varpi)]$, for all $\varpi\in[\mu,\upsilon]$. Then $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h)$, if and only if, $\Upsilon\left(\frac{1}{\varpi}\right)\in LRSX([\mu,\upsilon],\mathcal{K}_C^+,h)$.

Proof. Since $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h)$, then, for $\varpi,z\in[\mu,\upsilon],\theta\in[0,1]$, we have

$$\Upsilon\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)\leq_p h(1-\theta)\Upsilon(\varpi)+h(\theta)\Upsilon(z).$$

Therefore, we have

$$\begin{gathered}\Upsilon_*\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)\leq h(1-\theta)\Upsilon_*(\varpi)+h(\theta)\Upsilon_*(z),\\ \Upsilon^*\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)\leq h(1-\theta)\Upsilon^*(\varpi)+h(\theta)\Upsilon^*(z).\end{gathered} \tag{20}$$

Consider $\eta(\varpi)=\Upsilon\left(\frac{1}{\varpi}\right)$. Taking $m=\frac{1}{\varpi}$ and $n=\frac{1}{z}$ to replace ϖ and z, respectively. Then, by applying (20) we have

$$\begin{aligned}\Upsilon_*\left(\frac{\frac{1}{\varpi z}}{\theta\frac{1}{\varpi}+(1-\theta)\frac{1}{z}}\right)&=\Upsilon_*\left(\frac{1}{(1-\theta)\varpi+\theta z}\right)\\&=\eta_*\big((1-\theta)\varpi+\theta z\big)\\&\leq h(\theta)\Upsilon_*\left(\frac{1}{z}\right)+h(1-\theta)\Upsilon_*\left(\frac{1}{\varpi}\right)\\&=h(\theta)\eta_*(z)+h(1-\theta)\eta_*(\varpi),\\ \Upsilon^*\left(\frac{\frac{1}{\varpi z}}{\theta\frac{1}{\varpi}+(1-\theta)\frac{1}{z}}\right)&=\Upsilon^*\left(\frac{1}{(1-\theta)\varpi+\theta z}\right)\\&=\eta^*\big((1-\theta)\varpi+\theta z\big)\\&\leq h(\theta)\Upsilon^*\left(\frac{1}{z}\right)+h(1-\theta)\Upsilon^*\left(\frac{1}{\varpi}\right)\\&=h(\theta)\eta^*(z)+h(1-\theta)\eta^*(\varpi).\end{aligned}$$

It follows that

$$\left[\Upsilon_*\left(\frac{\frac{1}{\varpi z}}{\theta\frac{1}{\varpi}+(1-\theta)\frac{1}{z}}\right),\Upsilon^*\left(\frac{\frac{1}{\varpi z}}{\theta\frac{1}{\varpi}+(1-\theta)\frac{1}{z}}\right)\right]=\left[\eta_*\big((1-\theta)\varpi+\theta z\big),\eta^*\big((1-\theta)\varpi+\theta z\big)\right]$$

$$\leq_p h(\theta)[\eta_*(z),\eta^*(z)]+h(1-\theta)[\eta_*(\varpi),\eta^*(\varpi)].$$

that is

$$\eta\big((1-\theta)\varpi+\theta z\,\big)\leq_p h(\theta)\eta(z)+h(1-\theta)\eta(\varpi)$$

This concludes that $\eta(\varpi)\in LRSX([\mu,\upsilon],\mathcal{K}_C^+,h)$,.
Conversely, let $\eta\in LRSX([\mu,\upsilon],\mathcal{K}_C^+,h)$,. Then, for all $\varpi,z\in[\mu,\upsilon]$, $\theta\in[0,1]$, we have

$$\eta(\theta\varpi+(1-\theta)z\,)\leq_p h(\theta)\eta(\varpi)+h(1-\theta)\eta(z),$$

By using same steps as above, we have

$$\begin{aligned}\eta_*\left(\theta\frac{1}{\varpi}+(1-\theta)\frac{1}{z}\right)&=\Upsilon_*\left(\frac{1}{\theta\frac{1}{\varpi}+(1-\theta)\frac{1}{z}}\right)=\Upsilon_*\left(\frac{\varpi z}{(1-\theta)\varpi+\theta z}\right)\\&\leq h(\theta)\eta_*\left(\frac{1}{\varpi}\right)+h(1-\theta)\eta_*\left(\frac{1}{z}\right)\\&=h(\theta)\Upsilon_*(\varpi)+h(1-\theta)\Upsilon_*(z)\\\eta^*\left(\theta\frac{1}{\varpi}+(1-\theta)\frac{1}{z}\right)&=\Upsilon_*\left(\frac{1}{\theta\frac{1}{\varpi}+(1-\theta)\frac{1}{z}}\right)=\Upsilon_*\left(\frac{\varpi z}{(1-\theta)\varpi+\theta z}\right)\\&\leq h(\theta)\eta^*\left(\frac{1}{\varpi}\right)+h(1-\theta)\eta^*\left(\frac{1}{z}\right)\\&=h(\theta)\Upsilon_*(\varpi)+h(1-\theta)\Upsilon_*(z)\end{aligned}$$

that is

$$\Upsilon\left(\frac{\varpi z}{\theta\varpi+(1-\theta)z}\right)\leq_p h(1-\theta)\Upsilon(\varpi)+h(\theta)\Upsilon(z),$$

the proof the theorem has been completed.
Remark 2.14. If $h(\theta)=\theta$, and $\Upsilon_*(\varpi)=\Upsilon^*(\varpi)$, then from Theorem 2.14, we obtain Lemma 2.1 of [36].

3. Hermite-Hadamard inequalities for harmonically *h*-convex fuzzy interval-valued functions

In this section, we will prove two types of inequalities. First one is $H{\cdot}H$ and their variant forms, and the second one is $H{\cdot}H$ Fejér inequalities for $\mathcal{H}{-}h$-convex IVFs where the integrands are IVFs. The family of Lebesgue measurable IVFs is denoted by $L([\mu,\upsilon],\mathcal{K}_C^+)$ in the following.
Theorem 3.1. Let $\Upsilon:[\mu,\upsilon]\to\mathcal{K}_C^+$ be a IVF such that $\Upsilon(\varpi)=[\Upsilon_*(\varpi),\Upsilon^*(\varpi)]$, for all $\varpi\in[\mu,\upsilon]$. If $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h)$ and $\Upsilon\in L([\mu,\upsilon],\mathcal{K}_C^+)$, then

$$\begin{aligned}\frac{1}{\beta h\left(\frac{1}{2}\right)}\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)&\leq_p\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\left[\mathcal{J}^\beta_{\frac{1}{\mu}-}(\Upsilon\text{o}\delta)\left(\frac{1}{\upsilon}\right)+\mathcal{J}^\beta_{\frac{1}{\upsilon}+}(\Upsilon\text{o}\delta)\left(\frac{1}{\mu}\right)\right]\\&\leq_p[\Upsilon(\mu)+\Upsilon(\upsilon)]\int_0^1\theta^{\beta-1}[h(\theta)+h(1-\theta)]\,d\theta.\end{aligned}\tag{21}$$

If $\Upsilon(\varpi)$ is concave IVF then

$$\begin{aligned}\frac{1}{\beta h\left(\frac{1}{2}\right)}\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)&\geq_p\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\left[\mathcal{J}^\beta_{\frac{1}{\mu}-}(\Upsilon\text{o}\delta)\left(\frac{1}{\upsilon}\right)+\mathcal{J}^\beta_{\frac{1}{\upsilon}+}(\Upsilon\text{o}\delta)\left(\frac{1}{\mu}\right)\right]\\&\geq_p[\Upsilon(\mu)+\Upsilon(\upsilon)]\int_0^1\theta^{\beta-1}[h(\theta)+h(1-\theta)]d\theta.\end{aligned}\tag{22}$$

where $\delta(\varpi)=\frac{1}{\varpi}$.

Proof. Let $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h)$,. Then, by hypothesis, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\leq_p\Upsilon\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)+\Upsilon\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right).$$

Therefore, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\leq\Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)+\Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right),$$
$$\frac{1}{h\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\leq\Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)+\Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right).$$

Consider $\eta(\varpi)=\Upsilon\left(\frac{1}{\varpi}\right)$. By Theorem 2.13, we have $\eta(\varpi)\in LRSX([\mu,\upsilon],\mathcal{K}_C^+,h)$, then above inequality, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\eta_*\left(\frac{\mu+\upsilon}{2\mu\upsilon}\right)\leq\eta_*\left(\frac{\theta\mu+(1-\theta)\upsilon}{\mu\upsilon}\right)+\eta_*\left(\frac{(1-\theta)\mu+\theta\upsilon}{\mu\upsilon}\right),$$
$$\frac{1}{h\left(\frac{1}{2}\right)}\eta^*\left(\frac{\mu+\upsilon}{2\mu\upsilon}\right)\leq\eta^*\left(\frac{\theta\mu+(1-\theta)\upsilon}{\mu\upsilon}\right)+\eta^*\left(\frac{(1-\theta)\mu+\theta\upsilon}{\mu\upsilon}\right).$$

Multiplying both sides by $\theta^{\beta-1}$ and integrating the obtained result with respect to θ over (0,1), we have

$$\frac{1}{\beta h\left(\frac{1}{2}\right)}\int_0^1\theta^{\beta-1}\eta_*\left(\frac{\mu+\upsilon}{2\mu\upsilon}\right)d\theta$$
$$\leq\int_0^1\theta^{\beta-1}\eta_*\left(\frac{\theta\mu+(1-\theta)\upsilon}{\mu\upsilon}\right)d\theta+\int_0^1\theta^{\beta-1}\eta_*\left(\frac{(1-\theta)\mu+\theta\upsilon}{\mu\upsilon}\right)d\theta,$$
$$\frac{1}{\beta h\left(\frac{1}{2}\right)}\int_0^1\theta^{\beta-1}\eta^*\left(\frac{\mu+\upsilon}{2\mu\upsilon}\right)d\theta$$
$$\leq\int_0^1\theta^{\beta-1}\eta^*\left(\frac{\theta\mu+(1-\theta)\upsilon}{\mu\upsilon}\right)d\theta+\int_0^1\theta^{\beta-1}\eta^*\left(\frac{(1-\theta)\mu+\theta\upsilon}{\mu\upsilon}\right)d\theta.$$

Let $\varpi=\frac{(1-\theta)\mu+\theta\upsilon}{\mu\upsilon}$ and $z=\frac{\theta\mu+(1-\theta)\upsilon}{\mu\upsilon}$. Then we have

$$\frac{1}{\beta h\left(\frac{1}{2}\right)}\eta_*\left(\frac{\mu+\upsilon}{2\mu\upsilon}\right)\leq\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\int_{\frac{1}{\upsilon}}^{\frac{1}{\mu}}\left(\frac{1}{\mu}-z\right)^{\beta-1}\eta_*(z)\,dz+\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\int_{\frac{1}{\upsilon}}^{\frac{1}{\mu}}\left(\varpi-\frac{1}{\upsilon}\right)^{\beta-1}\eta_*(\varpi)\,d\varpi$$
$$\frac{1}{\beta h\left(\frac{1}{2}\right)}\eta_*\left(\frac{\mu+\upsilon}{2\mu\upsilon}\right)\leq\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\int_{\frac{1}{\upsilon}}^{\frac{1}{\mu}}\left(\frac{1}{\mu}-z\right)^{\beta-1}\eta^*(z)\,dz+\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\int_{\frac{1}{\upsilon}}^{\frac{1}{\mu}}\left(\varpi-\frac{1}{\upsilon}\right)^{\beta-1}\eta^*(\varpi)\,d\varpi,$$

$$=\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\left[\mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}\eta_*\left(\frac{1}{\upsilon}\right)+\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}\eta_*\left(\frac{1}{\mu}\right)\right]$$
$$=\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\left[\mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}\eta^*\left(\frac{1}{\upsilon}\right)+\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}\eta^*\left(\frac{1}{\mu}\right)\right],$$

It follows that

$$\frac{1}{\beta h\left(\frac{1}{2}\right)}\left[\eta_*\left(\frac{\mu+\upsilon}{2\mu\upsilon}\right),\eta^*\left(\frac{\mu+\upsilon}{2\mu\upsilon}\right)\right]$$

$$\leq_I \Gamma(\beta+1)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^-}\eta_*\left(\frac{1}{\upsilon}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\eta_*\left(\frac{1}{\mu}\right),\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^-}\eta^*\left(\frac{1}{\upsilon}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\eta^*\left(\frac{1}{\mu}\right)\right].$$

That is,

$$\frac{1}{\beta h\left(\frac{1}{2}\right)}\,\eta\left(\frac{\mu+\upsilon}{2\mu\upsilon}\right)\leq_p \Gamma(\beta+1)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^-}\eta\left(\frac{1}{\upsilon}\right)\tilde{+}\,\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\eta\left(\frac{1}{\mu}\right)\right]. \tag{23}$$

In a similar way as above, we have

$$\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^-}\eta\left(\frac{1}{\upsilon}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\eta\left(\frac{1}{\mu}\right)\right]\leq_p\left[\eta\left(\frac{1}{\mu}\right)+\eta\left(\frac{1}{\upsilon}\right)\right]\int_0^1\theta^{\beta-1}[h(\theta)+h(1-\theta)]. \tag{24}$$

Combining (23) and (24), we have

$$\frac{1}{\beta h\left(\frac{1}{2}\right)}\eta\left(\frac{\mu+\upsilon}{2\mu\upsilon}\right)\leq_p \Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^-}\eta\left(\frac{1}{\upsilon}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\eta\left(\frac{1}{\mu}\right)\right]\leq_p\left[\eta\left(\frac{1}{\mu}\right)+\eta\left(\frac{1}{\upsilon}\right)\right]\int_0^1\theta^{\beta-1}[h(\theta)+h(1-\theta)]d\theta,$$

that is

$$\frac{1}{\beta h\left(\frac{1}{2}\right)}\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\leq_p \Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\frac{1}{\mu}^-}(\Upsilon\text{o}\delta)\left(\frac{1}{\upsilon}\right)+\mathcal{J}^{\beta}_{\frac{1}{\upsilon}^+}(\Upsilon\text{o}\delta)\left(\frac{1}{\mu}\right)\right]$$

$$\leq_p[\Upsilon(\mu)+\Upsilon(\upsilon)]\int_0^1\theta^{\beta-1}[h(\theta)+h(1-\theta)]d\theta.$$

Hence, the required result.

Remark 3.2. Followings results can be obtained through inequality (21):

On fixing $h(\theta)=\theta$, the following $H{\cdot}H$ inequality is obtained, which is also new one;

$$\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\leq_p\frac{\Gamma(\beta+1)}{2(\upsilon-\mu)^{\beta}}\left[\mathcal{J}^{\beta}_{\frac{1}{\mu}^-}(\Upsilon\text{o}\delta)\left(\frac{1}{\upsilon}\right)+\mathcal{J}^{\beta}_{\frac{1}{\upsilon}^+}(\Upsilon\text{o}\delta)\left(\frac{1}{\mu}\right)\right]\leq_p\frac{\Upsilon(\mu)+\Upsilon(\upsilon)}{2}.$$

On fixing $h(\theta)=\theta$ and $\beta=1$, the following $H{\cdot}H$ inequality is obtained, which is also new one;

$$\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\leq_p\frac{\mu\upsilon}{\upsilon-\mu}\int_\mu^\upsilon\frac{\Upsilon(\varpi)}{\varpi^2}d\varpi\leq_p\frac{\Upsilon(\mu)+\Upsilon(\upsilon)}{2}. \tag{25}$$

On fixing $h(\theta)=\theta$ and $\Upsilon_*(\varpi)=\Upsilon^*(\varpi)$, the following $H{\cdot}H$ inequality is obtained, see [36]:

$$\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\leq\frac{\Gamma(\beta+1)}{2(\upsilon-\mu)^{\beta}}\left[\mathcal{J}^{\beta}_{\frac{1}{\mu}^-}(\Upsilon\text{o}\delta)\left(\frac{1}{\upsilon}\right)+\mathcal{J}^{\beta}_{\frac{1}{\upsilon}^+}(\Upsilon\text{o}\delta)\left(\frac{1}{\mu}\right)\right]\leq\frac{\Upsilon(\mu)+\Upsilon(\upsilon)}{2}. \tag{26}$$

On fixing $h(\theta)=\theta$ and $\Upsilon_*(\varpi)=\Upsilon^*(\varpi)$ with $\beta=1$, the following $H{\cdot}H$ inequality is obtained, see [19].

$$\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\le\frac{\mu\upsilon}{\upsilon-\mu}\int_\mu^\upsilon\frac{\Upsilon(\varpi)}{\varpi^2}d\varpi\le\frac{\Upsilon(\mu)+\Upsilon(\upsilon)}{2}. \tag{27}$$

For the product of $\mathcal{H}-h$-convex IVFs, we now have some $H{\cdot}H$ inequalities. These inequalities are modifications of previously published inequalities [38,34,43].

Theorem 3.3. Let $\Upsilon,\mathfrak{G}:[\mu,\upsilon]\to\mathcal{K}_C^+$ be a IVFs such that $\Upsilon(\varpi)=[\Upsilon_*(\varpi),\Upsilon^*(\varpi)]$ and $\mathfrak{G}(\varpi)=[\mathfrak{G}_*(\varpi),\mathfrak{G}^*(\varpi)]$ for all $\varpi\in[\mu,\upsilon]$, respectively. If $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h_1)$, $\mathfrak{G}\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h_2)$, and $\Upsilon\times\mathfrak{G}\in L([\mu,\upsilon],\mathcal{K}_C^+)$, then

$$\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\left[\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}\Upsilon\mathrm{o}\delta\left(\frac{1}{\mu}\right)\times\mathfrak{G}\mathrm{o}\delta\left(\frac{1}{\mu}\right)+\mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}\Upsilon\mathrm{o}\delta\left(\frac{1}{\upsilon}\right)\times\mathfrak{G}\mathrm{o}\delta\left(\frac{1}{\upsilon}\right)\right]$$
$$\le_p M(\mu,\upsilon)\int_0^1\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_1(\theta)h_2(\theta)d\theta$$
$$+M(\mu,\upsilon)\int_0^1\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_1(\theta)h_2(1-\theta)d\theta.$$

Where $M(\mu,\upsilon)=\Upsilon(\mu)\times\mathfrak{G}(\mu)+\Upsilon(\upsilon)\times\mathfrak{G}(\upsilon)$, $N(\mu,\upsilon)=\Upsilon(\mu)\times\mathfrak{G}(\upsilon)+\Upsilon(\upsilon)\times\mathfrak{G}(\mu)$, and $M(\mu,\upsilon)=[M_*(\mu,\upsilon),M^*(\mu,\upsilon)]$ and $N(\mu,\upsilon)=[N_*(\mu,\upsilon),N^*(\mu,\upsilon)]$.

Proof. Since $\Upsilon,\mathfrak{G}$ are $\mathcal{H}-h_1$ and $\mathcal{H}-h_2$-convex IVFs then, we have

$$\Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\le h_1(1-\theta)\Upsilon_*(\mu)+h_1(\theta)\Upsilon_*(\upsilon)$$
$$\Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\le h_1(1-\theta)\Upsilon^*(\mu)+h_1(\theta)\Upsilon^*(\upsilon).$$

and

$$\mathfrak{G}_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\le h_2(1-\theta)\mathfrak{G}_*(\mu)+h_2(\theta)\mathfrak{G}_*(\upsilon)$$
$$\mathfrak{G}^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\le h_2(1-\theta)\mathfrak{G}^*(\mu)+h_2(\theta)\mathfrak{G}^*(\upsilon).$$

From the definition of $\mathcal{H}-h$-convex IVFs it follows that $0\le_p\Upsilon(\varpi)$ and $0\le_p\mathfrak{G}(\varpi)$, so

$$\begin{aligned}
&\Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\\
&\le(h_1(1-\theta)\Upsilon_*(\mu)+h_1(\theta)\Upsilon_*(\upsilon))(h_2(1-\theta)\mathfrak{G}_*(\mu)+\theta\mathfrak{G}_*(\upsilon))\\
&=h_1(1-\theta)h_2(1-\theta)\Upsilon_*(\mu)\times\mathfrak{G}_*(\mu)+h_1(\theta)h_2(\theta)\Upsilon_*(\upsilon)\times\mathfrak{G}_*(\upsilon)\\
&\quad+h_1(1-\theta)\,h_2(\theta)\,\Upsilon_*(\mu)\times\mathfrak{G}_*(\upsilon)+h_1(\theta)\,h_2(1-\theta)\,\Upsilon_*(\upsilon)\times\mathfrak{G}_*(\mu)\\
&\Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\\
&\le(h_1(1-\theta)\Upsilon^*(\mu)+h_1(\theta)\Upsilon^*(\upsilon))(h_2(1-\theta)\mathfrak{G}^*(\mu)+h_2(\theta)\mathfrak{G}^*(\upsilon))\\
&=h_1(1-\theta)h_2(1-\theta)\Upsilon^*(\mu)\times\mathfrak{G}^*(\mu)+h_1(\theta)h_2(\theta)\Upsilon^*(\upsilon)\times\mathfrak{G}^*(\upsilon)\\
&\quad+h_1(1-\theta)\,h_2(\theta)\,\Upsilon^*(\mu)\times\mathfrak{G}^*(\upsilon)+h_1(\theta)\,h_2(1-\theta)\,\Upsilon^*(\upsilon)\times\mathfrak{G}^*(\mu).
\end{aligned} \tag{28}$$

Analogously, we have

$$\begin{aligned}
&\Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\mathfrak{G}_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\\
&\qquad\le h_1(\theta)h_2(\theta)\Upsilon_*(\mu)\times\mathfrak{G}_*(\mu)+h_1(1-\theta)h_2(1-\theta)\Upsilon_*(\upsilon)\times\mathfrak{G}_*(\upsilon)\\
&\qquad\quad+h_1(\theta)\,h_2(1-\theta)\,\Upsilon_*(\mu)\times\mathfrak{G}_*(\upsilon)+h_1(1-\theta)\,h_2(\theta)\,\Upsilon_*(\upsilon)\times\mathfrak{G}_*(\mu)\\
&\Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\\
&\qquad\le h_1(\theta)h_2(\theta)\Upsilon^*(\mu)\times\mathfrak{G}^*(\mu)+h_1(1-\theta)h_2(1-\theta)\Upsilon^*(\upsilon)\times\mathfrak{G}^*(\upsilon)\\
&\qquad\quad+h_1(\theta)\,h_2(1-\theta)\,\Upsilon^*(\mu)\times\mathfrak{G}^*(\upsilon)+h_1(1-\theta)\,h_2(\theta)\,\Upsilon^*(\upsilon)\times\mathfrak{G}^*(\mu).
\end{aligned}\tag{29}$$

Adding (28) and (29), we have

$$\begin{aligned}
&\Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\\
&\qquad\qquad+\Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\\
&\quad\le[h_1(\theta)h_2(\theta)+h_1(1-\theta)h_2(1-\theta)][\Upsilon_*(\mu)\times\mathfrak{G}_*(\mu)+\Upsilon_*(\upsilon)\times\mathfrak{G}_*(\upsilon)]\\
&\quad\quad+[h_1(\theta)h_2(1-\theta)+h_1(1-\theta)h_2(\theta)][\Upsilon_*(\upsilon)\times\mathfrak{G}_*(\mu)+\Upsilon_*(\mu)\times\mathfrak{G}_*(\upsilon)]\\
&\Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\\
&\qquad\qquad+\Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\\
&\quad\le[h_1(\theta)h_2(\theta)+h_1(1-\theta)h_2(1-\theta)][\Upsilon^*(\mu)\times\mathfrak{G}^*(\mu)+\Upsilon^*(\upsilon)\times\mathfrak{G}^*(\upsilon)]\\
&\quad\quad+[h_1(\theta)h_2(1-\theta)+h_1(1-\theta)h_2(\theta)][\Upsilon^*(\upsilon)\times\mathfrak{G}^*(\mu)+\Upsilon^*(\mu)\times\mathfrak{G}^*(\upsilon)].
\end{aligned}\tag{30}$$

Taking the result of multiplying (30) by $\theta^{\beta-1}$ and integrating it with respect to θ over (0, 1), we get

$$\begin{aligned}
&\int_0^1\theta^{\beta-1}\Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)d\theta\\
&\qquad\qquad+\int_0^1\theta^{\beta-1}\Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)d\theta\\
&\qquad\le M_*(\mu,\upsilon)\int_0^1\theta^{\beta-1}[h_1(\theta)h_2(\theta)+h_1(1-\theta)h_2(1-\theta)]\,d\theta\\
&\qquad\quad+N_*(\mu,\upsilon)\int_0^1\theta^{\beta-1}[h_1(\theta)h_2(1-\theta)+h_1(1-\theta)h_2(\theta)]\,d\theta\\
&\int_0^1\theta^{\beta-1}\Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)d\theta\\
&\qquad\qquad+\int_0^1\theta^{\beta-1}\Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)d\theta\\
&\qquad\le M^*(\mu,\upsilon)\int_0^1\theta^{\beta-1}[h_1(\theta)h_2(\theta)+h_1(1-\theta)h_2(1-\theta)]d\theta\\
&\qquad\quad+N^*(\mu,\upsilon)\int_0^1\theta^{\beta-1}[h_1(\theta)h_2(1-\theta)+h_1(1-\theta)h_2(\theta)]d\theta.
\end{aligned}$$

It follows that,

$$\begin{aligned}
&\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\left[\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}\Upsilon_*\left(\frac{1}{\mu}\right)\times\mathfrak{G}_*\left(\frac{1}{\mu}\right)+\mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}\Upsilon_*\left(\frac{1}{\upsilon}\right)\times\mathfrak{G}_*\left(\frac{1}{\upsilon}\right)\right]\\
&\qquad\le M_*(\mu,\upsilon)\int_0^1\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_1(\theta)h_2(\theta)d\theta\\
&\qquad\quad+N_*(\mu,\upsilon)\int_0^1\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_1(\theta)h_2(1-\theta)d\theta\\
&\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\left[\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}\Upsilon^*\left(\frac{1}{\mu}\right)\times\mathfrak{G}^*\left(\frac{1}{\mu}\right)+\mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}\Upsilon^*\left(\frac{1}{\upsilon}\right)\times\mathfrak{G}^*\left(\frac{1}{\upsilon}\right)\right]\\
&\qquad\le M^*(\mu,\upsilon)\int_0^1\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_1(\theta)h_2(\theta)d\theta\\
&\qquad\quad+N^*(\mu,\upsilon)\int_0^1\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_1(\theta)h_2(1-\theta)d\theta.
\end{aligned}$$

that is

$$\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^{+}}\Upsilon_{*}\left(\frac{1}{\mu}\right)\times\mathfrak{G}_{*}\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^{-}}\Upsilon_{*}\left(\frac{1}{\upsilon}\right)\times\mathfrak{G}_{*}\left(\frac{1}{\upsilon}\right),\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^{+}}\Upsilon^{*}\left(\frac{1}{\mu}\right)\times\mathfrak{G}^{*}\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^{-}}\Upsilon^{*}\left(\frac{1}{\upsilon}\right)\times\mathfrak{G}^{*}\left(\frac{1}{\upsilon}\right)\right]\leq_{I}\left[M_{*}(\mu,\upsilon),M^{*}(\mu,\upsilon)\right]\int_{0}^{1}\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_{1}(\theta)h_{2}(\theta)d\theta+\left[N_{*}(\mu,\upsilon),N^{*}(\mu,\upsilon)\right]\int_{0}^{1}\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_{1}(\theta)h_{2}(1-\theta)d\theta.$$

Thus,

$$\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^{+}}\Upsilon\text{o}\delta\left(\frac{1}{\mu}\right)\times\mathfrak{G}\text{o}\delta\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^{-}}\Upsilon\text{o}\delta\left(\frac{1}{\upsilon}\right)\times\mathfrak{G}\text{o}\delta\left(\frac{1}{\upsilon}\right)\right]$$

$$\leq_{p} M(\mu,\upsilon)\int_{0}^{1}\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_{1}(\theta)h_{2}(\theta)d\theta$$

$$+N(\mu,\upsilon)\int_{0}^{1}\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_{1}(\theta)h_{2}(1-\theta)d\theta.$$

As a result, the theorem has been proven.

Theorem 3.4. Let $\Upsilon,\mathfrak{G}:[\mu,\upsilon]\to\mathcal{K}_{C}^{+}$ be a IVF such that $\Upsilon(\varpi)=[\Upsilon_{*}(\varpi),\Upsilon^{*}(\varpi)]$ and $\mathfrak{G}(\varpi)=[\mathfrak{G}_{*}(\varpi),\mathfrak{G}^{*}(\varpi)]$ for all $\varpi\in[\mu,\upsilon]$, respectively. If $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_{C}^{+},h_{1})$ and $\mathfrak{G}\in LRHSX([\mu,\upsilon],\mathcal{K}_{C}^{+},h_{2})$ with $h_{1}\left(\frac{1}{2}\right)h_{2}\left(\frac{1}{2}\right)\neq 0$, and $\Upsilon\times\mathfrak{G}\in L([\mu,\upsilon],\mathcal{K}_{C}^{+})$, then

$$\frac{1}{\beta\, h_{1}\left(\frac{1}{2}\right)h_{2}\left(\frac{1}{2}\right)}\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\times\mathfrak{G}\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)$$

$$\leq_{p}\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^{+}}\Upsilon\text{o}\delta\left(\frac{1}{\mu}\right)\times\mathfrak{G}\text{o}\delta\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^{-}}\Upsilon\text{o}\delta\left(\frac{1}{\upsilon}\right)\times\mathfrak{G}\text{o}\delta\left(\frac{1}{\upsilon}\right)\right]$$

$$+M(\mu,\upsilon)\int_{0}^{1}\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_{1}(\theta)h_{2}(1-\theta)d\theta$$

$$+N(\mu,\upsilon)\int_{0}^{1}\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_{1}(\theta)h_{2}(\theta)d\theta.$$

Where $M(\mu,\upsilon)=\Upsilon(\mu)\times\mathfrak{G}(\mu)+\Upsilon(\upsilon)\times\mathfrak{G}(\upsilon)$, $N(\mu,\upsilon)=\Upsilon(\mu)\times\mathfrak{G}(\upsilon)+\Upsilon(\upsilon)\times\mathfrak{G}(\mu)$, and $M(\mu,\upsilon)=[M_{*}(\mu,\upsilon),M^{*}(\mu,\upsilon)]$ and $N(\mu,\upsilon)=[N_{*}(\mu,\upsilon),N^{*}(\mu,\upsilon)]$.

Proof. Consider $\Upsilon,\mathfrak{G}:[\mu,\upsilon]\to\mathcal{K}_{C}^{+}$ are $\mathcal{H}-h_{1}$ and $\mathcal{H}-h_{2}$-convex IVFs. Then by hypothesis, we have

$$\Upsilon_{*}\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\times\mathfrak{G}_{*}\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)$$
$$\Upsilon^{*}\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\times\mathfrak{G}^{*}\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)$$

$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix} \Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \\ +\Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right) \end{bmatrix}$$

$$+\ h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix} \Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \\ +\Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right) \end{bmatrix}$$

$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix} \Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \\ +\Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right) \end{bmatrix}$$

$$+\,h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix} \Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \\ +\Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right) \end{bmatrix},$$

$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix} \Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \\ +\Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right) \end{bmatrix}$$

$$+\ h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix} \left(h_1(\theta)\Upsilon_*(\mu)+h_1(1-\theta)\Upsilon_*(\upsilon)\right) \\ \times\left(h_2(1-\theta)\mathfrak{G}_*(\mu)+h_2(\theta)\mathfrak{G}_*(\upsilon)\right) \\ +\left(h_1(1-\theta)\Upsilon_*(\mu)+h_1(\theta)\Upsilon_*(\upsilon)\right) \\ \times\left(h_2(\theta)\mathfrak{G}_*(\mu)+h_2(1-\theta)\mathfrak{G}_*(\upsilon)\right) \end{bmatrix}$$

$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix} \Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \\ +\Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right) \end{bmatrix}$$

$$+\,h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix} \left(h_1(\theta)\Upsilon^*(\mu)+h_1(1-\theta)\Upsilon^*(\upsilon)\right) \\ \times\left(h_2(1-\theta)\mathfrak{G}^*(\mu)+h_2(\theta)\mathfrak{G}^*(\upsilon)\right) \\ +\left(h_1(1-\theta)\Upsilon^*(\mu)+h_1(\theta)\Upsilon^*(\upsilon)\right) \\ \times\left(h_2(\theta)\mathfrak{G}^*(\mu)+h_2(1-\theta)\mathfrak{G}^*(\upsilon)\right) \end{bmatrix},$$

$$\begin{aligned} &= h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \\ +\Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right) \end{bmatrix} \\ &\quad+\ h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \{h_1(\theta)h_2(1-\theta)+h_1(1-\theta)h_2(\theta)\}M_*(\mu,\upsilon) \\ +\{h_1(\theta)h_2(\theta)+h_1(1-\theta)h_2(1-\theta)\}N_*(\mu,\upsilon) \end{bmatrix} \\ &= h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \\ +\Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right) \end{bmatrix} \\ &\quad+\ h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \{h_1(\theta)h_2(1-\theta)+h_1(1-\theta)h_2(\theta)\}M^*(\mu,\upsilon) \\ +\{h_1(\theta)h_2(\theta)+h_1(1-\theta)h_2(1-\theta)\}N^*(\mu,\upsilon) \end{bmatrix}. \end{aligned} \tag{31}$$

Inequality (31) is multiplied by $\theta^{\beta-1}$ and integrated over $(0,1)$,

$$\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\times\mathfrak{G}_*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\int_0^1\theta^{\beta-1}d\theta$$
$$\le\left[\int_0^1\theta^{\beta-1}\Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)d\theta+\int_0^1\theta^{\beta-1}\Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)d\theta\right]$$
$$+\begin{bmatrix}M_*(\mu,\upsilon)\int_0^1\theta^{\beta-1}\{h_1(\theta)h_2(1-\theta)+h_1(1-\theta)h_2(\theta)\}\,d\theta\\+N_*(\mu,\upsilon)\int_0^1\{h_1(\theta)h_2(\theta)+h_1(1-\theta)h_2(1-\theta)\}d\theta\end{bmatrix}$$
$$\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\times\mathfrak{G}^*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\int_0^1\theta^{\beta-1}d\theta$$
$$\le\left[\int_0^1\theta^{\beta-1}\Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)d\theta+\int_0^1\theta^{\beta-1}\Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\times\mathfrak{G}^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)d\theta\right]$$
$$+\begin{bmatrix}M^*(\mu,\upsilon)\int_0^1\theta^{\beta-1}\{h_1(\theta)h_2(1-\theta)+h_1(1-\theta)h_2(\theta)\}d\theta\\+N^*(\mu,\upsilon)\int_0^1\theta^{\beta-1}\{h_1(\theta)h_2(\theta)+h_1(1-\theta)h_2(1-\theta)\}d\theta\end{bmatrix}$$

Taking $\varpi=\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}$ and $z=\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}$, then we get

$$\frac{1}{\beta\,h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Upsilon_*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\times\mathfrak{G}_*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)$$
$$\le\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\left[\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}\Upsilon_*\mathrm{o}\delta\left(\frac{1}{\mu}\right)\times\mathfrak{G}_*\mathrm{o}\delta\left(\frac{1}{\mu}\right)+\mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}\Upsilon_*\mathrm{o}\delta\left(\frac{1}{\upsilon}\right)\times\mathfrak{G}_*\mathrm{o}\delta\left(\frac{1}{\upsilon}\right)\right]$$
$$+\begin{bmatrix}M_*(\mu,\upsilon)\int_0^1\theta^{\beta-1}\{h_1(\theta)h_2(1-\theta)+h_1(1-\theta)h_2(\theta)\}\,d\theta\\+N_*(\mu,\upsilon)\int_0^1\{h_1(\theta)h_2(\theta)+h_1(1-\theta)h_2(1-\theta)\}d\theta\end{bmatrix}$$
$$\frac{1}{\beta\,h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Upsilon^*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\times\mathfrak{G}^*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)$$
$$\le\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\left[\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}\Upsilon^*\mathrm{o}\delta\left(\frac{1}{\mu}\right)\times\mathfrak{G}^*\mathrm{o}\delta\left(\frac{1}{\mu}\right)+\mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}\Upsilon^*\mathrm{o}\delta\left(\frac{1}{\upsilon}\right)\times\mathfrak{G}^*\mathrm{o}\delta\left(\frac{1}{\upsilon}\right)\right]$$
$$+\begin{bmatrix}M^*(\mu,\upsilon)\int_0^1\theta^{\beta-1}\{h_1(\theta)h_2(1-\theta)+h_1(1-\theta)h_2(\theta)\}d\theta\\+N^*(\mu,\upsilon)\int_0^1\theta^{\beta-1}\{h_1(\theta)h_2(\theta)+h_1(1-\theta)h_2(1-\theta)\}d\theta\end{bmatrix},$$

that is

$$\frac{1}{\beta\,h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\times\mathfrak{G}\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\le_p\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta\left[\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}\Upsilon\mathrm{o}\delta\left(\frac{1}{\mu}\right)\times\mathfrak{G}\mathrm{o}\delta\left(\frac{1}{\mu}\right)+\mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}\Upsilon\mathrm{o}\delta\left(\frac{1}{\upsilon}\right)\times\right.$$
$$\left.\mathfrak{G}\mathrm{o}\delta\left(\frac{1}{\upsilon}\right)\right]\qquad+M(\mu,\upsilon)\int_0^1\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_1(\theta)h_2(1-\theta)d\theta$$
$$+N(\mu,\upsilon)\int_0^1\left[\theta^{\beta-1}+(1-\theta)^{\beta-1}\right]h_1(\theta)h_2(\theta)d\theta.$$

As a result, the desired outcome has been achieved.

We now give $H{\cdot}H$ Fejér inequalities for $\mathcal{H}{-}h$-convex IVFs. Firstly, we obtain the second $H{\cdot}H$ Fejér inequality for $\mathcal{H}{-}h$-convex IVF.

Theorem 3.5. Let $\Upsilon:[\mu,\upsilon]\to\mathcal{K}_C^+$ be a IVF such that $\Upsilon(\varpi)=[\Upsilon_*(\varpi),\Upsilon^*(\varpi)]$ for all $\varpi\in[\mu,\upsilon]$, respectively. If $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h)$, $\Upsilon\in L([\mu,\upsilon],\mathcal{K}_C^+)$ and $\Omega:[\mu,\upsilon]\to\mathbb{R}$, $\Omega\left(\frac{1}{\frac{1}{\mu}+\frac{1}{\upsilon}-\frac{1}{\varpi}}\right)=\Omega(\varpi)\geq 0$, then

$$\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\frac{1}{\upsilon}^+}(\Omega o\delta)\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\frac{1}{\mu}^-}(\Omega o\delta)\left(\frac{1}{\upsilon}\right)\right]$$

$$\leq_p \frac{\Upsilon(\mu)+\Upsilon(\upsilon)}{2}\int_0^1\theta^{\beta-1}\{h(\theta)+h(1-\theta)\}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)d\theta. \tag{32}$$

If $\Upsilon\in LRHSV([\mu,\upsilon],\mathcal{K}_C^+,h)$, then inequality (32) is reversed such that

$$\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\frac{1}{\upsilon}^+}(\Omega o\delta)\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\frac{1}{\mu}^-}(\Omega o\delta)\left(\frac{1}{\upsilon}\right)\right]$$

$$\geq_p \frac{\Upsilon(\mu)+\Upsilon(\upsilon)}{2}\int_0^1\theta^{\beta-1}\{h(\theta)+h(1-\theta)\}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)d\theta.$$

Proof. Let $\Upsilon\in LRHSX([\mu,\upsilon],\mathcal{K}_C^+,h)$ and $\theta^{\beta-1}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\geq 0$. Then, we have

$$\begin{aligned}
&\theta^{\beta-1}\Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\\
&\qquad\leq \theta^{\beta-1}\big(h(1-\theta)\Upsilon_*(\mu)+h(\theta)\Upsilon_*(\upsilon)\big)\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\\
&\theta^{\beta-1}\Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\Omega(\theta\mu+(1-\theta)\upsilon)\\
&\qquad\leq \theta^{\beta-1}\big(h(1-\theta)\Upsilon^*(\mu)+h(\theta)\Upsilon^*(\upsilon)\big)\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right).
\end{aligned} \tag{33}$$

And

$$\begin{aligned}
&\theta^{\beta-1}\Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\\
&\qquad\leq \theta^{\beta-1}\big(h(\theta)\Upsilon_*(\mu)+h(1-\theta)\Upsilon_*(\upsilon)\big)\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\\
&\theta^{\beta-1}\Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\\
&\qquad\leq \theta^{\beta-1}\big(h(\theta)\Upsilon^*(\mu)+h(1-\theta)\Upsilon^*(\upsilon)\big)\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right).
\end{aligned} \tag{34}$$

After adding (33) and (34), and integrating over $[0,1]$, we get

$$\int_0^1 \theta^{\beta-1} \Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) d\theta$$

$$+\int_0^1 \theta^{\beta-1} \Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right) \Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) d\theta$$

$$\leq \int_0^1 \begin{bmatrix} \theta^{\beta-1}\Upsilon_*(\mu)\{h(\theta)+h(1-\theta)\}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \\ +\theta^{\beta-1}\Upsilon_*(\upsilon)\{h(1-\theta)+h(\theta)\}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \end{bmatrix} d\theta,$$

$$\int_0^1 \theta^{\beta-1} \Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) d\theta$$

$$+\int_0^1 \theta^{\beta-1} \Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right) \Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) d\theta$$

$$\leq \int_0^1 \begin{bmatrix} \theta^{\beta-1}\Upsilon^*(\mu)\{h(\theta)+h(1-\theta)\}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \\ +\theta^{\beta-1}\Upsilon^*(\upsilon)\{h(1-\theta)+h(\theta)\}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) \end{bmatrix} d\theta,$$

$$= \Upsilon_*(\mu)\int_0^1 \theta^{\beta-1}\{h(\theta)+h(1-\theta)\}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) d\theta$$

$$+\Upsilon_*(\upsilon)\int_0^1 \theta^{\beta-1}\{h(1-\theta)+h(\theta)\}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) d\theta,$$

$$= \Upsilon^*(\mu)\int_0^1 \theta^{\beta-1}\{h(\theta)+h(1-\theta)\}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) d\theta$$

$$+\Upsilon^*(\upsilon)\int_0^1 \theta^{\beta-1}\{h(1-\theta)+h(\theta)\}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) d\theta.$$

that is

$$\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^\beta \left[\mathcal{J}^\beta_{\frac{1}{\upsilon}^+}(\Omega o\delta)\left(\frac{1}{\mu}\right) + \mathcal{J}^\beta_{\frac{1}{\mu}^-}(\Omega o\delta)\left(\frac{1}{\upsilon}\right)\right]$$

$$\leq_p \frac{\Upsilon(\mu)+\Upsilon(\upsilon)}{2}\int_0^1 \theta^{\beta-1}\{h(\theta)+h(1-\theta)\}\Omega\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right) d\theta. \quad (35)$$

As a result, the desired result has been achieved.

Following result obtain the first interval fractional $H \cdot H$ Fejér inequality.

Theorem 3.6. Let $\Upsilon: [\mu, \upsilon] \to \mathcal{K}_C^+$ be a IVF such that $\Upsilon(\varpi) = [\Upsilon_*(\varpi), \Upsilon^*(\varpi)]$ for all $\varpi \in [\mu, \upsilon]$, respectively. If $\Upsilon \in LRHSX([\mu, \upsilon], \mathcal{K}_C^+, h)$, $\Upsilon \in L([\mu, \upsilon], \mathcal{K}_C^+)$ and $\Omega: [\mu, \upsilon] \to \mathbb{R}$, $\Omega\left(\frac{1}{\frac{1}{\mu}+\frac{1}{\upsilon}-\frac{1}{\varpi}}\right) = \Omega(\varpi) \geq 0$, then

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\left[\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}(\Omega o\delta)\left(\frac{1}{\mu}\right) + \mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}(\Omega o\delta)\left(\frac{1}{\upsilon}\right)\right] \leq_p \left[\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}(\Upsilon\Omega o\delta)\left(\frac{1}{\mu}\right) + \mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}(\Upsilon\Omega o\delta)\left(\frac{1}{\upsilon}\right)\right]. \quad (36)$$

If $\Upsilon \in LRHSV([\mu, \upsilon], \mathcal{K}_C^+, h)$, then inequality (36) is reversed such that

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\left[\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}(\Omega o\delta)\left(\frac{1}{\mu}\right) + \mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}(\Omega o\delta)\left(\frac{1}{\upsilon}\right)\right] \geq_p \left[\mathcal{J}^\beta_{\left(\frac{1}{\upsilon}\right)^+}(\Upsilon\Omega o\delta)\left(\frac{1}{\mu}\right) + \mathcal{J}^\beta_{\left(\frac{1}{\mu}\right)^-}(\Upsilon\Omega o\delta)\left(\frac{1}{\upsilon}\right)\right].$$

Proof. Since $\Upsilon \in LRHSX([\mu, \upsilon], \mathcal{K}_C^+, h)$, then we have

$$\Upsilon_*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right) \leq h\left(\frac{1}{2}\right)\left(\Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)+\Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\right)$$
$$\Upsilon^*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right) \leq h\left(\frac{1}{2}\right)\left(\Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)+\Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\right), \tag{37}$$

Multiplying both sides by (37) by $\theta^{\beta-1}\Omega\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)$ and then integrating the resultant with respect to θ over $[0,1]$, we obtain

$$\begin{aligned}
&\Upsilon_*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\int_0^1 \theta^{\beta-1}\Omega\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)d\theta \\
&\qquad \leq h\left(\frac{1}{2}\right)\begin{pmatrix} \int_0^1 \theta^{\beta-1}\Upsilon_*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\Omega\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)d\theta \\ +\int_0^1 \theta^{\beta-1}\Upsilon_*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\Omega\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)d\theta \end{pmatrix}, \\
&\Upsilon^*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\int_0^1 \theta^{\beta-1}\Omega\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)d\theta \\
&\qquad \leq h\left(\frac{1}{2}\right)\begin{pmatrix} \int_0^1 \theta^{\beta-1}\Upsilon^*\left(\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}\right)\Omega\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)d\theta \\ +\int_0^1 \theta^{\beta-1}\Upsilon^*\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)\Omega\left(\frac{\mu\upsilon}{(1-\theta)\mu+\theta\upsilon}\right)d\theta \end{pmatrix}.
\end{aligned} \tag{38}$$

Let $\varpi=\frac{\mu\upsilon}{\theta\mu+(1-\theta)\upsilon}$. Then, we have

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\Upsilon_*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\int_{\frac{1}{\upsilon}}^{\frac{1}{\mu}}\left(\varpi-\frac{1}{\upsilon}\right)^{\beta-1}\Omega\left(\frac{1}{\varpi}\right)d\varpi \\
&\qquad \leq \left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\int_{\frac{1}{\upsilon}}^{\frac{1}{\mu}}\left(\varpi-\frac{1}{\upsilon}\right)^{\beta-1}\Upsilon_*\left(\frac{1}{\frac{1}{\mu}+\frac{1}{\upsilon}-\frac{1}{\varpi}}\right)\Omega\left(\frac{1}{\varpi}\right)d\varpi \\
&\qquad\quad +\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\int_{\mu}^{\frac{1}{\mu}}\left(\varpi-\frac{1}{\upsilon}\right)^{\beta-1}\Upsilon_*\left(\frac{1}{\varpi}\right)\Omega\left(\frac{1}{\varpi}\right)d\varpi \\
&\qquad = \left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\int_{\frac{1}{\upsilon}}^{\frac{1}{\mu}}\left(\frac{1}{\mu}-\varpi\right)^{\beta-1}\Upsilon_*(\varpi)\Omega\left(\frac{1}{\frac{1}{\mu}+\frac{1}{\upsilon}-\frac{1}{\varpi}}\right)d\varpi \\
&\qquad\quad +\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\int_{\frac{1}{\upsilon}}^{\frac{1}{\mu}}\left(\varpi-\frac{1}{\upsilon}\right)^{\beta-1}\Upsilon_*\left(\frac{1}{\varpi}\right)\Omega\left(\frac{1}{\varpi}\right)d\varpi \\
&\qquad = 2\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\Upsilon_*\Omega\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^-}\Upsilon_*\Omega\left(\frac{1}{\upsilon}\right)\right], \\
&\frac{1}{h\left(\frac{1}{2}\right)}\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\Upsilon_*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\int_{\frac{1}{\upsilon}}^{\frac{1}{\mu}}(\varpi-\frac{1}{\upsilon})^{\beta-1}\Omega_*\left(\frac{1}{\varpi}\right)d\varpi \\
&\qquad \leq 2\Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\Upsilon^*\Omega\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^-}\Upsilon^*\Omega\left(\frac{1}{\upsilon}\right)\right].
\end{aligned} \tag{39}$$

From (39), we have

$$\Gamma(\beta)\frac{1}{2h\left(\frac{1}{2}\right)}\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\Upsilon_*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right),\ \Upsilon^*\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\right]\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\Omega\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^-}\Omega\left(\frac{1}{\upsilon}\right)\right]$$
$$\leq_I \Gamma(\beta)\left(\frac{\mu\upsilon}{\upsilon-\mu}\right)^{\beta}\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\Upsilon_*\Omega\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^-}\Upsilon_*\Omega\left(\frac{1}{\upsilon}\right),\ \mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\Upsilon^*\Omega\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^-}\Upsilon^*\Omega\left(\frac{1}{\upsilon}\right)\right],$$

that is

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^{+}}(\Omega o\delta)\left(\frac{1}{\mu}\right)+\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^{-}}(\Omega o\delta)\left(\frac{1}{\upsilon}\right)\right]\leq_{p}\left[\mathcal{J}^{\beta}_{\left(\frac{1}{\upsilon}\right)^{+}}(\Upsilon\Omega o\delta)\left(\frac{1}{\mu}\right)\right.$$
$$\left.+\mathcal{J}^{\beta}_{\left(\frac{1}{\mu}\right)^{-}}(\Upsilon\Omega o\delta)\left(\frac{1}{\upsilon}\right)\right].$$

The theorem has been proved.

Remark 3.7. From Theorem 3.5 and Theorem 3.6, following result can be obtained:

On fixing $h(\theta)=\theta$ and $\beta=1$ then following $H{\cdot}H$ inequality is obtained, which is also new one:

$$\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\int_{\mu}^{\upsilon}\frac{\Omega(\varpi)}{\varpi^{2}}d\varpi\leq_{p}\int_{\mu}^{\upsilon}\frac{\Upsilon(\varpi)}{\varpi^{2}}\Omega(\varpi)d\varpi\leq_{p}\frac{\Upsilon(\mu)+\Upsilon(\upsilon)}{2}\int_{\mu}^{\upsilon}\frac{\Omega(\varpi)}{\varpi^{2}}d\varpi.$$

On fixing $h(\theta)=\theta$ and $\Omega(\varpi)=1$, the inequality (21) is obtained.

On fixing $h(\theta)=\theta$, $\Omega(\varpi)=1$ and $\beta=1$, the following $H{\cdot}H$ inequality is obtained:

$$\Upsilon\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right)\leq_{p}\frac{\mu\upsilon}{\upsilon-\mu}\int_{\mu}^{\upsilon}\frac{\Upsilon(\varpi)}{\varpi^{2}}d\varpi\leq_{p}\frac{\Upsilon(\mu)+\Upsilon(\upsilon)}{2}.$$

4. Conclusion and future plan

The concept of left and right $\mathcal{H}-h$-convex IVFs has been discussed. We've also demonstrated that this type of convexity includes a few other types of classical convexity. The left and right $\mathcal{H}-h$-convex IVF is a harmonically convex function with distinct approaches that has been suggested. Furthermore, we have developed several novel classical and fractional integral inequalities of the Hermite-Hadamard and related Hermite-Hadamard type inequalities using the concept of left and right $\mathcal{H}$-h-convex IVF. The findings in this article can be used to identify different types of inequality. Furthermore, these results are universal and may be used to generate other, potentially useful, and fascinating inequalities using various fractional operators. In near future, we will generalize the class of left and right $\mathcal{H}-h$-convex IVFs and with the help of this class, we will try to obtain some new version of different type inequalities.

Conflict of interests

The authors declare that they have no competing interests.

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