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Research article

An existence result involving both the generalized proportional Riemann-Liouville and Hadamard fractional integral equations through generalized Darbo's fixed point theorem

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Abstract: In this paper, we propose and prove an extension and generalization, which extends and generalizes the Darbo's fixed point theorem (DFPT) in the context of measure of noncompactness (MNC). Thereafter, we use DFPT to investigate the existence of solutions to mixed-type fractional integral equations (FIE), which include both the generalized proportional (κ, τ)-Riemann-Liouville and Hadamard fractional integral equations. We've included a suitable example to strengthen the article.

Keywords: (κ, τ)-type generalized proportional fractional integral equation; generalized proportional Hadamard fractional (GPHF) integral equation; MNC; DFPT
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1. Introduction

The MNC is one of the most powerful tool of modern mathematical analysis, which was introduced by Kuratowski [1] in 1930. It was generalized by Banas [2] for solving functional equations, which is applicable to numerous mathematical problems. Darbo [3] has generalized of Schauder fixed point theorem (SFPT) and Banach contraction principle, using the concept of MNC.

In the present time, the fixed point theory (FPT) have so many applications in several area of mathematics along with FPT can be apply for the existence of solutions of FIE. It is still continue to earned the attention of researchers in various applications of functional calculus in science and technology.

Fractional calculus is a very powerful tool to achieve differentiation and integration with real or complex number order of operators. For recent research on fractional calculus, we refer the reader to (see [4–12]). On the other hand, there are various known forms of fractional integrals and their applications. For example: Katugampola [13] introduced a new FIE, which generalize Riemann-Liouville and Hadamard FIE into single form, Mubeen and Habibullah [14] have introduced the κ-fractional integral of Riemann-Liouville by using κ-gamma function defined by Diaz and Pariguan [15], Mehmet et al. [16] generalized a new FIE known as (κ, τ)-Riemann-Liouville FIE. Jarad et al. [17] have given the concept of generalized proportional integral operator, which has been specify certain probability density functions and has interested applications in statistics.

Inspired and motivated by ([16, 17]), in the context of MNC, we generalize the DFPT and a new FIE involving both the generalized proportional (κ, τ)-Riemann-Liouville and Hadamard. Thereafter, we use DFPT to investigate the existence of solutions of mixed-type FIE, which include both the generalized proportional (κ, τ)-Riemann-Liouville and Hadamard.

We have used the notations in this paper.

- Ξ: Banach space with the norm $\| \,.\, \|_{\Xi}$.
- $\bar{\aleph}$: closure of $\aleph$.
- $Conv\aleph$: convex closure of $\aleph$.
- $\mathfrak{M}_{\Xi}$: subset of all nonempty and bounded subsets of Ξ.
- $\mathfrak{N}_{\Xi}$: subset which contains all relatively compact sets.
- $\mathbb{R}$: $(-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{N}$ be set of natural numbers.

Banas and Lecko [18] have defined MNC as follows:

Definition 1.1. *A mapping $\Lambda : \mathfrak{M}_{\Xi} \to \mathbb{R}_+$ is said to be MNC in Ξ if it satisfies the following conditions:*

(N_1) The family ker $\Lambda = \{\aleph \in \mathfrak{M}_{\Xi} : \Lambda(\aleph) = 0\}$ is nonempty and ker $\Lambda \subset \mathfrak{N}_{\Xi}$.

(N_2) $\aleph_1 \subset \aleph_2 \implies \Lambda(\aleph_1) \le \Lambda(\aleph_2)$.

(N_3) $\Lambda\left(\bar{\aleph}\right) = \Lambda(\aleph) = \Lambda(Conv\aleph)$.

(N_4) $\Lambda(k\aleph_1 + (1-k)\aleph_2) \le k\Lambda(\aleph_1) + (1-k)\Lambda(\aleph_2)$, *for* $k \in [0, 1]$.

(N_5) If $(\aleph_n)$ is a sequence of closed sets from $\mathfrak{M}_{\Xi}$, such that $\aleph_{n+1} \subset \aleph_n$ for $n = 1, 2, 3, ...$ and if $\lim_{n\to\infty} \Lambda(\aleph_n) = 0$, then $\aleph_\infty = \bigcap_{n=1}^{\infty} \aleph_n \neq \emptyset$.

Example 1.2. *Let $\Xi = C(I)$ be the space of real continuous functions on I, where $I = [a, b]$. Then $g : \mathfrak{M}_{\Xi} \to \mathbb{R}_+$ is defined as*

$$\|g\| = \sup\{\|g(t)\| : t \in I\}, \; g \in \Xi.$$

Then g satisfies all the properties of MNC.

Remark 1.3. *Since* $\Lambda(\aleph_\infty) = \bigcup\left(\bigcap_{n=1}^{\infty} \aleph_n\right) \le \Lambda(\aleph_n)$, $\Lambda(\aleph_\infty) = 0$. *So* $\aleph_\infty \in ker\Lambda$.

Theorem 1.4. (Schauder) [19] A mapping $\Im : \aleph \to \aleph$ which is compact and continuous have a fixed point ($\mathbb{FP}$), where $\aleph$ is a nonempty, bounded, closed and convex (NBCC) subset of a Banach space Ξ.

Theorem 1.5. (Darbo) [20] Let $\Im : \aleph \to \aleph$ be a continuous mapping and Λ is an MNC. If for any nonempty subset ς of $\aleph$, there exists a $k \in [0, 1)$ having the inequality,

$$\Lambda(\Im\varsigma) \le k\,\Lambda(\varsigma),$$

then the mapping $\Im$ have a $\mathbb{FP}$ in $\aleph$.

Definition 1.6. *Let $\aleph$ be bounded subset of metric space Ξ. Then for bounded set $\aleph$, the Hausdorff MNC Λ is defined as*

$$\Lambda(\aleph) = \inf\{\epsilon > 0 : \aleph \text{ has finite } \epsilon - \text{net in } \Xi\}.$$

Definition 1.7. *[21] The functions $F : \mathbb{R}_+^2 \to \mathbb{R}$, which are continuous known by C-class function if it satisfies*

(F_1) $F(\omega, \nu) \le \omega$.

(F_2) $F(\omega, \nu) = \omega \implies \omega = 0$ *or* $\nu = 0$ *for all* $\omega, \nu \in \mathbb{R}$.

Example 1.8. (i) $F(\omega, \nu) = \omega - \nu$.
(ii) $F(\omega, \nu) = k\omega$, *where* $0 \le k < 1$.

Definition 1.9. *Suppose Ψ is the set of continuous functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy*

(a) $\psi(t) = 0$ *if and only if* $t = 0$.

(b) ψ *is non decreasing.*

(c) $\psi(t) < t$ *for every* $t > 0$.

Let $\Xi = C(I)$ be the space of real continuous functions on I, where $I = [a, b]$. Then Ξ is a Banach space with the norm

$$\|\varkappa\| = \sup\{|\varkappa(\varsigma)| : \varsigma \in I\},\ \varkappa \in \Xi.$$

Let Υ be a non-empty bounded subset of Ξ then, for $\varkappa \in \Upsilon$ and $\epsilon > 0$, $\omega(\varkappa, \epsilon)$ be the modulus of the continuity of $\varkappa$ and defined as

$$\omega(\varkappa, \epsilon) = \sup\{|\varkappa(\varsigma_1) - \varkappa(\varsigma_2)| : \varsigma_1, \varsigma_2 \in I, |\varsigma_1 - \varsigma_2| \le \epsilon\}.$$

Again, we define

$$\omega(\Upsilon, \epsilon) = \sup\{\omega(\varkappa, \epsilon) : \varkappa \in \Upsilon\},$$

$$\omega_0(\Upsilon) = \lim_{\epsilon \to 0} \omega(\Upsilon, \epsilon).$$

Hence the function ω_0 is a MNC in Ξ in such a way that the Hausdorff MNC Λ is given by $\Lambda(\Upsilon) = \frac{1}{2}\omega_0(\Upsilon)$ (see [2]).

2. Generalization of Darbo's fixed point theorem

Theorem 2.1. Let $(\Xi, \| \, . \, \|)$ be a Banach space. Suppose $\mathfrak{I} : \aleph \to \aleph$ is a continuous, nondecreasing and bounded (CNB) mapping fulfills the following inequality

$$\vartheta\left(\int_0^{\Lambda(\mathfrak{I}\aleph)} \pi(\varsigma)d\varsigma\right) \leq F\left(\vartheta\left(\int_0^{\Lambda(\aleph)} \pi(\varsigma)d\varsigma\right), \psi\left(\vartheta\left(\int_0^{\Lambda(\aleph)} \pi(\varsigma)d\varsigma\right)\right)\right) \tag{2.1}$$

for each bounded $\aleph$ of Ξ, where $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous functions, $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, continuous functions with $\vartheta(\varsigma) = 0$ if and only if $\varsigma = 0$, $\psi : \mathbb{R}_+ \to [0, 1)$ is a continuous function and Λ is MNC. Then $\mathfrak{I}$ contains at least one $\mathbb{FP}$.

Proof. Assume that $\aleph_n$ with $\aleph_0 = \aleph$ and $\aleph_{n+1} = conv(\mathfrak{I}\aleph_n)$ for all $n \geq 0$.
Also, $\mathfrak{I}\aleph_0 = \mathfrak{I}\aleph \subseteq \aleph = \aleph_0$, $\aleph_1 = conv(\mathfrak{I}\aleph_0) \subseteq \aleph = \aleph_0$. Continuing in the similar manner gives

$$\aleph_0 \supset \aleph_1 \supset, ..., \supset \aleph_n \supset, ... \tag{2.2}$$

Let $\Lambda(\aleph_n) > 0$ for $n \in \mathbb{N}$. We claim $\left\{\left(\int_0^{\Lambda(\mathfrak{I}\aleph_n)} \pi(\varsigma)d\varsigma\right)\right\}$ is positive decreasing sequence. By using (2.1), we have

$$\begin{aligned}\vartheta\left(\int_0^{\Lambda(\aleph_{n+1})} \pi(\varsigma)d\varsigma\right) &= \vartheta\left(\int_0^{\Lambda(conv(\mathfrak{I}\aleph_n))} \pi(\varsigma)d\varsigma\right) = \vartheta\left(\int_0^{\Lambda(\mathfrak{I}\aleph_n)} \pi(\varsigma)d\varsigma\right) \\ &\leq F\left(\vartheta\left(\int_0^{\Lambda(\aleph_n)} \pi(\varsigma)d\varsigma\right), \psi\left(\vartheta\left(\int_0^{\Lambda(\aleph_n)} \pi(\varsigma)d\varsigma\right)\right)\right) \\ &\leq \vartheta\left(\int_0^{\Lambda(\mathfrak{I}\aleph_n)} \pi(\varsigma)d\varsigma\right).\end{aligned}$$

Since ϑ is nondecreasing function, we get

$$\vartheta\left(\int_0^{\Lambda(\aleph_{n+1})} \pi(\varsigma)d\varsigma\right) \leq \vartheta\left(\int_0^{\Lambda(\mathfrak{I}\aleph_n)} \pi(\varsigma)d\varsigma\right).$$

Then $\left\{\left(\int_0^{\Lambda(\mathfrak{I}\aleph_n)} \pi(\varsigma)d\varsigma\right)\right\}$ is positive decreasing and bounded below, so it converges to $r = \left\{\left(\int_0^{\Lambda(\mathfrak{I}\aleph_n)} \pi(\varsigma)d\varsigma\right)\right\}$. Now suppose $r > 0$, then from Eq (2.1), we have

$$\begin{aligned}0 \leq \vartheta\left(\int_0^{\Lambda(\aleph_{n+1})} \pi(\varsigma)d\varsigma\right) = \vartheta\left(\int_0^{\Lambda(conv(\mathfrak{I}\aleph_n))} \pi(\varsigma)d\varsigma\right) &= \vartheta\left(\int_0^{\Lambda(\mathfrak{I}\aleph_n)} \pi(\varsigma)d\varsigma\right) \\ &\leq F\left(\vartheta\left(\int_0^{\Lambda(\aleph_n)} \pi(\varsigma)d\varsigma\right), \psi\left(\vartheta\left(\int_0^{\Lambda(\aleph_n)} \pi(\varsigma)d\varsigma\right)\right)\right),\end{aligned}$$

i.e., $\vartheta\left(\int\limits_0^{\Lambda(\Im\aleph_n)}\pi(\varsigma)d\varsigma\right) \le F\left(\vartheta\left(\int\limits_0^{\Lambda(\aleph_n)}\pi(\varsigma)d\varsigma\right), \psi\left(\vartheta\left(\int\limits_0^{\Lambda(\aleph_n)}\pi(\varsigma)d\varsigma\right)\right)\right)$. Taking the limit $n \to \infty$ on both the sides of this inequality, we have

$$\vartheta(r) \le F\left(\vartheta(r), \psi(\vartheta(r))\right) \le \vartheta(r)$$

which means

$$F\left(\vartheta(r), \psi(\vartheta(r))\right) = \vartheta(r),$$

so from (F_2), we get $\vartheta(r) = 0$, hence $r = 0$. which implies $\Lambda(\aleph_n) \to 0$.
Since $\aleph_n$ is nested sequence, so by the (N_5) property of (MNC), we conclude that $\aleph_\infty = \cap_{n=1}^{\infty}\aleph_n$ is NBCC of Ξ. Also, we know that $\aleph_\infty \in ker\Lambda$. Therefore $\aleph_\infty$ is compact and invariant under the mapping $\Im$. Hence by the SFPT, $\Im$ have a $\mathbb{FP}$ in $\aleph_\infty \subset \Xi$. □

Corollary 2.2. *If we take $\pi(\varsigma) = 1$ for $\varsigma \in [0, \infty)$ in Theorem (2.1), then we have*

$$\vartheta\left(\Lambda(\Im\aleph)\right) \le F\left(\vartheta\left(\Lambda(\aleph)\right), \psi\left(\vartheta\left(\Lambda(\Im\aleph)\right)\right)\right).$$

Corollary 2.3. *Take $F(\omega, \nu) = \omega\nu$ in Corollary(2.2), then we have*

$$\vartheta\left(\Lambda(\Im\aleph)\right) \le \psi\left(\vartheta\left(\Lambda(\Im\aleph)\right)\right)\vartheta\left(\Lambda(\aleph)\right).$$

It is extension of DFPT extended by Ghaemi and Samadi [23].

Corollary 2.4. *Take $F(\omega, \nu) = \omega - \nu$ in Corollary(2.2), then we have*

$$\vartheta\left(\Lambda(\Im\aleph)\right) \le \vartheta\left(\Lambda(\aleph)\right) - \psi\left(\vartheta\left(\Lambda(\Im\aleph)\right)\right).$$

It is generalization of DFPT generalize by Parvaneh et al. [22].

Corollary 2.5. *If we take $\psi(\tau) = k,\ 0 \le k < 1,\ \vartheta(\tau) = \tau$ in Corollary (2.3), then we have*

$$\Lambda(\Im\aleph) \le k\left(\Lambda(\aleph)\right).$$

Then it is DFPT [20].

Corollary 2.6. *If we take $\vartheta(\tau) = e^\tau$ and $\psi(\tau) = \tau - \tau^k,\ 0 \le k < 1$ in Corollary (2.4), then we have DFPT [20]*

$$\Lambda(\Im\aleph) \le k\left(\Lambda(\aleph)\right).$$

Corollary 2.7. *If we take $\vartheta(\tau) = \tau$ and $\psi(\tau) = \tau - k\tau,\ 0 \le k < 1$ in Corollary (2.4), then we have DFPT [20]*

$$\Lambda(\Im\aleph) \le k\left(\Lambda(\aleph)\right).$$

Corollary 2.8. *If we take $F(\omega, \nu) = k\omega$, where $0 \le k < 1,\ \vartheta(\tau) = \tau$ in Corollary (2.2), then we have*

$$\Lambda(\Im\aleph) \le k\left(\Lambda(\aleph)\right).$$

Then it is also DFPT [20].

Remark 2.9. *Hence it can be seen that the Theorem 2.2 is the generalization of the DFPT.*

3. Existence solutions of generalized proportional (κ,τ)-Riemann-Liouville and Hadamard FIE

We define a new generalization of Mehmet et al. [16] and known as the (κ,τ)-type generalized proportional FIE integral equation of order $\varpi > 0$ and defined as

$$\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{a}\Theta\right)(\varsigma) = \frac{(\tau+1)^{1-\frac{\varpi}{\kappa}}}{\rho^{\varpi}\kappa\Gamma_{\kappa}(\varpi)}\int_{a}^{\varsigma}\exp\left[\frac{(\rho-1)\left(\varsigma^{\tau+1}-\eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma^{\tau+1}-\eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\eta^{\tau}\Theta(\eta)d\eta,$$

where $\rho \in (0,1],\ \tau \in \mathbb{R}^{+}/\{-1\}$ and $\varpi,\kappa > 0$.

Also, motivated by Hadamard [9], we define a new generalization of Hadamard FIE, which is known as GPHF integral equation of order $\varpi > 0$ and defined as

$$\left({}^{H}\mathfrak{J}^{\varpi,\rho}_{a}\Theta\right)(\varsigma) = \frac{1}{\rho^{\varpi}\Gamma(\varpi)}\int_{a}^{\varsigma}\exp\left[\frac{(\rho-1)\left(\log(\varsigma)-\log(\eta)\right)}{\rho}\right]\left(\log(\varsigma)-\log(\eta)\right)^{\varpi-1}\frac{\Theta(\eta)}{\eta}d\eta,$$

where $\rho \in (0,1]$ and $\varsigma \in [a,b]$.

The present study, we have considered the following FIE:

$$\Theta(\varsigma) = F\left(\varsigma, \pounds(\varsigma,\Theta(\varsigma)), \left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{a=1}\Theta\right)(\varsigma), \left({}^{H}\mathfrak{J}^{\varpi,\rho}_{a=1}\Theta\right)(\varsigma)\right), \tag{3.1}$$

where $\varpi > 1,\ \kappa > 0,\ \rho \in (0,1],\ \tau \in \mathbb{R}^{+}/\{-1\}$ and $\varsigma \in I = [a,b],\ a > 0,\ \ b = T$.

Let $\mathcal{B}_{\nu_o} = \{\Theta \in \Xi :\ \ \|\Theta\| \le \nu_0\}$. We consider the following assumptions to solve the Eq (3.1):

(i) $F : I \times \mathbb{R}^3 \to \mathbb{R},\ \pounds : I \times \mathbb{R} \to \mathbb{R}$ be continuous and there exists constants $\wp_1,\ \wp_2,\ \wp_3 \ge 0$, which satisfy

$$\left|F(\varsigma,\pounds,I_1,J_1) - F(\varsigma,\bar{\pounds},\bar{I}_1,\bar{J}_1)\right| \le \wp_1\left|\pounds-\bar{\pounds}\right| + \wp_2\left|I_1-\bar{I}_1\right| + \wp_3\left|J_1-\bar{J}_1\right|,\ \varsigma \in I,\ \pounds, I_1, J_1, \bar{\pounds}, \bar{I}_1, \bar{J}_1 \in \mathbb{R}$$

and

$$\left|\pounds(\varsigma,P_1) - \pounds(\varsigma,P_2)\right| \le \wp_4\left|P_1-P_2\right|,\ where\ P_1,P_2 \in \mathbb{R}.$$

(ii) There exists $\nu_0 \in \mathbb{R}_+$, which satisfy

$$F = \sup\{|F(\varsigma,\pounds,I_1,J_1| : \varsigma \in I, \pounds \in [-L,L], I_1 \in [-\mathbf{I},\mathbf{I}], J_1 \in [-\mathbf{J},\mathbf{J}]\} \le \nu_0, \wp_1\wp_4 < 1,$$

$$L = \sup\{|\pounds(\varsigma,\Theta(\varsigma))| : \varsigma \in I, \Theta(\varsigma) \in [-\nu_0,\nu_0]\},$$

$$\mathbf{I} = \sup\{|\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma)| : \varsigma \in I, \Theta(\varsigma) \in [-\nu_0,\nu_0]\},$$

and

$$\mathbf{J} = \sup\{|\left({}^{H}\mathfrak{J}^{\varpi,\rho}_{1}\Theta\right)(\varsigma)| : \varsigma \in I, \Theta(\varsigma) \in [-\nu_0,\nu_0]\}.$$

(iii) $|F(\varsigma,0,0,0)| = 0,\ \ \pounds(\varsigma,0) = 0.$

(iv) For a positive solution $\nu_0 \in \mathbb{R}_+$ having inequality,
$\wp_1\wp_4\nu_0 + \wp_2\frac{\nu_0\exp\left[\frac{(\rho-1)T^{\tau+1}}{\rho}\right](\tau+1)^{-\frac{\varpi}{\kappa}}}{\varpi\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}-1}\Gamma(\frac{1}{\kappa})}\left(T^{(\tau+1)}-1\right)^{\frac{\varpi}{\kappa}} + \wp_3\frac{\nu_0\exp\left[\frac{(\rho-1)\log T}{\rho}\right]}{\rho^{\varpi}\Gamma(\varpi+1)}(\log T)^{\varpi} \le \nu_0.$

Theorem 3.1. *If the conditions* (i)–(iv) *holds, then the Eq (3.1) have a solution in* $\Xi = C(I)$.

Proof. Let the operator $\Im : \mathcal{B}_{\nu_o} \to \Xi$ is define as

$$(\Im\Theta)(\varsigma) = F\left(\varsigma, \pounds(\varsigma, \Theta(\varsigma)), \left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma), \left({}^{H}\beth^{\varpi,\rho}_{1}\Theta\right)(\varsigma)\right).$$

Step 1: First, we have to prove $\Im$ maps $\mathcal{B}_{\nu_o}$ into $\mathcal{B}_{\nu_o}$. Let $\Im \in \mathcal{B}_{\nu_o}$, we have

$$\begin{aligned}|(\Im\Theta)(\varsigma)| &= |F\left(\varsigma, \pounds(\varsigma, \Theta(\varsigma)), \left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma), \left({}^{H}\beth^{\varpi,\rho}_{1}\Theta\right)(\varsigma)\right) - F(\varsigma, 0, 0, 0)| + |F(\varsigma, 0, 0, 0)| \\ &\leq \wp_1|\pounds(\varsigma, \Theta(\varsigma)) - 0| + \wp_2|\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma) - 0| + \wp_3|\left({}^{H}\beth^{\varpi,\rho}_{1}\Theta\right)(\varsigma) - 0| + |F(\varsigma, 0, 0, 0)|\end{aligned}$$

where,

$$\begin{aligned}|\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma) - 0| &= \left|\frac{(\tau+1)^{1-\frac{\varpi}{\kappa}}}{\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}}\Gamma(\frac{\varpi}{\kappa})}\int_1^{\varsigma}\exp\left[\frac{(\rho-1)\left(\varsigma^{\tau+1}-\eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma^{\tau+1}-\eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\eta^{\tau}\Theta(\eta)d\eta\right| \\ &= \frac{(\tau+1)^{1-\frac{\varpi}{\kappa}}}{\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}}\Gamma(\frac{\varpi}{\kappa})}\left|\int_1^{\varsigma}\exp\left[\frac{(\rho-1)\left(\varsigma^{\tau+1}-\eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma^{\tau+1}-\eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\eta^{\tau}\Theta(\eta)d\eta\right| \\ &\leq \frac{\nu_0(\tau+1)^{1-\frac{\varpi}{\kappa}}\exp\left[\frac{(\rho-1)T^{\tau+1}}{\rho}\right]}{\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}}\Gamma(\frac{\varpi}{\kappa})}\int_1^{\varsigma}\left(\varsigma^{\tau+1}-\eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\eta^{\tau}d\eta \\ &\leq \frac{\nu_0\exp\left[\frac{(\rho-1)T^{\tau+1}}{\rho}\right](\tau+1)^{-\frac{\varpi}{\kappa}}}{\varpi\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}-1}\Gamma(\frac{1}{\kappa})}\left(T^{(\tau+1)}-1\right)^{\frac{\varpi}{\kappa}}\end{aligned}$$

and

$$\begin{aligned}|\left({}^{H}\beth^{\varpi,\rho}_{1}\Theta\right)(\varsigma) - 0| &= \left|\frac{1}{\rho^{\varpi}\Gamma(\varpi)}\int_1^{\varsigma}\exp\left[\frac{(\rho-1)(\log(\varsigma)-\log(\eta))}{\rho}\right](\log(\varsigma)-\log(\eta))^{\varpi-1}\frac{\Theta(\eta)}{\eta}d\eta\right| \\ &\leq \frac{\nu_0\exp\left[\frac{(\rho-1)\log T}{\rho}\right]}{\rho^{\varpi}\Gamma(\varpi)}\int_1^{\varsigma}(\log(\varsigma)-\log(\eta))^{\varpi-1}\frac{d\eta}{\eta} \\ &\leq \frac{\nu_0\exp\left[\frac{(\rho-1)\log T}{\rho}\right]}{\rho^{\varpi}\Gamma(\varpi+1)}(\log T)^{\varpi}.\end{aligned}$$

Therefore if $\|\Theta\| < \nu_0$ then

$$\|\Im\Theta\| < \wp_1\wp_4\nu_0 + \wp_2\frac{\nu_0\exp\left[\frac{(\rho-1)T^{\tau+1}}{\rho}\right](\tau+1)^{-\frac{\varpi}{\kappa}}}{\varpi\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}-1}\Gamma(\frac{1}{\kappa})}\left(T^{(\tau+1)}-1\right)^{\frac{\varpi}{\kappa}} + \wp_3\frac{\nu_0\exp\left[\frac{(\rho-1)\log T}{\rho}\right]}{\rho^{\varpi}\Gamma(\varpi+1)}(\log T)^{\varpi}.$$

So by the assumption (iv), $\Im$ maps $\mathcal{B}_{\nu_0}$ into $\mathcal{B}_{\nu_0}$.

Step 2: Now, we have to prove $\Im$ is continuous on $\mathcal{B}_{\nu_o}$. Let $\epsilon > 0$ and $\Theta, \bar{\Theta} \in \mathcal{B}_{\nu_o}$ such that $\| \Theta - \bar{\Theta} \| < \epsilon$, we have

$$\begin{aligned}\left|(\Im\Theta)(\varsigma) - \left(\Im\bar{\Theta}\right)(\varsigma)\right| \leq \Big|&F\left(\varsigma, \pounds(\varsigma, \Theta(\varsigma)), \left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma), \left({}^{H}\beth^{\varpi,\rho}_{1}\Theta\right)(\varsigma)\right) \\ &- F\left(\varsigma, \pounds(\varsigma, \bar{\Theta}(\varsigma)), \left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\bar{\Theta}\right)(\varsigma), \left({}^{H}\beth^{\varpi,\rho}_{1}\bar{\Theta}\right)(\varsigma)\right)\Big|\end{aligned}$$

$$\leq \wp_1\left|\pounds(\varsigma,\Theta(\varsigma))-\pounds(\varsigma,\bar{\Theta}(\varsigma))\right|+\wp_2\left|\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma)-\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\bar{\Theta}\right)(\varsigma)\right|$$
$$+\wp_3\left|\left({}^{H}\mathfrak{I}^{\varpi,\rho}_{1}\Theta\right)(\varsigma)-\left({}^{H}\mathfrak{I}^{\varpi,\rho}_{1}\bar{\Theta}\right)(\varsigma)\right|.$$

Also,

$$\left|\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma)-\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\bar{\Theta}\right)(\varsigma)\right|$$
$$=\left|\frac{(\tau+1)^{1-\frac{\varpi}{\kappa}}}{\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}}\Gamma(\frac{\varpi}{\kappa})}\int_1^{\varsigma}\exp\left[\frac{(\rho-1)\left(\varsigma^{\tau+1}-\eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma^{\tau+1}-\eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\eta^{\tau}\left(\Theta(\eta)-\bar{\Theta}(\eta)\right)d\eta\right|$$
$$\leq\frac{(\tau+1)^{1-\frac{\varpi}{\kappa}}}{\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}}\Gamma(\frac{\varpi}{\kappa})}\int_1^{\varsigma}\exp\left[\frac{(\rho-1)\left(\varsigma^{\tau+1}-\eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma^{\tau+1}-\eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\eta^{\tau}\left|\Theta(\eta)-\bar{\Theta}(\eta)\right|d\eta$$
$$<\frac{\epsilon(\tau+1)^{-\frac{\varpi}{\kappa}}\exp\left[\frac{(\rho-1)T^{\tau+1}}{\rho}\right]}{\varpi\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}-1}\Gamma(\frac{1}{\kappa})}\left(T^{(\tau+1)}-1\right)^{\frac{\varpi}{\kappa}},$$

and

$$\left|\left({}^{H}\mathfrak{I}^{\varpi,\rho}_{1}\Theta\right)(\varsigma)-\left({}^{H}\mathfrak{I}^{\varpi,\rho}_{1}\bar{\Theta}\right)(\varsigma)\right|$$
$$=\left|\frac{1}{\rho^{\varpi}\Gamma(\varpi)}\int_1^{\varsigma}\exp\left[\frac{(\rho-1)(\log(\varsigma)-\log(\eta))}{\rho}\right](\log(\varsigma)-\log(\eta))^{\varpi-1}(\Theta(\varsigma)-\bar{\Theta}(\varsigma))\frac{d\eta}{\eta}\right|$$
$$\leq\frac{1}{\rho^{\varpi}\Gamma(\varpi)}\int_1^{\varsigma}\exp\left[\frac{(\rho-1)(\log(\varsigma)-\log(\eta))}{\rho}\right](\log(\varsigma)-\log(\eta))^{\varpi-1}\left|\Theta(\eta)-\bar{\Theta}(\eta)\right|\frac{d\eta}{\eta}$$
$$<\frac{\epsilon\exp\left[\frac{(\rho-1)\log T}{\rho}\right]}{\rho^{\varpi}\Gamma(\varpi+1)}(\log T)^{\varpi}.$$

Hence $\|\Theta-\bar{\Theta}\|<\epsilon$, gives that

$$\left|(\mathfrak{I}\Theta)(\varsigma)-\left(\mathfrak{I}\bar{\Theta}\right)(\varsigma)\right|<\wp_1\wp_4\epsilon+\wp_2\frac{\epsilon(\tau+1)^{-\frac{\varpi}{\kappa}}\exp\left[\frac{(\rho-1)T^{\tau+1}}{\rho}\right]}{\varpi\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}-1}\Gamma(\frac{1}{\kappa})}\left(T^{(\tau+1)}-1\right)^{\frac{\varpi}{\kappa}}+\wp_3\frac{\epsilon\exp\left[\frac{(\rho-1)\log T}{\rho}\right]}{\rho^{\varpi}\Gamma(\varpi+1)}(\log T)^{\varpi}.$$

If $\epsilon\to 0$, we get $\left|(\mathfrak{I}\Theta)(\varsigma)-\left(\mathfrak{I}\bar{\Theta}\right)(\varsigma)\right|\to 0$. Hence $\mathfrak{I}$ become continuous on $\mathcal{B}_{\nu_o}$.

Step 3: Finally, to show an estimate of $\mathfrak{I}$ with respect to ω_0, suppose $\Upsilon(\neq\phi)\subseteq B_{\nu_0}$.
For an arbitrary $\epsilon>0$ and choose $\Theta\in\Upsilon$ and $\varsigma_1,\varsigma_2\in I$ such that $|\varsigma_2-\varsigma_1|\leq\epsilon$ and $\varsigma_2\geq\varsigma_1$, then,

$$\left|(\mathfrak{I}\Theta)(\varsigma_2)-(\mathfrak{I}\Theta)(\varsigma_1)\right|=\Big|F\left(\varsigma_2,\pounds(\varsigma_2,\Theta(\varsigma_2)),\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma_2),\left({}^{H}\mathfrak{I}^{\varpi,\rho}_{1}\Theta\right)(\varsigma_2)\right)$$
$$-F\left(\varsigma_1,\pounds(\varsigma_1,\Theta(\varsigma_1)),\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma_1),\left({}^{H}\mathfrak{I}^{\varpi,\rho}_{1}\Theta\right)(\varsigma_1)\right)\Big|$$
$$\leq\Big|F\left(\varsigma_2,\pounds(\varsigma_2,\Theta(\varsigma_2)),\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma_2),\left({}^{H}\mathfrak{I}^{\varpi,\rho}_{1}\Theta\right)(\varsigma_2)\right)$$
$$-F\left(\varsigma_2,\pounds(\varsigma_2,\Theta(\varsigma_2)),\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma_1),\left({}^{H}\mathfrak{I}^{\varpi,\rho}_{1}\Theta\right)(\varsigma_2)\right)\Big|$$
$$+\Big|F\left(\varsigma_2,\pounds(\varsigma_2,\Theta(\varsigma_2)),\left({}^{\tau}_{\kappa}I^{\varpi,\rho}_{1}\Theta\right)(\varsigma_1),\left({}^{H}\mathfrak{I}^{\varpi,\rho}_{1}\Theta\right)(\varsigma_1)\right)$$

$$
\begin{aligned}
&-F\left(\varsigma_2, \pounds(\varsigma_1, \Theta(\varsigma_1)), \left({}^{\tau}_{\kappa}\mathcal{I}_1^{\varpi,\rho}\Theta\right)(\varsigma_1), \left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right)\Big| \\
&+\Big|F\left(\varsigma_2, \pounds(\varsigma_1, \Theta(\varsigma_1)), \left({}^{\tau}_{\kappa}\mathcal{I}_1^{\varpi,\rho}\Theta\right)(\varsigma_1), \left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right) \\
&-F\left(\varsigma_1, \pounds(\varsigma_1, \Theta(\varsigma_1)), \left({}^{\tau}_{\kappa}\mathcal{I}_1^{\varpi,\rho}\Theta\right)(\varsigma_1), \left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right)\Big| \\
\le\ &\wp_3\left|\left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right| + \wp_2\left|\left({}^{\tau}_{\kappa}\mathcal{I}_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{\tau}_{\kappa}\mathcal{I}_1^{\varpi,\rho}\Theta\right)\right. \\
+\ &\wp_1\Big|\pounds(\varsigma_2, \Theta(\varsigma_2)) - \pounds(\varsigma_1, \Theta(\varsigma_1))\Big| + \omega_F(\mathcal{I}, \epsilon)(\varsigma_1)\Big| \\
\le\ &\wp_3\left|\left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right| + \wp_2\left|\left({}^{\tau}_{\kappa}\mathcal{I}_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{\tau}_{\kappa}\mathcal{I}_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right| \\
&+\wp_1\wp_4\Big|\Theta(\varsigma_2) - \Theta(\varsigma_1)\Big| + \omega_F(\mathcal{I}, \epsilon),
\end{aligned}
$$

where $\omega_F(\mathcal{I}, \epsilon) = \sup\{|F(\varsigma_2, \pounds, \mathcal{I}_1, \mathcal{J}_1) - F(\varsigma_1, \pounds, \mathcal{I}_1, \mathcal{J}_1)| : |\varsigma_2 - \varsigma_1| \le \epsilon; \varsigma_1, \varsigma_2 \in \mathcal{I}\}$. Also,

$$
\begin{aligned}
&\left|\left({}^{\tau}_{\kappa}\mathcal{I}_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{\tau}_{\kappa}\mathcal{I}_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right| \\
&= \frac{(\tau+1)^{1-\frac{\varpi}{\kappa}}}{\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}}\Gamma(\frac{\varpi}{\kappa})}\left|\int_1^{\varsigma_2} \exp\left[\frac{(\rho-1)\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\eta^{\tau}\Theta(\eta)d\eta\right. \\
&\quad -\left.\int_1^{\varsigma_1} \exp\left[\frac{(\rho-1)\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\eta^{\tau}\Theta(\eta)d\eta\right| \\
&\quad + \frac{(\tau+1)^{1-\frac{\varpi}{\kappa}}}{\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}}\Gamma(\frac{\varpi}{\kappa})}\left|\int_1^{\varsigma_1} \exp\left[\frac{(\rho-1)\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\eta^{\tau}\Theta(\eta)d\eta\right. \\
&\le \frac{(\tau+1)^{1-\frac{\varpi}{\kappa}}}{\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}}\Gamma(\frac{\varpi}{\kappa})}\int_{\varsigma_1}^{\varsigma_2} \exp\left[\frac{(\rho-1)\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\eta^{\tau}|\Theta(\eta)|d\eta \\
&\quad + \frac{(\tau+1)^{1-\frac{\varpi}{\kappa}}}{\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}}\Gamma(\frac{\varpi}{\kappa})}\int_1^{\varsigma_1}\left|\left(\exp\left[\frac{(\rho-1)\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\right.\right. \\
&\quad - \left.\exp\left[\frac{(\rho-1)\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma_1^{\tau+1} - \eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\right)\eta^{\tau}\Theta(\eta)\Big|d\eta \\
&\le \frac{\exp\left[\frac{(\rho-1)T^{\tau+1}}{\rho}\right](\tau+1)^{-\frac{\varpi}{\kappa}}}{\varpi\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}-1}\Gamma(\frac{1}{\kappa})}\|\Theta\|\left(T^{(\tau+1)} - 1\right)^{\frac{\varpi}{\kappa}} \\
&\quad + \|\Theta\|\frac{(\tau+1)^{1-\frac{\varpi}{\kappa}}}{\rho^{\frac{\varpi}{\kappa}}\kappa^{\frac{\varpi}{\kappa}}\Gamma(\frac{\varpi}{\kappa})}\int_1^{\varsigma_1}\left(\exp\left[\frac{(\rho-1)\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\right. \\
&\quad - \left.\exp\left[\frac{(\rho-1)\left(\varsigma_2^{\tau+1} - \eta^{\tau+1}\right)}{\rho}\right]\left(\varsigma_1^{\tau+1} - \eta^{\tau+1}\right)^{\frac{\varpi}{\kappa}-1}\right)\eta^{\tau}\Big|d\eta.
\end{aligned}
$$

As $\epsilon \to 0$, then $\varsigma_2 \to \varsigma_1$, and we have $\left|\left({}^{\tau}_{\kappa}I_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{\tau}_{\kappa}I_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right| \to 0$. And

$$\begin{aligned}
&\left|\left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right| \\
&\leq \frac{1}{\rho^{\varpi}\Gamma(\varpi)}\left|\int_1^{\varsigma_2} \exp\left[\frac{(\rho-1)(\log(\varsigma_2)-\log(\eta))}{\rho}\right](\log(\varsigma_2)-\log(\eta))^{\varpi-1}\frac{\Theta(\eta)}{\eta}d\eta\right. \\
&\quad \left. - \int_1^{\varsigma_1} \exp\left[\frac{(\rho-1)(\log(\varsigma_2)-\log(\eta))}{\rho}\right](\log(\varsigma_2)-\log(\eta))^{\varpi-1}\frac{\Theta(\eta)}{\eta}d\eta\right| \\
&\quad + \frac{1}{\rho^{\varpi}\Gamma(\varpi)}\left|\int_1^{\varsigma_1} \exp\left[\frac{(\rho-1)(\log(\varsigma_2)-\log(\eta))}{\rho}\right](\log(\varsigma_2)-\log(\eta))^{\varpi-1}\frac{\Theta(\eta)}{\eta}d\eta\right. \\
&\quad \left. - \int_1^{\varsigma_1} \exp\left[\frac{(\rho-1)(\log(\varsigma_1)-\log(\eta))}{\rho}\right](\log(\varsigma_1)-\log(\eta))^{\varpi-1}\frac{\Theta(\eta)}{\eta}d\eta\right| \\
&\leq \frac{1}{\rho^{\varpi}\Gamma(\varpi)}\int_{\varsigma_1}^{\varsigma_2} \exp\left[\frac{(\rho-1)(\log(\varsigma_2)-\log(\eta))}{\rho}\right](\log(\varsigma_2)-\log(\eta))^{\varpi-1}\frac{|\Theta(\eta)|}{\eta}d\eta \\
&\quad + \frac{1}{\rho^{\varpi}\Gamma(\varpi)}\int_1^{\varsigma_1}\left|\left(\exp\left[\frac{(\rho-1)(\log(\varsigma_2)-\log(\eta))}{\rho}\right](\log(\varsigma_2)-\log(\eta))^{\varpi-1}\right.\right. \\
&\quad \left.\left. - \exp\left[\frac{(\rho-1)(\log(\varsigma_2)-\log(\eta))}{\rho}\right](\log(\varsigma_1)-\log(\eta))^{\varpi-1}\right)\frac{\Theta(\eta)}{\eta}\right|d\eta \\
&\leq \frac{\exp\left[\frac{(\rho-1)\log T}{\rho}\right]}{\rho^{\varpi}\Gamma(\varpi+1)}\|\Theta\|(\log T)^{\varpi} \\
&\quad + \|\Theta\|\frac{1}{\rho^{\varpi}\Gamma(\varpi)}\int_1^{\varsigma_1}\left|\left(\exp\left[\frac{(\rho-1)(\log(\varsigma_2)-\log(\eta))}{\rho}\right](\log(\varsigma_2)-\log(\eta))^{\varpi-1}\right.\right. \\
&\quad \left.\left. - \exp\left[\frac{(\rho-1)(\log(\varsigma_2)-\log(\eta))}{\rho}\right](\log(\varsigma_1)-\log(\eta))^{\varpi-1}\right)\frac{1}{\eta}\right|d\eta.
\end{aligned}$$

As $\epsilon \to 0$, then $\varsigma_2 \to \varsigma_1$ and we have $\left|\left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right| \to 0$. Therefore

$$\begin{aligned}
\left|(\Im\Theta)(\varsigma_2) - (\Im\Theta)(\varsigma_1)\right| &\leq \wp_3\left|\left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right| + \wp_2\left|\left({}^{\tau}_{\kappa}I_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{\tau}_{\kappa}I_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right| \\
&\quad + \wp_1\wp_4\omega(\Theta,\epsilon) + \omega_F(I,\epsilon),
\end{aligned}$$

gives

$$\begin{aligned}
\omega(\Im\Theta,\epsilon) &\leq \wp_3\left|\left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{H}\daleth_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right| + \wp_2\left|\left({}^{\tau}_{\kappa}I_1^{\varpi,\rho}\Theta\right)(\varsigma_2) - \left({}^{\tau}_{\kappa}I_1^{\varpi,\rho}\Theta\right)(\varsigma_1)\right| \\
&\quad + \wp_1\wp_4\omega(\Theta,\epsilon) + \omega_F(I,\epsilon).
\end{aligned}$$

Using the uniform continuity of F on $I \times [-£, £] \times [-\mathcal{I}, \mathcal{I}] \times [-\mathcal{J}, \mathcal{J}]$, we get $\omega_F(I,\epsilon) \to 0$ as $\epsilon \to 0$. Taking $\sup_{\Im\in\Upsilon}$ and $\epsilon \to 0$ we get,

$$\omega_0(\Im\Upsilon) \leq \wp_1\wp_4\omega_0(\Upsilon),$$

Thus, by DFPT $\Im$ has a $\mathbb{FP}$ in $\Upsilon \subseteq \mathcal{B}_{\nu_0}$. Hence the Eq (3.1) have a solution in Ξ. □

Example 3.2. *Consider the following FIE*

$$\Theta(\varsigma) = \varsigma^2 + \frac{\Theta(\varsigma)}{1+\varsigma^4} + \frac{\left({}^{\frac{1}{5}}_{\frac{1}{5}}\mathcal{I}_1^{5,\frac{1}{5}}\Theta\right)(\varsigma)}{5^{75}} + \frac{\left({}^{H}\mathfrak{I}_1^{5,\frac{1}{5}}\Theta\right)(\varsigma)}{5^{10}} \tag{3.2}$$

for $\varsigma \in [1,2] = I$.

Here

$$\left({}^{\frac{1}{5}}_{\frac{1}{5}}\mathcal{I}_1^{5,\frac{1}{5}}\Theta\right)(\varsigma) = \frac{5^{74}}{6^{24}\Gamma(25)}\int_1^{\varsigma} \exp[-(\varsigma^{\frac{3}{2}} - \eta^{\frac{3}{2}})](\varsigma^{\frac{3}{2}} - \eta^{\frac{3}{2}})\eta^{\frac{1}{2}}\Theta(\varsigma)d\eta,$$

and

$$\left({}^{H}\mathfrak{I}_1^{5,\frac{1}{5}}\Theta\right)(\varsigma) = \frac{3125}{\Gamma(5)}\int_1^{\varsigma} \exp\left[-4\left(\log(\varsigma) - \log(\eta)\right)\right]\left(\log(\varsigma) - \log(\eta)\right)^4 \frac{\Theta(\eta)}{\eta}d\eta.$$

Also $F(\varsigma, £, \mathcal{I}_1, \mathcal{J}_1) = \varsigma^2 + £ + \frac{\mathcal{I}_1}{5^{75}} + \frac{\mathcal{J}_1}{5^{10}}$ and $£(\varsigma, \Theta) = \frac{\Theta(\varsigma)}{1+\varsigma^4}$. It is obvious that F, $£$ are continuous satisfying

$$|£(\varsigma, P_1) - £(\varsigma, P_2)| \leq \frac{|P_1 - P_2|}{2}$$

and $\left|F(\varsigma, £, \mathcal{I}_1, \mathcal{J}_1) - F(\varsigma, \bar{£}, \bar{\mathcal{I}}_1, \bar{\mathcal{J}}_1)\right| \leq \left|£ - \bar{£}\right| + \frac{1}{5^{75}}\left|\mathcal{I}_1 - \bar{\mathcal{I}}_1\right| + \frac{1}{5^{10}}\left|\mathcal{J}_1 - \bar{\mathcal{J}}_1\right|$.
Therefore $\wp_1 = 1$, $\wp_2 = \frac{1}{5^{75}}$, $\wp_3 = \frac{1}{5^{10}}$ *and* $\wp_4 = \frac{1}{2}$, $\wp_1\wp_4 = \frac{1}{2(5^{10})} < 1$ If $\| \Theta \| \leq \nu_0$ then

$$L = \frac{\nu_0}{2},\ \mathbf{I} = \frac{\nu_0[\exp(-4(2^{\frac{6}{5}}))]5^{74}(2^{\frac{6}{5}} - 1)^{25}}{(6^{25})\Gamma(5)},\quad \mathbf{J} = \frac{\nu_0 5^5 \exp[-4(\log 2)](\log 2)^4}{\Gamma(6)}.$$

Further,

$$|F(\varsigma, £, \mathcal{I}_1, \mathcal{J}_1| \leq \frac{\nu_0}{2} + \frac{\nu_0[\exp(-4(2^{\frac{6}{5}}))]5^{74}(2^{\frac{6}{5}} - 1)^{25}}{(6^{25})\Gamma(5)} + \frac{\nu_0 5^5 \exp[-4(\log 2)](\log 2)^4}{\Gamma(6)} \leq \nu_0.$$

If we choose $\nu_0 = 5$, then we have

$$L = \frac{5}{2},\ \mathbf{I} = \frac{5^{74}[\exp(-4(2^{\frac{6}{5}}))](2^{\frac{6}{5}} - 1)^{25}}{(6^{25})\Gamma(5)},\ \mathbf{J} = \frac{5^6 \exp[-4(\log 2)](\log 2)^4}{\Gamma(6)}.$$

$$F \leq 5,\ \sigma_1\sigma_4 < 1.$$

As we see, all the assumption of Theorem 3.1 from (*i*)–(*iv*) are fulfills. So by the Theorem 3.1 we conclude that the Eq (3.1) have a solution in Ξ.

4. Conclusions

In the present paper, we have generalized the DFPT and introduced a new class of (κ, τ)-fractional integral operators, which can be reduce to another related operators by choosing suitable values of κ, τ and ρ. Then, we established the existence of solution involving both the generalized proportional FIE of Riemann-Liouville and Hadamard, using DFPT. Finally, obtained result is illustrated by an example.

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Conflict of interest

The authors declare that they have no competing interests.

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