AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(8): 15436–15452.
DOI: 10.3934/math.2022845
Received: 03 January 2022
Revised: 08 June 2022
Accepted: 16 June 2022
Published: 20 June 2022

Research article

A_α matrix of commuting graphs of non-abelian groups

Bilal A. Rather[1], Fawad Ali[2,*], Nasim Ullah[3], Al-Sharef Mohammad[3], Anwarud Din[4] and Sehra[5]

[1] Department of Mathematical Sciences, College of Science, United Arab Emirate University, Al Ain 15551, Abu Dhabi, UAE
[2] Institute of Numerical Sciences, Kohat University of Science and Technology, Kohat 26000, KPK, Pakistan
[3] Department of Electrical Engineering, College of Engineering Taif University, Al-Hawiyah, Taif P.O. Box 888, Saudi Arabia
[4] Department of Mathematics, Sun Yat-Sen University, Guangzhou, China
[5] Department of Mathematics, Shaheed Benazir Bhutto Women University, Peshawar 25000, Pakistan

* **Correspondence:** Email: fawadali@math.qau.edu.pk.

Abstract: For a finite group $\mathcal{G}$ and a subset $X \neq \emptyset$ of $\mathcal{G}$, the commuting graph, indicated by $G = C(\mathcal{G}, X)$, is the simple connected graph with vertex set X and two distinct vertices x and y are edge connected in G if and only if they commute in X. The A_α matrix of G is specified as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, $\alpha \in [0, 1]$, where $D(G)$ is the diagonal matrix of vertex degrees while $A(G)$ is the adjacency matrix of G. In this article, we investigate the A_α matrix for commuting graphs of finite groups and we also find the A_α eigenvalues of the dihedral, the semidihedral and the dicyclic groups. We determine the upper bounds for the largest A_α eigenvalue for these graphs. Consequently, we get the adjacency eigenvalues, the Laplacian eigenvalues, and the signless Laplacian eigenvalues of these graphs for particular values of α. Further, we show that these graphs are Laplacian integral.

Keywords: A_α matrix; commuting graph; adjacency matrix; Laplacian matrix; signless Laplacian matrix; non-abelian groups
Mathematics Subject Classification: 15A18, 05C50, 05C25

1. Introduction

In this article, all our graphs are simple, finite and connected. A graph $G = G(V(G), E(G))$ is an ordered pair consisting of the vertex set $V(G)$ and the edge set $E(G)$. The cardinality of $V(G)$ is referred to as the *order* of G represented by n and the cardinality of $E(G)$ is called the *size* of G, symbolized by

m. The set of vertices adjacent to $v \in V(G)$, indicated by $N(v)$, is known as the *neighborhood* of v. The cardinality of $N(v)$ is the *degree* of a vertex v, symbolized by $d(v)$. A graph G is called *regular* if the degree of every vertex is same. For other notations and terminology, see [14].

Let $M_n(\mathbb{R})$ be the set of square matrices over the field $\mathbb{R}$ and $M^* := \{M \in M_n(\mathbb{R}) : M^T = M\}$, where M^* is the set of real symmetric matrices. The adjacency matrix of G, symbolized by $A(G)$, is described in the following way:

$$A(G) = (a_{ij})_n = \begin{cases} 1, & \text{if } i \text{ adjacent } j; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $A(G)$ is in M^*, so all its eigenvalues are in $\mathbb{R}$ and can be indexed from the largest to the least as: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The multiset of eigenvalues of $A(G)$ is the *spectrum* of G, and λ_1 is called the *spectral radius* (or *spectral norm*) of G.

Let $D(G) = diag(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees of G. The matrices $L(G) = A(G) - D(G)$ and $Q(G) = A(G) + D(G)$ are known as the Laplacian matrix and the signless Laplacian matrix of G, respectively. The matrix $L(G)$ is positive semi-definite and the matrix $Q(G)$ is positive semi-definite (definite) and their spectrum is in $\mathbb{R}$. Laplacian eigenvalues are denoted by $\lambda_1^L \geq \lambda_2^L \geq \cdots \geq \lambda_n^L = 0$ while the signless Laplacian eigenvalues are denoted by $\lambda_1^Q \geq \lambda_2^Q \geq \cdots \geq \lambda_n^Q$. It is well known that $\lambda_n^L = 0$ is always the eigenvalue of $L(G)$ and $\lambda_{n-1}^L > 0$ for connected graphs, which is referred to as the algebraic connectivity of G.

Nikiforov [23] suggested examining the convex combinations $A_\alpha(G)$ of the adjacency matrix $A(G)$ and the diagonal matrix $D(G)$ specified by $A_\alpha(G) = (1-\alpha)A(G) + \alpha D(G)$, where $0 \leq \alpha \leq 1$. Obviously, $A_0(G) = A(G), A_1(G) = D(G)$ and $Q(G) = 2A_{\frac{1}{2}}(G) = A(G)+D(G)$. Also $A_\alpha(G)-A_\gamma(G) = (\alpha-\gamma)L(G) = (\alpha-\gamma)(D(G)-A(G))$, where $L(G)$ is the Laplacian matrix of G. Thus, $A_\alpha(G)$ matrix merges the spectral theories of $A(G)$, $L(G)$ and $Q(G)$ as well as their uncountably many combinations. Recent articles on the spectral properties of the A matrix can be obtained in [21, 23–25, 27, 33] and the references in those articles.

The matrix $A_\alpha(G)$ belongs to the class M^*, so its eigenvalues can be arranged in decreasing order as $\lambda_1^\alpha \geq \lambda_2^\alpha \geq \cdots \geq \lambda_n^\alpha$, where λ_1^α is called the α *spectral radius* (or $\lambda(A_\alpha)$ generalized adjacency spectral radius) of G. For a connected graph G, the matrix $A_\alpha(G)$ (for $\alpha \neq 1$) is an irreducible and non-negative. Consequently, according to the Perron-Frobenius theorem, $\lambda(A_\alpha(G))$ is the simple eigenvalue, and $\lambda(A_\alpha(G))$ has only one positive unit eigenvector X, which is known as the generalized adjacency Perron vector of G.

Consider $\mathcal{G}$ is a finite group with n elements and e is the identity element. If X is a non empty subset of $\mathcal{G}$, then the *commuting graph* of $\mathcal{G}$ related to X, is indicated by $C(\mathcal{G}, X)$, and is defined with X as the vertex set and two different vertices x and y are edge connected in $C(\mathcal{G}, X)$ if and only if they commute in X. The commuting graphs of matrix rings and semirings over the finite fields were studied in [1, 15]. The metric dimension, the resolving polynomial, the clique number and the chromatic number of the commuting graphs of the dihedral groups were discussed in [4, 11]. Recent results on the commuting graphs of the generalized dihedral groups can be found in [12, 20]. The adjacency spectrum of commuting graphs were studied in [5, 13], the Laplacian as well as the signless Laplacian spectra of the commuting graphs on the dihedral groups were explored in [3, 32]. For other spectral properties and energies of commuting graphs, see [16, 18].

2. A_α eigenvalues of commuting graphs of finite non-abelian groups

Any column vector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ can be regarded as a function defined by $V(G)$ which associates every vertex v_i to x_i, i.e., $X(v_i) = x_i$ for all $i = 1, \ldots, n$. Also, the quadratic form of the A_α matrix is:

$$\langle A_\alpha X, X\rangle = (2\alpha - 1) \sum_{u \in V(G)} x_u^2 d(u) + (1-\alpha) \sum_{uv \in E(G)} (x_u + x_v)^2,$$

and λ is the A_α eigenvalue of G that corresponds to the eigenvector X whenever $X \neq 0$ and for every $v_i \in V(G)$, we have:

$$\lambda x_i = \alpha d(v_i) x_i + (1-\alpha) \sum_{v_i v_j \in E(G)} x_j, \tag{2.1}$$

or equivalently

$$(\lambda - \alpha d(v_i)) x_i = (1-\alpha) \sum_{v_i v_j \in E(G)} x_j. \tag{2.2}$$

Equations (2.1) and (2.2) are known as eigenequations for the matrix A_α of G.

Our first result is helpful in finding some A_α eigenvalues of G with some special structure.

Theorem 2.1. *Suppose G is a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $B = \{v_1, v_2, \ldots, v_k\}$ is the set of vertices of G satisfying $N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \ldots, k\}$. Then the following hold.*

- **(i)** *If B is an independent set of G, then $b\alpha$ is an A_α eigenvalue of G with multiplicity at least $k-1$, where b is the degree of v_i, for $i = 1, 2, \ldots, k$.*
- **(ii)** *If B is a clique of G, then $\alpha(\omega + \beta) - 1$ is an A_α eigenvalue of G having multiplicity at least $k-1$, whereas β is the total number of vertices in $V(G) \setminus B$, that are edge connected to every vertex of clique.*

Proof. **(i)** Since $\{v_1, v_2, \ldots, v_k\}$ is the independent set of G sharing the same neighbourhood, so $d(v_1) = d(v_2) = \cdots = d(v_k) = b$ (say). We first index the independent vertices, so that the A_α matrix of G can be put as:

$$A_\alpha(G) = \left(\begin{array}{cccc|c} b\alpha & 0 & \ldots & 0 & \\ 0 & b\alpha & \ldots & 0 & B_{k\times(n-k)} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & b\alpha & \\ \hline & & B^T & & C_{(n-k)\times(n-k)} \end{array}\right). \tag{2.3}$$

Let $X_{i-1} = \Big(-1, x_{i2}, x_{i3}, \ldots, x_{ik}, \underbrace{0, 0, 0, \ldots, 0}_{n-k}\Big)^T$ be the vector in $\mathbb{R}^n$ such that

$$x_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } 2 \le i \le k; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $X_1, X_2, \ldots, X_{k-1}$ are the linearly independent vectors and we note that all the rows of $B_{k\times(n-k)}$ are identical. Therefore, we have

$$A_\alpha(G) X_1 = \begin{pmatrix} -b\alpha & b\alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}^T = b\alpha X_1.$$

Thus, it follows that X_1 is the eigenvector that corresponds to the A_α eigenvalue $b\alpha$. Similarly, $X_2, X_3, \ldots, X_{p-1}$ are the eigenvectors of the A_α matrix corresponding to the eigenvalue $b\alpha$.

(ii) By hypothesis, $\{v_1, v_2, \ldots, v_k\}$ forms the clique of G with the same neighbourhood, so $d(v_1) = d(v_2) = \cdots = d(v_k) = \omega - 1 + \beta$, whereas β is the number of vertices in $|N(v_1) \setminus B|$, which are adjacent to every vertex of the clique. We first label the vertices of the clique, hence, the A_α matrix of G may be expressed as:

$$A_\alpha(G) = \left(\begin{array}{cccc|c} (\omega-1+\beta)\alpha & 1-\alpha & \ldots & 1-\alpha & \\ 1-\alpha & (\omega-1+\beta)\alpha & \ldots & 1-\alpha & B_{k\times(n-k)} \\ \vdots & \vdots & \ddots & \vdots & \\ 1-\alpha & 1-\alpha & \cdots & (\omega-1+\beta)\alpha & \\ \hline & & B^T & & C_{(n-k)\times(n-k)} \end{array}\right). \tag{2.4}$$

Let $X_1, X_2, \ldots, X_{k-1}$ be the linearly independent vectors defined as in **(i)**. Then, we have

$$A_\alpha(G)X_1 = \begin{pmatrix} -(\omega+\beta)\alpha+1 & (\omega+\beta)\alpha-1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}^T = (\alpha(\omega+\beta)-1)X_1.$$

Likewise, $X_2, \ldots, X_{p-1}$ are the eigenvectors of $A_\alpha(G)$ that corresponds to the eigenvalue $\alpha(\omega+\beta)-1$. This proves the result. □

Assume that a matrix M has a kind of symmetry and may be put in the form

$$M = \begin{pmatrix} X & Y & Y & \cdots & Y & Y \\ Y^T & \mathcal{B} & C & \cdots & C & C \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y^T & C & C & \cdots & \mathcal{B} & C \\ Y^T & C & C & \cdots & C & \mathcal{B} \end{pmatrix}, \tag{2.5}$$

where $X \in R^{n_1\times n_1}$, $Y \in R^{n_1\times n_2}$ and $\mathcal{B}, C \in R^{n_2\times n_2}$, so that $n = n_1 + \eta n_2$, where η is the number of copies of $\mathcal{B}$. Then the spectrum of M can be found by the following result.

Lemma 2.2. *[17] Assume that M is a matrix of the form presented in (2.5), having $\eta \geq 1$ copies of the block matrix $\mathcal{B}$ and $Spec(M)$ is the spectrum of M. Then $Spec(M) = Spec(\mathcal{B} - C)^{\eta-1} \cup Spec(M')$, whereas $M' = \begin{pmatrix} X & \sqrt{\eta}Y \\ \sqrt{\eta}Y^T & \mathcal{B} + (\eta-1)C \end{pmatrix}_{n_1+n_2}$, and $Spec(\mathcal{B} - C)^{\eta-1}$ represents the set of eigenvalues of matrix $\mathcal{B} - C$ each with multiplicity $\eta - 1$.*

All our groups are assumed to be finite with identity element represented by e. For notations and definitions, we follow [22]. The presentation of the dihedral group D_{2n}, $n > 2$, is provided as:

$$D_{2n} = \langle a, b : a^n = e = b^2, aba = b \rangle.$$

Clearly, the last condition is equivalent to $ab = ba^{-1} = ba^{n-1}$. Similarly, the presentation of the semidihedral group SD_{8n} of order $8n$ and the dicyclic group Q_{4n} having order $4n$ are given by:

$$SD_{8n} = \langle a, b : a^{4n} = e = b^2, ab = ba^{2n-1} \rangle,$$

and

$$Q_{4n} = \langle a, b := b^4 a^{2n} = e, a^n = b^2, ba^{-1} = ab \rangle.$$

The center of $\mathcal{G}$, symbolized by $Z(\mathcal{G})$, is specified by:

$$Z(\mathcal{G}) = \{z \in \mathcal{G} : za = az \text{ for each } a \in \mathcal{G}\}.$$

Any finite cyclic group $\mathcal{G}$ of order n can be written as the group $\mathbb{Z}_n$ of integers modulo n. Clearly, the commuting graph $G = C(\mathbb{Z}_n, \mathbb{Z}_n)$ is the complete graph K_n, as every element of $\mathbb{Z}_n$ commutes with every other element. The A_α spectrum of $C(\mathbb{Z}_n, \mathbb{Z}_n)$ is $\{(\alpha n - 1)^{[n-1]}, n - 1\}$, where $[n - 1]$ represents the algebraic multiplicity of eigenvalue. It easily follows that $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$, for even n and $Z(D_{2n}) = \{e\}$, for odd n. Also,the center of the dicyclic group is $Z(Q_{4n}) = \{e, a^n\}$. For the commuting graph [4] $G = C(D_{2n}, Z(D_{2n}))$, G is K_1, for odd n and G is K_2, for even n. So, the commuting graphs $C(\mathcal{G}, Z(\mathcal{G}))$ have simple structures and it will be interesting to investigate the commuting graphs with the non trivial structures.

The next result can be found in [4], which gives the structure of D_{2n}, where X is D_{2n} itself.

Lemma 2.3. *The commuting graph of the dihedral group D_{2n} is*

$$C(D_{2n}, D_{2n}) = \begin{cases} K_1 \vee (K_{n-1} \cup \overline{K}_n), & \text{if } n \text{ is odd;} \\ K_2 \vee (K_{n-2} \cup \frac{n}{2} K_2), & \text{if } n \text{ is even.} \end{cases}$$

In the following result, we find the A_α eigenvalues of the commuting graphs of the dihedral group.

Theorem 2.4. *The following properties hold for the commuting graph $C(D_{2n}, D_{2n})$ of D_{2n}.*

- **(i)** *If n is odd, then the A_α spectrum of $C(D_{2n}, D_{2n})$ comprises the eigenvalues α and $\alpha n - 1$ having multiplicities $n - 1$, $n - 2$, respectively, and the three zeros of polynomial* (2.6).
- **(ii)** *If n is even, then the A_α spectrum of $C(D_{2n}, D_{2n})$ comprises the simple eigenvalue $2\alpha n - 1$, the eigenvalue $2\alpha + 1$ having multiplicity $\frac{n}{2} - 1$, the eigenvalues $\alpha n - 1$ and $4\alpha - 1$ having algebraic multiplicities $n - 3$ and $\frac{n}{2}$, respectively, and the three zeros of polynomial* (2.7).

Proof. **(i)** By Lemma 2.3, the commuting graph of D_{2n} for odd n is $C(D_{2n}, D_{2n}) = K_1 \vee (K_{n-1} \cup \overline{K}_n)$. Let $\{v_1, v_2, \ldots, v_n, u, u_1, u_2, \ldots, u_{n-1}\}$ be the vertex set of $C(D_{2n}, D_{2n})$, where v_i's are the pendent vertices, u is the vertex of degree $2n - 1$ and u_i's are the vertices of degree $n - 1$. As v_i's are independent vertices sharing the vertex u, so by Theorem 2.1, we get the A_α eigenvalue α with algebraic multiplicity $n - 1$. Also, u_i's are the vertices of clique sharing the vertex u and by **(ii)** of Theorem 2.1 with $\beta = 1$, we obtain the A_α eigenvalue $\alpha(n - 1 + 1) - 1 = \alpha n - 1$ with algebraic multiplicity $n - 2$. In this way, we get $2n - 3$ A_α eigenvalues and the remaining three A_α eigenvalues of $C(D_{2n}, D_{2n})$ can be found by using eigenequation (2.1). Let X be the eigenvector of $A_\alpha(C(D_{2n}, D_{2n}))$ with $x_i = X(v_i)$, for $i = 1, 2, 3, \ldots, 2n$. Then, it follows that every component of X that corresponds to every pendent vertex is equal to x_1, component of X that corresponds to the vertex u is x_2 and the components of X that corresponds to the vertices u_i's is equal to x_3. Therefore, from the eigenequation $A_\alpha X = \lambda X$, we have

$$\begin{aligned} \lambda x_1 &= \alpha x_1 + (1 - \alpha) x_2, \\ \lambda x_2 &= \underbrace{(1 - \alpha) x_1 + (1 - \alpha) x_1 + \cdots + (1 - \alpha) x_1}_{n} + \alpha(2n - 1) x_2 + \underbrace{(1 - \alpha) x_3 + \cdots + (1 - \alpha) x_3}_{n-1}, \end{aligned}$$

$$\lambda x_3 = (1-\alpha)x_2 + \alpha(n-1)x_3 + \underbrace{(1-\alpha)x_3 + (1-\alpha)x_3 + \cdots + (1-\alpha)x_3}_{n-2},$$

and the coefficient matrix for the right side of the above equations is:

$$\begin{pmatrix} \alpha & 1-\alpha & 0 \\ n(1-\alpha) & \alpha(2n-1) & (n-1)(1-\alpha) \\ 0 & 1-\alpha & \alpha+n-2 \end{pmatrix}.$$

The characteristic polynomial of the above matrix is given by:

$$\begin{aligned} &x^3 - x^2(\alpha + 2\alpha n + n - 2) + x\left(-\alpha^2 + 2\alpha n^2 + 3\alpha^2 n - 2\alpha n + (1-\alpha)^2(1-n) - n\right) \\ &\quad - \left(\alpha + \alpha^2 n^2 + 2\alpha n^2 - n^2 + \alpha^2 n - 6\alpha n + 2n\right). \end{aligned} \tag{2.6}$$

(ii) If n is even, then by Lemma 2.3, the commuting graph of D_{2n} is $K_2 \vee \left(K_{n-2} \cup \frac{n}{2}K_2\right)$. Let $v_1, v_2, \ldots, v_{n-2}, v, u, u_{11}, u_{12}, u_{21}, u_{22}, \ldots, u_{\frac{n}{2}1}, u_{\frac{n}{2}2}$ be the vertex labelling of $C(D_{2n}, D_{2n})$, where v_i's are the vertices of K_{n-2}, u and v are the degree $2n-1$ vertices and $u_{i1}, u_{i2},\ i = 1, 2, \ldots, \frac{n}{2}$ are the vertices of the degree 3. Since v_i's form the clique and share the same neighbourhood $\{u, v\}$, so by **(ii)** of Theorem 2.1, $\alpha(n-2+2) - 1 = \alpha n - 1$ is the A_α eigenvalue of $C(D_{2n}, D_{2n})$ having multiplicity $n-3$. Also, $\{u, v\}$ are the vertices of K_2 with $\beta = n - 2 + 2 \times \frac{n}{2} = 2(n-1)$ and by Theorem 2.1, is shown here that $2\alpha n - 1$ is the simple A_α eigenvalue of $C(D_{2n}, D_{2n})$. Similarly, we see that $4\alpha - 1$ is the A_α eigenvalue of $C(D_{2n}, D_{2n})$ having multiplicity $\frac{n}{2}$. The remaining $\frac{n}{2} + 2$ A_α eigenvalues of $C(D_{2n}, D_{2n})$ can be obtained by using Equation (2.1). If X is the eigenvector of $A_\alpha(C(D_{2n}, D_{2n}))$, then it is evident that every component of X that corresponds to v_i's is equal to x_1, the components of X that corresponds to u and v is x_2 and the components of X that corresponds to u_{i1} as well as u_{i2} is equal to $x_i + 2$, for $i = 1, 2, \ldots, \frac{n}{2}$. Thus from eigenequation (2.1), we have:

$$\begin{aligned} \lambda x_1 &= \alpha(n-1)x_1 + (n-3)(1-\alpha)x_1 + (2-2\alpha)x_2, \\ \lambda x_2 &= (1-\alpha)(n-2)x_1 + (\alpha(2n-2)+1)x_2 + 2(1-\alpha)x_3 + 2(1-\alpha)x_4 + \cdots + 2(1-\alpha)x_{\frac{n}{2}+2}, \\ \lambda x_3 &= 2(1-\alpha)x_2 + 3\alpha x_3 + (1-\alpha)x_3, \\ \lambda x_4 &= 2(1-\alpha)x_2 + 3\alpha x_4 + (1-\alpha)x_4, \\ &\ \vdots \\ \lambda x_{\frac{n}{2}+2} &= 2(1-\alpha)x_2 + 3\alpha x_{\frac{n}{2}+2} + (1-\alpha)x_{\frac{n}{2}+2}, \end{aligned}$$

and the coefficient matrix of the right ride of the above system of equations is

$$\left(\begin{array}{cc|cccc} 2\alpha+n-3 & 2(1-\alpha) & 0 & 0 & \ldots & 0 \\ (n-2)(1-\alpha) & \alpha(2n-2)+1 & 2(1-\alpha) & 2(1-\alpha) & \ldots & 2(1-\alpha) \\ \hline 0 & 2(1-\alpha) & 2\alpha+1 & 0 & \ldots & 0 \\ 0 & 2(1-\alpha) & 0 & 2\alpha+1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 2(1-\alpha) & 0 & 0 & \ldots & 2\alpha+1 \end{array}\right)_{\left(\frac{n}{2}+2\right)\times\left(\frac{n}{2}+2\right)}.$$

Now, applying Lemma 2.2 to the above matrix with

$$X = \begin{pmatrix} 2\alpha + n - 3 & 2(1-\alpha) \\ (n-2)(1-\alpha) & \alpha(2n-2)+1 \end{pmatrix}, Y = \begin{pmatrix} 0 \\ 2-2\alpha \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 2\alpha + 1 \end{pmatrix}, C = (0)$$

and note that $\eta = \frac{n}{2}$, we get the A_α eigenvalue $2\alpha + 1$ with algebraic multiplicity $\frac{n}{2} - 1$. The other three A_α eigenvalues of $C(D_{2n}, D_{2n})$ are the eigenvalues of the following matrix:

$$\begin{pmatrix} 2\alpha + n - 3 & 2(1-\alpha) & 0 \\ (n-2)(1-\alpha) & \alpha(2n-2)+1 & 2(1-\alpha)\sqrt{\frac{n}{2}} \\ 0 & 2(1-\alpha)\sqrt{\frac{n}{2}} & 2\alpha+1 \end{pmatrix},$$

and its characteristic polynomial is

$$\begin{aligned} & x^3 - x^2(2\alpha + 2\alpha n + n - 1) + x\left(-4\alpha + 2\alpha n^2 + 4\alpha^2 n + 4\alpha n - 2n - 1\right) \\ & - 2\alpha - 2\alpha^2 n^2 - 6\alpha n^2 + 2n^2 - 8\alpha^2 n + 22\alpha n - 5n - 1. \end{aligned} \tag{2.7}$$

□

The next lemma gives the structure of the commuting graph of the semidihedral group SD_{8n}.

Lemma 2.5. *[32] The structure of the commuting graph of SD_{8n} is given as:*

$$C(SD_{8n}, D_{8n}) = \begin{cases} K_4 \vee (K_{4n-4} \cup nK_4), & \text{if } n \text{ is odd;} \\ K_2 \vee (K_{4n-2} \cup 2nK_2), & \text{if } n \text{ is even.} \end{cases}$$

In the subsequent result, we find the A_α eigenvalues of the commuting graph of the semidihedral group.

Theorem 2.6. *For the commuting graph $C(SD_{8n}, SD_{8n})$ of the semidihedral group SD_{8n}, the subsequent properties hold.*

(i) *If n is odd, then the A_α spectrum of $C(SD_{8n}, SD_{8n})$ comprises the eigenvalues $4\alpha n - 1$, $8\alpha n - 1$, $8\alpha - 1$, $4\alpha + 3$ with algebraic multiplicities $4n - 5$, 3, $3n$, $n - 1$, respectively, and the eigenvalues of the following matrix:*

$$\begin{pmatrix} 4\alpha + 4n - 5 & 4(1-\alpha) & 0 \\ (4n-4)(1-\alpha) & 8\alpha n - \alpha + 3 & 4\sqrt{n}(1-\alpha) \\ 0 & 4\sqrt{n}(1-\alpha) & 4\alpha + 3 \end{pmatrix}.$$

(ii) *If n is even, then the A_α spectrum of $C(SD_{8n}, SD_{8n})$ comprises the simple eigenvalue $8\alpha n - 1$, the eigenvalues $4\alpha n - 1$, $4\alpha - 1$, $2\alpha + 1$ having algebraic multiplicities $4n - 3$, $2n$, $2n - 1$, respectively, and the three eigenvalues of the sequel matrix:*

$$\begin{pmatrix} 2\alpha + 4n - 3 & 2(1-\alpha) & 0 \\ (4n-2)(1-\alpha) & 8\alpha n - 2\alpha + 1 & 2\sqrt{2n}(1-\alpha) \\ 0 & 2\sqrt{2n}(1-\alpha) & 2\alpha + 1 \end{pmatrix}.$$

Proof. By using Theorem 2.1, Lemmas 2.2 and 2.5 and proceeding as in **(ii)** of Theorem 2.4, this result can be proved. □

In the next result, we will determine the A_α eigenvalues of the commuting graph of the dicyclic group Q_{4n}.

Theorem 2.7. *The A_α spectrum of $C(Q_{4n}, Q_{4n})$ consists of the simple eigenvalue $4\alpha n - 1$, and the eigenvalues $2\alpha n - 1$, $4\alpha - 1$, $2\alpha + 1$ with algebraic multiplicities $2n - 3$, n, $n - 1$, respectively, and the rest three eigenvalues are the zeros of polynomial* (2.8).

Proof. The commuting graph $C(Q_{4n}, Q_{4n})$ [5] of Q_{4n} is $C(Q_{4n}, Q_{4n}) = K_2 \vee (K_{2n-2} \cup nK_2)$. Let $v_1, v_2, \dots, v_{2n-2}, v, u, u_{11}, u_{12}, u_{21}, u_{22}, \dots, u_{n1}, u_{n2}$ be the vertex indexing of $C(Q_{4n}, Q_{4n})$, where v_i's are the vertices of the degree $2n - 1$, v and u are the vertices of the degree $4n - 1$ and u_{i1}, u_{i2} are the vertices of degree 3, for $i = 1, 2, \dots, n$. Since v_i's form the clique and share the same neighbourhood $\{u, v\}$, so by Theorem 2.1, we have $2n\alpha - 1$ is the A_α eigenvalue of $C(Q_{4n}, Q_{4n})$ with algebraic multiplicity $2n - 3$. Likewise, u and v form the clique K_2 and share the same neighbourhood with $\beta = 2n - 2 + 2n = 4n - 2$ and again using Theorem 2.1, we obtain the simple A_α eigenvalue $4\alpha n - 1$. Similarly, for $i = 1, 2, \dots, n$ considering the vertices u_{i1} and u_{i2} with their neighbourhood $\{u, v\}$, we obtain the A_α eigenvalue $4\alpha - 1$ of $C(Q_{4n}, Q_{4n})$ with algebraic multiplicity n. The other $n + 2$, A_α eigenvalues of $C(Q_{4n}, Q_{4n})$ can be found by using Eq (2.1). If X is the eigenvector of $A_\alpha(C(Q_{4n}, Q_{4n}))$, then it is clear that every component of X corresponding to v_i's is equal to x_1, the components of X corresponding to u and v is x_2 and the components of X corresponding to u_{i1} and u_{i2} is equal to $x_i + 2$, for $i = 1, 2, \dots, n$. Therefore, by eigenequation (2.1), we have

$$\begin{aligned}
\lambda x_1 =& \alpha(2n-1)x_1 + (2n-3)(1-\alpha)x_1 + 2(1-\alpha)x_2,\\
\lambda x_2 =& (2n-2)(1-\alpha)x_1 + (\alpha(4n-2)+1)x_2 + 2(1-\alpha)x_3 + 2(1-\alpha)x_4 + \cdots + 2(1-\alpha)x_{n+2},\\
\lambda x_3 =& 2(1-\alpha)x_2 + (2\alpha+1)x_3,\\
\lambda x_4 =& 2(1-\alpha)x_2 + (2\alpha+1)x_4,\\
&\vdots\\
\lambda x_{n+2} =& 2(1-\alpha)x_2 + (2\alpha+1)x_{n+2},
\end{aligned}$$

and the coefficient matrix of the right side of the above system of equations is

$$\left(\begin{array}{cc|cccc}
2\alpha+2n-3 & 2(1-\alpha) & 0 & 0 & \dots & 0\\
(2n-2)(1-\alpha) & \alpha(4n-2)+1 & 2(1-\alpha) & 2(1-\alpha) & \dots & 2(1-\alpha)\\
\hline
0 & 2(1-\alpha) & 2\alpha+1 & 0 & \dots & 0\\
0 & 2(1-\alpha) & 0 & 2\alpha+1 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 2(1-\alpha) & 0 & 0 & \dots & 2\alpha+1
\end{array}\right)_{(n+2)\times(n+2)}.$$

Now, applying Lemma 2.2 to the above matrix with

$$X = \begin{pmatrix} 2\alpha+2n-3 & 2(1-\alpha)\\ (2n-2)(1-\alpha) & \alpha(4n-2)+1 \end{pmatrix}, Y = \begin{pmatrix} 0\\ 2-2\alpha \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 2\alpha+1 \end{pmatrix}, C = (0)$$

and note that $\eta = n$, we obtain the A_α eigenvalue $2\alpha + 1$ with algebraic multiplicity $n - 1$. The other three A_α eigenvalues of $C(Q_{4n}, Q_{4n})$ are the eigenvalues of the subsequent matrix:

$$\begin{pmatrix} 2\alpha + 2n - 3 & 2(1-\alpha) & 0 \\ (2n-2)(1-\alpha) & \alpha(4n-2) + 1 & 2(1-\alpha)\sqrt{n} \\ 0 & 2(1-\alpha)\sqrt{n} & 2\alpha + 1 \end{pmatrix},$$

and its characteristic polynomial is given as:

$$\begin{aligned} x^3 - x^2(2\alpha + 4\alpha n + 2n - 1) + x\left(-4\alpha + 8\alpha n^2 + 8\alpha^2 n + 8\alpha n - 4n - 1\right) \\ - 2\alpha - 8\alpha^2 n^2 - 24\alpha n^2 + 8n^2 - 16\alpha^2 n + 44\alpha n - 10n - 1. \end{aligned} \tag{2.8}$$

□

As A_α matrix merges the spectral theories of the adjacency matrix, the Laplacian matrix, and the signless Laplacian matrix. Thus for $\alpha = 0$, we find the adjacency spectrum of the commuting graphs of D_{2n}, SD_{8n} and Q_{4n} as already obtained by [5, 13], by using different techniques. Similarly, for $\alpha = \frac{1}{2}$, we have $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$, so we get the signless Laplacian spectrum of SD_{8n}, previously obtained in [32], but there is an error in the eigenvalues and with their multiplicities. Also, using the fact that $A_{\alpha_1}(G) - A_{\alpha_2}(G) = (\alpha_1 - \alpha_2)L(G)$, we can find the Laplacian spectrum of the commuting graphs of D_{2n}, SD_{8n} and Q_{4n}.

Theorem 2.8. *Suppose $C(\mathcal{G})$ is a commuting graph of a finite group $\mathcal{G}$ and $\sigma(\mathcal{G})$ be its Laplacian spectrum. Then the following hold.*

(i) *The Laplacian spectrum of $C(D_{2n}, D_{2n})$ is*

$$\sigma = \begin{cases} \{0, 1^{[n]}, n^{[n-2]}, 2n\}, & \text{if } n \text{ is odd;} \\ \{0, 2^{[\frac{n}{2}]}, 4^{[\frac{n}{2}]}, n^{[n-2]}, 2n^{[2]}\}, & \text{if } n \text{ is even.} \end{cases}$$

(ii) *The Laplacian spectrum of $C(SD_{8n}, D_{8n})$ is*

$$\sigma = \begin{cases} \{0, 4^{[n]}, 8^{[3n]}, (4n)^{[4n-5]}, (8n)^{[4]}\}, & \text{if } n \text{ is odd;} \\ \{0, 2^{[2n]}, 4^{[2n]}, (4n)^{[4n-3]}, (8n)^{[2]}\}, & \text{if } n \text{ is even.} \end{cases}$$

(iii) *The Laplacian spectrum of $C(Q_{4n}, Q_{4n})$ is*

$$\sigma = \{0, 2^{[n]}, 4^{[n]}, (2n)^{[2n-3]}, (4n)^{[2]}\}.$$

A matrix $M \in M_n(\mathbb{F})$ over the field $\mathbb{F}$ is called the *integral* if its spectrum consists of only integers. Similarly, the Laplacian matrix $L(G)$ of G is integral if all the eigenvalues of $L(G)$ are integers. Next, we have the immediate consequence of Theorem 2.8 about the Laplacian integral graphs.

Theorem 2.9. *The commuting graphs of the dihedral group, the semidihedral group and the dicyclic group are Laplacian integral graphs.*

3. Bounds for the A_α spectral radius of commuting graphs of non-abelian groups

If a matrix has the special type of symmetry, so that its block representation can be written as:

$$M = \begin{pmatrix} \mathcal{M}_{1,1} & \mathcal{M}_{1,2} & \cdots & \mathcal{M}_{1,d} \\ \mathcal{M}_{2,1} & \mathcal{M}_{2,2} & \cdots & \mathcal{M}_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}_{d,1} & \mathcal{M}_{d,2} & \cdots & \mathcal{M}_{d,d} \end{pmatrix}_{n\times n},$$

the rows and the columns of M are partitioned according to a partition $\mathcal{P} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_l\}$ of the set $S = \{1, 2, \ldots, d\}$. The quotient matrix Q (see [14]) of M is a $d \times d$ matrix whose entries are the average column (row) sums of the blocks $\mathcal{M}_{i,j}$ of M. The partition $\mathcal{P}$ is known as *regular* (equitable) if every block $\mathcal{M}_{i,j}$ of M has the constant column (row) sum and in such case Q is called the *regular quotient matrix*. Generally, the eigenvalues of Q interlace the eigenvalues of M. However, for the regular partition $\mathcal{P}$ of S, any eigenvalue of the matrix Q is the eigenvalue of the matrix M.

Next, we state a result which is crucial in establishing bounds for the A_α spectral radius.

Theorem 3.1. *[19] Assume that M_1 and M_2 are the Hermitian matrices of order n such that $M_3 = M_1 + M_2$ and $\lambda_1(M_i) \geq \lambda_2(M_i) \geq \cdots \geq \lambda_n(M_i)$, $i = 1, 2, 3$ be their eigenvalues. Then*

$$\lambda_k(M_3) \leq \lambda_j(M_1) + \lambda_{k-j+1}(M_2), \quad n \geq k \geq j \geq 1,$$
$$\lambda_k(M_3) \geq \lambda_j(M_1) + \lambda_{k-j+n}(M_2), \quad n \geq j \geq k \geq 1,$$

where λ_i is the i-th largest eigenvalue. Both the inequalities are equalities [31] iff there exists a unit vector which is the eigenvector to every of the three eigenvalues involved.

The following result is a consequence of Theorem 3.1 and this can be found in [14].

Corollary 3.2. *[14] Let $M \in M^*$ be such that $M = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$, and $\lambda_n(M)$ and $\lambda_1(M)$ be the smallest and the largest eigenvalues of M, respectively. Then*

$$\lambda_1(M) + \lambda_n(M) \leq \lambda_1(A) + \lambda_1(B).$$

Now, we give the bounds for the A_α eigenvalues of commuting graphs of non-abelian groups.

Theorem 3.3. *Let λ_1^α be the A_α spectral radius of $\mathcal{C}(D_{2n}, D_{2n})$. Then*

$$\lambda_1^\alpha \leq \begin{cases} \frac{1}{2}\left(2\alpha n + n - 2 + \sqrt{n^2 + 4n\alpha - 4n\alpha^2 - 4\alpha n^2 + 4n^2\alpha^2}\right) + \sqrt{n}(1-\alpha), & \text{if } n \text{ is odd;} \\ \frac{1}{2}\left(2\alpha n + n - 2 + \sqrt{n^2 + 8n\alpha - 8n\alpha^2 - 4\alpha n^2 + 4n^2\alpha^2}\right) + \sqrt{2n}(1-\alpha), & \text{if } n \text{ is even.} \end{cases}$$

Proof. For odd n, let $\{u, v_1, v_2, \ldots, v_{n-1}, u_1, u_2, \ldots, u_n\}$ be the vertices of $\mathcal{C}(D_{2n}, D_{2n})$, where u is the vertex of degree $2n-1$, v_i's are the vertices of degree $n-1$ and u_i's are pendent vertices. Under this labelling, the A_α matrix of $C(D_{2n}, D_{2n})$ is $A_\alpha(C(D_{2n}, D_{2n})) = A + B$, where block representation of A is

$$A = \left(\begin{array}{c|cccc|ccc} \alpha(2n-1) & 1-\alpha & \dots & 1-\alpha & 0 & \dots & 0 \\ \hline 1-\alpha & \alpha(n-1) & \dots & 1-\alpha & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1-\alpha & 1-\alpha & \dots & \alpha(n-1) & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \alpha \end{array}\right),$$

and its regular quotient matrix is

$$Q = \begin{pmatrix} \alpha(2n-1) & (1-\alpha)(n-1) & 0 \\ 1-\alpha & n-2+\alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

The eigenvalues of Q are $\{\alpha, \frac{1}{2}\left(2\alpha n + n - 2 \pm \sqrt{-4\alpha^2 n + 4\alpha n + 4\alpha^2 n^2 - 4\alpha n^2 + n^2}\right)\}$.

Also, the matrix B is

$$B = \begin{pmatrix} 0 & \mathbf{0}_{1\times(n-1)} & (1-\alpha)J_{1\times n} \\ \mathbf{0}_{(n-1)\times 1} & \mathbf{0}_{n-1} & \mathbf{0}_{(n-1)\times n} \\ (1-\alpha)J_{n\times 1} & \mathbf{0}_{n\times(n-1)} & \mathbf{0}_n \end{pmatrix},$$

where J is the matrix of all ones. The regular quotient matrix of B is

$$\begin{pmatrix} 0 & 0 & n(1-\alpha) \\ 0 & 0 & 0 \\ 1-\alpha & 0 & 0 \end{pmatrix},$$

and its eigenvalues are $\{0, \pm\sqrt{n}(1-\alpha)\}$. Therefore, by Theorem 3.1, the inequality

$$\lambda_1^\alpha(C(D_{2n}, D_{2n})) \le \lambda(A) + \lambda(B),$$

implies that

$$\lambda_1^\alpha(C(D_{2n}, D_{2n})) \le \frac{1}{2}\left(2\alpha n + n - 2 + \sqrt{n^2 - 4n\alpha^2 + 4n\alpha + 4n^2\alpha^2 - 4n^2\alpha}\right) + \sqrt{n}(1-\alpha).$$

For even n, with vertex labelling as in Theorem 2.4, the A_α matrix of $C(D_{2n}, D_{2n})$ can be put as $A_\alpha(C(D_{2n}, D_{2n})) = A + B$, where

$$A = \left(\begin{array}{ccc|cc|cc|c|cc} \alpha(n-1) & \dots & 1-\alpha & 1-\alpha & 1-\alpha & 0 & 0 & \dots & 0 & 0 \\ 1-\alpha & \dots & 1-\alpha & 1-\alpha & 1-\alpha & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1-\alpha & \dots & \alpha(n-1) & 1-\alpha & 1-\alpha & 0 & 0 & \dots & 0 & 0 \\ \hline 1-\alpha & \dots & 1-\alpha & \alpha(2n-1) & 1-\alpha & 1-\alpha & 1-\alpha & \dots & 1-\alpha & 1-\alpha \\ 1-\alpha & \dots & 1-\alpha & 1-\alpha & \alpha(2n-1) & 1-\alpha & 1-\alpha & \dots & 1-\alpha & 1-\alpha \\ \hline 0 & \dots & 0 & 1-\alpha & 1-\alpha & 3\alpha & 1-\alpha & \dots & 0 & 0 \\ 0 & \dots & 0 & 1-\alpha & 1-\alpha & 1-\alpha & 3\alpha & \dots & 0 & 0 \\ \hline \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1-\alpha & 1-\alpha & 0 & 0 & \dots & 3\alpha & 1-\alpha \\ 0 & \dots & 0 & 1-\alpha & 1-\alpha & 0 & 0 & \dots & 1-\alpha & 3\alpha \end{array}\right),$$

and the regular quotient matrix of A is

$$Q=\begin{pmatrix} n-3+2\alpha & 2(1-\alpha) & 0 & 0 & \dots & 0\\ (n-2)(1-\alpha) & \alpha(2n-2)+1 & 0 & 0 & \dots & 0\\ 0 & 0 & 2\alpha+1 & 0 & \dots & 0\\ 0 & 0 & 0 & 2\alpha+1 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \dots & 2\alpha+1 \end{pmatrix}_{n+2}.$$

Now, by Lemma 2.2, with $X=\begin{pmatrix} n-3+2\alpha & 2(1-\alpha)\\ (n-2)(1-\alpha) & \alpha(2n-2)+1\end{pmatrix}$, $Y=(0)$, $\mathcal{B}=(2\alpha+1)$ and $C=(0)$, we get the eigenvalue $2\alpha+1$ with algebraic multiplicity $n-1$ and the other three eigenvalues of Q are the eigenvalues of the sequel matrix:

$$M'=\begin{pmatrix} n-3+2\alpha & 2(1-\alpha) & 0\\ (n-2)(1-\alpha) & \alpha(2n-2)+1 & 0\\ 0 & 0 & 2\alpha+1\end{pmatrix}.$$

The eigenvalues of M' are $\{2\alpha+1, \frac{1}{2}\left(2\alpha n+n-2\pm\sqrt{n^2+8n\alpha-4n^2\alpha-8n\alpha^2+4n^2\alpha^2}\right)\}$.

Similarly,

$$B=\begin{pmatrix} \mathbf{0}_{n-2} & \mathbf{0}_{(n-2)\times 2} & \mathbf{0}_{(n-2)\times n}\\ \mathbf{0}_{2\times(n-2)} & \mathbf{0}_{2\times 2} & (1-\alpha)J_{2\times n}\\ \mathbf{0}_{n\times(n-2)} & (1-\alpha)J_{2\times n} & \mathbf{0}_{n\times n}\end{pmatrix},$$

and its quotient matrix is $\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & n(1-\alpha)\\ 0 & 2(1-\alpha) & 0\end{pmatrix}$, whose eigenvalues are $\{0, \sqrt{2n}(1-\alpha)\}$. Therefore, by Theorem 3.1, we obtain

$$\lambda_1^\alpha(C(D_{2n},D_{2n}))\le\frac{1}{2}\left(2\alpha n+n-2+\sqrt{n^2+8n\alpha-4n^2\alpha-8n\alpha^2+4n^2\alpha^2}\right)+\sqrt{2n}(1-\alpha).$$

□

Likewise to Theorem 3.3, we have the subsequent results for the commuting graphs of SD_{8n} and Q_{4n}.

Theorem 3.4. *Let λ_1^α be the A_α spectral radius of the commuting graph $C(SD_{8n},SD_{8n})$. Then*

$$\lambda_1^\alpha\le\begin{cases} 4\alpha n+2n-1+2\sqrt{n^2+4n\alpha-4n\alpha^2-4\alpha n^2+4n^2\alpha^2}+4\sqrt{n}(1-\alpha), & \text{if } n \text{ is odd};\\ 4\alpha n+2n-1+2\sqrt{n^2+2n\alpha-2n\alpha^2-4\alpha n^2+4n^2\alpha^2}+2\sqrt{2n}(1-\alpha), & \text{if } n \text{ is even}.\end{cases}$$

Theorem 3.5. *Let λ_1^α be the A_α spectral radius of $C(Q_{4n},Q_{4n})$. Then*

$$\lambda_1^\alpha\le 2\alpha n+n-1+\sqrt{n^2+4n\alpha-4n\alpha^2-4\alpha n^2+4n^2\alpha^2}+2\sqrt{n}(1-\alpha).$$

Finally, we obtain the upper bounds for the A_α spectral radius and the least A_α eigenvalue of commuting graphs of non-abelian groups.

Theorem 3.6. *Let λ_1^α and λ_n^α be the A_α spectral radius and the smallest A_α eigenvalue of the commuting graph $C(D_{2n}, D_{2n})$. Then*

$$\lambda_1^\alpha + \lambda_n^\alpha \leq \begin{cases} n + \alpha + \alpha n - 2 + \sqrt{\alpha^2(n^2 - n + 1) + n(1 - 2\alpha)}, & \text{if } n \text{ is odd;} \\ \alpha n + 2\alpha + \frac{1}{2}\left(n + \sqrt{n(-8\alpha^2 + 8\alpha + 4\alpha^2 n - 4\alpha n + n)}\right), & \text{if } n \text{ is even.} \end{cases}$$

Proof. Labelling the vertices as in Theorem 2.4, the A_α matrix of $C(D_{2n}, D_{2n})$ for odd n can be written as $A_\alpha(C(D_{2n}, D_{2n})) = \begin{pmatrix} A_{n+1} & C_{(n+1)\times(n-1)} \\ C^T & B_{n-1} \end{pmatrix}$, where

$$A = \left(\begin{array}{cccc|c} \alpha & 0 & \dots & 0 & 1-\alpha \\ 0 & \alpha & \dots & 0 & 1-\alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha & 1-\alpha \\ \hline 1-\alpha & 1-\alpha & \dots & \alpha & \alpha(2n-1) \end{array}\right), B = \begin{pmatrix} \alpha(n-1) & 1-\alpha & \dots & 1-\alpha \\ 1-\alpha & \alpha(n-1) & \dots & 1-\alpha \\ \vdots & \vdots & \ddots & \vdots \\ 1-\alpha & 1-\alpha & \cdots & \alpha(n-1) \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1-\alpha & 1-\alpha & \cdots & 1-\alpha \end{pmatrix}.$$

By applying Lemma 2.2 to the matrix B, with $X = (0), Y = (0), \mathcal{B} = \alpha(n-1)$, and $C = (1-\alpha)$. Consequently, $\alpha n - 1$ and $n + \alpha - 2$ are its only distinct eigenvalues. Also, the regular quotient matrix of A is

$$\begin{pmatrix} \alpha & 1-\alpha \\ n(1-\alpha) & \alpha(2n-1) \end{pmatrix},$$

and its eigenvalues are $\alpha n \pm \sqrt{\alpha^2(n^2 - n + 1) + n(1 - 2\alpha)}$. Therefore, by Corollary 3.2, we have

$$\lambda_1^\alpha + \lambda_n^\alpha \leq n + \alpha + \alpha n - 2 + \sqrt{\alpha^2(n^2 - n + 1) + n(1 - 2\alpha)}.$$

For even n, indexing the vertices as in Theorem 2.4, the A_α matrix of $C(D_{2n}, D_{2n})$ can be written as $A_\alpha(C(D_{2n}, D_{2n})) = \begin{pmatrix} A_n & C_n \\ C^T & B_n \end{pmatrix}$, where

$$A = \left(\begin{array}{cccc|cc} \alpha(n-1) & 1-\alpha & \dots & 1-\alpha & 1-\alpha & 1-\alpha \\ 1-\alpha & \alpha(n-1) & \dots & 1-\alpha & 1-\alpha & 1-\alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1-\alpha & 1-\alpha & \dots & \alpha(n-1) & 1-\alpha & 1-\alpha \\ \hline 1-\alpha & 1-\alpha & \dots & 1-\alpha & \alpha(2n-1) & 1-\alpha \\ 1-\alpha & 1-\alpha & \dots & 1-\alpha & 1-\alpha & \alpha(2n-1) \end{array}\right),$$

$$B = \left(\begin{array}{cc|cc|c|cc} 3\alpha & 1-\alpha & 0 & 0 & \dots & 0 & 0 \\ 1-\alpha & 3\alpha & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 3\alpha & 1-\alpha & \dots & 0 & 0 \\ 0 & 0 & 1-\alpha & 3\alpha & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & \dots & 3\alpha & 1-\alpha \\ 0 & 0 & 0 & 0 & \dots & 1-\alpha & 3\alpha \end{array}\right), C = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1-\alpha & 1-\alpha & \dots & 1-\alpha \\ 1-\alpha & 1-\alpha & \dots & 1-\alpha \end{pmatrix}.$$

Now, the quotient matrix of A is $\begin{pmatrix} n-3+2\alpha & 2(1-\alpha) \\ (n-2)(1-\alpha) & \alpha(2n-2)+1 \end{pmatrix}$ and its eigenvalues are $\frac{1}{2}\left(2\alpha n + n - 2 \pm \sqrt{n(-8\alpha^2 + 8\alpha + 4\alpha^2 n - 4\alpha n + n)}\right)$. Similarly, the regular quotient of B is

$$Q = \begin{pmatrix} 2\alpha+1 & 0 & \dots & 0 \\ 0 & 2\alpha+1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2\alpha+1 \end{pmatrix}_{\frac{n}{2}}$$

and we have $2\alpha + 1$ is the eigenvalue of Q with multiplicity $\frac{n}{2}$. Thus, by Corollary 3.2, we obtain

$$\lambda_1^\alpha + \lambda_n^\alpha \leq 2\alpha + 1 + \frac{1}{2}\left(2\alpha n + n - 2 \pm \sqrt{n(-8\alpha^2 + 8\alpha + 4\alpha^2 n - 4\alpha n + n)}\right).$$

□

Following the proof of Theorem 3.6, we have the similar results for the commuting graphs of the semidihedral and the dicyclic groups.

Theorem 3.7. *Let λ_1^α and λ_n^α be the A_α spectral radius and the smallest A_α eigenvalue of the commuting graph $C(SD_{8n}, SD_{8n})$. Then*

$$\lambda_1^\alpha + \lambda_n^\alpha \leq \begin{cases} 4\alpha n + 2n + 4\alpha + 2 + 2\sqrt{n^2 + 4n\alpha - 4n^2\alpha - 4n\alpha^2 + 4n^2\alpha^2}, & \text{if } n \text{ is odd;} \\ 4\alpha n + 2n + 2\alpha + 2\sqrt{n^2 + 2n\alpha - 4n^2\alpha - 2n\alpha^2 + 4n^2\alpha^2}, & \text{if } n \text{ is even.} \end{cases}$$

Theorem 3.8. *Let λ_1^α and λ_n^α be the A_α spectral radius and the smallest A_α eigenvalue of the commuting graph $C(Q_{4n}, Q_{4n})$. Then*

$$\lambda_1^\alpha + \lambda_n^\alpha \leq 2\alpha n + n + 2\alpha + \sqrt{n^2 + 4n\alpha - 4n^2\alpha - 4n\alpha^2 + 4n^2\alpha^2}.$$

4. Conclusions

In this article, the adjacency eigenvalues, the Laplacian eigenvalues, the signless Laplacian eigenvalues, and the generalized adjacency eigenvalues of graphs are given, including the bounds on the smallest and largest eigenvalues. The A_α matrix makes it very interesting to study the eigenvalues of well-known matrices in a very natural setting. Spectral properties of the graph defined by algebraic structures (groups, rings, modules, vector spaces, and others) have attracted many researchers, and various interesting problems have been solved both in combinatorics and algebra; for some recent developments, see [2, 5–10, 26–30, 32]. However, the A_α spectrum of all commuting and non-commuting graphs of groups remains open at large.

Acknowledgments

This research was supported by Taif University Researchers Supporting Project number (TURSP-2020/144), Taif University, Taif, Saudi Arabia.

The authors would like to gratefully acknowledge anonymous referee's constructive comments and suggestions, which greatly helped to improve the readability of the manuscript.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. A. Abdollahi, Commuting graph of full matrix rings over finite fields, *Linear Algebra Appl.*, **428** (2008), 2947–2954. https://doi.org/10.1016/j.laa.2008.01.036
2. Abdussakir, Sudarman, M. N. Jauhari, F. Ali, Survey on topological indices and graphs associated with a commutative ring, *J. Phys.: Conf. Ser.*, **1562** (2020), 012008.
3. A. Abdussakir, R. R. Elvierayani, M. Nafisah, On the spectra of commuting and non commuting graph on dihedral groups, *Cauchy*, **4** (2017), 176–182.
4. F. Ali, M. Salman, S. Huang, On the commuting graph of dihedral group, *Commun. Algebra*, **44** (2016), 2389–2401.
5. F. Ali, Y. Li, The connectivity and the spectral radius of commuting graphs on certain finite groups, *Linear Multilinear Algebra*, **69** (2019), 2945–2958. http://doi.org/10.1080/03081087.2019.1700893 .
6. F. Ali, B. A. Rather, N. Fatima, M. Sarfraz, A. Ullah, K. A. M. Alharbi, et al., On the topological indices of commuting graphs for finite non-Abelian groups, *Symmetry*, **14** (2022), 1266.
7. F. Ali, S. Fatima, W. Wang, On the power graph of certain of certain finite groups, *Linear Multilinear Algebra*, 2020. https://doi.org/10.1080/03081087.2020.1856028
8. F. Ali, B. A. Rather, A. Din, T. Saeed, A. Ullah, Power graphs of finite groups determined by Hosoya properties, *Entropy*, **24** (2022), 213. https://doi.org/10.3390/e24020213
9. D. F. Anderson, T. Asir, A. Badawi, T. T. Chelvam, *Graphs from rings*, Springer Nature Switzerland, 2021.
10. M. Ashraf, J. H. Asaloon, A. M. Alanazi, A. Alamer, An ideal-based dot total graph of a commutative ring, *Mathematics*, **9** (2021), 3072. https://doi.org/10.3390/math9233072
11. T. T. Chelvam, K. Selvakumar, S. Raja, Commuting graph on dihedral group, *J. Math. Comput. Sci.*, **2** (2011), 402–406.
12. J. Chen, L. Tang, The commuting graphs on dicyclic groups, *Algebra Colloq.*, **27** (2020), 799–806. https://doi.org/10.1142/S1005386720000668

13. T. Cheng, M. Dehmer, F. Emmert-Strein, Y. Li, W. Liu, Properties of commuting graphs over semidihedral groups, *Symmetry*, **13** (2021). http://doi.org/10.3390/sym13010103

14. D. M. Cvetković, P. Rowlison, S. Simić, *An introduction to theory of graph spectra*, UK: Cambridge University Press, 2011. https://doi.org/10.1017/CBO9780511801518

15. D. Dolžan, P. Oblak, Commuting graph of matrices over semirings, *Linear Algebra Appl.*, **435** (2011), 1657–1665. https://doi.org/10.1016/j.laa.2010.04.014

16. W. N. T. Fasfous, R. K. Nath, R. Sharafdini, Various spectra and energy of commuting graphs of finite rings, *Hacettepe J. Math. Stat.*, **49** (2020), 1915–1925.

17. E. Fritscher, V. Trevisan, Exploring symmetries to decompose matrices and graphs preserving the spectrum, *SIAM J. Matrix Anal. Appl.*, **37** (2016), 260–289. https://doi.org/10.1137/15M1013262

18. M. Ghorbani, Z. G. Alkhansari, A. Z. Bashi, On the eigenvalue of non commuting graphs of groups, *Alg. Struc. Appl.*, **4** (2017), 27–38. https://doi.org/10.29252/ASTA.4.2.27

19. R. Horn, C. Johnson, *Matrix analysis*, 2 Eds., Cambridge University Press, 2013.

20. V. Kakkar, G. S. Rawat, On commuting graph of generalized dihedral groups, *Discrete Math. Algorithms Appl.*, **11** (2019), 1950024. http://doi.org/10.1142/S1793830919500241 .

21. D. Li, Y. Chen, J. Meng, The A_α spectral radius of trees and unicyclic graphs with given degree sequence, *Appl. Math. Comput.*, **363** (2019), 124622. https://doi.org/10.1016/j.amc.2019.124622

22. W. K. Nicholson, *Introduction to abstract algebra*, 4 Eds., John Wiley and sons, New Jersey, 2012.

23. V. Nikiforov, Merging the $A-$and $Q-$spectral theories, *Appl. Anal. Discrete Math.*, **11** (2017), 18–107.

24. V. Nikiforov, G. Pasten, O. Rojo, R. L. Soto, On the A_α spectra of trees, *Linear Algebra Appl.*, **520** (2017), 286–305. https://doi.org/10.1016/j.laa.2017.01.029

25. S. Pirzada, B. A. Rather, H. A. Ganie, R. Shaban, On $\alpha-$adjacency energy of graphs, *AKCE Int. J. Graphs Comb.*, **18** (2021), 39–46. https://doi.org/10.1080/09728600.2021.1917973

26. S. Pirzada, B. A. Rather, T. A. Chishti, U. Samee, On normalized Laplacian spectrum of zero divisor graphs of commutative ring $\mathbb{Z}_n$, *Electron. J. Graph Theory Appl.*, **9** (2021), 331–345. http://dx.doi.org/10.5614/ejgta.2021.9.2.7

27. B. A. Rather, On distribution of Laplacian eigenvalues of graphs, 2021. https://doi.org/10.48550/arXiv.2107.09161 .

28. B. A. Rather, M. Aijaz, F. Ali, N. Mlaiki, A. Ullah, On distance signless Laplacian eigenvalues of zero divisor graph of commutative rings, *AIMS Math.*, **7** (2022), 12635–12649. https://doi.org/10.3934/math.2022699

29. B. A. Rather, S. Pirzada, T. A. Chishti, A. M. A. Alghamdi, On normalized Laplacian eigenvalues of power graphs associated to finite cyclic groups, *Discrete Math. Algorithms Appl.*, 2022. https://doi.org/10.1142/S1793830922500707

30. B. A. Rather, S. Pirzada, T. A. Naikoo, On distance signless Laplacian spectra of power graphs of the integer modulo group, *Art Discrete Appl. Math.*, 2022. https://doi.org/10.26493/2590-9770.1393.2be

31. W. So, Commutativity and spectra of Hermitian matrices, *Linear Algebra Appl.*, **212-213** (1994), 121–129. https://doi.org/10.1016/0024-3795(94)90399-9

32. M. Torktaz, A. R. Ashrafi, Spectral properties of the commutating graphs of certain finite groups, *AKCE Int. J. Graphs Comb.*, **16** (2019), 300–309. https://doi.org/10.1016/j.akcej.2018.09.006

33. C. Wang, S. Wang, The A_α-spectral radii of graphs with given connectivity, *Mathematics*, **7** (2019), 44. http://doi.org/10.3390/math7010044

AIMS Press

© 2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)