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*Research article*

# Picture fuzzy topological spaces and associated continuous functions

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**Abstract:** This paper describes a study of picture fuzzy topological spaces. We prove some basic results related to picture fuzzy sets together with the introduction of new notions such as the rank, picture fuzzy base and picture fuzzy sub-base of picture fuzzy topological spaces. With the help of these notions, we present a method to design picture fuzzy topological spaces. Furthermore, we introduce the concept of continuity to picture fuzzy topological spaces and find a necessary and sufficient condition for a picture fuzzy continuous function between two picture fuzzy topological spaces.

**Keywords:** fuzzy sets; picture fuzzy sets; picture fuzzy topological spaces; picture fuzzy topological bases; picture fuzzy topological sub-bases; continuity
**Mathematics Subject Classification:** 03E72, 08A72, 54A40

## 1. Introduction

Zadeh [1] presented the concept of fuzzy set which has many applications in decision-making, business, economy, data mining etc. A fuzzy subset $\aleph$ on a universe $X$ is an object of the form $\{x, \mu_{\aleph}(x): x \in X\}$, where $\mu_{\aleph}: X \to [0,1]$ is called a membership function of $\aleph$ and $\mu_{\aleph}(x)$ is known as a degree of membership of $x$ in $\aleph$. Clearly, fuzzy set theory is generalization of classical set theory. In classical sets, the corresponding membership function of a set $\aleph$ is the characteristic function of $\aleph$ that is equal to 1 $if\ x \in \aleph$ and 0 if $x \notin \aleph$. After the introduction of fuzzy sets, many new theories have been developed to deal with uncertainty and imprecision. Some of these theories are the expansions of the fuzzy set theory, while others have endeavored to manage uncertainty and imprecision in some other suitable way. It has been found that only the membership value is insufficient to express certain types of information. Therefore, a component called non-membership value has been introduced for proper illustration of the information. In 1986, Atanassov [2] presented the generalization fuzzy sets by defining intuitionistic fuzzy sets (IFSs). An intuitionistic fuzzy subset $\aleph$ on a universe $X$ is an object of the form $\{x, \mu_{\aleph}(x), \rho_{\aleph}(x): x \in X\}$, where $\mu_{\aleph}: X \to [0,1]$ and $\rho_{\aleph}: X \to [0,1]$ are the membership and non-membership functions respectively, such that $\mu_{\aleph}(x) + \rho_{\aleph}(x) \leq 1$ for all $x \in X$.

The IFS considers only the degree of membership and non-membership during the analysis. However, in certain cases, the degree of the neutrality also plays a critical role in the decision-making process. For instance, in the case of voting to address this into the analysis, Cuong [3] defined a new extension of the fuzzy set and IFS, named the picture fuzzy set, which considers the degree of neutrality along with the membership grades. A picture fuzzy subset $\aleph$ of $X$ is defined as $\aleph = \{x, \mu_{\aleph}(x), \rho_{\aleph}(x), \sigma_{\aleph}(x): x \in X\}$, where $\mu_{\aleph}: X \to [0,1]$, $\rho_{\aleph}: X \to [0,1]$ and $\sigma_{\aleph}: X \to [0,1]$ are positive, neutral and negative membership functions respectively satisfying $\mu_{\aleph}(x) + \rho_{\aleph}(x) + \sigma_{\aleph}(x) \leq 1$. For all $x \in X$, $1 - \left(\mu_{\aleph}(x) + \rho_{\aleph}(x) + \sigma_{\aleph}(x)\right)$ is termed the degree of refusal of $x$. Picture fuzzy sets have a variety of applications in decision-making environments. Chellamani et al. [4] used picture fuzzy soft graphs to design a decision-making scheme. In [5], the authors defined a multiple attribute decision-making technique based on picture fuzzy nano topologies. Yang et al. [6] developed an adjustable soft discernibility matrix with the help of picture fuzzy soft sets and presented its applications in decision making. A new similarity measure between picture fuzzy sets was proposed in [7]. The authors verified the efficacy of the suggested similarity measure through a numerical example**.** Joshi [8] presented an innovative decision-making process for a picture fuzzy environment with the help of the concept of R-Norm and the VIKOR technique. More development on picture fuzzy sets can be seen in [9–14]. In daily life, picture fuzzy set theory provides more than one choice for any decision. As examples:

1) Suppose a person is suffering from some disease. Then the positive, neutral and negative membership functions can be associated with curability, treatment and bitterness of disease respectively. Refusal can be related to the insufficient economic condition of the patient meaning that he cannot afford the hospital expenses and refuses to be hospitalized.
2) Suppose a person has an allegation of a crime. Then the positive, neutral and negative membership functions can be associated with the maximum punishment, moderate punishment and release of the accused person respectively. Refusal can be related to the dismissal of the case due to reconciliation.

The development of crisp topology originated from classical analysis, and it has numerous applications in many research areas like data mining, machine learning, quantum gravity and data analysis [15–21]. The notion of topology discloses the relationship between spatial elements and features. It narrates some specific spatial mappings and designs datasets with superior quality control and better information integrity. Homeomorphisms are isomorphisms of the category of topological spaces and they have a vital role in the theory. These are in fact a nice sort of continuous functions that provides a way of detecting whether two topological spaces are the same from the point of view of the topological structure. The concept of continuity is the topological part of a homeomorphism [15]. Furthermore, continuity reformulates the term of closeness in the mathematical models. Therefore, continuity is one of the most important properties of a function between two topological spaces. In the literature, there is a wide variety of the functions stated by the researchers such as bijective linear transformations for vector spaces, isomorphism between groups, rings, fields, modules, graphs, linear orders and diffeomorphisms between manifolds.

In the recent years topology has not only firmly established itself as an important part of the general mathematical structure, but also of physicists mathematical arsenal. The notion of the fuzzy topological space was introduced by Chang [22] in 1968. He generalized some fundamental concepts of topology, such as open sets, closed sets, neighborhoods and continuity. Based on this concept, many studies have been conducted in general theoretical areas and for different applications. Lowen changed the fundamental characteristic of topology and presented one more definition of a fuzzy topological space [23,24]. In [25], Lowen presented the concept of fuzzy continuous functions and generalized some results of continuity as related to classical topology. The author also introduced the notion of fuzzy compactness. Warren [26] defined fuzzy neighbourhood bases and proved some more results of fuzzy continuous functions. Another paper [27] dealing with fuzzy continuous functions was appeared in 1977. The author also introduced initial and final fuzzy topologies and developed the fuzzy Tychonoff theorem. For more on fuzzy topological spaces and related continuous functions, we suggest reading [28–32]. In 1995, the concept of intuitionistic fuzzy topological space was introduced by Coker. He discussed counterparts of some general topological concepts like compactness and continuity [33]. Ramadan et al. [34] introduced the notions of fuzzy almost continuous functions, and fuzzy compactness in intuitionistic fuzzy topological space. They also investigated some fundamental characteristics of these notions. Turanli and Coker [35] studied the concept of fuzzy connectedness in intuitionistic fuzzy topological spaces. For more details on intuitionistic fuzzy topological spaces, we refer the readers to [36–39]. Kramosil and Michálek [40] presented a comprehensive study on fuzzy metrics and statistical metric spaces. In [41,42], the authors introduced the concept of fuzzy soft topological space. The authors also developed a decision-making method based on proposed fuzzy soft topological space.

Keeping in mind the above literature and the importance of the picture fuzzy sets, as well as topological spaces, we reveal the study of picture fuzzy topological spaces. The major contributions of this paper are as follows:

1) The definition of some new notions such as the rank, picture fuzzy base and picture fuzzy sub-base of picture fuzzy topological spaces for picture fuzzy sets.
2) The design of various picture fuzzy topological spaces based on the stated notions
3) The definition of the notion of continuity in picture fuzzy topological spaces and introduction of a necessary and sufficient condition for a picture fuzzy continuous function between two picture fuzzy topological spaces

## 2. Picture fuzzy topology and some basic results

In this section, we define the notion of a picture fuzzy topological space. Moreover, we present some basic results for picture fuzzy sets which are useful to design picture fuzzy topological spaces. Let us rewrite the set operations over picture fuzzy subsets [3]:

**Definition 1 [3]** Let $\aleph_1$ and $\aleph_2$ be two picture fuzzy subsets of $X$, then

i) $\aleph_1 \subseteq \aleph_2$ if and only if $\mu_{\aleph_1}(x) \leq \mu_{\aleph_2}(x), \rho_{\aleph_1}(x) \leq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) \geq \sigma_{\aleph_2}(x)$.

ii) $$\aleph_1 \cup \aleph_2 = \left\{ \begin{array}{c} x, \max\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right), \min\left(\rho_{\aleph_1}(x), \rho_{\aleph_2}(x)\right), \\ \min\left(\sigma_{\aleph_1}(x), \sigma_{\aleph_2}(x)\right) : x \in X \end{array} \right\}.$$

iii) $$\aleph_1 \cap \aleph_2 = \left\{ \begin{array}{c} x, \min\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right), \min\left(\rho_{\aleph_1}(x), \rho_{\aleph_2}(x)\right), \\ \max\left(\sigma_{\aleph_1}(x), \sigma_{\aleph_2}(x)\right) : x \in X \end{array} \right\}.$$

iv) ${\aleph_1}^c = \{x, \sigma_{\aleph_1}(x), \rho_{\aleph_1}(x), \mu_{\aleph_1}(x) : x \in X\}$.

v) $\aleph_1 = \aleph_2$ if and only if $\aleph_1 \subseteq \aleph_2$ and $\aleph_2 \subseteq \aleph_1$.

It should be noted that, in the case of an infinite union and intersection, the maximum (max) and minimum (min) are replaced by the supremum (sup) and infimum (inf) respectively.

Throughout this paper, we denote the family of all picture fuzzy subsets of $X$ by $\Gamma_X$ and $I_X, O_X \in \Gamma_X$ represent $\{x, 1,0,0 : x \in X\}$ and $\{x, 0,0,1 : x \in X\}$ respectively.

**Remark 1:** Let $\aleph \in \Gamma_X$; it is important to mention that $\aleph \subseteq I_X$ but $O_X \nsubseteq \aleph$ in general. Therefore, a question arises; for which family of picture fuzzy subsets $O_X \subseteq \aleph$? We will give the answer later (see (ii) of Theorem 10).

**Definition 2:** Let $X$ denote a non-empty crisp set. A family $\Im$ of picture fuzzy subsets of $X$ is called a picture fuzzy topology on $X$ if

i) $I_X, O_X \in \Im$,

ii) for any $S_1, S_2 \in \Im$, we have $S_1 \cap S_2 \in \Im$,

iii) for any $\{S_i\}_{i \in \Omega} \subseteq \Im$, we have $\bigcup_{i \in \Omega} S_i \in \Im$, where $\Omega$ is an arbitrary index set.

If $\Im$ forms a picture fuzzy topology on $X$, then the pair $(X, \Im)$ denotes picture fuzzy topological space.

The following theorem is an immediate consequence of Definition 1.

**Theorem 1.** For all $\aleph_1, \aleph_2, \aleph_3 \in \Gamma_X$, the following conditions hold:

i) $\aleph_1 \cup \aleph_2 = \aleph_2 \cap \aleph_1 = \aleph_1$ if $\aleph_1 = \aleph_2$.

ii) $\aleph_1 \cup \aleph_2 = \aleph_2 \cup \aleph_1$ and $\aleph_1 \cap \aleph_2 = \aleph_2 \cap \aleph_1$.

iii) $\aleph_1 \cup (\aleph_2 \cup \aleph_3) = (\aleph_1 \cup \aleph_2) \cup \aleph_3$ and $\aleph_1 \cap (\aleph_2 \cap \aleph_3) = (\aleph_1 \cap \aleph_2) \cap \aleph_3$.

**Remark 2:** The converse of Theorem 1 (i) is not true.

Consider two picture fuzzy subsets $\aleph_1 = \{x, \mu_{\aleph_1}(x), \rho_{\aleph_1}(x), \sigma_{\aleph_1}(x) : x \in X\}$ and $\aleph_2 = \{x, \mu_{\aleph_2}(x), \rho_{\aleph_2}(x), \sigma_{\aleph_2}(x) : x \in X\}$ such that $\mu_{\aleph_1}(x) = \mu_{\aleph_2}(x), \rho_{\aleph_1}(x) \leq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) = \sigma_{\aleph_2}(x)$.

Then, clearly $\aleph_1 \cup \aleph_2 = \aleph_2 \cap \aleph_1 = \aleph_1$, but $\aleph_1 \neq \aleph_2$.

Remark 2 motivates us to define a new relation on $\Gamma_X$.

**Definition 3:** A picture fuzzy subset $\aleph_1$ is balanced with respect to the picture fuzzy subset $\aleph_2$ if and only if $\mu_{\aleph_1}(x) = \mu_{\aleph_2}(x), \rho_{\aleph_1}(x) \leq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) = \sigma_{\aleph_2}(x)$. If such is the case, then we write $\aleph_1 \equiv \aleph_2$.

The relation $\equiv$ is significant in designing various types of picture fuzzy topologies (see Theorem 13 and Example 8). Also, it is easy to verify that $\equiv$ is a partial order relation on $\Gamma_X$.

The following result is an extension of Theorem 1 (i)

**Theorem 2** For the picture fuzzy subsets $\aleph_1$ and $\aleph_2$ of $X$, we have $\aleph_1 \cup \aleph_2 = \aleph_2 \cap \aleph_1 = \aleph_1$ if and only if $\aleph_1 \equiv \aleph_2$.

The proof is an immediate consequence of Definition 3.

The following theorem shows that the union and intersection are binary operations on $\Gamma_X$.

**Theorem 3** Let $\aleph_1 = \{x, \mu_{\aleph_1}(x), \rho_{\aleph_1}(x), \sigma_{\aleph_1}(x): x \in X\}$ and $\aleph_2 = \{x, \mu_{\aleph_2}(x), \rho_{\aleph_2}(x), \sigma_{\aleph_2}(x): x \in X\}$ be two picture fuzzy subsets of $X$. Then $\aleph_1 \cup \aleph_2$ and $\aleph_1 \cap \aleph_2$ are the picture fuzzy subsets of $X$.

**Proof.** We know

$$\aleph_1 \cup \aleph_2 = \{x, \mu_{\aleph_1 \cup \aleph_2}(x), \rho_{\aleph_1 \cup \aleph_2}(x), \sigma_{\aleph_1 \cup \aleph_2}(x): x \in X\}$$
$$= \left\{\begin{matrix} x, \max\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right), \min\left(\rho_{\aleph_1}(x), \rho_{\aleph_2}(x)\right), \\ \min\left(\sigma_{\aleph_1}(x), \sigma_{\aleph_2}(x)\right): x \in X \end{matrix}\right\}$$

and

$$\aleph_1 \cap \aleph_2 = \{x, \mu_{\aleph_1 \cap \aleph_2}(x), \rho_{\aleph_1 \cap \aleph_2}(x), \sigma_{\aleph_1 \cap \aleph_2}(x): x \in X\}$$
$$= \left\{\begin{matrix} x, \min\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right), \min\left(\rho_{\aleph_1}(x), \rho_{\aleph_2}(x)\right), \\ \max\left(\sigma_{\aleph_1}(x), \sigma_{\aleph_2}(x)\right): x \in X \end{matrix}\right\}$$

For all pairs $\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right)$, $\left(\rho_{\aleph_1}(x), \rho_{\aleph_2}(x)\right)$ and $\left(\sigma_{\aleph_1}(x), \sigma_{\aleph_2}(x)\right)$, there are the following eight possibilities:

i) $\mu_{\aleph_1}(x) \geq \mu_{\aleph_2}(x)$, $\rho_{\aleph_1}(x) \geq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) \geq \sigma_{\aleph_2}(x)$
ii) $\mu_{\aleph_1}(x) \geq \mu_{\aleph_2}(x)$, $\rho_{\aleph_1}(x) \leq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) \geq \sigma_{\aleph_2}(x)$
iii) $\mu_{\aleph_1}(x) \geq \mu_{\aleph_2}(x)$, $\rho_{\aleph_1}(x) \geq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) \leq \sigma_{\aleph_2}(x)$
iv) $\mu_{\aleph_1}(x) \geq \mu_{\aleph_2}(x)$, $\rho_{\aleph_1}(x) \leq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) \leq \sigma_{\aleph_2}(x)$
v) $\mu_{\aleph_1}(x) \leq \mu_{\aleph_2}(x)$, $\rho_{\aleph_1}(x) \leq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) \leq \sigma_{\aleph_2}(x)$
vi) $\mu_{\aleph_1}(x) \leq \mu_{\aleph_2}(x)$, $\rho_{\aleph_1}(x) \geq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) \leq \sigma_{\aleph_2}(x)$
vii) $\mu_{\aleph_1}(x) \leq \mu_{\aleph_2}(x)$, $\rho_{\aleph_1}(x) \leq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) \geq \sigma_{\aleph_2}(x)$
viii) $\mu_{\aleph_1}(x) \leq \mu_{\aleph_2}(x)$, $\rho_{\aleph_1}(x) \geq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) \geq \sigma_{\aleph_2}(x)$

It can be easily verified that $\mu_{\aleph_1 \cup \aleph_2} + \rho_{\aleph_1 \cup \aleph_2} + \sigma_{A \cup B} \leq 1$ and $\mu_{\aleph_1 \cap \aleph_2} + \rho_{\aleph_1 \cap \aleph_2} + \sigma_{\aleph_1 \cap \aleph_2} \leq 1$ for all the above possibilities.

The following theorem shows that the distributive laws for union and intersection holds in $\Gamma_X$.

**Theorem 4** For all picture fuzzy subsets $\aleph_1, \aleph_2$ and $\aleph_3$ of $X$

i) $\aleph_1 \cup (\aleph_2 \cap \aleph_3) = (\aleph_1 \cup \aleph_2) \cap (\aleph_1 \cup \aleph_3)$
ii) $\aleph_1 \cap (\aleph_2 \cup \aleph_3) = (\aleph_1 \cap \aleph_2) \cup (\aleph_1 \cap \aleph_3)$

**Proof.** We shall prove the first part here. The second part can be obtained similarly.

Let $x \in X$; then

$$\mu_{\aleph_1 \cup (\aleph_2 \cap \aleph_3)}(x) = max\left(\mu_{\aleph_1}(x), \mu_{\aleph_2 \cap \aleph_3}(x)\right)$$
$$= max\left(\mu_{\aleph_1}(x), min\left(\mu_{\aleph_2}(x), \mu_{\aleph_3}(x)\right)\right) \quad (1)$$

and

$$\mu_{(\aleph_1\cup\aleph_2)\cap(\aleph_1\cup\aleph_3)}(x) = min\left(\mu_{\aleph_1\cup\aleph_2}(x), \mu_{\aleph_1\cup\aleph_3}(x)\right)$$
$$= min\left(max\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right), max\left(\mu_{\aleph_1}(x), \mu_{\aleph_3}(x)\right)\right) \quad (2)$$

Now there are three possibilities

a) If $max\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x), \mu_{\aleph_3}(x)\right) = \mu_{\aleph_1}(x)$, then (1) and (2) yield
$$\mu_{\aleph_1\cup(\aleph_2\cap\aleph_3)}(x) = \mu_{\aleph_1}(x) = \mu_{(\aleph_1\cup\aleph_2)\cap(\aleph_1\cup\aleph_3)}(x)$$

b) If $max\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x), \mu_{\aleph_3}(x)\right) = \mu_{\aleph_2}(x)$, then from (1) and (2), we obtain
$$\mu_{\aleph_1\cup(\aleph_2\cap\aleph_3)}(x) = max\left(\mu_{\aleph_1}(x), \mu_{\aleph_3}(x)\right) = \mu_{(\aleph_1\cup\aleph_2)\cap(\aleph_1\cup\aleph_3)}(x)$$

c) If $max\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x), \mu_{\aleph_3}(x)\right) = \mu_{\aleph_3}(x)$, then (1) and (2) imply that
$$\mu_{\aleph_1\cup(\aleph_2\cap\aleph_3)}(x) = max\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right) = \mu_{(\aleph_1\cup\aleph_2)\cap(\aleph_1\cup\aleph_3)}(x)$$

Thus, in all three cases we have
$$\mu_{\aleph_1\cup(\aleph_2\cap\aleph_3)}(x) = \mu_{(\aleph_1\cup\aleph_2)\cap(\aleph_1\cup\aleph_3)}(x)$$

Similarly, we can obtain that
$$\rho_{\aleph_1\cup(\aleph_2\cap\aleph_3)}(x) = \rho_{(\aleph_1\cup\aleph_2)\cap(\aleph_1\cup\aleph_3)}(x)$$

and
$$\sigma_{\aleph_1\cup(\aleph_2\cap\aleph_3)}(x) = \sigma_{(\aleph_1\cup\aleph_2)\cap(\aleph_1\cup\aleph_3)}(x)$$

Hence,
$$\aleph_1 \cup (\aleph_2 \cap \aleph_3) = (\aleph_1 \cup \aleph_2) \cap (\aleph_1 \cup \aleph_3).$$

## 3. Equivalence classes in $\Gamma_X$ and rank of picture fuzzy topology

In this section, we define an important equivalence relation ∥ on $\Gamma_X$. We show that the picture fuzzy subsets of the same equivalence class under ∥ satisfy some basic properties of inclusion, intersection and union (See Theorem 11). We also classify picture fuzzy topological spaces into different ranks on the basis of the newly defined equivalence relation ∥.

We begin this section with the following theorem.

**Theorem 5** Let $\aleph_1 = \left\{x, \mu_{\aleph_1}(x), \rho_{\aleph_1}(x), \sigma_{\aleph_1}(x) : x \in X\right\}$ and $\aleph_2 = \left\{x, \mu_{\aleph_2}(x), \rho_{\aleph_2}(x), \sigma_{\aleph_2}(x) : x \in X\right\}$ be two fuzzy subsets of $X$. Then

i) $\aleph_1 \cap \aleph_2 \subseteq \aleph_1$ and $\aleph_1 \cap \aleph_2 \subseteq \aleph_2$

ii) $\aleph_1 \subseteq \aleph_2$ if and only if $\aleph_1 \cap \aleph_2 = \aleph_1$

**Proof.** We know
$$\aleph_1 \cup \aleph_2 = \left\{x, \mu_{\aleph_1\cup\aleph_2}(x), \rho_{\aleph_1\cup\aleph_2}(x), \sigma_{\aleph_1\cup\aleph_2}(x) : x \in X\right\}$$

$$= \left\{ \begin{matrix} x, \max\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right), \min\left(\rho_{\aleph_1}(x), \rho_{\aleph_2}(x)\right), \\ \min\left(\sigma_{\aleph_1}(x), \sigma_{\aleph_2}(x)\right): x \in X \end{matrix} \right\}$$

and

$$\aleph_1 \cap \aleph_2 = \left\{x, \mu_{\aleph_1 \cap \aleph_2}(x), \rho_{\aleph_1 \cap \aleph_2}(x), \sigma_{\aleph_1 \cap \aleph_2}(x): x \in X\right\}$$

$$= \left\{ \begin{matrix} x, \min\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right), \min\left(\rho_{\aleph_1}(x), \rho_{\aleph_2}(x)\right), \\ \max\left(\sigma_{\aleph_1}(x), \sigma_{\aleph_2}(x)\right): x \in X \end{matrix} \right\}$$

i) Since $\mu_{\aleph_1 \cap \aleph_2}(x) = \min\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right)$, $\rho_{\aleph_1 \cap \aleph_2}(x) = \min\left(\rho_{\aleph_1}(x), \rho_{\aleph_2}(x)\right)$ and $\sigma_{\aleph_1 \cap \aleph_2}(x) = \max\left(\sigma_{\aleph_1}(x), \sigma_{\aleph_2}(x)\right)$,

$$\mu_{\aleph_1 \cap \aleph_2}(x) \leq \mu_{\aleph_1}(x),\ \ \mu_{\aleph_2}(x),$$

$$\rho_{\aleph_1 \cap \aleph_2}(x) \leq \rho_{\aleph_1}(x), \rho_{\aleph_2}(x)$$

and

$$\sigma_{\aleph_1 \cap \aleph_2}(x) \geq \sigma_{\aleph_1}(x), \sigma_{\aleph_2}(x).$$

Thus, $\aleph_1 \cap \aleph_2 \subseteq \aleph_1$ and $\aleph_1 \cap \aleph_2 \subseteq \aleph_2$.
ii) Let $\aleph_1 \subseteq \aleph_2$; then $\mu_{\aleph_1}(x) \leq \mu_{\aleph_2}(x)$, $\rho_{\aleph_1}(x) \leq \rho_{\aleph_2}(x)$ and $\sigma_{\aleph_1}(x) \geq \sigma_{\aleph_2}(x)$. This reveals that

$$\mu_{\aleph_1 \cap \aleph_2}(x) = \mu_{\aleph_1}(x),\ \rho_{\aleph_1 \cap \aleph_2}(x) = \rho_{\aleph_1}(x) \text{ and } \sigma_{\aleph_1 \cap \aleph_2}(x) = \sigma_{\aleph_1}(x),$$

implying that $\aleph_1 \cap \aleph_2 = \aleph_1$.
Conversely, suppose that $\aleph_1 \cap \aleph_2 = \aleph_1$. Since $\aleph_1 \cap \aleph_2 \subseteq \aleph_2$, therefore $\aleph_1 \subseteq \aleph_2$.

The following examples show that it is not necessary for picture fuzzy subsets $\aleph_1$ and $\aleph_2$ to satisfy the following:

i) $\aleph_1 \subseteq \aleph_1 \cup \aleph_2$ and $\aleph_2 \subseteq \aleph_1 \cup \aleph_2$ (3)
ii) $\aleph_1 \subseteq \aleph_2$ if and only if $\aleph_1 \cup \aleph_2 = \aleph_2$ (4)

**Example 1** Consider two picture fuzzy subsets of $X = \{a, b, c\}$ given as

$$\aleph_1 = \left\{ \begin{matrix} (a, 0.50, 0.20, 0.25), (b, 0.40, 0.10, 0.50), \\ (c, 0.20, 0.30, 0.45) \end{matrix} \right\},$$

and

$$\aleph_2 = \left\{ \begin{matrix} (a, 0.40, 0.30, 0.10), (b, 0.20, 0.60, 0.10), \\ (c, 0.30, 0.20, 0.15) \end{matrix} \right\}$$

Then

$$\aleph_1 \cup \aleph_2 = \left\{ \begin{matrix} (a, 0.50, 0.20, 0.10), (b, 0.40, 0.10, 0.10), \\ (c, 0.30, 0.20, 0.15) \end{matrix} \right\}.$$

Clearly, neither $\aleph_1 \subseteq \aleph_1 \cup \aleph_2$ nor $\aleph_2 \subseteq \aleph_1 \cup \aleph_2$.
**Example 2** Consider two picture fuzzy subsets of $X = \{a, b, c\}$ given as

$$\aleph_1 = \begin{Bmatrix} (a, 0.30, 0.20, 0.25), (b, 0.10, 0.30, 0.50), \\ (c, 0.20, 0.20, 0.45) \end{Bmatrix}$$

and

$$\aleph_2 = \begin{Bmatrix} (a, 0.40, 0.30, 0.10), (b, 0.20, 0.60, 0.10), \\ (c, 0.30, 0.20, 0.15) \end{Bmatrix}.$$

Then

$$\aleph_1 \cup \aleph_2 = \begin{Bmatrix} (a, 0.40, 0.20, 0.10), (b, 0.20, 0.30, 0.10), \\ (c, 0.30, 0.20, 0.15) \end{Bmatrix}.$$

Clearly, $\aleph_1 \cup \aleph_2 \neq \aleph_2$.

**Example 3** Consider

$$\aleph_1 = \begin{Bmatrix} (a, 0.30, 0.20, 0.25), (b, 0.10, 0.30, 0.50), \\ (c, 0.20, 0.20, 0.45) \end{Bmatrix}$$

and

$$\aleph_2 = \begin{Bmatrix} (a, 0.40, 0.15, 0.10), (b, 0.20, 0.25, 0.10), \\ (c, 0.30, 0.20, 0.15) \end{Bmatrix}.$$

Then, $\aleph_1 \cup \aleph_2 = \aleph_2$ but $\aleph_1 \nsubseteq \aleph_2$.

Thus, a curiosity arises to explore the sufficient condition for the validation of the statement given in Eqs (3) and (4). The upcoming three theorems resolve this curiosity.

**Theorem 7** $\aleph_1 \subseteq \aleph_1 \cup \aleph_2$ if $\rho_{\aleph_1}(x) \leq \rho_{\aleph_2}(x)$ for all $x \in X$.

**Proof:** Suppose that $\rho_{\aleph_1}(x) \leq \rho_{\aleph_2}(x)$. Then

$$\begin{aligned} \mu_{\aleph_1 \cup \aleph_2}(x) &= max\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right) \\ &\geq \mu_{\aleph_1}(x), \rho_{\aleph_1 \cup \aleph_2}(x) \\ &= \min\left(\rho_{\aleph_1}(x), \rho_{\aleph_2}(x)\right) \\ &= \rho_{\aleph_1}(x) \end{aligned}$$

and

$$\sigma_{\aleph_1 \cup \aleph_2}(x) = min\left(\sigma_{\aleph_1}(x), \sigma_{\aleph_2}(x)\right) \leq \sigma_{\aleph_1}(x).$$

Thus, $\aleph_1 \subseteq \aleph_1 \cup \aleph_2$.

**Theorem 8** $\aleph_1 \subseteq \aleph_1 \cup \aleph_2$ and $\aleph_2 \subseteq \aleph_1 \cup \aleph_2$ if $\rho_{\aleph_1}(x) = \rho_{\aleph_2}(x)$ for all $x \in X$.

**Proof.** The proof is an immediate consequence of Theorem 7.

**Theorem 9** $\aleph_1 \subseteq \aleph_2 \Leftrightarrow \aleph_1 \cup \aleph_2 = \aleph_2$ if $\rho_{\aleph_1}(x) = \rho_{\aleph_2}(x)$ for all $x \in X$.

**Proof.** Let $\rho_{\aleph_1}(x) = \rho_{\aleph_2}(x)$ for all $x \in X$. Then

$$\aleph_1 \subseteq \aleph_2 \Leftrightarrow \max\left(\mu_{\aleph_1}(x), \mu_{\aleph_2}(x)\right) = \mu_{\aleph_2}(x),$$

$$\min\left(\rho_{\aleph_1}(x), \rho_{\aleph_2}(x)\right) = \rho_{\aleph_1}(x) = \rho_{\aleph_2}(x)$$

and

$$\min\left(\sigma_{\aleph_1}(x), \sigma_{\aleph_2}(x)\right) = \sigma_{\aleph_2}(x) \Leftrightarrow \aleph_1 \cup \aleph_2 = \aleph_2.$$

Since the Theorems 8 and 9, reveal that a sufficient condition for the picture fuzzy subsets $\aleph_1$ and $\aleph_2$ to satisfy conditions (3) and (4) is $\rho_{\aleph_1}(x) = \rho_{\aleph_2}(x)$. This leads us to define a new relation on $\Gamma_X$.

**Definition 4** A picture fuzzy subset $\aleph_1 = \left\{x, \mu_{\aleph_1}(x), \rho_{\aleph_1}(x), \sigma_{\aleph_1}(x): x \in X\right\}$ is $\rho$-equivalent to a picture fuzzy subset $\aleph_2 = \left\{x, \mu_{\aleph_2}(x), \rho_{\aleph_2}(x), \sigma_{\aleph_2}(x): x \in X\right\}$ if and only if $\mu_{\aleph_1}(x) = \rho_{\aleph_2}(x)$ for all $x \in X$. If $\aleph_1$ is $\rho$-equivalent to $\aleph_2$, then we write $\aleph_1 \parallel \aleph_2$.

It is easy to verify that $\parallel$ is an equivalence relation and hence all equivalences classes under $\parallel$ form a partition of the set $\Gamma_X$ of all picture fuzzy subsets of $X$.

**Theorem 10** For all $\aleph_1, \aleph_2 \in \Gamma_X$, the following conditions hold:

i) $\aleph_1 = \aleph_2$ if and only if $\aleph_1 \parallel \aleph_2$ and $\aleph_1 \equiv \aleph_2$.

ii) $\aleph_1 \subseteq I_X$ and $O_X \subseteq \aleph_1$ if and only if $\aleph_1 \parallel I_X$.

The proof is straightforward

The final conclusion of this section can be summarized in the following theorem.

**Theorem 11** Let $\aleph_1, \aleph_2 \in \Gamma_X$; then the following conditions hold:

i) $\aleph_1 \cap \aleph_2 \subseteq \aleph_1$ and $\aleph_1 \cap \aleph_2 \subseteq \aleph_2$.

ii) $\aleph_1 \subseteq \aleph_2$ if and only if $\aleph_1 \cap \aleph_2 = \aleph_1$.

iii) $\aleph_1 \subseteq \aleph_1 \cup \aleph_2$ and $\aleph_2 \subseteq \aleph_1 \cup \aleph_2$if $\aleph_1 \parallel \aleph_2$.

iv) If $\aleph_1 \parallel \aleph_2$ then $\aleph_1 \subseteq \aleph_2 \Leftrightarrow \aleph_1 \cup \aleph_2 = \aleph_2$.

**Proof.** (i) By using Definition 1 (iii), the proof is straightforward.

(ii) The proof is an immediate consequence of Definition 1 (iii).

(iii) In view of Theorem 8, we have $\aleph_1 \subseteq \aleph_1 \cup \aleph_2$ and $\aleph_2 \subseteq \aleph_1 \cup \aleph_2$ if $\rho_{\aleph_1}(x) = \rho_{\aleph_2}(x)$ for all $x \in X$. Also, Definition 4 reveals that $\aleph_1 \parallel \aleph_2$ if and only if $\rho_{\aleph_1}(x) = \rho_{\aleph_2}(x)$ for all $x \in X$. Thus, $\aleph_1 \subseteq \aleph_1 \cup \aleph_2$ and $\aleph_2 \subseteq \aleph_1 \cup \aleph_2$ if $\aleph_1 \parallel \aleph_2$.

(iv) In view of Theorem 9, we have $\aleph_1 \subseteq \aleph_2 \Leftrightarrow \aleph_1 \cup \aleph_2 = \aleph_2$ if $\rho_{\aleph_1}(x) = \rho_{\aleph_2}(x)$ for all $x \in X$. Also, Definition 4 reveals that $\aleph_1 \parallel \aleph_2$ if and only if $\rho_{\aleph_1}(x) = \rho_{\aleph_2}(x)$ for all $x \in X$. Thus, $\aleph_1 \subseteq \aleph_2 \Leftrightarrow \aleph_1 \cup \aleph_2 = \aleph_2$ if $\aleph_1 \parallel \aleph_2$.

Theorem 11 plays a vital role in the construction of picture topological spaces. It holds for the picture fuzzy subsets of the same equivalence class, we classify picture fuzzy topologies on the basis of the number equivalence classes. This leads us to define an important notion called the rank of picture fuzzy topology.

**Definition 5** A natural number $n$ is called a rank of the picture fuzzy topology $\mathfrak{I}$ if it has $n$ distinct equivalence classes under $\parallel$.

## 4. Picture fuzzy bases and sub-bases and construction of picture fuzzy topological spaces

In this section, we first define the notions of picture fuzzy bases and sub-bases of picture fuzzy topology. Next, we construct some picture fuzzy topologies of different picture fuzzy sub-bases and ranks.

**Definition 6** Let $\boldsymbol{\mathcal{B}} = \{\beta_i \in \Gamma_X\}$ such that $I_X, O_X \notin \boldsymbol{\mathcal{B}}$. Consider another sub-collection $\mathfrak{I}$ of $\Gamma_X$ with the following properties:

i) $I_X, O_X \in \mathfrak{I}$.

ii) $S_i \in \mathfrak{I} \setminus \{I_X, O_X\}$ and $S_i \nparallel O_X$ if and only if there exists $\beta_i \in \boldsymbol{\mathcal{B}}$ such that if $S_i = \bigcup_{i\in\Omega}\beta_i$.

iii) $S_i \in \mathfrak{I} \setminus \{I_X, O_X\}$ and $S_i \parallel O_X$ if and only there exists $\beta_i \in \boldsymbol{\mathcal{B}}$ such that $S_i = O_X \cup \{\bigcup_{i\in\Omega}\beta_i\}$.

If $\mathfrak{I}$ forms a picture fuzzy topology on $X$, then $\boldsymbol{\mathcal{B}}$ is called a picture fuzzy base for $\mathfrak{I}$.

**Theorem 12** $\boldsymbol{\mathcal{B}}$ is a picture fuzzy base for the sub-collection $\mathfrak{I}$ (given in definition 6) of $\Gamma_X$ if it is closed under a finite intersection.

**Proof:** Suppose that $\boldsymbol{\mathcal{B}}$ is closed under a finite intersection. We want to show that the sub-collection $\mathfrak{I}$ (given in definition 6) of $\Gamma_X$ is a picture fuzzy topology on $X$.

(1) The definition 6 shows that $I_X, O_X \in \mathfrak{I}$.

(2) Let $S_1, S_2 \in \mathfrak{I} \setminus \{I_X, O_X\}$, then there are three cases

(i) $S_1 \nparallel S_2$ (ii) $S_1 \parallel O_X$ and $S_2 \nparallel O_X$ (iii) $S_1, S_2 \parallel O_X$.

Let (i) be true, then $S_1 \cap S_2 = (\beta_1 \cup \beta_2 \cup \ldots \cup \beta_r \cup \ldots) \cap (\beta_1' \cup \beta_2' \cup \ldots \cup \beta_s' \cup \ldots)$

$$= (\beta_1 \cap \beta_1') \cup (\beta_1 \cap \beta_2') \cup \ldots \cup (\beta_1 \cap \beta_s') \cup \ldots \cup (\beta_2 \cap \beta_1') \cup (\beta_2 \cap \beta_2') \cup \ldots \cup (\beta_2 \cap \beta_s') \cup \ldots \cup (\beta_r \cap \beta_1') \cup (\beta_r \cap \beta_2') \cup \ldots \cup (\beta_r \cap \beta_s') \cup \ldots$$

$= \bigcup_{i,j\in\Omega}(\beta_i \cap \beta_j')$, where $\beta_i, \beta_j' \in \boldsymbol{\mathcal{B}}$.

Since $\boldsymbol{\mathcal{B}}$ is closed under a finite intersection, therefore each $\beta_i \cap \beta_j' \in \boldsymbol{\mathcal{B}}$.

Thus, $S_1 \cap S_2 = \bigcup_{i,j\in\Omega}(\beta_i \cap \beta_j') \in \mathfrak{I}$.

Similarly, $S_1 \cap S_2 \in \mathfrak{I}$ for (ii) and (iii) as well.

(3) Suppose $\{S_i\}_{i\in\Omega} \subseteq \mathfrak{I} \setminus \{O_X, I_X\}$; then there are the following two possibilities:

(i) $S_i \nparallel O_X$ for all $i \in \Omega$. This implies that $\bigcup_{i\in\Omega} S_i = \bigcup_{j\in\Omega'}\beta_j \in \mathfrak{I}$

(ii) Some of the members of $\{S_i\}_{i\in\Omega}$ are $\rho$-equivalent to $O_X$; then

$$\bigcup_{i\in\Omega} S_i = O_X \cup \left(\bigcup_{j\in\Omega'}\beta_j\right) \in \mathfrak{I}.$$

Moreover, $O_X \cap S_i = O_X$, $I_X \cup S_i = I_X$ and $O_X \cup S_i = I_X \cap S_i$ together with (2) and (3) imply that $S_1 \cap S_2 \in \mathfrak{I}$ for all $S_1, S_2 \in \mathfrak{I}$ and $\bigcup_{i\in\Omega} S_i \in \mathfrak{I}$ for all $\{S_i\}_{i\in\Omega} \subseteq \mathfrak{I}$.

**Definition 7** A collection $\mathfrak{S}$ of picture fuzzy subsets of $X$ is called a picture fuzzy sub-base of the picture fuzzy topology $\mathfrak{I}$ if

i) all finite intersections of the elements of $\mathfrak{S}$ evolves a base for some picture fuzzy topology $\mathfrak{I}$;

ii) $\aleph_1 \cap \aleph_2 \in \mathfrak{S}$ implies $\aleph_1 \subseteq \aleph_2$ for all $\aleph_1, \aleph_2 \in \mathfrak{S}$.

It should be that, we want to see the minimum elements in a sub-base that is why we impose Condition (ii) in its definition.

**Remark 3** Every collection $\mathfrak{S} = \{\aleph_i : \aleph_i \in \Gamma_X\}$ of a picture fuzzy subset of $X$, satisfying Condition (ii) of Definition 7, yields some picture fuzzy topology $\mathfrak{I}$. Because the collection $\boldsymbol{\mathcal{B}}$, evolved by all finite intersections of the elements of $\mathfrak{S}$, is closed under finite intersections, Theorem 12 guarantees its role as a base for the picture fuzzy topology $\mathfrak{I}$.

**Remark 4** The arbitrary union and finite intersection of picture fuzzy subsets from one equivalence class belong to the same equivalence class. This, together with the definitions of a picture fuzzy base and picture fuzzy sub-base implies that the sub-base of the picture fuzzy topology $\mathfrak{I}$ of rank $n$ contains at least $n-1$ elements.

*4.1. Construction of picture fuzzy topological spaces*

All of the elements in picture fuzzy topologies of rank 1 belong to the same equivalence class. Therefore, Theorem 12 is valid for these elements. For this reason, we found that it is relatively easy to construct the picture fuzzy topologies of rank 1 as compared to those of higher ranks.

The following facts regarding the picture fuzzy topology $\mathfrak{I}$ of rank 1 are easy to prove:

i) $\{I_X = \{x, 1,0,0\}, O_X = \{x, 0,0,1\}\}$ is the smallest picture fuzzy topology of rank 1. It is a trivial picture fuzzy topology.

ii) The picture fuzzy topology of rank 1 with the sub-base $\mathfrak{S} = \{S_1\}$ is $\{I_X, O_X, S_1\}$.

iii) The picture fuzzy topology of rank 1 with $\mathfrak{S} = \{S_1, S_2, \dots, S_n\}$ such that $S_1 \subset S_2 \subset \cdots \subset S_n$ is $\{I_X, O_X, S_1, S_2, \dots, S_n\}$.

The upcoming example shows the structure of the picture fuzzy topology of rank 2 on $X = \{a, b, c\}$ with the sub-base $\mathfrak{S} = \{S_1, S_2\}$ such that neither $S_1 \subset S_2$ nor $S_2 \subset S_1$.

**Example 4** Consider $\mathfrak{S} = \left\{ \begin{array}{c} S_1 = \left\{ \begin{array}{c} (a, 0.25,0.20,0.30), (b, 0.35,0.10,0.45), \\ (c, 0.30,0.35,0.10) \end{array} \right\}, \\ S_2 = \left\{ \begin{array}{c} (a, 0.45,0.20,0.35), (b, 0.25,0.10,0.40), \\ (c, 0.50,0.35,0.05) \end{array} \right\} \end{array} \right\}$.

Then all finite intersections of the elements of $\mathfrak{S}$ give

$$\boldsymbol{\mathcal{B}} = \left\{ \begin{array}{c} S_1 = \left\{ \begin{array}{c} (a, 0.25,0.20,0.30), (b, 0.35,0.10,0.45), \\ (c, 0.30,0.35,0.10) \end{array} \right\}, \\ S_2 = \left\{ \begin{array}{c} (a, 0.45,0.20,0.35), (b, 0.25,0.10,0.40), \\ (c, 0.50,0.35,0.05) \end{array} \right\}, \\ S_1 \cap S_2 = S_3 = \left\{ \begin{array}{c} (a, 0.25,0.20,0.35), (b, 0.25,0.10,0.45), \\ (c, 0.30,0.35,0.10) \end{array} \right\} \end{array} \right\}.$$

The picture fuzzy topology obtained from $\boldsymbol{\mathcal{B}}$ is

$$\mathfrak{J} = \left\{ \begin{array}{c} I_X = \left\{ \begin{array}{c} (a, 1.00,0.00,0.00), (b, 1.00,0.00,0.00), \\ (c, 1.00,0.00,0.00) \end{array} \right\}, \\ O_X = \left\{ \begin{array}{c} (a, 0.00,0.00,1.00), (b, 0.00,0.00,1.00), \\ (c, 0.00,0.00,1.00) \end{array} \right\} \\ S_1 = \left\{ \begin{array}{c} (a, 0.25,0.20,0.30), (b, 0.35,0.10,0.45), \\ (c, 0.30,0.35,0.10) \end{array} \right\}, \\ S_2 = \left\{ \begin{array}{c} (a, 0.45,0.20,0.35), (b, 0.25,0.10,0.40), \\ (c, 0.50,0.35,0.05) \end{array} \right\}, \\ S_1 \cap S_2 = S_3 = \left\{ \begin{array}{c} (a, 0.25,0.20,0.35), (b, 0.25,0.10,0.45), \\ (c, 0.30,0.35,0.10) \end{array} \right\}, \\ S_1 \cup S_2 = S_4 = \left\{ \begin{array}{c} (a, 0.45,0.20,0.30), (b, 0.35,0.10,0.40), \\ (c, 0.50,0.35,0.05) \end{array} \right\}, \\ O_X \cup S_1 = S_5 = \left\{ \begin{array}{c} (a, 0.25,0.00,0.30), (b, 0.35,0.00,0.45), \\ (c, 0.30,0.00,0.10) \end{array} \right\}, \\ O_X \cup S_2 = S_6 = \left\{ \begin{array}{c} (a, 0.45,0.00,0.35), (b, 0.25,0.00,0.40), \\ (c, 0.50,0.00,0.05) \end{array} \right\}, \\ O_X \cup S_3 = S_7 = \left\{ \begin{array}{c} (a, 0.25,0.00,0.35), (b, 0.25,0.00,0.45), \\ (c, 0.30,0.00,0.10) \end{array} \right\}, \\ O_X \cup S_4 = S_8 = \left\{ \begin{array}{c} (a, 0.45,0.00,0.30), (b, 0.35,0.00,0.40), \\ (c, 0.50,0.00,0.05) \end{array} \right\} \end{array} \right\}.$$

In the following example, we construct the picture fuzzy topology of rank 3 on $X = \{a, b, c\}$ with the sub-base $\mathfrak{S} = \{S_1, S_2\}$ such that $S_1 \subset S_2$ but $S_1 \not\equiv S_2$.

**Example 5** Consider $\mathfrak{S} = \left\{ \begin{array}{c} S_1 = \left\{ \begin{array}{c} (a, 0.30,0.20,0.45), (b, 0.20,0.25,0.40), \\ (c, 0.30,0.35,0.10) \end{array} \right\}, \\ S_2 = \left\{ \begin{array}{c} (a, 0.35,0.30,0.35), (b, 0.25,0.30,0.30), \\ (c, 0.50,0.40,0.05) \end{array} \right\} \end{array} \right\}.$

The base $\boldsymbol{\mathcal{B}}$ evolved from $\mathfrak{S}$ is $\mathfrak{S}$ itself. Thus, the picture fuzzy topology constructed from $\boldsymbol{\mathcal{B}}$ is

$$\mathfrak{J} = \left\{ \begin{array}{c} I_X = \left\{ \begin{array}{c} (a, 1.00,0.00,0.00), (b, 1.00,0.00,0.00), \\ (c, 1.00,0.00,0.00) \end{array} \right\}, \\ O_X = \left\{ \begin{array}{c} (a, 0.00,0.00,1.00), (b, 0.00,0.00,1.00), \\ (c, 0.00,0.00,1.00) \end{array} \right\} \\ S_1 = \left\{ \begin{array}{c} (a, 0.30,0.20,0.45), (b, 0.20,0.25,0.40), \\ (c, 0.30,0.35,0.10) \end{array} \right\}, \\ S_2 = \left\{ \begin{array}{c} (a, 0.35,0.30,0.35), (b, 0.25,0.30,0.30), \\ (c, 0.50,0.40,0.05) \end{array} \right\}, \\ S_1 \cup S_2 = S_3 = \left\{ \begin{array}{c} (a, 0.35,0.20,0.35), (b, 0.25,0.25,0.30), \\ (c, 0.50,0.35,0.05) \end{array} \right\}, \\ O_X \cup S_1 = S_4 = \left\{ \begin{array}{c} (a, 0.30,0.00,0.45), (b, 0.20,0.00,0.40), \\ (c, 0.30,0.00,0.10) \end{array} \right\}, \\ O_X \cup S_2 = S_5 = \left\{ \begin{array}{c} (a, 0.35,0.00,0.35), (b, 0.25,0.00,0.30), \\ (c, 0.50,0.00,0.05) \end{array} \right\} \end{array} \right\}.$$

The succeeding example describes the design of the picture fuzzy topology of rank 3 on $X = \{a, b, c\}$ with the sub-base $\mathfrak{S} = \{S_1, S_2\}$ such that $S_1 \subset S_2$ and $S_1 \equiv S_2$.

**Example 6** Consider $\mathfrak{S} = \left\{ \begin{matrix} S_1 = \left\{ \begin{matrix} (a, 0.35,0.20,0.25), (b, 0.20,0.15,0.30), \\ (c, 0.20,0.35,0.15) \end{matrix} \right\}, \\ S_2 = \left\{ \begin{matrix} (a, 0.35,0.30,0.25), (b, 0.20,0.25,0.30), \\ (c, 0.20,0.40,0.15) \end{matrix} \right\} \end{matrix} \right\}$.

The base $\boldsymbol{\mathcal{B}}$ evolved from $\mathfrak{S}$ is $\mathfrak{S}$ itself. Now, we obtain the following picture fuzzy topology from $\boldsymbol{\mathcal{B}}$**:**

$$\mathfrak{I} = \left\{ \begin{matrix} I_X = \left\{ \begin{matrix} (a, 1.00,0.00,0.00), (b, 1.00,0.00,0.00), \\ (c, 1.00,0.00,0.00) \end{matrix} \right\}, \\ O_X = \left\{ \begin{matrix} (a, 0.00,0.00,1.00), (b, 0.00,0.00,1.00), \\ (c, 0.00,0.00,1.00) \end{matrix} \right\} \\ S_1 = \left\{ \begin{matrix} (a, 0.35,0.20,0.25), (b, 0.20,0.15,0.30), \\ (c, 0.20,0.35,0.15) \end{matrix} \right\}, \\ S_2 = \left\{ \begin{matrix} (a, 0.35,0.30,0.25), (b, 0.20,0.25,0.30), \\ (c, 0.20,0.40,0.15) \end{matrix} \right\}, \\ O_X \cup S_1 = S_3 = \left\{ \begin{matrix} (a, 0.35,0.00,0.25), (b, 0.20,0.00,0.30), \\ (c, 0.20,0.00,0.15) \end{matrix} \right\} \end{matrix} \right\}.$$

Next, we form a picture fuzzy topology of rank 3 on $X = \{a, b, c\}$ with the sub-base $\mathfrak{S} = \{S_1, S_2\}$ such that neither $S_1 \subset S_2$ nor $S_2 \subset S_1$.

**Example 7** Consider $\mathfrak{S} = \left\{ \begin{matrix} S_1 = \left\{ \begin{matrix} (a, 0.10,0.35,0.30), (b, 0.20,0.25,0.40), \\ (c, 0.50,0.40,0.05) \end{matrix} \right\}, \\ S_2 = \left\{ \begin{matrix} (a, 0.25,0.30,0.35), (b, 0.25,0.30,0.30), \\ (c, 0.30,0.35,0.10) \end{matrix} \right\} \end{matrix} \right\}$.

Then

$$\boldsymbol{\mathcal{B}} = \left\{ \begin{matrix} S_1 = \left\{ \begin{matrix} (a, 0.10,0.35,0.30), (b, 0.20,0.25,0.40), \\ (c, 0.50,0.40,0.05) \end{matrix} \right\}, \\ S_2 = \left\{ \begin{matrix} (a, 0.25,0.30,0.35), (b, 0.25,0.30,0.30), \\ (c, 0.30,0.35,0.10) \end{matrix} \right\}, \\ S_1 \cap S_2 = S_3 = \left\{ \begin{matrix} (a, 0.10,0.30,0.35), (b, 0.20,0.25,0.40), \\ (c, 0.30,0.35,0.10) \end{matrix} \right\} \end{matrix} \right\}$$

is the base designed from $\mathfrak{S}$, which further gives

$$\mathfrak{I} = \{I_X, O_X, S_1, S_2, S_3, \dots, S_{10}\}$$

where
$$I_X = \left\{ \begin{matrix} (a, 1.00,0.00,0.00), (b, 1.00,0.00,0.00), \\ (c, 1.00,0.00,0.00) \end{matrix} \right\};$$

$$O_X = \left\{ \begin{matrix} (a, 0.00,0.00,1.00), (b, 0.00,0.00,1.00), \\ (c, 0.00,0.00,1.00) \end{matrix} \right\};$$

$$S_1 = \left\{ \begin{matrix} (a, 0.10,0.35,0.30), (b, 0.20,0.25,0.40), \\ (c, 0.50,0.40,0.05) \end{matrix} \right\};$$

$$S_2 = \begin{Bmatrix} (a, 0.25, 0.30, 0.35), (b, 0.25, 0.30, 0.30), \\ (c, 0.30, 0.35, 0.10) \end{Bmatrix}$$

$$S_3 = S_1 \cap S_2 = \begin{Bmatrix} (a, 0.10, 0.30, 0.35), (b, 0.20, 0.25, 0.40), \\ (c, 0.30, 0.35, 0.10) \end{Bmatrix};$$

$$S_4 = S_1 \cup S_2 = \begin{Bmatrix} (a, 0.25, 0.30, 0.30), (b, 0.25, 0.25, 0.30), \\ (c, 0.50, 0.35, 0.05) \end{Bmatrix}$$

$$S_5 = S_1 \cup S_3 = \begin{Bmatrix} (a, 0.10, 0.30, 0.30), (b, 0.20, 0.25, 0.40), \\ (c, 0.50, 0.35, 0.05) \end{Bmatrix};$$

$$S_6 = S_2 \cup S_3 = \begin{Bmatrix} (a, 0.25, 0.30, 0.35), (b, 0.25, 0.25, 0.30), \\ (c, 0.30, 0.35, 0.10) \end{Bmatrix}$$

$$S_7 = O_X \cap S_1 = \begin{Bmatrix} (a, 0.10, 0.00, 0.30), (b, 0.20, 0.00, 0.40), \\ (c, 0.50, 0.00, 0.05) \end{Bmatrix};$$

$$S_8 = O_X \cap S_2 = \begin{Bmatrix} (a, 0.25, 0.00, 0.35), (b, 0.25, 0.00, 0.30), \\ (c, 0.30, 0.00, 0.10) \end{Bmatrix}$$

$$S_9 = O_X \cap S_3 = \begin{Bmatrix} (a, 0.10, 0.00, 0.35), (b, 0.20, 0.00, 0.40), \\ (c, 0.30, 0.00, 0.10) \end{Bmatrix};$$

$$S_{10} = O_X \cap S_4 = \begin{Bmatrix} (a, 0.25, 0.00, 0.30), (b, 0.25, 0.00, 0.30), \\ (c, 0.50, 0.00, 0.05) \end{Bmatrix}$$

*4.2. Condition for the smallest picture fuzzy topologies containing $n$ distinct elements*

In the case of a crisp set topology, $\{\emptyset, S_1, S_2, S_3, \dots, S_{n-2}, X\}$ is the smallest topology containing $S_1, S_2, S_3, \dots, S_{n-2}$ if $S_1 \subset S_2 \subset S_3 \subset \cdots \subset S_{n-2}$. It should be noted that the Pythagorean fuzzy topology follows the same pattern, that is, "$\{O_X, S_1, S_2, S_3, \dots, S_{n-2}, I_X\}$ is the smallest Pythagorean fuzzy topology containing $S_1, S_2, S_3, \dots, S_{n-2}$ if and only if $S_1 \subset S_2 \subset S_3 \subset \cdots \subset S_{n-2}$. It is also true for fuzzy topology and intuitionistic fuzzy topology because Pythagorean fuzzy topology is the generalization of both. But it is easy to verify that it does not hold for picture fuzzy topology of rank greater than one. The following theorem presents a condition for $\{O_X, S_1, S_2, S_3, \dots, S_{n-2}, I_X\}$ to be the smallest picture fuzzy topology containing $S_1, S_2, S_3, \dots, S_{n-2}$.

**Theorem 13** $\{O_X, S_1, S_2, S_3, \dots, S_{n-2}, I_X\}$ is the smallest picture fuzyy topology containing $S_1, S_2, S_3, \dots, S_{n-2}$ if $S_1 \equiv S_2 \equiv S_3 \equiv \cdots \equiv S_{n-2}$.

**Proof.** The proof is an immediate consequence of Theorem 2.

In the next example, we show that the converse of Theorem 13 does not hold. We construct a picture fuzzy topology of rank 3 with the sub-base $\mathfrak{S} = \{S_1, S_2, S_3, S_4\}$, where $S_1 \parallel S_2, S_1 \subseteq S_2$, $S_3 \parallel S_4, S_3 \subseteq S_4, S_2 \nparallel S_3, S_1 \equiv S_3$ and $S_2 \equiv S_4$.

**Example 8** Consider that

$$\mathfrak{S} = \begin{Bmatrix} S_1 = \{(a, 0.10, 0.15, 0.40), (b, 0.20, 0.10, 0.35), (c, 0.20, 0.15, 0.20)\}, \\ S_2 = \{(a, 0.30, 0.15, 0.35), (b, 0.25, 0.10, 0.30), (c, 0.30, 0.15, 0.10)\}, \\ S_3 = \{(a, 0.10, 0.10, 0.40), (b, 0.20, 0.05, 0.35), (c, 0.20, 0.15, 0.20)\}, \\ S_4 = \{(a, 0.30, 0.10, 0.35), (b, 0.25, 0.05, 0.30), (c, 0.30, 0.15, 0.10)\}, \end{Bmatrix};$$

then all finite intersections of the elements of $\mathfrak{S}$ evolve

$$\boldsymbol{\mathcal{B}} = \begin{Bmatrix} S_1 = \{(a, 0.10, 0.15, 0.40), (b, 0.20, 0.10, 0.35), (c, 0.20, 0.15, 0.20)\}, \\ S_2 = \{(a, 0.30, 0.15, 0.35), (b, 0.25, 0.10, 0.30), (c, 0.30, 0.15, 0.10)\}, \\ S_3 = \{(a, 0.10, 0.10, 0.40), (b, 0.20, 0.05, 0.35), (c, 0.20, 0.15, 0.20)\}, \\ S_4 = \{(a, 0.30, 0.10, 0.35), (b, 0.25, 0.05, 0.30), (c, 0.30, 0.15, 0.10)\}, \end{Bmatrix}.$$

The topology constructed from $\boldsymbol{\mathcal{B}}$ is

$$\mathfrak{J} = \begin{Bmatrix} I_X = \{(a, 1.00, 0.00, 0.00), (b, 1.00, 0.00, 0.00), (c, 1.00, 0.00, 0.00)\}, \\ O_X = \{(a, 0.00, 0.00, 1.00), (b, 0.00, 0.00, 1.00), (c, 0.00, 0.00, 1.00)\}, \\ S_1 = \{(a, 0.10, 0.15, 0.40), (b, 0.20, 0.10, 0.35), (c, 0.20, 0.15, 0.20)\}, \\ S_2 = \{(a, 0.30, 0.15, 0.35), (b, 0.25, 0.10, 0.30), (c, 0.30, 0.15, 0.10)\}, \\ S_3 = \{(a, 0.10, 0.10, 0.40), (b, 0.20, 0.05, 0.35), (c, 0.20, 0.15, 0.20)\}, \\ S_4 = \{(a, 0.30, 0.10, 0.35), (b, 0.25, 0.05, 0.30), (c, 0.30, 0.15, 0.10)\}, \end{Bmatrix}.$$

Clearly**,** $\mathfrak{J} = \{I_X, O_X, S_1, S_2, S_3, S_4\}$ is the smallest picture fuzzy topology containing $S_1, S_2, S_3$ and $S_4$ but $S_1 \equiv S_2 \equiv S_3 \equiv S_4$ is not true.

## 5. Continuity in picture fuzzy topological spaces

Continuity is one of the key characteristics of a function defined between two topological spaces. In this section, we firstly define the image and pre-image of picture fuzzy subsets of $X$ and $Y$ respectively, under some function $h: X \to Y$ and prove some basic properties related to it. Then, we introduce the notion of a picture fuzzy continuous function and develop a necessary and sufficient condition for picture fuzzy continuous functions between two picture fuzzy topological spaces.

**Definition 8** Let $h: X \to Y$ be a function and $\aleph \in \Gamma_X$ and $\mathcal{K} \in \Gamma_Y$. The image of $\aleph$ under $h$ is denoted by $h(\aleph)$ such that $h(\aleph) = \{y, \mu_{h(\aleph)}(y), \rho_{h(\aleph)}(y), \sigma_{h(\aleph)}(y) : y \in Y\}$, where

$$\mu_{h(\aleph)}(y) = \begin{cases} \text{Sup}\,\{\mu_\aleph(z) : z \in h^{-1}(y)\}, & if\ h^{-1}(y) \neq \varphi \\ 0, & if\ h^{-1}(y) = \varphi \end{cases},$$

$$\rho_{h(\aleph)}(y) = \begin{cases} \text{Inf}\,\{\rho_\aleph(z) : z \in h^{-1}(y)\}, & if\ h^{-1}(y) \neq \varphi \\ 0, & if\ h^{-1}(y) = \varphi \end{cases}$$

and

$$\sigma_{h(\aleph)}(y) = \begin{cases} \text{Inf}\,\{\sigma_\aleph(z) : z \in h^{-1}(y)\}, & if\ h^{-1}(y) \neq \varphi \\ 1, & if\ h^{-1}(y) = \varphi \end{cases}.$$

The pre-image of $\mathcal{K}$ under $h$ is

$$h^{-1}(\mathcal{K}) = \{x, \mu_{h^{-1}(\mathcal{K})}(x), \rho_{h^{-1}(\mathcal{K})}(x), \sigma_{h^{-1}(\mathcal{K})}(x) : x \in X\},$$

where

$$\mu_{h^{-1}(\mathcal{K})}(x) = \mu_{\mathcal{K}}(h(x)),\ \rho_{h^{-1}(\mathcal{K})}(x) = \rho_{\mathcal{K}}(h(x)) \text{ and } \sigma_{h^{-1}(\mathcal{K})}(x) = \sigma_{\mathcal{K}}(h(x)).$$

It is easy to verify that both $h(\aleph)$ and $h^{-1}(\mathcal{K})$ are picture fuzzy subsets of $X$ and $Y$ respectively.

In the following theorem, we present some fundamental results related to the image and pre-image.

**Theorem 14** Let $h: X \to Y$ be a function, then

i) $h^{-1}(\mathcal{K}^c) = [h^{-1}(\mathcal{K})]^c$ for all $\mathcal{K} \in \Gamma_Y$.

ii) $[h(\aleph)]^c \subseteq h(\aleph^c)$ for all $\aleph \in \Gamma_X$.

iii) For all $\mathcal{K}_1, \mathcal{K}_2 \in \Gamma_Y$, $\mathcal{K}_1 \subseteq \mathcal{K}_2$ implies $h^{-1}(\mathcal{K}_1) \subseteq h^{-1}(\mathcal{K}_2)$

iv) For all $\aleph_1, \aleph_2 \in \Gamma_X$, $\aleph_1 \subseteq \aleph_2$ implies $h(\aleph_1) \subseteq h(\aleph_2)$

v) $h\left(h^{-1}(\mathcal{K})\right) \subseteq \mathcal{K}$ for all $\mathcal{K} \in \Gamma_Y$.

vi) For all $\aleph \in \Gamma_X, \aleph \subseteq h^{-1}\left(h(\aleph)\right)$ if $h$ is injective. (Note that in case of crisp, fuzzy and intuitionistic sets $\aleph \subseteq h^{-1}\left(h(\aleph)\right)$ for all $h$.

**Proof.** i). Suppose $x \in X$, then for any $\mathcal{K} \in \Gamma_Y$, we have

$$\mu_{h^{-1}(\mathcal{K}^c)}(x) = \mu_{\mathcal{K}^c}\left(h(x)\right) = \sigma_{\mathcal{K}}\left(h(x)\right) = \sigma_{h^{-1}(\mathcal{K})}(x) = \mu_{[h^{-1}(\mathcal{K})]^c}(x)$$

Similarly, we can show that

$$\rho_{h^{-1}(\mathcal{K}^c)}(x) = \rho_{[h^{-1}(\mathcal{K})]^c}(x) \text{ and } \sigma_{h^{-1}(\mathcal{K}^c)}(x) = \sigma_{[h^{-1}(\mathcal{K})]^c}(x).$$

ii). Let $y \in Y$ such that $h^{-1}(y) \neq \emptyset$, then for $\aleph \in \Gamma_X$, we have

$$\mu_{[h(\aleph)]^c}(y) = \sigma_{h(\aleph)}(y) = \operatorname{Inf}\{\sigma_{\aleph}(z): z \in h^{-1}(y)\}$$

and

$$\mu_{h(\aleph^c)}(y) = \operatorname{Sup}\{\mu_{\aleph^c}(z): z \in h^{-1}(y)\} = \operatorname{Sup}\{\sigma_{\aleph}(z): z \in h^{-1}(y)\}$$

Therefore,

$$\mu_{[h(\aleph)]^c}(y) \leq \mu_{h(\aleph^c)}(y) \text{ for all } y \in Y \text{ such that } h^{-1}(y) \neq \emptyset.$$

Also clearly $\mu_{[h\aleph]^c}(v) = \mu_{h(\mathcal{F}^c)}(y) = 0$ for all $y \in Y$ such that $h^{-1}(y) = \emptyset$.
Similarly, it can be proved that

$$\rho_{[h(\aleph)]^c}(y) = \rho_{h(\aleph^c)}(y) \text{ and } \sigma_{[h(\aleph)]^c}(y) \geq \sigma_{h(\aleph^c)}(y) \text{ for all } y \in Y.$$

Thus, $[h(\aleph)]^c \subseteq h(\aleph^c)$.
iii) Let $\mathcal{K}_1, \mathcal{K}_2 \in \Gamma_Y$ such that $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Then for $x \in X$, we have

$$\begin{aligned}\mu_{h^{-1}(\mathcal{K}_1)}(x) &= \mu_{\mathcal{K}_1}\left(h(x)\right)\\ &\leq \mu_{\mathcal{K}_2}\left(h(x)\right)\\ &= \mu_{h^{-1}(\mathcal{K}_2)}(x)\end{aligned}$$

In a similar fashion, it can be shown that

$$\rho_{h^{-1}(\mathcal{K}_1)}(x) \leq \rho_{h^{-1}(\mathcal{K}_2)}(x) \text{ and } \sigma_{h^{-1}(\mathcal{K}_1)}(x) \geq \sigma_{h^{-1}(\mathcal{K}_2)}(x).$$

Thus, $h^{-1}(\mathcal{K}_1) \subseteq h^{-1}(\mathcal{K}_2)$.
iv). Let $\aleph_1, \aleph_2 \in \Gamma_X$ such that $\aleph_1 \subseteq \aleph_2$.
Suppose $y \in Y$ and $h^{-1}(y) \neq \emptyset$, then

$$\begin{aligned}\mu_{h(\aleph_1)}(y) &= \operatorname{Sup}\left\{\mu_{\aleph_1}(z): z \in h^{-1}(y)\right\}\\ &\leq \operatorname{Sup}\left\{\mu_{\aleph_2}(z): z \in h^{-1}(y)\right\}\\ &= \mu_{h(\aleph_2)}(y)\end{aligned}$$

Therefore, $\mu_{h(\aleph_1)}(y) \leq \mu_{h(\aleph_2)}(y)$ if $h^{-1}(y) \neq \emptyset$,.
Clearly, $\mu_{h(\aleph_1)}(y) = \mu_{h(\aleph_2)}(y) = 0$ for $y \in Y$ and $h^{-1}(y) = \emptyset$. Thus, $\mu_{h(\aleph_1)}(y) \leq \mu_{h(\aleph_2)}(y)$ for all $y \in Y$.
Similarly, we can show that

$$\rho_{h(\aleph_1)}(y) \leq \rho_{h(\aleph_2)}(y) \text{ and } \sigma_{h(\aleph_1)}(y) \geq \sigma_{h(\aleph_2)}(y) \text{ for all } y \in Y.$$

Hence, $h(\aleph_1) \subseteq h(\aleph_2)$.

v). Let $\mathcal{K} \in \Gamma_Y$ and $y \in Y$ such that $h^{-1}(y) = \emptyset$, then

$$\mu_{h\left(h^{-1}(\mathcal{K})\right)}(y) = \text{Sup}\left\{\mu_{h^{-1}(\mathcal{K})}(z): z \in h^{-1}(y)\right\}$$

$$= \text{Sup}\left\{\mu_{\mathcal{K}}\big(h(z)\big): z \in h^{-1}(y)\right\}$$

$$= \mu_{\mathcal{K}}(y)$$

Also if $h^{-1}(y) = \emptyset$, then $\mu_{h\left(h^{-1}(\mathcal{K})\right)}(y) = 0 \leq \mu_{\mathcal{K}}(y)$.
Similarly, we can show that $\rho_{h\left(h^{-1}(\mathcal{K})\right)}(y) \leq \rho_{\mathcal{K}}(y)$ and $\sigma_{h\left(h^{-1}(\mathcal{K})\right)}(y) \geq \sigma_{\mathcal{K}}(y)$ for all $y \in Y$.
Thus, $h\big(h^{-1}(\mathcal{K})\big) \subseteq \mathcal{K}$.
vi). Let $h : X \to Y$ be an injective function and $\aleph \in \Gamma_X$ , then for all $x \in X$

$$\rho_{h^{-1}(h(\aleph))}(x) = \rho_{h(\aleph)}\big(h(x)\big)$$

$$= \text{Inf}\big\{\rho_{\aleph}(z): z \in h^{-1}\big(h(x)\big)\big\}$$

$$= \rho_{\aleph}(x) \text{ (Since } h \text{ is injective function)}$$

Similarly, we can prove $\mu_{h^{-1}(h(\aleph))}(x) \geq \mu_{\aleph}(x)$ and $\sigma_{h^{-1}(h(\aleph))}(x) \leq \sigma_{\aleph}(x)$.
Thus, $\aleph \subseteq h^{-1}\big(h(\aleph)\big)$.

Next, we define the neighbourhood of a picture fuzzy subset. Instead of the neighbourhood of a fuzzy point, Chang [22] defined the notion of neighbourhood of a fuzzy subset. We also follow the idea presented by Chang.

**Definition 9**. Let $(X, \mathcal{T})$ be a topological space and $\aleph \in \Gamma_X$. Then, $\mathcal{A} \in \Gamma_X$ is called a neighbourhood of $\aleph$ if $\aleph \subseteq S \subseteq \mathcal{A}$ for some $S \in \mathcal{T}$.

The following proposition is easy to prove.

**Proposition 1** Let$(X, \mathcal{T})$ be a picture fuzzy topological space. Then $S \in \mathcal{T}$ if and only if for each $\aleph \subseteq S$, there exists a neighbourhood $\mathcal{A}$ of $\aleph$ contained in $S$.

We now define the notion of a picture fuzzy continuous function between two picture fuzzy topological spaces.

**Definition 10**. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two picture fuzzy topological spaces and $\aleph \in \Gamma_X$. A function $h : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called a picture fuzzy continuous function if for each neithbourhood $\mathcal{B}$ of $h(\aleph)$ there exists a neighbourhood $\mathcal{A}$ of $\aleph$ such that $h(\mathcal{A}) \subseteq \mathcal{B}$.

The following theorem shows that no non-injective picture fuzzy continuous function exists.

**Theorem 15.** The continuity of $h: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ implies the injection of $h: X \to Y$.

**Proof.** Suppose $h$ is non-injective function, then $h(x_1) = h(x_2)$ for $x_1 \neq x_2$. Consider $\aleph \in \Gamma_X$ such that

a) $\mu_{\aleph}(x_i) = 1$ and $\rho_{\aleph}(x_i) = 0$ for all $x_i \in X - \{x_1\}$

b) $\sigma_{\aleph}(x_j) = 0$ for all $x_j \in X$

c) $\mu_{\aleph}(x_1) + \rho_{\aleph}(x_1) = 1$ and $\mu_{\aleph}(x_1), \rho_{\aleph}(x_1) \in (0,1)$.

d) $\rho_{\aleph}(x_1)$ is greater than every $S \in \mathcal{T}_1$ such that $\rho_S(x_1) < 1$.

Then $\aleph$ is not contained in any $S \in \mathcal{T}_1$ for which $\rho_S(x_1) < 1$. If $S' \in \mathcal{T}_1$ such that $\rho_{S'}(x_1) = 1$, then obviously $\mu_{S'}(x_1) = 0$ and $\mu_{\aleph}(x_1) > 0$. Therefore, $\aleph$ is not contained in any $S' \in \mathcal{T}_1$ such that $\rho_{S'}(x_1) = 1$. Moreover, $I_Y$ is a neighbourhood of $h(\aleph)$. But, there does not exist any neighbourhood of $\aleph$. This contradicts the continuity of $h$. Hence, $h$ is injective.

**Theorem 16.** Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two picture fuzzy topological spaces. Then, the following statements are equivalent.

(i) $h: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is a picture fuzzy continuous function.

(ii) For all $\aleph \in \Gamma_X$and each neighbourhood $\mathcal{B}$ of $h(\aleph)$, there exists a neighbourhood $\mathcal{A}$ of $\aleph$ such that $h(\aleph') \subseteq \mathcal{B}$ for all $\aleph' \in \Gamma_X$ contained in $\mathcal{A}$.

(iii) For all $\aleph \in \Gamma_X$ and each neighbourhood $\mathcal{B}$ of $h(\aleph)$ there exists a neighbourhood $\mathcal{A}$ of $\aleph$ such that $\aleph' \subseteq h^{-1}(\mathcal{B})$ for all $\aleph' \in \Gamma_X$ contained in $\mathcal{A}$; hence $\mathcal{A} \subseteq h^{-1}(\mathcal{B})$.

(iv) For all $\aleph \in \Gamma_X$ and each neighbourhood $\mathcal{B}$ of $h(\aleph)$, $h^{-1}(\mathcal{B})$ is a neighbourhood of $\aleph$.

**Proof** (i) $\Rightarrow$ (ii) Let (i) hold and suppose that $\aleph \in \Gamma_X$ and $\mathcal{B}$ is a neighbourhood of $h(\aleph)$. Then there exists a neighbourhood $\mathcal{A}$ of $\aleph$ such that $h(\mathcal{A}) \subseteq \mathcal{B}$. Suppose that $\aleph' \in \Gamma_X$ is contained in $\mathcal{A}$, then by (iv) of Theorem 14 and our assumption, we have $h(\aleph') \subseteq h(\mathcal{A}) \subseteq \mathcal{B}$.

(ii)$\Rightarrow$ (iii) Suppose that (ii) is true. Since $h$ is injective, the application of Theorem 14 (vi) gives $\aleph' \subseteq h^{-1}(h(\aleph'))$. By our assumption, we have $h(\aleph') \subseteq \mathcal{B}$, then the application of (iii) of Theorem 14 yields $h^{-1}(h(\aleph')) \subseteq h^{-1}(\mathcal{B})$. Thus, ultimately $\aleph' \subseteq h^{-1}(h(\aleph'))$ and $h^{-1}(h(\aleph')) \subseteq h^{-1}(\mathcal{B})$ together imply $\aleph' \subseteq h^{-1}(\mathcal{B})$. Since $\aleph'$ was taken to be arbitrary, every picture fuzzy subset of $X$ that is contained in $\mathcal{A}$ is contained in $h^{-1}(\mathcal{B})$ as well. Hence, $\mathcal{A} \subseteq h^{-1}(\mathcal{B})$.

(iii) $\Rightarrow$ ( iv) Suppose that for all $\aleph \in \Gamma_X$ and each neighbourhood $\mathcal{B}$ of $h(\aleph)$ there exists a neighbourhood $\mathcal{A}$ of $\aleph$ such that $\mathcal{A} \subseteq h^{-1}(\mathcal{B})$. By the definition of neighbourhood, we have $\aleph \subseteq S \subseteq \mathcal{A}$ for some $S \in \mathcal{T}_1$. Thus, ultimately $\aleph \subseteq S \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq h^{-1}(\mathcal{B})$ together imply $h^{-1}(\mathcal{B})$ is a neighbourhood of $\aleph$.

(iv)$\Rightarrow$ (i) Assume that all $\aleph \in \Gamma_X$ and each neighbourhood $\mathcal{B}$ of $h(\aleph)$, $h^{-1}(\mathcal{B})$ is a neighbourhood of $\aleph$. Therefore, $\aleph \subseteq S \subseteq h^{-1}(\mathcal{B})$ for some $S \in \mathcal{T}_1$. By using (v) of Theorem 14, we get $h(h^{-1}(\mathcal{B})) \subseteq \mathcal{B}$, implying that $h$ is a pciture fuzzy continuous function.

The following theorem describes a necessary and sufficient condition for picture fuzzy continuous functions between two picture fuzzy topological spaces.

**Theorem 17** Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two picture fuzzy topological spaces. Then $h: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is picture fuzzy continuous if and only if for every $T \in \mathcal{T}_2$, $h^{-1}(T) \in \mathcal{T}_1$ and $h$ is injective.

**Proof.** Suppose that $h: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is a picture fuzzy continuous, then by Theorem 15, we obtain $h$ is injective. Let $T \in \mathcal{T}_2$, we want to show that $h^{-1}(T) \in \mathcal{T}_1$. For this, let $\aleph \subseteq h^{-1}(T)$, then (iv) and (v) of Theorem 14 together imply that $h(\aleph) \subseteq h(h^{-1}(T)) \subseteq T$. Since $T \in \mathcal{T}_2$, therefore Proposition 1 guarantees that $h(\aleph)$ has a neighbourhood $\mathcal{B}$ such that $\mathcal{B} \subseteq T$. The hypothesis allows us to apply (iv) of Theorem 16, which gives that $h^{-1}(\mathcal{B})$ is a neighbourhood of $\aleph$. Moreover, (iii) of Theorem 16 yields that $h^{-1}(\mathcal{B}) \subseteq h^{-1}(T)$. Lastly, by the application of Theorem 15, we conclude that $h^{-1}(T) \in \mathcal{T}_1$.

Conversely, suppose that for every $T \in \mathcal{T}_2$, $h^{-1}(T) \in \mathcal{T}_1$. Let $\aleph \in \Gamma_X$ and $\mathcal{B}$ be a neighbourhood of $h(\aleph)$. Then $h(\aleph) \subseteq T' \subseteq \mathcal{B}$ for some $T' \in \mathcal{T}_2$. By assumption $h^{-1}(T') \in \mathcal{T}_1$ and the application of Theorem 14 yields $\aleph \subseteq h^{-1}\big(h(\aleph)\big) \subseteq h^{-1}(T') \subseteq h^{-1}(\mathcal{B})$. This means that $h^{-1}(\mathcal{B})$ is a neighbourhoof of $\aleph$ which further implies that $h$ is a picture fuzzy continuous function.

The following theorem shows that we can generate a picture fuzzy topology on $X$ if there exists a picture fuzzy topology on $Y$ and a function $h: X \to Y$. The topology generated in this way is the weakest topology on $X$ for which $h$ is a picture fuzzy continuous.

**Theorem 18** Let $h: X \to Y$ be an injective function and $\mathcal{T}_2$ be a picture fuzzy topology on $Y$. Then $h^{-1}(\mathcal{T}_2) = \{h^{-1}(T_i): T_i \in \mathcal{T}_2\}$ forms the weakest picture fuzzy topology among those picture fuzzy topologies on $X$ for which $h$ is a picture fuzzy continuous function.

**Proof.** First, we show that $h^{-1}(\mathcal{T}_2) = \{h^{-1}(T_i): T_i \in \mathcal{T}_2\}$ is a picture fuzzy topology on $X$.

(i) Let $S_1, S_2 \in h^{-1}(\mathcal{T}_2)$, then $S_1 = h^{-1}(T_1)$ and $S_2 = h^{-1}(T_2)$ for some $T_1, T_2 \in \mathcal{T}_2$.

For $x \in X$, we have

$$\mu_{S_1 \cap S_2}(x) = min\{\mu_{S_1}(x), \mu_{S_2}(x)\} = min\{\mu_{h^{-1}(T_1)}(x), \mu_{h^{-1}(T_2)}(x)\}$$
$$= min\{\mu_{T_1}\big(h(x)\big), \mu_{T_2}\big(h(x)\big)\} = \mu_{T_1 \cap T_2}\big(h(x)\big) = \mu_{h^{-1}(T_1 \cap T_2)}(x).$$

Similarly, we can acquire $\rho_{S_1 \cap S_2}(x) = \rho_{h^{-1}(T_1 \cap T_2)}(x)$ and $\sigma_{S_1 \cap S_2}(x) = \sigma_{h^{-1}(T_1 \cap T_2)}(x)$.

Thus, $S_1 \cap S_2 \in h^{-1}(\mathcal{T}_2)$.

(ii) Let $\{S_i\}_{i \in \Omega} \subseteq h^{-1}(\mathcal{T}_2)$, then for all $\in \Omega$, there exists $T_i \in \mathcal{T}_2$ such that $S_i = h^{-1}(T_i)$.

For $x \in X$, we have

$$\mu_{\cup_{i \in \Omega} S_i}(x) = \sup_{i \in \Omega}\{\mu_{S_i}(x)\}$$
$$= \sup_{i \in \Omega}\{\mu_{h^{-1}(T_i)}(x)\}$$
$$= \sup_{i \in \Omega}\{\mu_{T_i}\big(h(x)\big)\}$$
$$= \mu_{\cup_{i \in \Omega} T_i}\big(h(x)\big)$$
$$= \mu_{h^{-1}(\cup_{i \in \Omega} T_i)}(x).$$

Similarly, it can be proved that

$$\rho_{\cup_{i \in \Omega} S_i}(x) = \rho_{h^{-1}(\cup_{i \in \Omega} T_i)}(x) \text{ and } \sigma_{\cup_{i \in \Omega} S_i}(x) = \sigma_{h^{-1}(\cup_{i \in \Omega} T_i)}(x).$$

Thus, we obtain $\cup_{i \in \Omega} S_i \in h^{-1}(\mathcal{T}_2)$.

(iii) It is easy to prove that $h^{-1}(O_Y) = O_X$ and $h^{-1}(I_Y) = I_X$. Thus, $O_X, I_X \in h^{-1}(\mathcal{T}_2)$.

By Theorem 17, we have

$$h: \big(X, h^{-1}(\mathcal{T}_2)\big) \to (Y, \mathcal{T}_2) \text{ is a picture fuzzy continuous function.}$$

Assume that $h: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is also a picture fuzzy continuous function. Suppose $S_j \in h^{-1}(\mathcal{T}_2)$, then $S_j = h^{-1}\big(T_j\big)$ for some $T_j \in \mathcal{T}_2$. Then, the application of Theorem 17 yields $S_j \in \mathcal{T}_1$. Hence, $h^{-1}(\mathcal{T}_2) \subseteq \mathcal{T}_1$.

Since $\mathcal{T}_2$ is a picture fuzzy topology on $Y$, therefore $T_1 \cap T_2 \in \mathcal{T}_2$

## 6. Conclusions

In the present study, we extended the notion of topological spaces and its properties to the picture fuzzy set. In it, we defined the rank of the picture fuzzy topology, picture fuzzy base and picture fuzzy sub-base. Some examples are also provided to state their concept. Based on the concept of the fuzzy sub-base and base, we developed a technique to form picture fuzzy topological spaces. The desirable properties of the topological space are derived in details. Later on, we stated the concept of the picture fuzzy continuous function and derived the necessary and sufficient conditions for a function to become a picture fuzzy continuous function between the two-picture fuzzy topological spaces. In future research work, we will extend the work to some other fuzzy environment and derive some more properties for different topological spaces. Further, we will expand our work to consider the topological space to be compact as well as the corresponding results.

## Conflicts of interest

The authors declare no conflicts of interest regarding the publication of this article.

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