AIMS Mathematics

AIMS Mathematics, 7(8): 14756–14784.
DOI: 10.3934/math.2022811
Received: 04 April 2022
Revised: 20 May 2022
Accepted: 20 May 2022
Published: 08 June 2022

http://www.aimspress.com/journal/Math

*Research article*

# Some fixed point results for $\xi$-chainable neutrosophic and generalized neutrosophic cone metric spaces with application

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**Abstract:** This manuscript is concerned for introducing novel concepts of $\xi$-chainable neutrosophic metric space and generalized neutrosophic cone metric spaces. We use four self-mappings to establish common fixed point theorem in the sense of $\xi$-chainable neutrosophic metric space and three self-mappings to establish common fixed point results in the sense of generalized neutrosophic metric spaces. Certain properties of $\xi$-chainable neutrosophic metric space and generalized neutrosophic metric spaces are defined and their examples are presented. An application to fuzzy Fredholm integral equation of second kind is developed to verify the validity of proposed results. These results boost the approaches of existing literature of fuzzy metric spaces and fuzzy fixed theory.

**Keywords:** cone metric space; intuitionistic generalized fuzzy cone metric space; generalized neutrosophic metric spaces; $\xi$-chainable neutrosophic metric space; fixed point
**Mathematics Subject Classification:** 47H10, 54H25

## 1. Introduction

In the field of fixed point theory, the notion of metric spaces (MS) and the Banach contraction principle play crucial roles. The axiomatic clarity of MS attracts many researchers. There have been a lot of generalizations to the MS so far. This demonstrates the allurement and scope of the definition of the MS. The notion of cone metric space (CMS) was proposed by Haung and Zhang [1] and to investigate some fixed point results for contractive mappings. The classical techniques cannot deal with uncertain analysis problems. To deal with such problems, Zadeh [2] introduced the notion of fuzzy sets (FSs). This concept succeeded in shifting a lot of mathematical structures from crisp set theory to fuzzy set theory. In this connectedness, Kramosil and Michalek [3] introduced the notion of fuzzy metric spaces (FMS), in which they used the idea of continuous triangular norms (CTN) and Garbiec [4] gave the fuzzy interpretation of Banach contraction principle in FMS. It deals with membership function only in the case of FMS. To generalize the idea of FMS, the notion of intuitionistic fuzzy metric spaces (IFMS) by using the idea of CTN and continuous triangular conorm (CTCN) was initiated by Park [5] that deals with membership and non-membership functions. The concept of intuitionistic fuzzy sets is initiated by Atanassov [6]. Recently, Kirisci [7] generalized the concept of IFMS by using the idea of neutrosophic sets (NSs), which was given by Smarandache [8] and coined the notion of neutrosophic metric space (NMS), in which we deal with membership (truthiness), indeterminacy (naturalness) and non-membership (falsity) functions. Riaz and Hashmi [18] gave fixed points of fuzzy neutrosophic soft mapping with decision-making.

Combining the idea of CMS and FSs, Oner [11] coined the notion of fuzzy cone metric space and proved fuzzy cone Banach contraction principle. Ali [12] proposed the notion of intuitionistic fuzzy cone metric space. Recently, Jeyaraman [13] tossed the notion of intuitionistic generalized fuzzy cone metric space (IGFCMS) and proved some common fixed point results and Ali [14] introduced the idea of $\epsilon$-chainable intuitionistic fuzzy metric spaces. Congxin [20] investigates the fuzzy Fredholm integral equation of second kind. For further interesting notions of fuzzy fixed point theory, see [9,10,15–17,19,23–28]. Al-Omeri et al. [21] introduced neutrosophic fixed point theorems and cone metric spaces. Sowndrara et al. [22] proposed fixed point results for contraction theorems in neutrosophic metric spaces.

Rasham et al. [23–28] introduced several interesting results for fixed point theory using several mappings. Riaz et al. [29,30] introduced the novel concepts of linear Diophantine fuzzy sets and spherical linear Diophantine fuzzy sets. Sitara et al. [31] established decision-making analysis based on q-rung picture fuzzy graph structures. Akram et al. [32] proposed new decision-making approach under complex spherical fuzzy prioritized weighted aggregation operators. Ashraf and Abdullah [33] introduced spherical aggregation operators and their application in multi-attribute group decision-making. Liu et al. [34] introduced cosine similarity measures and distance measures between complex q-rung orthopair fuzzy sets. Riaz et al. [34] introduced multi-criteria decision making based on bipolar picture fuzzy operators and new distance measures. Al-Omeri et al. [35] proved several interesting fixed point results in generalization of neutrosophic metric spaces. Ishtiaq et al. [36] introduced the notion of orthogonal neutrosophic metric spaces and established some fixed point results. Also, we refer [38–42] for more detail.

Main objectives of the manuscript are as follows:

1) To introduce novel concepts of $\xi$-chainable neutrosophic metric space and generalized neutrosophic cone metric spaces.
2) To prove fixed point theorem for four self-mappings in the sense of $\xi$-chainable neutrosophic metric space.
3) To prove fixed point results for three self-mappings in the sense of generalized neutrosophic metric spaces.

4) Application to fuzzy Fredholm integral equation of second kind is given to verify the validity of proposed fixed point results.
5) To enhance existing literature of fuzzy metric spaces and fuzzy fixed theory.

This manuscript is organized as follows. In Section 2, some rudimentary concepts of CTN, CTCN, cone, cone metric space, NSs, IFMS, and IFGCMS are presented. In Section 3, the notion of GNCMS is introduced and some fixed point theorems are established using three self-mappings in the sense of GNCMS and non-trivial examples also are presented. In Section 4, the notion of $\xi$-chainable NMS is introduced and fixed point theorems are established by using four self-mappings. Lastly, the conclusion is given in Section 5.

## 2. Preliminaries

In this section, we review some elementary concepts including, CTN, CTCN, cone, cone metric space, NSs, IFMS, and IFGCMS. These concepts are essential for the analysis in the whole manuscript.

**Definition 2.1.** [5] A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous triangular norm (CTN) if:

A. $\pi * \mu = \mu * \pi$, for all, $\pi, \mu \in [0, 1]$;
B. $*$ is continuous;
C. $\pi * 1 = \pi$, for all, $\pi \in [0, 1]$;
D. $(\pi * \mu) * \rho = \pi * (\mu * \rho)$, for all, $\pi, \mu, \rho \in [0, 1]$;
E. If $\pi \leq \rho$ and $\mu \leq \sigma$, with $\pi, \mu, \rho, \sigma \in [0, 1]$, then $\pi * \mu \leq \rho * \sigma$.

**Example 2.2.** [5,16] Some fundamental examples of continuous triangular norms (CTNs) are: $\pi * \mu = \pi \cdot \mu, \pi * \mu = \min\{\pi, \mu\}$ and $\pi * \mu = \max\{\pi + \mu - 1, 0\}$.

**Definition 2.3.** [5] A binary operation $\circ$ : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous triangular conorm (CTCN) if it meets the below assertions:

a. $\pi \circ \mu = \mu \circ \pi$, for all, $\pi, \mu \in [0, 1]$;
b. $\circ$ is continuous;
c. $\pi \circ 0 = 0$, for all, $\pi \in [0, 1]$;
d. $(\pi \circ \mu) \circ \rho = \pi \circ (\mu \circ \rho)$, for all, $\pi, \mu, \rho \in [0, 1]$;
e. If $\pi \leq \rho$ and $\mu \leq \sigma$, with $\pi, \mu, \rho, \sigma \in [0, 1]$, then $\pi \circ \mu \leq \rho \circ \sigma$.

**Example 2.4.** [5] $\pi \circ \mu = \max\{\pi, \mu\}$ and $\pi \circ \mu = \min\{\pi + \mu, 1\}$ are examples of CTCNs.

**Definition 2.5.** [1] Let a real Banach space $X$ and a subset $P$ of $X$, $P$ is named to be cone if and only if

c1) $P \neq \emptyset, P \neq \{0\}$, and $P$ is closed,
c2) $a, b \in R, a, b \geq 0, \vartheta, \zeta \in P, a\vartheta + b\zeta \in P$,
c3) $\vartheta \in P$ and $-\vartheta \in P$ implies $\vartheta = 0$.

**Definition 2.6.** [1] Let a set $X \neq \emptyset$ and assume that a mapping $\delta: X \times X \rightarrow Y$ ($Y$ is a Banach space) fulfills:

1) $\delta(\vartheta, \zeta) \succ 0$ for all $\vartheta, \zeta \in X$ and $\delta(\vartheta, \zeta) = 0$ if and only if $\vartheta = \zeta$;
2) $\delta(\vartheta, \zeta) = \delta(\zeta, \vartheta)$ for all $\vartheta, \zeta \in X$;
3) $\delta(\vartheta, \zeta) \preccurlyeq \delta(\vartheta, \lambda) + \delta(\lambda, \zeta)$ for all $\vartheta, \zeta, \lambda \in X$.

Then $\delta$ is called a cone metric on $X$, and $(X, \delta)$ is called a cone metric space.

**Definition 2.7.** [15] Let a set $X \neq \emptyset$ and $\vartheta \in X$. A neutrosophic set (NS) $G$ in $X$ is categorized by three components:

(i) truth-membership function $M_G(\vartheta)$,
(ii) indeterminacy-membership function $N_G(\vartheta)$,
(iii) falsity-membership function $O_G(\vartheta)$.

Functions $M_G(\vartheta)$, $N_G(\vartheta)$ and $O_G(\vartheta)$ are real standard or non-standard subsets of $]0^-,1^+[$, that is, $M_G(\vartheta): X \to ]0^-,1^+[$, $N_G(\vartheta): X \to ]0^-,1^+[$ and $O_G(\vartheta): X \to ]0^-,1^+[$ such that

$$0^- \leq \sup M_G(\vartheta) + \sup N_G(\vartheta) + \sup O_G(\vartheta) \leq 3^+.$$

**Definition 2.8.** [5] Take $X \neq \emptyset$. Let $*$ be a CTN, $\circ$ be a CTCN, and $F, V$ be FSs on $X \times X \times (0, +\infty)$. If $(X, F, V, *, \circ)$ verifies the following for all $\vartheta, \zeta \in X$ and $s, t > 0$:

I. $F(\vartheta, \zeta, t) + V(\vartheta, \zeta, t) \leq 1$;
II. $F(\vartheta, \zeta, t) > 0$;
III. $F(\vartheta, \zeta, t) = 1 \text{ if and only if } \vartheta = \zeta$;
IV. $F(\vartheta, \zeta, t) = F(\zeta, \vartheta, t)$;
V. $F(\vartheta, \lambda, t + s) \geq F(\vartheta, \zeta, t) * F(\zeta, \lambda, s)$;
VI. $F(\vartheta, \zeta, \cdot): (0, +\infty) \to [0,1]$ is continuous and $\lim_{t \to +\infty} F(\vartheta, \zeta, t) = 1$ for all $t > 0$;
VII. $V(\vartheta, \zeta, t) > 0$;
VIII. $V(\vartheta, \zeta, t) = 0 \text{ if and only if } \vartheta = \zeta$;
IX. $V(\vartheta, \zeta, t) = V(\zeta, \vartheta, t)$;
X. $V(\vartheta, \lambda, t + s) \leq V(\vartheta, \zeta, t) \circ N(\zeta, \lambda, s)$;
XI. $V(\vartheta, \zeta, \cdot): (0, +\infty) \to [0,1]$ is continuous and $\lim_{t \to +\infty} V(\vartheta, \zeta, t) = 0$ for all $t > 0$;

then $(X, F, V, *, \circ)$ is an intuitionistic fuzzy metric space (IFMS).

**Definition 2.9.** [13] An intuitionistic generalized fuzzy cone metric space (IGFCMS) is a 6-tuple $(X, D, U, *, \circ, C)$ where $X$ is a arbitrary set, $*$ is a CTN $\circ$ is a CTCN, $C$ is a closed cone and $D$ and $U$ are NSs in $X^3 \times \text{int}(C)$ fulfilling the following circumstances:
for all $\vartheta, \zeta, \lambda, u \in X$ and $\boldsymbol{c}, \boldsymbol{c}' \in \text{int}(C)$,

F1. $D(\vartheta, \zeta, \lambda, \boldsymbol{c}) + U(\vartheta, \zeta, \lambda, \boldsymbol{c}) \leq 1$,
F2. $D(\vartheta, \zeta, \lambda, \boldsymbol{c}) > 0$,
F3. $D(\vartheta, \zeta, \lambda, \boldsymbol{c}) = 1$ if and only if $\vartheta = \zeta = \lambda$,
F4. $D(\vartheta, \zeta, \lambda, \boldsymbol{c}) = D(p\{\vartheta, \zeta, \lambda\}, c)$ where $p$ is a permutation function,
F5. $D(\vartheta, \zeta, \lambda, \boldsymbol{c} + \boldsymbol{c}') \geq D(\vartheta, \zeta, u, \boldsymbol{c}) * D(\vartheta, \lambda, \lambda, \boldsymbol{c}')$,
F6. $D(\vartheta, \zeta, \lambda, \cdot): \text{int}(C) \to [0, 1]$ is continuous,
F7. $U(\vartheta, \zeta, \lambda, \boldsymbol{c}) < 1$,
F8. $U(\vartheta, \zeta, \lambda, \boldsymbol{c}) = 1$ if and only if $\vartheta = \zeta = \lambda$,
F9. $U(\vartheta, \zeta, \lambda, \boldsymbol{c}) = U(p\{\vartheta, \zeta, \lambda\}, c)$ where $p$ is a permutation function,
F10. $U(\vartheta, \zeta, \lambda, \boldsymbol{c} + \boldsymbol{c}') \leq U(\vartheta, \zeta, u, c) \circ U(\vartheta, \lambda, \lambda, \boldsymbol{c}')$,
F11. $U(\vartheta, \zeta, \lambda, \cdot): \text{int}(C) \to [0, 1]$ is continuous.

The triplet $(D, U, O)$ is called an IGFCMS on $X$.

**Definition 2.10.** [7] Let $X \neq \emptyset$ and $*$ is a CTN and $\circ$ be a CTCN. $L, W$ and $Q$ are NSs on $X \times X \times (0, +\infty)$ is named neutrosophic metric on $X$, if for all $\vartheta, \zeta, \lambda \in X$, the below circumstances fulfil:

i. $L(\vartheta, \zeta, t) + W(\vartheta, \zeta, t) + Q(\vartheta, \zeta, t) \leq 3$ for all $t \in \mathbb{R}^+$;
ii. $L(\vartheta, \zeta, t) > 0$ for all $t > 0$;
iii. $L(\vartheta, \zeta, t) = 1$ for all $t > 0$, if and only if $\vartheta = \zeta$;
iv. $L(\vartheta, \zeta, t) = L(\zeta, \vartheta, t)$ for all $t > 0$;
v. $L(\vartheta, \lambda, t + s) \geq L(\vartheta, \zeta, t) * L(\zeta, \lambda, s)$ for all $t, s > 0$;

vi. $L(\vartheta,\zeta,\ \cdot):(0,+\infty)\rightarrow[0,1]$ is continuous and $\lim_{t\rightarrow+\infty}L(\vartheta,\zeta,t)=1$ for all $t>0$;
vii. $W(\vartheta,\zeta,t)<1$ for all $t>0$;
viii. $W(\vartheta,\zeta,t)=0$ for all $t>0$, if and only if $\vartheta=\zeta$;
ix. $W(\vartheta,\zeta,t)=W(\zeta,\vartheta,t)$ for all $t>0$;
x. $W(\vartheta,\lambda,t+s)\leq W(\vartheta,\zeta,t)\circ W(\zeta,\lambda,s)$ for all $t,s>0$;
xi. $W(\vartheta,\zeta,\ \cdot):(0,+\infty)\rightarrow[0,1]$ is continuous and $\lim_{t\rightarrow+\infty}W(\vartheta,\zeta,t)=0$ for all $t>0$;
xii. $Q(\vartheta,\zeta,t)<1$ for all $t>0$;
xiii. $Q(\vartheta,\zeta,t)=0$ for all $t>0$, if and only if $\vartheta=\zeta$;
xiv. $Q(\vartheta,\zeta,t)=Q(\zeta,\vartheta,t)$ for all $t>0$;
xv. $Q(\vartheta,\lambda,t+s)\leq Q(\vartheta,\zeta,t)\circ Q(\zeta,\lambda,s)$ for all $t,s>0$;
xvi. $Q(\vartheta,\zeta,\ \cdot):(0,+\infty)\rightarrow[0,1]$ is continuous and $\lim_{t\rightarrow+\infty}Q(\vartheta,\zeta,t)=0$ for all $t>0$.

Then $(X,L,W,Q,*,\circ)$ is called a neutrosophic metric space (NMS).

## 3. Generalized neutrosophic cone metric space (GNCMS)

In this section, the notion of GNCMS is given, some fixed point theorems are proved in the sense of GNCMS and non-trivial examples are also imparted.

**Definition 3.1.** A $GNCMS$ is a 7-tuple $(X,M,N,O,*,\circ,C)$ where $X$ is an arbitrary non-empty set, "$*$" is a CTN, "$\circ$" is a CTCN, $C$ is a closed cone and $M,N$ and $O$ are NSs in $X^3\times\mathrm{int}\,(C)$ fulfilling the following circumstances:

For all $\vartheta,\zeta,\lambda,u\in X$ and $c,c'\in\mathrm{int}(C)$,

(NC1) $M(\vartheta,\zeta,\lambda,c)+N(\vartheta,\zeta,\lambda,c)+O(\vartheta,\zeta,\lambda,c)\leq 3$,
(NC2) $M(\vartheta,\zeta,\lambda,c)>0$,
(NC3) $M(\vartheta,\zeta,\lambda,c)=1$ if and only if $\vartheta=\zeta=\lambda$,
(NC4) $M(\vartheta,\zeta,\lambda,c)=M(p\{\vartheta,\zeta,\lambda\},c)$ where $p$ is a permutation function,
(NC5) $M(\vartheta,\zeta,\lambda,c+c')\geq M(\vartheta,\zeta,u,c)*M(\vartheta,\lambda,\lambda,c')$,
(NC6) $M(\vartheta,\zeta,\lambda,\cdot):\mathrm{int}(C)\rightarrow[0,1]$ is continuous,
(NC7) $N(\vartheta,\zeta,\lambda,c)<1$,
(NC8) $N(\vartheta,\zeta,\lambda,c)=1$ if and only if $\vartheta=\zeta=\lambda$,
(NC9) $N(\vartheta,\zeta,\lambda,c)=N(p\{\vartheta,\zeta,\lambda\},c)$ where $p$ is a permutation function,
(NC10) $N(\vartheta,\zeta,\lambda,c+c')\leq N(\vartheta,\zeta,u,c)\circ N(\vartheta,\lambda,\lambda,c')$,
(NC11) $N(\vartheta,\zeta,\lambda,\cdot):\mathrm{int}(C)\rightarrow[0,1]$ is continuous,
(NC12) $O(\vartheta,\zeta,\lambda,c)<1$,
(NC13) $O(\vartheta,\zeta,\lambda,c)=1$ if and only if $\vartheta=\zeta=\lambda$,
(NC14) $O(\vartheta,\zeta,\lambda,c)=O(p\{\vartheta,\zeta,\lambda\},c)$ where $p$ is a permutation function,
(NC15) $O(\vartheta,\zeta,\lambda,c+c')\leq O(\vartheta,\zeta,u,c)\circ O(\vartheta,\lambda,\lambda,c')$,
(NC16) $O(\vartheta,\zeta,\lambda,\cdot):\mathrm{int}(C)\rightarrow[0,1]$ is continuous.

The triplet $(M,N,O)$ is known as GNCM on $X$. The functions $M(\vartheta,\zeta,\lambda,c)$, $N(\vartheta,\zeta,\lambda,c)$, and $O(\vartheta,\zeta,\lambda,c)$ indicate respectively, the degree of nearness, the degree of non-nearness and neutral functions between $\vartheta,\zeta$ and $\lambda$ with respect to $c$.

**Example 3.2.** Consider the metric space $X=[0,+\infty)$ with metric $d$. Let $C=\mathbb{R}^+$, define $*$ by $\pi*\mu=\pi\cdot\mu$, and $\circ$ by $\pi\circ\mu=\max\{\pi,\mu\}$. Define $M,N,O:X^3\times(0,+\infty)\rightarrow[0,1]$ by

$$M(\vartheta,\zeta,\lambda,c)=\frac{c}{c+\big(d(\vartheta,\zeta)+d(\zeta,\lambda)+d(\lambda,\vartheta)\big)},$$

$$N(\vartheta,\zeta,\lambda,c)=\frac{\left(d(\vartheta,\zeta)+d(\zeta,\lambda)+d(\lambda,\vartheta)\right)}{c+\left(d(\vartheta,\zeta)+d(\zeta,\lambda)+d(\lambda,\vartheta)\right)},$$
$$O(\vartheta,\zeta,\lambda,c)=\frac{\left(d(\vartheta,\zeta)+d(\zeta,\lambda)+d(\lambda,\vartheta)\right)}{c}$$

for all $\vartheta,\zeta,\lambda \in X$ and $c\in \mathrm{int}(C)$. Then, it is clear that $(X,M,N,O,*,\circ,C)$ is a $GNCMS$.

**Definition 3.3.** Let $(X,M,N,O,*,\circ,C)$ be a *GNCM*. Then the radius $r\in(0,1)$ and $c\in\mathrm{int}(C)$ the open ball $B(\vartheta,r,c)$ with centre $\vartheta$ and is defined by

$$B(\vartheta,r,c)=\{\zeta,\lambda\in X: M(\vartheta,\zeta,\lambda,c)>1-r, N(\vartheta,\zeta,\lambda,c)<r, O(\vartheta,\zeta,\lambda,c)<r\}.$$

**Definition 3.4.** Let $(X,M,N,O,*,\circ,C)$ be a *GNCM*. Then the radius $r\in(0,1)$ and $c\in\mathrm{int}(C)$ the closed ball $\bar{B}(\vartheta,r,c)$ with centre $\vartheta$ and is defined by

$$\bar{B}(\vartheta,r,c)=\{\zeta,\lambda\in X: M(\vartheta,\zeta,\lambda,c)\geq 1-r, N(\vartheta,\zeta,\lambda,c)\leq r, O(\vartheta,\zeta,\lambda,c)\leq r\}.$$

**Definition 3.5.** Let $(X,M,N,O,*,\circ,C)$ be a *GNCM*. Define $\tau=\{A\subseteq X:\ \vartheta\in A$ if and only if there exist $r\in(0,1)$ and $c\in\mathrm{int}(C)$ such that $B(\vartheta,r,c)\subset A\}$. Then $\tau$ is called a topology on $X$.

*Proof*. The proof of Definition 3.5 is easy to examine on the line as in [12].

**Remark 3.6.** A $GNCMS$ is symmetric.

**Definition 3.7.** Let $(X,M,N,O,*,\circ,C)$ be an $GNCMS$. A self mapping $T: X\to X$ is said to be $k-$ Neutrosopic cone contraction ($k-NCC$) if there exists $k\in(0,1)$ such that

$$\left(\frac{1}{M(T(\vartheta),T(\zeta),\ T(\lambda),c)}-1\right)\leq k\left(\frac{1}{M(\vartheta,\zeta,\lambda,c)}-1\right),$$
$$N(T(\vartheta),T(\zeta),T(\lambda),c)\leq kN(\vartheta,\zeta,\lambda,c),$$
$$O(T(\vartheta),T(\zeta),T(\lambda),c)\leq kO(\vartheta,\zeta,\lambda,c)$$

for all $\vartheta,\zeta,\lambda \in X$ and $c \in \mathrm{int}\,(C)$.

**Definition 3.8.** In $GNCMS$ $(X,M,N,O,*,\circ,C)$, the triplet $(M,N,O)$ is named to be triangular if, for all $\vartheta,\zeta,\lambda,u \in X$ and $c \in \mathrm{int}\,(C)$,

$$\left(\frac{1}{M(\vartheta,\zeta,\lambda,c)}-1\right)\leq\left(\frac{1}{M(\vartheta,\zeta,u,c)}-1\right)+\left(\frac{1}{M(u,\lambda,\lambda,c)}-1\right),$$
$$N(\vartheta,\zeta,\lambda,c)\leq N(\vartheta,\zeta,u,c)+N(u,\lambda,\lambda,c),$$
$$O(\vartheta,\zeta,\lambda,c)\leq O(\vartheta,\zeta,u,c)+O(u,\lambda,\lambda,c).$$

**Definition 3.9.** Let $(X,M,N,O,*,\circ,C)$ be a $GNCMS$, $\vartheta'\in X$ and $\{\vartheta_n\}$ be a sequence in $X$, then

A1) $\{\vartheta_n\}$ is said to be a convergent to $\vartheta'\in X$ if, for all $c\in\mathrm{int}\,(C)$,

$$\lim_{n\to+\infty}\left(\frac{1}{M(\vartheta_n,\vartheta',\vartheta',c)}-1\right)=0,\qquad \lim_{n\to+\infty}N(\vartheta_n,\vartheta',\vartheta',c)=0$$
$$\text{and}\ \lim_{n\to+\infty}O(\vartheta_n,\vartheta',\vartheta',c)=0$$

It is represented by $\lim_{n\to+\infty}\vartheta_n=\vartheta'$ or $\vartheta_n\to\vartheta'$ as $n\to+\infty$.

A2) $\{\vartheta_n\}$ is said to be a Cauchy sequence if, for all $c\in\mathrm{int}\,(C)$ and $m\in\mathbb{N}$,

$$\lim_{n\to+\infty}\left(\frac{1}{M(\vartheta_{n+m},\vartheta_n,\vartheta_n,c)}-1\right)=0,\qquad \lim_{n\to+\infty}N(\vartheta_{n+m},\vartheta_n,\vartheta_n,c)=0$$

and

$$\lim_{n\to+\infty}N(\vartheta_{n+m},\vartheta_n,\vartheta_n,c)=0$$

A3) $(X,M,N,O,*,\circ,C)$ is said to be a complete $GNCM$S if every Cauchy sequence in $X$ converges.

**Definition 3.10.** Let $(X,M,N,O,*,\circ,C)$ be a $GNCM$S. A sequence $\{\vartheta_n\}$ in $X$ is $k$-neutrosophic cone contractive ($k$-NCC) if their exists $k \in (0,1)$ such that

$$\left(\frac{1}{M(\vartheta_n,\vartheta_{n+1},\vartheta_{n+1},c)}-1\right) \le k\left(\frac{1}{M(\vartheta_{n-1},\vartheta_n,\vartheta_n,c)}-1\right),$$
$$N(\vartheta_n,\vartheta_{n+1},\vartheta_{n+1},c) \le kN(\vartheta_{n-1},\vartheta_n,\vartheta_n,c),$$
$$O(\vartheta_n,\vartheta_{n+1},\vartheta_{n+1},c) \le kO(\vartheta_{n-1},\vartheta_n,\vartheta_n,c)$$

for all $c \in \mathrm{int}\,(C)$.

From the help of [12], we have the following Theorem.

**Theorem 3.11.** Let $(X,M,N,O,*,\circ,C)$ be a complete $GNCMS$ in which $k$-NCC sequences are Cauchy. Let $T: X \to X$ be a $k$-NCC mapping. Then, $T$ has a unique fixed point.

**Remark 3.12.** The proof of Theorem 3.11 follows from Theorem 3.13 when $T = Q$ and $k_2 = k_3 = k_4 = 0$.

Next, let us prove some common fixed point theorems for two self-mappings satisfying generalized contractive conditions in a complete $GNCMS$.

**Theorem 3.13.** Let $(X,M,N,O,*,\circ,C)$ be a complete $GNCMS$ where $(M,N,O)$ be triangular. If $T,Q: X \to X$ satisfy the following: for all $\vartheta,\zeta,\lambda \in X$ and $c \in \mathrm{int}\,(C)$,

$$\left(\frac{1}{M(T\vartheta,Q\zeta,Q\lambda,c)}-1\right) \le \left\{\begin{array}{l} k_1\left(\frac{1}{M(\vartheta,\zeta,\lambda,c)}-1\right)+k_2\left(\frac{1}{M(\vartheta,\zeta,Q\lambda,c)}-1\right) \\ +k_3\left(\frac{1}{M(\vartheta,Q\zeta,\lambda,c)}-1\right)+k_4\left(\frac{1}{M(T\vartheta,\zeta,\lambda,c)}-1\right)\end{array}\right\}, \quad (1)$$

$$N(T\vartheta,Q\zeta,Q\lambda,c) \le \left\{\begin{array}{l} k_1N(\vartheta,\zeta,\lambda,c)+k_2N(\vartheta,\zeta,Q\lambda,c) \\ +k_3N(\vartheta,Q\zeta,\lambda,c)+k_4N(T\vartheta,\zeta,\lambda,c)\end{array}\right\} \quad (2)$$

$$O(T\vartheta,Q\zeta,Q\lambda,c) \le \left\{\begin{array}{l} k_1O(\vartheta,\zeta,\lambda,c)+k_2O(\vartheta,\zeta,Q\lambda,c) \\ +k_3O(\vartheta,Q\zeta,\lambda,c)+k_4O(T\vartheta,\zeta,\lambda,c)\end{array}\right\}, \quad (3)$$

where $k_i \in [0,1), i = 1,\dots,4$ and $k_1 + 2(k_2 + k_3) + k_4 < 1$. Then, $T$ and $Q$ have a unique common point.

*Proof.* Let $\vartheta_0 \in X$ be a random. Let $\{\vartheta_n\}$ be a $k-NCC$ sequence defined by

$$\vartheta_{3n+1} = T\vartheta_{3n},$$
$$\vartheta_{3n+2} = Q\vartheta_{3n+1}, \text{for } n \ge 0.$$

From (1), we have

$$\left(\frac{1}{M(\vartheta_{3n+1},\vartheta_{3n+2},\vartheta_{3n+2},c)}-1\right) \le \left(\frac{1}{M(T\vartheta_{3n},Q\vartheta_{3n+1},Q\vartheta_{3n+1},c)}-1\right)$$

$$\le \left\{\begin{array}{l} k_1\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c)}-1\right)+k_2\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+1},Q\vartheta_{3n+1},c)}-1\right) \\ +k_3\left(\frac{1}{M(\vartheta_{3n},Q\vartheta_{3n+1},\vartheta_{3n+1},c)}-1\right)+k_4\left(\frac{1}{M(T\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c)}-1\right)\end{array}\right\}$$

$$= \left\{\begin{array}{l} k_1\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c)}-1\right)+k_2\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+2},c)}-1\right) \\ +k_3\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+2},\vartheta_{3n+1},c)}-1\right)+k_4\left(\frac{1}{M(\vartheta_{3n+1},\vartheta_{3n+1},\vartheta_{3n+1},c)}-1\right)\end{array}\right\}$$

$$= \left\{\begin{array}{c} k_1\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c)}-1\right) \\ +k_2\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+2},c)}-1\right)+k_3\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+2},\vartheta_{3n+1},c)}-1\right)\end{array}\right\}$$

$$\leq \left\{\begin{array}{c} k_1\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c)}-1\right) \\ +k_2\left[\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c)}-1\right)+\left(\frac{1}{M(\vartheta_{3n+1},\vartheta_{3n+2},\vartheta_{3n+2},c)}-1\right)\right] \\ +k_3\left[\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c)}-1\right)+\left(\frac{1}{M(\vartheta_{3n+1},\vartheta_{3n+2},\vartheta_{3n+2},c)}-1\right)\right]\end{array}\right\}$$

$$= \left\{\begin{array}{c} (k_1+k_2+k_3)\left(\frac{1}{M(\vartheta_{3n+1},\vartheta_{3n+1},\vartheta_{3n+1},c)}-1\right) \\ +(k_2+k_3)\left(\frac{1}{M(\vartheta_{3n+1},\vartheta_{3n+2},\vartheta_{3n+2},c)}-1\right)\end{array}\right\}.$$

Therefore,

$$\left(\frac{1}{M(\vartheta_{3n+1},\vartheta_{3n+2},\vartheta_{3n+2},c)}-1\right) \leq \frac{k_1+k_2+k_3}{1-(k_2+k_3)}\left(\frac{1}{M(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c)}-1\right).$$

Similarly,

$$\left(\frac{1}{M(\vartheta_{3n+2},\vartheta_{3n+3},\vartheta_{3n+3},c)}-1\right) \leq \frac{k_1+k_2+k_3}{1-(k_2+k_3)}\left(\frac{1}{M(\vartheta_{3n+1},\vartheta_{3n+2},\vartheta_{3n+2},c)}-1\right),$$

$$\left(\frac{1}{M(\vartheta_{3n+3},\vartheta_{3n+4},\vartheta_{3n+4},c)}-1\right) \leq \frac{k_1+k_2+k_3}{1-(k_2+k_3)}\left(\frac{1}{M(\vartheta_{3n+2},\vartheta_{3n+3},\vartheta_{3n+3},c)}-1\right).$$

Put $M_n = \left(\frac{1}{M(\vartheta_n,\vartheta_{n+1},\vartheta_{n+1},c)}-1\right)$ and

$$k = \frac{k_1+k_2+k_3}{1-(k_2+k_3)}.$$

Then we get the inequalities:

$$M_{3n+1} \leq kM_{3n},$$
$$M_{3n+2} \leq kM_{3n+1},$$
$$M_{3n+3} \leq kM_{3n+2}.$$

These imply that

$$M_{n+1} \leq kM_n \text{ for } n = 0,1,2,\ldots, \tag{4}$$

From (2), we have

$$N(\vartheta_{3n+1},\vartheta_{3n+2},\vartheta_{3n+2},c) \leq N(T\vartheta_{3n},Q\vartheta_{3n+1},Q\vartheta_{3n+1},c)$$

$$\leq \left\{\begin{array}{c} k_1N(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c)+k_2N(\vartheta_{3n},\vartheta_{3n+1},Q\vartheta_{3n+1},c) \\ +k_3N(\vartheta_{3n},Q\vartheta_{3n+1},\vartheta_{3n+1},c)+k_4N(T\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c)\end{array}\right\}$$

$$= \left\{\begin{array}{c} k_1N(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c)+k_2N(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+2},c) \\ +k_3N(\vartheta_{3n},\vartheta_{3n+2},\vartheta_{3n+1},c)+k_4N(\vartheta_{3n+1},\vartheta_{3n+1},\vartheta_{3n+1},c)\end{array}\right\}$$

$$= \begin{Bmatrix} k_1 N(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) \\ +k_2 N(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+2}, c) + k_3 N(\vartheta_{3n}, \vartheta_{3n+2}, \vartheta_{3n+1}, c) \end{Bmatrix}$$

$$\leq \begin{Bmatrix} k_1 N(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) \\ +k_2 [N(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) + N(\vartheta_{3n}, \vartheta_{3n+2}, \vartheta_{3n+2}, c)] \\ +k_3 [N(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) + N(\vartheta_{3n}, \vartheta_{3n+2}, \vartheta_{3n+2}, c)] \end{Bmatrix}$$

$$= \begin{Bmatrix} (k_1 + k_2 + k_3) N(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) \\ +(k_2 + k_3) N(\vartheta_{3n}, \vartheta_{3n+2}, \vartheta_{3n+2}, c) \end{Bmatrix}.$$

Therefore,

$$N(\vartheta_{3n+1}, \vartheta_{3n+2}, \vartheta_{3n+2}, c) \leq \frac{(k_1 + k_2 + k_3)}{1 - (k_2 + k_3)} N(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c).$$

Similarly,

$$N(\vartheta_{3n}, \vartheta_{3n+3}, \vartheta_{3n+3}, c) \leq \frac{(k_1 + k_2 + k_3)}{1 - (k_2 + k_3)} N(\vartheta_{3n}, \vartheta_{3n+2}, \vartheta_{3n+2}, c),$$

$$N(\vartheta_{3n}, \vartheta_{3n+4}, \vartheta_{3n+4}, c) \leq \frac{(k_1 + k_2 + k_3)}{1 - (k_2 + k_3)} N(\vartheta_{3n}, \vartheta_{3n+3}, \vartheta_{3n+3}, c).$$

Putting $N_n = N(\vartheta_n, \vartheta_{n+1}, \vartheta_{n+1}, c)$, we have the following inequalities:
For $n = 0,1,2, \ldots,$

$$N_{3n+1} \leq k N_{3n},$$

$$N_{3n+2} \leq k N_{3n+1},$$

$$N_{3n+3} \leq k N_{3n+2}.$$

These imply that

$$N_{n+1} \leq k N_n, \quad \text{for } n = 0,1,2, \ldots, \tag{5}$$

From (3), we have

$$O(\vartheta_{3n+1}, \vartheta_{3n+2}, \vartheta_{3n+2}, c) \leq O(T\vartheta_{3n}, Q\vartheta_{3n+1}, Q\vartheta_{3n+1}, c)$$

$$\leq \begin{Bmatrix} k_1 O(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) + k_2 O(\vartheta_{3n}, \vartheta_{3n+1}, Q\vartheta_{3n+1}, c) \\ +k_3 O(\vartheta_{3n}, Q\vartheta_{3n+1}, \vartheta_{3n+1}, c) + k_4 O(T\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) \end{Bmatrix}$$

$$= \begin{Bmatrix} k_1 O(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) + k_2 O(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+2}, c) \\ +k_3 O(\vartheta_{3n}, \vartheta_{3n+2}, \vartheta_{3n+1}, c) + k_4 O(\vartheta_{3n+1}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) \end{Bmatrix}$$

$$= \begin{Bmatrix} k_1 O(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) \\ +k_2 O(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+2}, c) + k_3 O(\vartheta_{3n}, \vartheta_{3n+2}, \vartheta_{3n+1}, c) \end{Bmatrix}$$

$$\leq \begin{Bmatrix} k_1 O(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) \\ +k_2 [O(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) + O(\vartheta_{3n}, \vartheta_{3n+2}, \vartheta_{3n+2}, c)] \\ +k_3 [O(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) + O(\vartheta_{3n}, \vartheta_{3n+2}, \vartheta_{3n+2}, c)] \end{Bmatrix}$$

$$= \begin{Bmatrix} (k_1 + k_2 + k_3) O(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c) \\ +(k_2 + k_3) O(\vartheta_{3n}, \vartheta_{3n+2}, \vartheta_{3n+2}, c) \end{Bmatrix}.$$

Therefore,

$$O(\vartheta_{3n+1},\vartheta_{3n+2},\vartheta_{3n+2},c)\leq\frac{(k_1+k_2+k_3)}{1-(k_2+k_3)}O(\vartheta_{3n},\vartheta_{3n+1},\vartheta_{3n+1},c).$$

Similarly,

$$O(\vartheta_{3n},\vartheta_{3n+3},\vartheta_{3n+3},c)\leq\frac{(k_1+k_2+k_3)}{1-(k_2+k_3)}O(\vartheta_{3n},\vartheta_{3n+2},\vartheta_{3n+2},c),$$

$$O(\vartheta_{3n},\vartheta_{3n+4},\vartheta_{3n+4},c)\leq\frac{(k_1+k_2+k_3)}{1-(k_2+k_3)}O(\vartheta_{3n},\vartheta_{3n+3},\vartheta_{3n+3},c).$$

Putting $O_n=O(\vartheta_n,\vartheta_{n+1},\vartheta_{n+1},c)$ we have the following inequalities:
For $n=0,1,2,\ldots,$

$$O_{3n+1}\leq kO_{3n},$$

$$O_{3n+2}\leq kO_{3n+1},$$

$$O_{3n+3}\leq kO_{3n+2}.$$

These imply that

$$O_{n+1}\leq kO_n,\quad \text{for } n=0,1,2,\ldots, \tag{6}$$

The inequalities (4)–(6) make $\{\vartheta_n\}$ a $k$-NCC sequence.
Now $(M,N,O)$ is triangular and the space $(X,M,N,O,*,\circ,C)$ is symmetric. Therefore, we have

$$\left(\frac{1}{M(\vartheta_n,\vartheta_n,\vartheta_m,c)}-1\right)\leq\left\{\begin{array}{c}\left(\frac{1}{M(\vartheta_n,\vartheta_{n+1},\vartheta_{n+1},c)}-1\right)\\+\left(\frac{1}{M(\vartheta_{n+1},\vartheta_{n+2},\vartheta_{n+2},c)}-1\right)+\cdots+\left(\frac{1}{M(\vartheta_{m-1},\vartheta_m,\vartheta_m,c)}-1\right)\end{array}\right\}$$

$$=M_n+M_{n+1}+\cdots+M_{m-1}$$

$$\leq k^nM_0+k^{n+1}M_0+\cdots+k^{m-1}M_0$$

$$\leq\frac{k^n}{1-k}M_0\to 0 \text{ as } n\to+\infty.$$

$$N(\vartheta_n,\vartheta_n,\vartheta_m,c)\leq\left\{\begin{array}{c}N(\vartheta_n,\vartheta_{n+1},\vartheta_{n+1},c)\\+N(\vartheta_{n+1},\vartheta_{n+1},\vartheta_{n+2},c)+\cdots+N(\vartheta_{m-1},\vartheta_m,\vartheta_m,c)\end{array}\right\}$$

$$=N_n+N_{n+1}+\cdots+N_{m-1}$$

$$\leq k^nN_0+k^{n+1}N_0+\cdots+k^{m-1}N_0$$

$$\leq\frac{k^n}{1-k}N_0\to 0 \text{ as } n\to+\infty.$$

and

$$O(\vartheta_n,\vartheta_n,\vartheta_m,c)\leq\left\{\begin{array}{c}O(\vartheta_n,\vartheta_{n+1},\vartheta_{n+1},c)\\+O(\vartheta_{n+1},\vartheta_{n+1},\vartheta_{n+2},c)+\cdots+O(\vartheta_{m-1},\vartheta_m,\vartheta_m,c)\end{array}\right\}$$

$$=O_n+O_{n+1}+\cdots+O_{m-1}$$

$$\leq k^n O_0 + k^{n+1} O_0 + \cdots + k^{m-1} O_0$$

$$\leq \frac{k^n}{1-k} O_0 \to 0 \text{ as } n \to +\infty.$$

Therefore, a sequence $\{\vartheta_n\}$ is Cauchy. As $X$ is complete, there exists $\vartheta \in X$ such that

$$\lim_{n\to+\infty} \left(\frac{1}{M(\vartheta_n,\vartheta,\vartheta,c)} - 1\right) = 0, \quad \lim_{n\to+\infty} N(\vartheta_n,\vartheta,\vartheta,c) = 0 \text{ and } \lim_{n\to+\infty} O(\vartheta_n,\vartheta,\vartheta,c) = 0 \tag{7}$$

From (5)–(7), we deduce that

$$M_{n+1} = k^n M_0, \qquad N_{n+1} = k^n N_0, \quad O_{n+1} = k^n O_0 \quad \text{for } n = 0,1,2,\ldots,$$

$$\lim_{n\to+\infty} M_n = 0, \quad \lim_{n\to+\infty} N_n = 0 \text{ and } \lim_{n\to+\infty} O_n = 0 \tag{8}$$

Since $(M, N, O)$ is triangular,

$$\left(\frac{1}{M(\vartheta,\vartheta,T\vartheta,c)} - 1\right) \leq \left(\frac{1}{M(\vartheta,\vartheta,\vartheta_{3n+2},c)} - 1\right) + \left(\frac{1}{M(\vartheta_{3n+2},T\vartheta,T\vartheta,c)} - 1\right), \tag{9}$$

$$N(\vartheta,\vartheta,T\vartheta,c) \leq N(\vartheta,\vartheta,\vartheta_{3n+2},c) + N(\vartheta_{3n+2},T\vartheta,T\vartheta,c), \tag{10}$$

$$O(\vartheta,\vartheta,T\vartheta,c) \leq O(\vartheta,\vartheta,\vartheta_{3n+2},c) + O(\vartheta_{3n+2},T\vartheta,T\vartheta,c). \tag{11}$$

From (1), we have

$$\left(\frac{1}{M(\vartheta_{3n+2},T\vartheta,T\vartheta,c)} - 1\right) \leq \left(\frac{1}{M(Q\vartheta_{3n+1},T\vartheta,T\vartheta,c)} - 1\right)$$

$$\leq \left\{\begin{matrix} k_1\left(\frac{1}{M(\vartheta_{3n+1},\vartheta,\vartheta,c)} - 1\right) + k_2\left(\frac{1}{M(\vartheta_{3n+1},\vartheta,T\vartheta,c)} - 1\right) \\ +k_3\left(\frac{1}{M(\vartheta_{3n+1},T\vartheta,\vartheta,c)} - 1\right) + k_4\left(\frac{1}{M(Q\vartheta_{3n+1},\vartheta,\vartheta,c)} - 1\right)\end{matrix}\right\}$$

$$= \left\{\begin{matrix} k_1\left(\frac{1}{M(\vartheta_{3n+1},\vartheta,\vartheta,c)} - 1\right) + k_2\left(\frac{1}{M(\vartheta_{3n+1},\vartheta,T\vartheta,c)} - 1\right) \\ +k_3\left(\frac{1}{M(\vartheta_{3n+1},T\vartheta,\vartheta,c)} - 1\right) + k_4\left(\frac{1}{M(\vartheta_{3n+2},\vartheta,\vartheta,c)} - 1\right)\end{matrix}\right\}$$

$$\to (k_2 + k_3)\left(\frac{1}{M(\vartheta,\vartheta,T\vartheta,c)} - 1\right) \text{as } n \to +\infty.$$

Therefore,

$$\lim_{n\to+\infty} \sup\left(\frac{1}{M(\vartheta_{3n+2},T\vartheta,T\vartheta,c)} - 1\right) \leq (k_2 + k_3)\left(\frac{1}{M(\vartheta,\vartheta,T\vartheta,c)} - 1\right). \tag{12}$$

From (2), we have

$$N(\vartheta_{3n+2},T\vartheta,T\vartheta,c) \leq N(Q\vartheta_{3n+1},T\vartheta,T\vartheta,c)$$

$$\leq \left\{\begin{matrix} k_1 N(\vartheta_{3n+1},\vartheta,\vartheta,c) + k_2 N(\vartheta_{3n+1},\vartheta,T\vartheta,c) \\ +k_3 N(\vartheta_{3n+1},T\vartheta,\vartheta,c) + k_4 N(Q\vartheta_{3n+1},\vartheta,\vartheta,c)\end{matrix}\right\}$$

$$= \left\{\begin{matrix} k_1 N(\vartheta_{3n+1},\vartheta,\vartheta,c) + k_2 N(\vartheta_{3n+1},\vartheta,T\vartheta,c) \\ +k_3 N(\vartheta_{3n+1},T\vartheta,\vartheta,c) + k_4 N(\vartheta_{3n+2},\vartheta,\vartheta,c)\end{matrix}\right\}$$

$$\to (k_2 + k_3)\, N(\vartheta, \vartheta, T\vartheta, c) \text{ as } n \to +\infty.$$

Therefore,

$$\lim_{n\to+\infty} \sup N(\vartheta_{3n+2}, T\vartheta, T\vartheta, c) \leq (k_2 + k_3)\, N(\vartheta, \vartheta, T\vartheta, c). \tag{13}$$

From (3), we have

$$O(\vartheta_{3n+2}, T\vartheta, T\vartheta, c) \leq O(Q\vartheta_{3n+1}, T\vartheta, T\vartheta, c)$$

$$\leq \begin{Bmatrix} k_1 O(\vartheta_{3n+1}, \vartheta, \vartheta, c) + k_2 O(\vartheta_{3n+1}, \vartheta, T\vartheta, c) \\ +k_3 O(\vartheta_{3n+1}, T\vartheta, \vartheta, c) + k_4 O(Q\vartheta_{3n+1}, \vartheta, \vartheta, c) \end{Bmatrix}$$

$$= \begin{Bmatrix} k_1 O(\vartheta_{3n+1}, \vartheta, \vartheta, c) + k_2 O(\vartheta_{3n+1}, \vartheta, T\vartheta, c) \\ +k_3 O(\vartheta_{3n+1}, T\vartheta, \vartheta, c) + k_4 O(\vartheta_{3n+2}, \vartheta, \vartheta, c) \end{Bmatrix}$$

$$\to (k_2 + k_3)\, O(\vartheta, \vartheta, T\vartheta, c) \text{ as } n \to +\infty.$$

Therefore,

$$\lim_{n\to+\infty} \sup O(\vartheta_{3n+2}, T\vartheta, T\vartheta, c) \leq (k_2 + k_3)\, O(\vartheta, \vartheta, T\vartheta, c). \tag{14}$$

From (9) to (14), we obtain that

$$\left(\frac{1}{M(\vartheta, \vartheta, T\vartheta, c)} - 1\right) \leq (k_2 + k_3)\left(\frac{1}{M(\vartheta, \vartheta, T\vartheta, c)} - 1\right),$$

$$N(\vartheta, \vartheta, T\vartheta, c) \leq (k_2 + k_3)\, N(\vartheta, \vartheta, T\vartheta, c),$$

$$O(\vartheta, \vartheta, T\vartheta, c) \leq (k_2 + k_3)\, O(\vartheta, \vartheta, T\vartheta, c).$$

Since $k_2 + k_3 < 1$, we have

$$\left(\frac{1}{M(\vartheta, \vartheta, T\vartheta, c)} - 1\right) = 0, N(\vartheta, \vartheta, T\vartheta, c) = 0, O(\vartheta, \vartheta, T\vartheta, c) = 0.$$

Therefore $T\vartheta = \vartheta$.
In a similar way, we can examine that $Q\vartheta = \vartheta$. Then, $T\vartheta = Q\vartheta = \vartheta$.
Suppose $Tu = Qu = u$.
From (1), we have

$$\left(\frac{1}{M(\vartheta, u, u, c)} - 1\right) \leq \left(\frac{1}{M(T\vartheta, Qu, Qu, c)} - 1\right)$$

$$\leq \begin{Bmatrix} k_1\left(\frac{1}{M(\vartheta, u, u, c)} - 1\right) + k_2\left(\frac{1}{M(\vartheta, u, Qu, c)} - 1\right) \\ +k_3\left(\frac{1}{M(\vartheta, Qu, u, c)} - 1\right) + k_4\left(\frac{1}{M(T\vartheta, u, u, c)} - 1\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} k_1\left(\frac{1}{M(\vartheta, u, u, c)} - 1\right) + k_2\left(\frac{1}{M(\vartheta, u, u, c)} - 1\right) \\ +k_3\left(\frac{1}{M(\vartheta, u, u, c)} - 1\right) + k_4\left(\frac{1}{M(\vartheta, u, u, c)} - 1\right) \end{Bmatrix}$$

$$= (k_1 + k_2 + k_3 + k_4)\left(\frac{1}{M(\vartheta, u, u, c)} - 1\right).$$

This is,

$$\left(\frac{1}{M(\vartheta,u,u,c)}-1\right)\le(k_1+k_2+k_3+k_4)\left(\frac{1}{M(\vartheta,u,u,c)}-1\right).$$

Therefore,

$$\left(\frac{1}{M(\vartheta,u,u,c)}-1\right)=0,\ \text{since}\ \ k_1+k_2+k_3+k_4<1.$$

Hence, '$\vartheta$' is the unique common fixed point of $T$ and $Q$.

**Corollary 3.14.** Let $(X,M,N,O,*,\circ,C)$ be a complete $GNCMS$ where $(M,N,O)$ be triangular. If $T:X\to X$ for all $\vartheta,\zeta,\lambda\in X$ and $c\in\operatorname{int}(C)$,

$$\left(\frac{1}{M(T\vartheta,T\zeta,T\lambda,c)}-1\right)\le\left\{\begin{array}{c}k_1\left(\frac{1}{M(\vartheta,\zeta,\lambda,c)}-1\right)+k_2\left(\frac{1}{M(\vartheta,\zeta,T\lambda,c)}-1\right)\\+k_3\left(\frac{1}{M(\vartheta,T\zeta,\lambda,c)}-1\right)+k_4\left(\frac{1}{M(T\vartheta,\zeta,\lambda,c)}-1\right)\end{array}\right\},$$

$$N(T\vartheta,T\zeta,T\lambda,c)\le\left\{\begin{array}{c}k_1N(\vartheta,\zeta,\lambda,c)+k_2N(\vartheta,\zeta,T\lambda,c)\\+k_3N(\vartheta,T\zeta,\lambda,c)+k_4N(T\vartheta,\zeta,\lambda,c)\end{array}\right\},$$

$$O(T\vartheta,T\zeta,T\lambda,c)\le\left\{\begin{array}{c}k_1O(\vartheta,\zeta,\lambda,c)+k_2O(\vartheta,\zeta,T\lambda,c)\\+k_3O(\vartheta,T\zeta,\lambda,c)+k_4O(T\vartheta,\zeta,\lambda,c)\end{array}\right\}$$

where $k_i\in[0,1), i=1,\dots,4$ and $k_1+2(k_2+k_3)+k_4<1$. Then, $T$ has a unique fixed point.

*Proof.* Easy to prove on the lines of Theorem 3.13.

**Example 3.15.** Consider the metric space $X=[0,+\infty)$ with metric $d$ given by $d(\vartheta,\zeta)=|\vartheta-\zeta|$ for all $\vartheta,\zeta\in X$. Let $C=\mathbb{R}^+$, define $*$ by $\pi*\mu=\pi\cdot\mu$, and $\circ$ by $\pi\circ\mu=\max\{\pi,\mu\}$. Define $M,N,O:X^3\times(0,+\infty)\to[0,1]$ by

$$M(\vartheta,\zeta,\lambda,c)=\frac{c}{c+(|\vartheta-\zeta|+|\zeta-\lambda|+|\lambda-\vartheta|)},$$

$$N(\vartheta,\zeta,\lambda,c)=\frac{|\vartheta-\zeta|+|\zeta-\lambda|+|\lambda-\vartheta|}{c+|\vartheta-\zeta|+|\zeta-\lambda|+|\lambda-\vartheta|},$$

$$O(\vartheta,\zeta,\lambda,c)=\frac{|\vartheta-\zeta|+|\zeta-\lambda|+|\lambda-\vartheta|}{c}$$

for all $\vartheta,\zeta,\lambda\in X$ and $c\in\operatorname{int}(C)$. Then, it is clear that $(X,M,N,O,*,\circ)$ is a complete $GNCMS$ and that $(M,N,O)$ is triangular. Assume the self mappings $T,Q:X\to X$ are given by

$$T\vartheta=\begin{cases}\frac{1}{4}\vartheta, & \vartheta\in[0,2),\\ 0, & \vartheta\in[2,+\infty),\end{cases}$$

$$Q\vartheta=\begin{cases}\frac{1}{4}\vartheta, & \vartheta\in[0,2),\\ \frac{1}{\vartheta}, & \vartheta\in[2,+\infty),\end{cases}$$

Here, $T$ and $Q$ together satisfy the conditions (1)–(3) with $k_1=\frac{1}{2},k_2=\frac{1}{8},k_3=\frac{1}{16},k_4=\frac{1}{16}$. Therefore, $T$ and $Q$ have a unique common fixed point and it is $\vartheta=0$.

**Theorem 3.16.** Let $(X,M,N,O,*,\circ,C)$ be a complete $GNCMS$ where $(M,N,O)$ is triangular.

Assume that $T, Q, R: X \to X$ satisfy the following

$$\left(\frac{1}{M(T\vartheta, Q\zeta, R\lambda, c)} - 1\right) \leq k\left(\frac{1}{\Omega_1(\vartheta,\zeta,\lambda)} - 1\right), \tag{15}$$

$$N(T\vartheta, Q\zeta, R\lambda) \leq k\Omega_2(\vartheta,\zeta,\lambda), \tag{16}$$

$$O(T\vartheta, Q\zeta, R\lambda) \leq k\Omega_3(\vartheta,\zeta,\lambda), \tag{17}$$

for all $\vartheta,\zeta,\lambda \in X$ and $c \in \operatorname{int}(C)$ where $k \in (0,1)$ and

$$\Omega_1(\vartheta,\zeta,\lambda) = \min\{M(\vartheta, Q\zeta, R\lambda, c), M(T\vartheta, \zeta, R\lambda, c), M(T\vartheta, Q\zeta, \lambda, c)\},$$
$$\Omega_2(\vartheta,\zeta,\lambda) = \min\{N(\vartheta, Q\zeta, R\lambda, c), N(T\vartheta, \zeta, R\lambda, c), N(T\vartheta, Q\zeta, \lambda, c)\},$$
$$\Omega_3(\vartheta,\zeta,\lambda) = \min\{O(\vartheta, Q\zeta, R\lambda, c), O(T\vartheta, \zeta, R\lambda, c), O(T\vartheta, Q\zeta, \lambda, c)\}.$$

Then, $T, Q$ and $R$ have a unique common fixed point.

*Proof*. Let $\vartheta_0 \in X$ be arbitrary. Define the sequence $\{\vartheta_n\}$ as in Theorem 3.10. Then, from (15)–(17) for $n = 0, 1, 2, \ldots$,

$$\left(\frac{1}{M(\vartheta_{3n+1}, \vartheta_{3n+2}, \vartheta_{3n+2}, c)} - 1\right) \leq \frac{k}{1-k}\left(\frac{1}{M(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c)} - 1\right),$$

$$\left(\frac{1}{M(\vartheta_{3n+2}, \vartheta_{3n+3}, \vartheta_{3n+3}, c)} - 1\right) \leq \frac{k}{1-k}\left(\frac{1}{M(\vartheta_{3n+1}, \vartheta_{3n+2}, \vartheta_{3n+2}, c)} - 1\right),$$

$$\left(\frac{1}{M(\vartheta_{3n+3}, \vartheta_{3n+4}, \vartheta_{3n+4}, c)} - 1\right) \leq \frac{k}{1-k}\left(\frac{1}{M(\vartheta_{3n+2}, \vartheta_{3n+3}, \vartheta_{3n+3}, c)} - 1\right),$$

$$N(\vartheta_{3n+1}, \vartheta_{3n+2}, \vartheta_{3n+2}, c) \leq \frac{k}{1-k} N(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c),$$

$$N(\vartheta_{3n+2}, \vartheta_{3n+3}, \vartheta_{3n+3}, c) \leq \frac{k}{1-k} N(\vartheta_{3n+1}, \vartheta_{3n+2}, \vartheta_{3n+2}, c),$$

$$N(\vartheta_{3n+3}, \vartheta_{3n+4}, \vartheta_{3n+4}, c) \leq \frac{k}{1-k} N(\vartheta_{3n+2}, \vartheta_{3n+3}, \vartheta_{3n+3}, c),$$

$$O(\vartheta_{3n+1}, \vartheta_{3n+2}, \vartheta_{3n+2}, c) \leq \frac{k}{1-k} O(\vartheta_{3n}, \vartheta_{3n+1}, \vartheta_{3n+1}, c),$$

$$O(\vartheta_{3n+2}, \vartheta_{3n+3}, \vartheta_{3n+3}, c) \leq \frac{k}{1-k} O(\vartheta_{3n+1}, \vartheta_{3n+2}, \vartheta_{3n+2}, c),$$

$$O(\vartheta_{3n+3}, \vartheta_{3n+4}, \vartheta_{3n+4}, c) \leq \frac{k}{1-k} O(\vartheta_{3n+2}, \vartheta_{3n+3}, \vartheta_{3n+3}, c).$$

Therefore, we have

$$\left(\frac{1}{M(\vartheta_{n+1}, \vartheta_{n+2}, \vartheta_{n+2}, c)} - 1\right) \leq \frac{k}{1-k}\left(\frac{1}{M(\vartheta_{n}, \vartheta_{n+1}, \vartheta_{n+1}, c)} - 1\right),$$

$$N(\vartheta_{n+1}, \vartheta_{n+2}, \vartheta_{n+2}, c) \leq \frac{k}{1-k} N(\vartheta_{n}, \vartheta_{n+1}, \vartheta_{n+1}, c),$$

$$O(\vartheta_{n+1}, \vartheta_{n+2}, \vartheta_{n+2}, c) \leq \frac{k}{1-k} O(\vartheta_{n}, \vartheta_{n+1}, \vartheta_{n+1}, c).$$

Using these inequalities repeatedly, we obtain that

$$\left(\frac{1}{M(\vartheta_{n+1},\vartheta_{n+2},\vartheta_{n+2},c)}-1\right)\leq \frac{k^n}{1-k}\left(\frac{1}{M(\vartheta_0,\vartheta_1,\vartheta_1,c)}-1\right),$$

$$N(\vartheta_{n+1},\vartheta_{n+2},\vartheta_{n+2},c)\leq\frac{k^n}{1-k}N(\vartheta_0,\vartheta_1,\vartheta_1,c),$$

$$O(\vartheta_{n+1},\vartheta_{n+2},\vartheta_{n+2},c)\leq\frac{k^n}{1-k}O(\vartheta_0,\vartheta_1,\vartheta_1,c).$$

Therefore, $\{\vartheta_n\}$ is $k$-FCC and Cauchy and hence we can find an element $\vartheta\in X$ such that

$$\lim_{n\to+\infty}\left(\frac{1}{M(\vartheta_n,\vartheta,\vartheta,c)}-1\right)=0,\quad \lim_{n\to+\infty}N(\vartheta_0,\vartheta_1,\vartheta_1,c)=0,\ \lim_{n\to+\infty}O(\vartheta_0,\vartheta_1,\vartheta_1,c) \tag{18}$$

Since $(M,N,O)$ is triangular,

$$\left(\frac{1}{M(\vartheta,\vartheta,T\vartheta,c)}-1\right)\leq\left(\frac{1}{M(\vartheta,\vartheta,\vartheta_{3n+2},c)}-1\right)+\left(\frac{1}{M(\vartheta_{3n+2},T\vartheta,T\vartheta,c)}-1\right), \tag{19}$$

$$N(\vartheta,\vartheta,T\vartheta,c)\leq N(\vartheta,\vartheta,\vartheta_{3n+2},c)+N(\vartheta_{3n+2},T\vartheta,T\vartheta,c), \tag{20}$$

$$O(\vartheta,\vartheta,T\vartheta,c)\leq O(\vartheta,\vartheta,\vartheta_{3n+2},c)+O(\vartheta_{3n+2},T\vartheta,T\vartheta,c). \tag{21}$$

From (15)–(17), we obtain

$$\lim_{n\to+\infty}\sup\left(\frac{1}{M(\vartheta_{3n+2},T\vartheta,T\vartheta,c)}-1\right)\leq k\left(\frac{1}{M(\vartheta,\vartheta,T\vartheta,c)}-1\right), \tag{22}$$

$$\lim_{n\to+\infty}\sup N(\vartheta_{3n+2},T\vartheta,T\vartheta,c)\leq kN(\vartheta,\vartheta,T\vartheta,c), \tag{23}$$

$$\lim_{n\to+\infty}\sup O(\vartheta_{3n+2},T\vartheta,T\vartheta,c)\leq kO(\vartheta,\vartheta,T\vartheta,c). \tag{24}$$

From (19) to (24), we obtain that

$$\left(\frac{1}{M(\vartheta,\vartheta,T\vartheta,c)}-1\right)\leq k\left(\frac{1}{M(\vartheta,\vartheta,T\vartheta,c)}-1\right),$$

$$N(\vartheta,\vartheta,T\vartheta,c)\leq kN(\vartheta,\vartheta,T\vartheta,c),$$

$$O(\vartheta,\vartheta,T\vartheta,c)\leq kO(\vartheta,\vartheta,T\vartheta,c).$$

As $k<1$, we obtain

$$\left(\frac{1}{M(\vartheta,\vartheta,T\vartheta,c)}-1\right)=0, N(\vartheta,\vartheta,T\vartheta,c)=0, O(\vartheta,\vartheta,T\vartheta,c)=0.$$

Therefore, $T\vartheta=\vartheta$. In a similar way, we can bring that $Q\vartheta=\vartheta$ and $R\vartheta=\vartheta$. Suppose

$$Tu=Qu=Ru=u.$$

From (15), we have that

$$\left(\frac{1}{M(\vartheta,u,u,c)}-1\right)=\left(\frac{1}{M(T\vartheta,Qu,Ru,c)}-1\right)\leq k\left(\frac{1}{\Omega_1(\vartheta,\zeta,\lambda)}-1\right),$$

where,

$$\Omega_1(\vartheta,\zeta,\lambda)=\min\{M(\vartheta,Qu,Ru,c),M(T\vartheta,u,Ru,c),M(T\vartheta,Qu,u,c)\},$$

$$= \min\{M(\vartheta,u,u,c), M(\vartheta,u,u,c), M(\vartheta,u,u,c)\} = M(\vartheta,u,u,c).$$

From (16), we have

$$N(\vartheta,u,u,c) = N(T\vartheta, Qu, Ru, c) \le k\Omega_2(\vartheta,\zeta,\lambda)$$

where,

$$\Omega_2(\vartheta,\zeta,\lambda) = \min\{N(\vartheta,Qu,Ru,c), N(T\vartheta,u,Ru,c), N(T\vartheta,Qu,u,c)\}$$
$$= \min\{N(\vartheta,u,u,c), N(\vartheta,u,u,c), N(\vartheta,u,u,c)\} = N(\vartheta,u,u,c).$$

From (17), we have

$$O(\vartheta,u,u,c) = O(T\vartheta,Qu,Ru,c) \le k\Omega_2(\vartheta,\zeta,\lambda)$$

where,

$$\Omega_2(\vartheta,\zeta,\lambda) = \min\{O(\vartheta,Qu,Ru,c), O(T\vartheta,u,Ru,c), O(T\vartheta,Qu,u,c)\}$$
$$= \min\{O(\vartheta,u,u,c), O(\vartheta,u,u,c), O(\vartheta,u,u,c)\} = O(\vartheta,u,u,c).$$

Therefore,

$$\left(\frac{1}{M(\vartheta,u,u,c)} - 1\right) \le k\left(\frac{1}{M(\vartheta,u,u,c)} - 1\right),$$
$$N(\vartheta,u,u,c) \le kN(\vartheta,u,u,c),$$
$$O(\vartheta,u,u,c) \le kO(\vartheta,u,u,c).$$

Hence,

$$\left(\frac{1}{M(\vartheta,u,u,c)} - 1\right) = 0, N(\vartheta,u,u,c) = 0, O(\vartheta,u,u,c) = 0.$$

Therefore, $\vartheta = u$ and we can conclude that $T, Q$ and $R$ have a unique common fixed point.

**Example 3.17.** Consider $X = [0,+\infty)$ with for all $\vartheta,\zeta \in X$. Let $C = \mathbb{R}^+$. Define $*$ by $\pi * \mu = \pi \cdot \mu$ and $\circ$ by $\pi \circ \mu = \max\{\pi,\mu\}$. Define $M, N, O : X^3 \times (0,+\infty) \to [0,1]$ by

$$M(\vartheta,\zeta,\lambda,c) = \begin{cases} 1, & \text{if } \vartheta = \zeta \\ \dfrac{c}{c + (\max\{\vartheta,\zeta\} + \max\{\zeta,\lambda\} + \max\{\lambda,\vartheta\})}, & \text{if othewise,} \end{cases}$$

$$N(\vartheta,\zeta,\lambda,c) = \begin{cases} 0, & \text{if } \vartheta = \zeta \\ \dfrac{\max\{\vartheta,\zeta\} + \max\{\zeta,\lambda\} + \max\{\lambda,\vartheta\}}{c + (\max\{\vartheta,\zeta\} + \max\{\zeta,\lambda\} + \max\{\lambda,\vartheta\})}, & \text{if otherwise,} \end{cases}$$

$$O(\vartheta,\zeta,\lambda,c) = \begin{cases} 0, & \text{if } \vartheta = \zeta \\ \dfrac{\max\{\vartheta,\zeta\} + \max\{\zeta,\lambda\} + \max\{\lambda,\vartheta\}}{c}, & \text{if othewise} \end{cases}$$

for all $\vartheta,\zeta,\lambda \in X$ and $c \in \text{int}(C)$. Then, it is clear that $(X, M, N, O, *, \circ, C)$ be a complete $GNCMS$ and that $(M, N, O)$ is triangular. Define self-mappings $T, Q$ and $R$ from $X$ to $X$ by

$$T\vartheta = \begin{cases} \dfrac{1}{2}\vartheta - \dfrac{1}{4}, & \vartheta \in [0,1), \\ \dfrac{1}{2}\vartheta + \dfrac{3}{2}, & \vartheta \in [1,+\infty), \end{cases}$$

$$Q\vartheta = \begin{cases} \frac{1}{2}\vartheta - \frac{1}{4}, & \vartheta \in [0,1), \\ \frac{2}{3}\vartheta + 1, & \vartheta \in [1,+\infty), \end{cases}$$

$$R\vartheta = \begin{cases} \frac{1}{2}\vartheta - \frac{1}{4}, & \vartheta \in [0,1), \\ \frac{1}{3}\vartheta + 2, & \vartheta \in [1,+\infty). \end{cases}$$

Then,

$$\left(\frac{1}{M(T\vartheta, Q\zeta, R\lambda, c)} - 1\right) = \frac{1}{2}\left(\frac{1}{M(\vartheta, \zeta, \lambda, c)} - 1\right) \leq \frac{1}{4}\left(\frac{1}{\Omega_1(\vartheta, \zeta, \lambda)} - 1\right),$$

$$N(T\vartheta, Q\zeta, R\lambda, c) = \frac{1}{2}N(\vartheta, \zeta, \lambda, c) = \frac{1}{4}\Omega_2(\vartheta, \zeta, \lambda)$$

$$O(T\vartheta, Q\zeta, R\lambda, c) = \frac{1}{2}O(\vartheta, \zeta, \lambda, c) = \frac{1}{4}\Omega_2(\vartheta, \zeta, \lambda)$$

for all $\vartheta, \zeta, \lambda \in X$. Thus, $T, Q$ and $R$ together satisfy the conditions (15)–(17) with $k = \frac{1}{4}$. Therefore, $T, Q$ and $R$ have a unique common fixed point, and it is $\vartheta = 3$.

**Corollary 3.18.** Let $(X, M, N, O, *, \circ, C)$ be a complete $GNCMS$ where $(M, N, O)$ be triangular. If $T: X \to X$ satisfy the following

$$\left(\frac{1}{M(T\vartheta, T\zeta, T\lambda, c)} - 1\right) \leq k\left(\frac{1}{\Omega_1(\vartheta, \zeta, \lambda)} - 1\right),$$

$$N(T\vartheta, T\zeta, T\lambda) \leq k\Omega_2(\vartheta, \zeta, \lambda),$$

$$O(T\vartheta, T\zeta, T\lambda) \leq k\Omega_3(\vartheta, \zeta, \lambda)$$

for all $\vartheta, \zeta, \lambda \in X$ and $c \in \text{int}(C)$ where $k \in (0,1)$ and

$$\Omega_1(\vartheta, \zeta, \lambda) = \min\{M(\vartheta, T\zeta, T\lambda, c), M(T\vartheta, \zeta, T\lambda, c), M(T\vartheta, T\zeta, \lambda, c)\},$$

$$\Omega_2(\vartheta, \zeta, \lambda) = \min\{N(\vartheta, T\zeta, T\lambda, c), N(T\vartheta, \zeta, T\lambda, c), N(T\vartheta, T\zeta, \lambda, c)\},$$

$$\Omega_3(\vartheta, \zeta, \lambda) = \min\{O(\vartheta, T\zeta, T\lambda, c), O(T\vartheta, \zeta, T\lambda, c), O(T\vartheta, T\zeta, \lambda, c)\}.$$

Then, $T$ has a unique fixed point.

*Application to fuzzy Fredholm integral equation*

Let $X = C([e, g], \mathbb{R})$ be the set of all real valued continuous functions defined on $[e, g]$ and $C = \mathbb{R}^+$.

Now, we consider the fuzzy Fredholm integral equation:

$$\vartheta(l) = f(j) + \delta \int_e^g F(l, j)\vartheta(l)dj \text{ for all } l, j \in [e, g] \tag{25}$$

where $\delta > 0, f(j)$ is a fuzzy function of $j \in [e, g]$ and $F \in X$. Define $M, N$ and $O$ by

$$M(\vartheta(l), \zeta(l), \lambda(l), c) = \sup_{l \in [e,g]} \frac{c}{c + |\vartheta(l) - \zeta(l)| + |\zeta(l) - \lambda(l)| + |\lambda(l) - \vartheta(l)|}$$

for all $\vartheta, \zeta, \lambda \in X$ and $c \in \text{int}(C)$,

$$N(\vartheta(l), \zeta(l), \lambda(l), c) = \sup_{l \in [e,g]} \frac{|\vartheta(l) - \zeta(l)| + |\zeta(l) - \lambda(l)| + |\lambda(l) - \vartheta(l)|}{c + |\vartheta(l) - \zeta(l)| + |\zeta(l) - \lambda(l)| + |\lambda(l) - \vartheta(l)|}$$

for all $\vartheta, \zeta, \lambda \in X$ and $c \in \text{int}(C)$, and

$$O(\vartheta(l),\zeta(l),\lambda(l),c) = \sup_{l\in[e,g]} \frac{|\vartheta(l)-\zeta(l)|+|\zeta(l)-\lambda(l)|+|\lambda(l)-\vartheta(l)|}{c}$$

for all $\vartheta,\zeta,\lambda \in X$ and $c \in \text{int}(C)$, with CTN and CTCN respectively. Define $\pi * \mu = \pi.\mu$ and $\pi \circ \mu = \max\{\pi,\mu\}$. Then $(X,M,N,O,*,\circ,C)$ is a complete GNCMS and $(M,N,O)$ is triangular. Assume that $|F(l,j)\vartheta(l) - F(l,j)\zeta(l)| \leq |\vartheta(l)-\zeta(l)|$ , $|F(l,j)\zeta(l) - F(l,j)\lambda(l)| \leq |\zeta(l)-\lambda(l)|$ , and $|F(l,j)\lambda(l) - F(l,j)\vartheta(l)| \leq |\lambda(l)-\vartheta(l)|$, for $\vartheta,\zeta,\lambda \in X$, $0<k<1$ and for all $l,j \in [e,g]$. Also consider $\delta \int_e^g dj \leq k < 1$. Then the fuzzy integral equation (25) has a unique solution.

*Proof*. Define $T: X \to X$ by

$$T\vartheta(l) = f(j) + \delta \int_e^g F(l,j)\vartheta(l)dj \text{ for all } l,j \in [e,g]$$

Scrutinize that survival of a fixed point of the operator $T$ is come from the survival of solution of fuzzy integral equation.

Now for all $\vartheta,\zeta,\lambda \in X$, we obtain

$$\left(\frac{1}{M(T\vartheta(l),T\zeta(l),T\lambda(l),c)} - 1\right)$$

$$= \sup_{l\in[e,g]}\left(\frac{1}{\frac{c}{c+|T\vartheta(l)-T\zeta(l)|+|T\zeta(l)-T\lambda(l)|+|T\lambda(l)-T\vartheta(l)|}} - 1\right)$$

Now,

$$|T\vartheta(l)-T\zeta(l)| = \left|\left(f(j)+\delta\int_e^g F(l,j)\vartheta(l)dj\right) - \left(f(j)+\ \delta\int_e^g F(l,j)\zeta(l)dj\right)\right|$$

$$= \left|\delta\int_e^g F(l,j)\vartheta(l)dj -\ \delta\int_e^g F(l,j)\zeta(l)dj\right| = |F(l,j)\vartheta(l) -\ F(l,j)\zeta(l)|\left(\delta\int_e^g dj\right)$$

$$\leq k|F(l,j)\vartheta(l) -\ F(l,j)\zeta(l)| \leq |\vartheta(l)-\zeta(l)|. \quad (26)$$

$$|T\zeta(l)-T\lambda(l)| = \left|\left(f(j)+\delta\int_e^g F(l,j)\zeta(l)dj\right) - \left(f(j)+\ \delta\int_e^g F(l,j)\lambda(l)dj\right)\right|$$

$$= \left|\delta\int_e^g F(l,j)\zeta(l)dj -\ \delta\int_e^g F(l,j)\lambda(l)dj\right| = |F(l,j)\zeta(l) -\ F(l,j)\lambda(l)|\left(\delta\int_e^g dj\right)$$

$$\leq k|F(l,j)\zeta(l) -\ F(l,j)\lambda(l)| \leq |\zeta(l)-\lambda(l)|. \quad (27)$$

$$|T\lambda(l)-T\vartheta(l)| = \left|\left(f(j)+\delta\int_e^g F(l,j)\lambda(l)dj\right) - \left(f(j)+\ \delta\int_e^g F(l,j)\vartheta(l)dj\right)\right|$$

$$= \left|\delta\int_e^g F(l,j)\lambda(l)dj -\ \delta\int_e^g F(l,j)\vartheta(l)dj\right| = |F(l,j)\lambda(l) -\ F(l,j)\vartheta(l)|\left(\delta\int_e^g dj\right)$$

$$\leq k|F(l,j)\lambda(l) -\ F(l,j)\vartheta(l)| \leq |\lambda(l)-\vartheta(l)|. \quad (28)$$

Hence,

$$\left(\frac{1}{M(T\vartheta(l),T\zeta(l),T\lambda(l),c)}-1\right)$$
$$\leq k\sup_{l\in[e,g]}\left(\frac{1}{\frac{c}{c+|\vartheta(l)-\zeta(l)|+|\zeta(l)-\lambda(l)|+|\lambda(l)-\vartheta(l)|}}-1\right)$$
$$=k\left(\frac{1}{M(\vartheta(l),\zeta(l),\lambda(l),c)}-1\right).$$

Now,

$$N(T\vartheta(l),T\zeta(l),T\lambda(l),c)=\sup_{l\in[e,g]}\frac{|T\vartheta(l)-T\zeta(l)|+|T\zeta(l)-T\lambda(l)|+|T\lambda(l)-T\vartheta(l)|}{c+|T\vartheta(l)-T\zeta(l)|+|T\zeta(l)-T\lambda(l)|+|T\lambda(l)-T\vartheta(l)|}$$

By using (26)–(28), we can examine that

$$N(T\vartheta(l),T\zeta(l),T\lambda(l),c)\leq k\sup_{l\in[e,g]}\frac{|\vartheta(l)-\zeta(l)|+|\zeta(l)-\lambda(l)|+|\lambda(l)-\vartheta(l)|}{c+|\vartheta(l)-\zeta(l)|+|\zeta(l)-\lambda(l)|+|\lambda(l)-\vartheta(l)|}$$
$$=N(\vartheta(l),\zeta(l),\lambda(l),c),$$

and

$$O(T\vartheta(l),T\zeta(l),T\lambda(l),c)=\sup_{l\in[e,g]}\frac{|T\vartheta(l)-T\zeta(l)|+|T\zeta(l)-T\lambda(l)|+|T\lambda(l)-T\vartheta(l)|}{c}$$

By using (26)–(28), we can examine that

$$N(T\vartheta(l),T\zeta(l),T\lambda(l),c)\leq k\sup_{l\in[e,g]}\frac{|\vartheta(l)-\zeta(l)|+|\zeta(l)-\lambda(l)|+|\lambda(l)-\vartheta(l)|}{c}$$
$$=O(\vartheta(l),\zeta(l),\lambda(l),c).$$

Therefore, all the circumstances of Corollary 3.14 are fulfilled. Hence operator $T$ has a unique fixed point. This implies that fuzzy integral equation (25) has a unique solution.

## 4. $\xi$-chainable neutrosophic metric space

In this section, we introduce the notion of $\xi$-chainable NMS and prove a fixed point theorem by using four self-maps.

**Definition 4.1.** Let $(X,E,H,Z,*,\circ)$ be an NMS and $T,S:X\to X$ are mappings. A point $\vartheta\in X$ is called coincident point of $T$ and $S$ if and only if $T\vartheta=S\vartheta$.

**Definition 4.2.** A self-mapping pair $(T,S)$ of an NMS is said to be weakly compatible if they commute at the coincident points i.e., $T\vartheta=S\vartheta$ for some $\vartheta\in X$, then $TS\vartheta=ST\vartheta$.

In [7] definitions of convergent sequence, Cauchy sequence and completeness in the sense of NMS are defined as follows.

**Definition 4.3.** A sequence $\{\vartheta_n\}$ in an NMS $(X,E,H,Z,*,\circ)$ is said to be convergent to $\vartheta\in X$, if

$$\lim_{n\to+\infty}E(\vartheta_n,\vartheta,t)=1,\ \text{ for all } t>0,$$
$$\lim_{n\to+\infty}H(\vartheta_n,\vartheta,t)=0,\ \text{ for all } t>0,$$

$$\lim_{n\to+\infty} Z(\vartheta_n,\vartheta,t)=0,\ \text{ for all } t>0.$$

**Definition 4.4.** A sequence $\{\vartheta_n\}$ in an NMS $(X,E,H,Z,*,\circ)$ is said to be Cauchy if there exists $n\in\mathbb{N}$ such that

$$\lim_{n\to+\infty} E(\vartheta_n,\vartheta_{n+p},t)=1,$$
$$\lim_{n\to+\infty} H(\vartheta_n,\vartheta_{n+p},t)=0,$$
$$\lim_{n\to+\infty} T(\vartheta_n,\vartheta_{n+p},t)=0,$$

for all $t\geq 0, p\geq 1$.

**Definition 4.5.** An NMS $(X,E,H,Z,*,\circ)$ is said to complete if every Cauchy sequence is convergent in $X$.

**Definition 4.6.** Let $(X,E,H,Z,*,\circ)$ be an NMS. A finite sequence $\vartheta=\vartheta_0,\vartheta_1,\vartheta_2,\cdots,\vartheta_n=\zeta$ is called $\xi$-chain from $\vartheta$ to $\zeta$ if there exists a positive integer $\xi>0$ such that $E(\vartheta_i,\vartheta_{i-1},t)>1-\xi$, $H(\vartheta_i,\vartheta_{i-1},t)<1-\xi$ and $Z(\vartheta_i,\vartheta_{i-1},t)<1-\xi$ for all $t>0$ and $i=1,2,3,\ldots,n$.

An NMS $(X,E,H,Z,*,\circ)$ is called $\xi$-chainable if for any $\vartheta,\zeta\in X$, there exists an $\xi$-chain from $\vartheta$ to $\zeta$.

**Lemma 4.7.** Let $(X,E,H,Z,*,\circ)$ be an NMS, if $E(\vartheta,\zeta,kt)\geq E(\vartheta,\zeta,t), H(\vartheta,\zeta,kt)\leq H(\vartheta,\zeta,t)$ and $Z(\vartheta,\zeta,kt)\leq Z(\vartheta,\zeta,t)$ for a number $k\in(0,1)$ and for all $\vartheta,\zeta\in X, t>0$, then $\vartheta=\zeta$.

*Proof.* Since $E(\vartheta,\zeta,kt)\geq E(\vartheta,\zeta,t), H(\vartheta,\zeta,kt)\leq H(\vartheta,\zeta,t)$ and $Z(\vartheta,\zeta,kt)\leq Z(\vartheta,\zeta,t)$, by using results in [17], we obtain

$$E(\vartheta,\zeta,t)\geq E\left(\vartheta,\zeta,\frac{t}{k}\right), H(\vartheta,\zeta,t)\leq H\left(\vartheta,\zeta,\frac{t}{k}\right) \text{ and } Z(\vartheta,\zeta,t)\leq Z\left(\vartheta,\zeta,\frac{t}{k}\right).$$

By repeating application of above inequalities, we deduce

$$E(\vartheta,\zeta,t)\geq E\left(\vartheta,\zeta,\frac{t}{k}\right)\geq E\left(\vartheta,\zeta,\frac{t}{k^2}\right)\geq E\left(\vartheta,\zeta,\frac{t}{k^3}\right)\geq\cdots\geq E\left(\vartheta,\zeta,\frac{t}{k^n}\right)\geq\cdots,$$
$$H(\vartheta,\zeta,t)\leq H\left(\vartheta,\zeta,\frac{t}{k}\right)\leq H\left(\vartheta,\zeta,\frac{t}{k^2}\right)\leq H\left(\vartheta,\zeta,\frac{t}{k^3}\right)\leq\cdots\leq H\left(\vartheta,\zeta,\frac{t}{k^n}\right)\leq\cdots$$
$$Z(\vartheta,\zeta,t)\leq Z\left(\vartheta,\zeta,\frac{t}{k}\right)\leq Z\left(\vartheta,\zeta,\frac{t}{k^2}\right)\leq Z\left(\vartheta,\zeta,\frac{t}{k^3}\right)\leq\cdots\leq Z\left(\vartheta,\zeta,\frac{t}{k^n}\right)\leq\cdots$$

for $n\in\mathbb{N}$, by proceeding limit as $n\to+\infty$, we obtain $E(\vartheta,\zeta,t)=1, H(\vartheta,\zeta,t)=0$ and $Z(\vartheta,\zeta,t)=0$ for all $t>0$ and by using definition of an NMS we have $\vartheta=\zeta$.

**Lemma 4.8.** Let $(X,E,H,Z,*,\circ)$ be an NMS. For a number $k\in(0,1)$ and a sequence $\{\zeta_n\}$ such that

$$E(\zeta_{n+2},\zeta_{n+1},kt)\geq E(\zeta_{n+1},\zeta_n,t), \tag{29}$$

$$H(\zeta_{n+2},\zeta_{n+1},kt)\leq H(\zeta_{n+1},\zeta_n,t) \tag{30}$$

$$Z(\zeta_{n+2},\zeta_{n+1},kt)\leq Z(\zeta_{n+1},\zeta_n,t) \tag{31}$$

for all $t>0$ and $n\in\mathbb{N}$, $\{\zeta_n\}$ is a Cauchy sequence.

*Proof.* By using induction with (29)–(31), we obtain for all $t>0$ and $n=0,1,2,\cdots$,

$$E(\zeta_{n+1},\zeta_{n+2},t)\geq E\left(\zeta_1,\zeta_2,\tfrac{t}{k^n}\right), \tag{32}$$

$$H(\zeta_{n+1},\zeta_{n+2},t)\leq H\left(\zeta_1,\zeta_2,\tfrac{t}{k^n}\right), \tag{33}$$

$$Z(\zeta_{n+1},\zeta_{n+2},t)\leq Z\left(\zeta_1,\zeta_2,\tfrac{t}{k^n}\right). \tag{34}$$

Thus by using (32)–(34) and definition of NMS, for any $p \in \mathbb{N}$, we get

$$E(\zeta_n, \zeta_{n+p}, t) \geq E\left(\zeta_n, \zeta_{n+1}, \frac{t}{p}\right)^{p-\text{times}} * \cdots * E\left(\zeta_{n+p-1}, \zeta_{n+p}, \frac{t}{p}\right)$$

$$\geq E\left(\zeta_1, \zeta_2, \frac{t}{pk^{n-1}}\right)^{p-\text{times}} * \cdots * E\left(\zeta_1, \zeta_2, \frac{t}{pk^{n+p-2}}\right),$$

$$H(\zeta_n, \zeta_{n+p}, t) \leq H\left(\zeta_n, \zeta_{n+1}, \frac{t}{p}\right)^{p-\text{times}} \circ \cdots \circ H\left(\zeta_{n+p-1}, \zeta_{n+p}, \frac{t}{p}\right)$$

$$\leq H\left(\zeta_1, \zeta_2, \frac{t}{pk^{n-1}}\right)^{p-\text{times}} \circ \cdots \circ H\left(\zeta_1, \zeta_2, \frac{t}{pk^{n+p-2}}\right),$$

$$Z(\zeta_n, \zeta_{n+p}, t) \leq Z\left(\zeta_n, \zeta_{n+1}, \frac{t}{p}\right)^{p-\text{times}} \circ \cdots \circ Z\left(\zeta_{n+p-1}, \zeta_{n+p}, \frac{t}{p}\right)$$

$$\leq Z\left(\zeta_1, \zeta_2, \frac{t}{pk^{n-1}}\right)^{p-\text{times}} \circ \cdots \circ Z\left(\zeta_1, \zeta_2, \frac{t}{pk^{n+p-2}}\right).$$

Therefore, by using the definition of NMS, we obtain

$$\lim_{n\to+\infty} E(\zeta_n, \zeta_{n+p}, t) \geq 1^{p-\text{times}} * \cdots * 1 = 1,$$

$$\lim_{n\to+\infty} H(\zeta_n, \zeta_{n+p}, t) \leq 0^{p-\text{times}} \circ \cdots \circ 0 = 0,$$

$$\lim_{n\to+\infty} H(\zeta_n, \zeta_{n+p}, t) \leq 0^{p-\text{times}} \circ \cdots \circ 0 = 0.$$

Hence, $\{\zeta_n\}$ is a Cauchy sequence in $X$.

**Theorem 4.9.** Assume $A, B, S$ and $T$ are self mappings of a G-complete $\xi$-chainable NMS $(X, E, H, Z, *, \circ)$ with CTN $*$ and CTCN $\circ$ defined by $a * a \geq a$ and $(1-a) \circ (1-a) \leq (1-a)$ for all $a \in [0,1]$ fulfilling the below circumstances:

1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,
2) $A$ and $S$ are continuous.
3) The pairs $(A, S)$ and $(B, T)$ are W-compatible.
4) There exists $q \in (0,1)$ such that

$$E(A\vartheta, B\zeta, qt) \geq \Big\{E(S\vartheta, T\zeta, t) * E(S\vartheta, A\vartheta, t) * \frac{1}{2}[E(S\vartheta, T\zeta, t) + E(A\vartheta, T\zeta, t)] * \frac{1}{2}[E(A\vartheta, B\zeta, t) + E(S\vartheta, B\zeta, t)] * E(B\zeta, T\zeta, t) * E(A\vartheta, T\zeta, t) * E(S\vartheta, B\zeta, t)\Big\},$$

$$H(A\vartheta, B\zeta, qt) \leq \Big\{H(S\vartheta, T\zeta, t) \circ H(S\vartheta, A\vartheta, t) \circ \frac{1}{2}[H(S\vartheta, T\zeta, t) + H(A\vartheta, T\zeta, t)] \circ \frac{1}{2}[H(A\vartheta, B\zeta, t) + H(S\vartheta, B\zeta, t)] \circ H(B\zeta, T\zeta, t) \circ H(A\vartheta, T\zeta, t) \circ H(S\vartheta, B\zeta, t)\Big\},$$

$$Z(A\vartheta, B\zeta, qt) \leq \Big\{Z(S\vartheta, T\zeta, t) \circ Z(S\vartheta, A\vartheta, t) \circ \frac{1}{2}[Z(S\vartheta, T\zeta, t) + Z(A\vartheta, T\zeta, t)] \circ \frac{1}{2}[Z(A\vartheta, B\zeta, t) + Z(S\vartheta, B\zeta, t)] \circ Z(B\zeta, T\zeta, t) \circ Z(A\vartheta, T\zeta, t) \circ Z(S\vartheta, B\zeta, t)\Big\}$$

for all $\vartheta, \zeta \in X$ and $t > 0$. Then $A, B, S$ and $T$ have a common unique fixed point.

*Proof.* As $A(X) \subseteq T(X)$, for any point $\vartheta_0 \in X$ there exists $\vartheta_1 \in X$ such that $A\vartheta_0 = T\vartheta_1$. Since $B(X) \subseteq S(X)$, for $\vartheta_1$, we can pick $\vartheta_2 \in X$ such that $B\vartheta_1 = S\vartheta_2$. By induction, we obtain a sequence $\{\zeta_n\}$ in $X$ as follows:

$$\zeta_{2n-1} = T\vartheta_{2n-1} = A\vartheta_{2n-2} \text{ and } \zeta_{2n} = S\vartheta_{2n} = B\vartheta_{2n} \text{ for } n \in \mathbb{N}.$$

We can easily examine that $\{\zeta_n\}$ is a sequence in $X$ and by the completeness of $(X, E, H, Z, *, \circ)$, the sequence $\{\zeta_n\}$ converges to $z \in X$. Hence, the sequences $\{T\vartheta_{2n-1}\}, \{A\vartheta_{2n-2}\}, \{S\vartheta_{2n}\}$ and $\{B\vartheta_{2n}\}$ are also convergent to $z \in X$.

Since $X$ is $\xi$-chainable there exists $\xi$-chain from $\vartheta_n$ to $\vartheta_{n+1}$, that is,

$$\vartheta_n = \zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_l = \vartheta_{n+1}$$

such that

$$E(\zeta_i, \zeta_{i+1}, t) > 1 - \xi, H(\zeta_i, \zeta_{i+1}, t) < 1 - \xi \text{ and } Z(\zeta_i, \zeta_{i+1}, t) < 1 - \xi$$

for all $t > 0$ and $i = 1,2,3, \cdots, l$. Thus

$$\begin{aligned} E(\vartheta_n, \vartheta_{n+1}, t) &\geq E\left(\zeta_1, \zeta_2, \frac{t}{l}\right) * E\left(\zeta_2, \zeta_3, \frac{t}{l}\right) * \cdots * E\left(\zeta_{l-1}, \zeta_l, \frac{t}{l}\right) > (1-\xi) * (1-\xi) * \cdots * (1-\xi) \\ &\geq (1-\xi), \end{aligned}$$

$$\begin{aligned} H(\vartheta_n, \vartheta_{n+1}, t) &\leq H\left(\zeta_1, \zeta_2, \frac{t}{l}\right) \circ H\left(\zeta_2, \zeta_3, \frac{t}{l}\right) \circ \cdots \circ H\left(\zeta_{l-1}, \zeta_l, \frac{t}{l}\right) \\ &< (1-\xi) \circ (1-\xi) \circ \cdots \circ (1-\xi) \leq (1-\xi), \end{aligned}$$

$$\begin{aligned} Z(\vartheta_n, \vartheta_{n+1}, t) &\leq Z\left(\zeta_1, \zeta_2, \frac{t}{l}\right) \circ Z\left(\zeta_2, \zeta_3, \frac{t}{l}\right) \circ \cdots \circ Z\left(\zeta_{l-1}, \zeta_l, \frac{t}{l}\right) < (1-\xi) \circ (1-\xi) \circ \cdots \circ (1-\xi) \\ &\leq (1-\xi). \end{aligned}$$

For $m > n$,

$$\begin{aligned} E(\vartheta_n, \vartheta_m, t) &\geq E\left(\vartheta_n, \vartheta_{n+1}, \frac{t}{m-n}\right) * E\left(\vartheta_{n+1}, \vartheta_{n+2}, \frac{t}{m-n}\right) * \cdots * E\left(\vartheta_{m-1}, \vartheta_m, \frac{t}{m-n}\right) \\ &> (1-\xi) * (1-\xi) * \cdots * (1-\xi) \geq (1-\xi), \end{aligned}$$

$$\begin{aligned} H(\vartheta_n, \vartheta_m, t) &\leq H\left(\vartheta_n, \vartheta_{n+1}, \frac{t}{m-n}\right) \circ H\left(\vartheta_{n+1}, \vartheta_{n+2}, \frac{t}{m-n}\right) \circ \cdots \circ H\left(\vartheta_{m-1}, \vartheta_m, \frac{t}{m-n}\right) \\ &< (1-\xi) * (1-\xi) \circ \cdots \circ (1-\xi) \leq (1-\xi), \end{aligned}$$

$$\begin{aligned} Z(\vartheta_n, \vartheta_m, t) &\leq Z\left(\vartheta_n, \vartheta_{n+1}, \frac{t}{m-n}\right) \circ Z\left(\vartheta_{n+1}, \vartheta_{n+2}, \frac{t}{m-n}\right) \circ \cdots \circ Z\left(\vartheta_{m-1}, \vartheta_m, \frac{t}{m-n}\right) \\ &< (1-\xi) * (1-\xi) \circ \cdots \circ (1-\xi) \leq (1-\xi). \end{aligned}$$

Therefore $\{\vartheta_n\}$ is a Cauchy sequence in $X$, hence there exists $\vartheta \in X$ such that $\{\vartheta_n\}$ converges $\vartheta$. By using (2) $A\vartheta_{2n-2} \to A\vartheta, S\vartheta_{2n} \to S\vartheta$ as $n \to +\infty$. From the uniqueness of limits, we get $A\vartheta = z = S\vartheta$. Since the pair $(A, S)$ is W-compatible, $AS\vartheta = SA\vartheta$ and so $Az = Sz$. By using (2) we have $AS\vartheta_{2n} \to AS\vartheta$ and therefore, $AS\vartheta_{2n} \to Sz$. By the continuity of $S$, we have $SS\vartheta_{2n} \to Sz$. By using (4), we obtain

$$E(AS\vartheta_{2n}, B\vartheta_{2n-1}, qt) \geq \Big\{E(SS\vartheta_{2n}, T\vartheta_{2n-1}, t) * E(SS\vartheta_{2n}, AS\vartheta_{2n}, t) * \frac{1}{2}[E(SS\vartheta_{2n}, T\vartheta_{2n-1}, t) + E(AS\vartheta_{2n}, T\vartheta_{2n-1}, t)] * \frac{1}{2}[E(AS\vartheta_{2n}, B\vartheta_{2n-1}, t) + E(SS\vartheta_{2n}, B\vartheta_{2n-1}, t)] * E(B\vartheta_{2n-1}, T\vartheta_{2n-1}, t) * E(AS\vartheta_{2n}, T\vartheta_{2n-1}, t) * E(SS\vartheta_{2n}, B\vartheta_{2n-1}, t)\Big\},$$

$$H(AS\vartheta_{2n}, B\vartheta_{2n-1}, qt) \leq \Big\{H(SS\vartheta_{2n}, T\vartheta_{2n-1}, t) \circ H(SS\vartheta_{2n}, AS\vartheta_{2n}, t) \circ \frac{1}{2}[H(SS\vartheta_{2n}, T\vartheta_{2n-1}, t) + H(AS\vartheta_{2n}, T\vartheta_{2n-1}, t)] \circ \frac{1}{2}[H(AS\vartheta_{2n}, B\vartheta_{2n-1}, t) + H(SS\vartheta_{2n}, B\vartheta_{2n-1}, t)] \circ H(B\vartheta_{2n-1}, T\vartheta_{2n-1}, t) \circ H(AS\vartheta_{2n}, T\vartheta_{2n-1}, t) \circ H(SS\vartheta_{2n}, B\vartheta_{2n-1}, t)\Big\},$$

$$Z(AS\vartheta_{2n}, B\vartheta_{2n-1}, qt) \leq \Big\{Z(SS\vartheta_{2n}, T\vartheta_{2n-1}, t) \circ Z(SS\vartheta_{2n}, AS\vartheta_{2n}, t) \circ \frac{1}{2}[Z(SS\vartheta_{2n}, T\vartheta_{2n-1}, t) + Z(AS\vartheta_{2n}, T\vartheta_{2n-1}, t)] \circ \frac{1}{2}[Z(AS\vartheta_{2n}, B\vartheta_{2n-1}, t) + Z(SS\vartheta_{2n}, B\vartheta_{2n-1}, t)] \circ Z(B\vartheta_{2n-1}, T\vartheta_{2n-1}, t) \circ Z(AS\vartheta_{2n}, T\vartheta_{2n-1}, t) \circ Z(SS\vartheta_{2n}, B\vartheta_{2n-1}, t)\Big\}.$$

Taking the limit as $n \to +\infty$, we deduce

$$E(Sz, z, qt) \geq \Big\{E(Sz, z, t) * E(Sz, Sz, t) * \frac{1}{2}[E(Sz, z, t) + E(Sz, z, t)] * \frac{1}{2}[E(Sz, z, t) + E(Sz, z, t)] * E(z, z, t) * E(Sz, z, t) * E(Sz, z, t)\Big\},$$

$$H(Sz, z, qt) \leq \Big\{H(Sz, z, t) \circ H(Sz, Sz, t) \circ \frac{1}{2}[H(Sz, z, t) + H(Sz, z, t)] \circ \frac{1}{2}[H(Sz, z, t) + H(Sz, z, t)] \circ H(z, z, t) \circ H(Sz, z, t) \circ H(Sz, z, t)\Big\},$$

$$Z(Sz, z, qt) \leq \Big\{Z(Sz, z, t) \circ Z(Sz, Sz, t) \circ \frac{1}{2}[Z(Sz, z, t) + Z(Sz, z, t)] \circ \frac{1}{2}[Z(Sz, z, t) + Z(Sz, z, t)] \circ Z(z, z, t) \circ Z(Sz, z, t) \circ Z(Sz, z, t)\Big\}.$$

By Lemma 4.7, we obtain $Sz = z$, hence $Az = Sz = z$. Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that $Tv = Az = z$. By using (4), we get

$$\begin{aligned}
&E(A\vartheta_{2n}, Bv, qt)\\
&\qquad \geq \Big\{E(S\vartheta_{2n}, Tv, t) * E(S\vartheta_{2n}, A\vartheta_{2n}, t) * \frac{1}{2}[E(S\vartheta_{2n}, Tv, t) + E(A\vartheta_{2n}, Tv, t)]\\
&\qquad * \frac{1}{2}[E(A\vartheta_{2n}, Bv, t) + E(S\vartheta_{2n}, Bv, t)] * E(Bv, Tv, t) * E(A\vartheta_{2n}, Tv, t)\\
&\qquad * E(S\vartheta_{2n}, Bv, t)\Big\},\\
&H(A\vartheta_{2n}, Bv, qt)\\
&\qquad \leq \Big\{H(S\vartheta_{2n}, Tv, t) \circ H(S\vartheta_{2n}, A\vartheta_{2n}, t) \circ \frac{1}{2}[H(S\vartheta_{2n}, Tv, t) + H(A\vartheta_{2n}, Tv, t)]\\
&\qquad \circ \frac{1}{2}[H(A\vartheta_{2n}, Bv, t) + H(S\vartheta_{2n}, Bv, t)] \circ H(Bv, Tv, t) \circ H(A\vartheta_{2n}, Tv, t)\\
&\qquad \circ H(S\vartheta_{2n}, Bv, t)\Big\},\\
&Z(A\vartheta_{2n}, Bv, qt)\\
&\qquad \leq \Big\{Z(S\vartheta_{2n}, Tv, t) \circ Z(S\vartheta_{2n}, A\vartheta_{2n}, t) \circ \frac{1}{2}[Z(S\vartheta_{2n}, Tv, t) + Z(A\vartheta_{2n}, Tv, t)]\\
&\qquad \circ \frac{1}{2}[Z(A\vartheta_{2n}, Bv, t) + Z(S\vartheta_{2n}, Bv, t)] \circ Z(Bv, Tv, t) \circ Z(A\vartheta_{2n}, Tv, t)\\
&\qquad \circ Z(S\vartheta_{2n}, Bv, t)\Big\}.
\end{aligned}$$

Taking the limit $n \to +\infty$, we obtain

$$\begin{aligned}
E(z, Bv, qt) &\geq \Big\{E(z, Tv, t) * E(z, z, t) * \frac{1}{2}[E(z, Tv, t) + E(z, Tv, t)] * \frac{1}{2}[E(z, Bv, t) + E(z, Bv, t)]\\
&\quad * E(Bv, Tv, t) * E(z, Tv, t) * E(z, Bv, t)\Big\}
\end{aligned}$$

$$\begin{aligned}
&= \Big\{E(z, z, t) * E(z, z, t) * \frac{1}{2}[E(z, z, t) + E(z, z, t)] * \frac{1}{2}[E(z, Bv, t) + E(z, Bv, t)] * E(Bv, z, t)\\
&\quad * E(z, z, t) * E(z, Bv, t)\Big\} \geq E(Bv, z, t),
\end{aligned}$$

$$\begin{aligned}
H(z, Bv, qt) &\leq \Big\{H(z, Tv, t) \circ H(z, z, t) \circ \frac{1}{2}[H(z, Tv, t) + H(z, Tv, t)]\\
&\quad \circ \frac{1}{2}[H(z, Bv, t) + H(z, Bv, t)] \circ H(Bv, Tv, t) \circ H(z, Tv, t) \circ H(z, Bv, t)\Big\}
\end{aligned}$$

$$\begin{aligned}
&= \Big\{H(z, z, t) \circ H(z, z, t) \circ \frac{1}{2}[H(z, z, t) + H(z, z, t)] \circ \frac{1}{2}[H(z, Bv, t) + H(z, Bv, t)] \circ H(Bv, z, t)\\
&\quad \circ H(z, z, t) \circ H(z, Bv, t)\Big\} \leq H(Bv, z, t),
\end{aligned}$$

$$\begin{aligned}
Z(z, Bv, qt) &\leq \Big\{Z(z, Tv, t) \circ Z(z, z, t) \circ \frac{1}{2}[Z(z, Tv, t) + Z(z, Tv, t)] \circ \frac{1}{2}[Z(z, Bv, t) + Z(z, Bv, t)]\\
&\quad \circ Z(Bv, Tv, t) \circ Z(z, Tv, t) \circ Z(z, Bv, t)\Big\}
\end{aligned}$$

$$\begin{aligned}
&= \Big\{Z(z, z, t) \circ Z(z, z, t) \circ \frac{1}{2}[Z(z, z, t) + Z(z, z, t)] \circ \frac{1}{2}[Z(z, Bv, t) + Z(z, Bv, t)] \circ Z(Bv, z, t)\\
&\quad \circ Z(z, z, t) \circ Z(z, Bv, t)\Big\} \leq Z(Bv, z, t).
\end{aligned}$$

By Lemma 4.7, we get $Bv = z$, and therefore, we obtain $Tv = Bv = z$. Since $(B,T)$ is W-compatible, $TBv = BTv$ and hence $Tz = Bz$. By using (4), we deduce

$$E(A\vartheta_{2n}, Bz, qt) \geq \Big\{E(S\vartheta_{2n}, Tz, t) * E(S\vartheta_{2n}, A\vartheta_{2n}, t) * \frac{1}{2}[E(S\vartheta_{2n}, Tz, t) + E(A\vartheta_{2n}, Tz, t)] * \frac{1}{2}[E(A\vartheta_{2n}, Bz, t) + E(S\vartheta_{2n}, Bz, t)] * E(Bz, Tz, t) * E(A\vartheta_{2n}, Tz, t) * E(S\vartheta_{2n}, Bz, t)\Big\},$$

$$H(A\vartheta_{2n}, Bz, qt) \leq \Big\{H(S\vartheta_{2n}, Tz, t) \circ H(S\vartheta_{2n}, A\vartheta_{2n}, t) \circ \frac{1}{2}[H(S\vartheta_{2n}, Tz, t) + H(A\vartheta_{2n}, Tz, t)] \circ \frac{1}{2}[H(A\vartheta_{2n}, Bz, t) + H(S\vartheta_{2n}, Bz, t)] \circ H(Bz, Tz, t) \circ H(A\vartheta_{2n}, Tz, t) \circ H(S\vartheta_{2n}, Bz, t)\Big\},$$

$$Z(A\vartheta_{2n}, Bz, qt) \leq \Big\{Z(S\vartheta_{2n}, Tz, t) \circ Z(S\vartheta_{2n}, A\vartheta_{2n}, t) \circ \frac{1}{2}[Z(S\vartheta_{2n}, Tz, t) + Z(A\vartheta_{2n}, Tz, t)] \circ \frac{1}{2}[Z(A\vartheta_{2n}, Bz, t) + Z(S\vartheta_{2n}, Bz, t)] \circ Z(Bz, Tz, t) \circ Z(A\vartheta_{2n}, Tz, t) \circ Z(S\vartheta_{2n}, Bz, t)\Big\}.$$

Taking the limit $n \to +\infty$, we obtain

$$E(z, Bz, qt) \geq \Big\{E(z, Tz, t) * E(z, z, t) * \frac{1}{2}[E(z, Tz, t) + E(z, Tz, t)] * \frac{1}{2}[E(z, Bz, t) + E(z, Bz, t)] * E(Bz, Tz, t) * E(z, Tz, t) * E(z, Bz, t)\Big\}$$

$$= \Big\{E(z, z, t) * E(z, z, t) * \frac{1}{2}[E(z, z, t) + E(z, z, t)] * \frac{1}{2}[E(z, Bz, t) + E(z, Bz, t)] * E(Bz, z, t) * E(z, z, t) * E(z, Bz, t)\Big\} \geq E(Bz, z, t),$$

$$H(z, Bz, qt) \leq \Big\{H(z, Tz, t) \circ H(z, z, t) \circ \frac{1}{2}[H(z, Tz, t) + H(z, Tz, t)] \circ \frac{1}{2}[H(z, Bz, t) + H(z, Bz, t)] \circ H(Bz, Tz, t) \circ H(z, Tz, t) \circ H(z, Bz, t)\Big\}$$

$$= \Big\{H(z, z, t) \circ H(z, z, t) \circ \frac{1}{2}[H(z, z, t) + H(z, z, t)] \circ \frac{1}{2}[H(z, Bz, t) + H(z, Bz, t)] \circ H(Bz, z, t) \circ H(z, z, t) \circ H(z, Bz, t)\Big\} \leq H(Bz, z, t),$$

$$Z(z,Bz,qt) \leq \Big\{Z(z,Tz,t) \circ Z(z,z,t) \circ \frac{1}{2}[Z(z,Tz,t) + Z(z,Tz,t)] \circ \frac{1}{2}[Z(z,Bz,t) + Z(z,Bz,t)] \circ Z(Bz,Tz,t) \circ Z(z,Tz,t) \circ Z(z,Bz,t)\Big\}$$
$$= \Big\{Z(z,z,t) \circ Z(z,z,t) \circ \frac{1}{2}[Z(z,z,t) + Z(z,z,t)] \circ \frac{1}{2}[Z(z,Bz,t) + Z(z,Bz,t)] \circ Z(Bz,z,t) \circ Z(z,z,t) \circ Z(z,Bz,t)\Big\} \leq Z(Bz,z,t).$$

The above inequalities imply that $Bz = z$. Therefore, $Az = Sz = Bz = Tz = z$. Hence $A, B, S$ and $T$ have a common fixed point in $X$.
Now we are going to examine the uniqueness. For this, let $w$ be another common fixed point of $A, B, S$ and $T$. By using (4), we obtain

$$E(z,w,qt) = E(Az,Bw,qt)$$
$$\geq \Big\{E(Sz,Tw,t) * E(Sz,Az,t) * \frac{1}{2}[E(Sz,Tw,t) + E(Az,Tw,t)] * \frac{1}{2}[E(Az,Bw,t) + E(Sz,Bw,t)] * E(Bw,Tw,t) * E(Az,Tw,t) * E(Sz,Bw,t)\Big\}$$
$$\geq E(z,w,t),$$

$$H(z,w,qt) = H(Az,Bw,qt)$$
$$\leq \Big\{H(Sz,Tw,t) \circ H(Sz,Az,t) \circ \frac{1}{2}[H(Sz,Tw,t) + H(Az,Tw,t)] \circ \frac{1}{2}[H(Az,Bw,t) + H(Sz,Bw,t)] \circ H(Bw,Tw,t) \circ H(Az,Tw,t) \circ H(Sz,Bw,t)\Big\}$$
$$\leq H(z,w,t),$$

$$Z(z,w,qt) = Z(Az,Bw,qt)$$
$$\leq \Big\{Z(Sz,Tw,t) \circ Z(Sz,Az,t) \circ \frac{1}{2}[Z(Sz,Tw,t) + Z(Az,Tw,t)] \circ \frac{1}{2}[Z(Az,Bw,t) + Z(Sz,Bw,t)] \circ Z(Bw,Tw,t) \circ Z(Az,Tw,t) \circ Z(Sz,Bw,t)\Big\}$$
$$\leq Z(z,w,t).$$

By Lemma 4.7, we obtain that $z = w$. Hence *A*, *B*, *S* and *T* have a unique common fixed point in *X*.

## 5. Conclusions

We introduced the novel concepts of generalized neutrosophic cone metric space and $\xi$-chainable neutrosophic metric space. We investigated common fixed point results for two and three self-mappings in the sense of generalized neutrosophic cone metric space and common fixed point results for four self mappings in the sense of $\xi$-chainable neutrosophic metric space. The uniqueness of solution is investigated by using fuzzy Fredholm integral equation of second kind. This new setting has many applications in fuzzy analysis and it will open new doors to generalize common fixed point results. Also this work can be extended to other self-mappings.

## Conflict of interest

The authors declare no conflict of interest.

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