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The Cauchy problem for coupled system of the generalized Camassa-Holm equations

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Abstract: Local well-posedness for the Cauchy problem of coupled system of generalized Camassa-Holm equations in the Besov spaces is established by employing the Littlewood-Paley theory and a priori estimate of solution to transport equation. Furthermore, the blow-up criterion of solutions to the problem is illustrated. Our main new contribution is that the effects of dissipative coefficient λ and exponent b in the nonlinear terms to the solutions are analyzed. To the best of our knowledge, the results in Theorems 1.1 and 1.2 are new.

Keywords: local well-posedness; coupled system; generalized Camassa-Holm equations; blow-up criterion
Mathematics Subject Classification: 35G25, 35L15

1. Introduction

The main purpose of this work is to investigate the following coupled system of generalized Camassa-Holm equations

$$\begin{cases} m_t + u^{b-1}vm_x + (b+1)u_xv^{b-1}m + \lambda m = 0, & t>0,\ x\in\mathbb{R}, \\ n_t + v^{b-1}un_x + (b+1)v_xu^{b-1}n + \lambda n = 0, & t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $m(t,x) = (1-\partial_x^2)u(t,x)$, $n(t,x) = (1-\partial_x^2)v(t,x)$, $b \in \mathbb{R}$. $\lambda(u-u_{xx})$ and $\lambda(v-v_{xx})$ stand for the dissipative terms. $\lambda \in \mathbb{R}^+$ is the dissipative coefficient. $(t,x) \in \mathbb{R}^+\times\mathbb{R}$ and $(u_0, v_0) \in B^s_{p,r}(\mathbb{R})\times B^s_{p,r}(\mathbb{R})(s > \max(\frac{5}{2}, 2+\frac{1}{p}))$.

With the rapid development of science and technology, the problem of shallow water wave propagation has become one of the hot issues in the theory of water wave motion. The study of shallow water wave equation originates from the phenomenon of solitary wave on free surface of shallow water. Solitary wave is an important physical phenomenon of shallow water wave. The

Camassa-Holm equation is a typical shallow water wave equation. Solitary wave has the advantages of not changing waveform and high fidelity in the process of propagation. Therefore, the soliton theory is extensively applied to the engineering technology fields such as nonlinear optics, hydrodynamics, optical fiber communication and so on. In practical application, the soliton successfully explains the problems of density pit and infrared outward movement in laser target shooting. In modern optical fiber soliton communication technology, optical solitons with constant amplitude, pulse and shape are used to transmit signals in order to increase the transmission rate. Nowadays, shallow water wave equation has become an important model and mainstay of solitary wave theory. This paper is devoted to investigating the well-posedness and properties of solutions to the Cauchy problem for coupled system of generalized shallow water wave equations. A more comprehensive understanding of the dynamic properties of shallow water wave equations is illustrated, which lays a foundation for the further study of properties of solitons.

Recently, local well-posedness for the Cauchy problem of generalized Camassa-Holm equation attracts more attention (see [9, 11, 16, 18, 22, 26, 28, 29, 32, 36, 37, 44, 49]). Let us state a brief overview of several related works. Ferraioli and Freire [9] introduce a two-component system depending on the parameter b (problem (1.1) with $\lambda = 0$). The authors show that the system admits one-peakon solutions with non-constant amplitude in the case $b = 2$. Problem (1.1) is invariant under the transformation $(u, v) \to (v, u)$. Taking $v = u$ and $\lambda = 0$ in problem (1.1), we obtain the generalized Camassa-Holm equation which is derived in [36, 37], namely

$$m_t + u^b m_x + (b + 1)u_x u^{b-1} m = 0, \tag{1.2}$$

where $u(t, x)$ represents the fluid velocity at time t in x direction, $m(t, x) = (1 - \partial_x^2)u(t, x)$. It is shown in [1] that Eq (1.2) asserts peakon and multi-peakon solutions. Yan [44] studies wave breaking and global existence for a family of peakon equations with high order nonlinearity. Himonas and Holliman [16] investigate properties of solutions to Eq (1.2) by embedding the equation into a two-parameter family system. Freire et al. [11] illustrate blow-up phenomenon of a Camassa-Holm type equation with quadratic and cubic nonlinearities. Himonas and Thompson [18] prove persistence properties and unique continuation of solutions to the problem, which contains the Camassa-Holm equation, Degasperis-Procesi equation and Novikov equation as special case. Zhang and Liu [49] derive local well-posedness for the Cauchy problem in the Besov space $B^s_{p,r}(\mathbb{R})(s > \max(1 + \frac{1}{p}, \frac{3}{2}))$. The blow-up criteria of solutions is presented. Li et al. [22] demonstrate non-uniform dependence for higher dimensional Camassa-Holm equations in the Besov spaces. Ming et al. [29] establish local well-posedness for the Cauchy problem of a dissipative shallow water equation in $B^s_{p,r}(\mathbb{R})(s > \max(1+\frac{1}{p}, \frac{3}{2}))$ and blow-up mechanisms of solutions in the Sobolev space $H^s(\mathbb{R})(s > \frac{3}{2})$. Guo [14] considers long time behaviors of solutions to a generalized Camassa-Holm equation with $k + 1$ order nonlinearities in the case that the compactly supported initial potential keeps the same sign. Linares and Ponce [26] prove unique continuation properties for solutions to the Camassa-Holm equation and related models. Mutlubas and Freire [32] study the Cauchy problem and continuation of periodic solutions for a generalized Camassa-Holm equation.

When $b = 1$ in Eq (1.2), we have the classical Camassa-Holm equation [3]

$$m_t + um_x + 2u_x m = 0, \tag{1.3}$$

where $m(t, x) = (1 - \partial_x^2)u(t, x)$. Equation (1.3) describes unidirectional propagation of waves at free surface of shallow water under the influence of gravity. Equation (1.3) admits peakon solution $u(t, x) =$

$ce^{-|x-ct|}(c \neq 0)$, which is a feature observed for traveling waves of the largest amplitude (see [6, 7]). Equation (1.3) also has breaking waves (see [48]). The solutions remain bounded while their slopes become unbounded in finite time. Equation (1.3) is completely integrable. In other words, the equation can be transformed into a linear flow by means of an associated isospectrum problem in the sense of infinite dimensional Hamiltonian system (see [8]). Local well-posedness for the Cauchy problem of Eq (1.3) in $H^s(\mathbb{R})(s > \frac{3}{2})$ and blow-up criteria of solutions are presented in [23]. Persistence properties of solutions to the Cauchy problem in $H^s(\mathbb{R})(s > \frac{5}{2})$ are investigated (see [17]). In other words, the strong solution with compact support initial values is not compactly supported at any later time unless it is the zero solution. Yan et al. [47] consider the Cauchy problem of the generalized Camassa-Holm equation in $B^s_{p,r}(\mathbb{R})(s > \max(1+\frac{1}{p}, \frac{3}{2}))$. Wu and Yin [43] discuss the dissipative Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x + \lambda(u - u_{xx}) = 2u_x u_{xx} + uu_{xxx},$$

where $\lambda(u-u_{xx})$ $(\lambda > 0)$ is the dissipative term. The authors derive global existence and blow-up criteria of solutions in $H^s(\mathbb{R})(s > \frac{3}{2})$. Novruzova and Hagverdiyevb [34] investigate behaviors of solutions to the dissipative Camassa-Holm equation with arbitrary dispersion coefficient. Wang et al. [41] study the Cauchy problem of the higher order μ-Camassa-Holm equation. Making use of the Green function of operator $(\mu - \partial_x^2)^{-2}$, the authors verify local well-posedness for the Cauchy problem in $H^s(\mathbb{S})(s > \frac{7}{2})$. Global existence of strong solutions and weak solutions are illustrated. Ji and Zhou [21] obtain wave breaking and global solutions to the dissipative Camassa-Holm type equation in the periodic case. Chen and Guan [4] obtain global solutions to the generalized Camassa-Holm equation. The readers may refer to references [5,10,12,24,25,30,31,35,38,40,42,50] for local well-posedness of the Cauchy problem and global existence of solutions to the generalized Camassa-Holm models.

Taking $b = 2$ in Eq (1.2), we arrive at the classical Novikov equation

$$m_t + u^2 m_x + 3uu_x m = 0, \tag{1.4}$$

which has been discussed by many scholars (see detailed illustrations in [13, 15, 19, 20, 33, 45, 46]). It is worth to notice that the nonlinear terms in the Camassa-Holm equation are quadratic. So it is of great interests to investigate the integrable equations with cubic or higher order nonlinear terms. A remarkable difference between the Novikov equation and Camassa-Holm equation is that the Nivikov equation has cubic nonlinearity. Equation (1.4) is derived in a symmetry classification of nonlocal partial differential equations. It is shown that Eq (1.4) possesses a bi-Hamiltonian structure and peakon solutions $u(t, x) = iA\sqrt{c}e^{-|x-ct-x_0|}$ $(c > 0)$, where x_0 is a constant (see [19, 20]). Moreover, well-posedness for the Cauchy problem of Eq (1.4) in the Sobolev space and Besov space has been investigated. More precisely, local well-posedness for the Cauchy problem of the Novikov equation in $H^s(\mathbb{R})(s > \frac{3}{2})$ is demonstrated in [15]. Local well-posedness for the Cauchy problem of dissipative Novikov equation is proved by making use of the semigroup theory (see [45]). Yan et al. [46] investigate local well-posedness for the Cauchy problem of the Novikov equation in $B^s_{p,r}(\mathbb{R})(s > \max(1 + \frac{1}{p}, \frac{3}{2}))$. Guo [13] studies persistence properties of solution to the Cauchy problem of Eq (1.4).

In general, the system (1.1) may be considered as a two-component generalization of both Camassa-Holm equation and Novikov equation when $\lambda = 0$. Zhou et al. [51] investigate the Cauchy problem of a generalized coupled Camassa-Holm system by applying transport equation theory and the Friedrichs regularization method. Local well-posedness for the Cauchy problem in the critical Besov spaces and blow-up criterion of solutions are established. For the case $b = 2$, $\lambda = 0$ in problem (1.1), Tang and Liu [39] obtain local well-posedness for the Cauchy problem of two-component Novikov system in the critical Besov spaces $B^{\frac{5}{2}}_{2,1}(\mathbb{R}) \times B^{\frac{5}{2}}_{2,1}(\mathbb{R})$ by employing priori estimates of solutions to transport equation in $B^{\frac{3}{2}}_{2,\infty}(\mathbb{R})$ and the Osgood lemma. The solution map $(u_0, v_0) \to (u, v)$ is Holder continuous in $B^{\frac{5}{2}}_{2,1}(\mathbb{R}) \times B^{\frac{5}{2}}_{2,1}(\mathbb{R})$ equipped with the weak topology.

Inspired by the works in [9, 29, 39, 47, 49, 51], we investigate local well-posedness for the Cauchy problem (1.1) and blow-up dynamic of solutions in the Besov spaces $B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R})(s > \max(\frac{5}{2}, 2 + \frac{1}{p}))$. This is different from the regularity index $s > \max(\frac{3}{2}, 1 + \frac{1}{p})$ of solutions to the Camassa-Holm equation (see [47]) and Novikov equation (see [29]). It is worth to mention that problem (1.1) contains the problem studied in [39] as special case. We extend parts of results in [29, 49] for single shallow water equation to the dissipative shallow water system. To the best of authors' knowledge, the results in Theorems 1.1 and 1.2 are new.

Setting $m(t, x) = (1 - \partial_x^2)u(t, x)$, $m_0(x) = (1 - \partial_x^2)u_0(x)$, $n(t, x) = (1 - \partial_x^2)v(t, x)$, $n_0(x) = (1 - \partial_x^2)v_0(x)$, we rewrite problem (1.1) in the form

$$\begin{cases} (\partial_t + u^{b-1} v\partial_x)m = -(b + 1)u_x v^{b-1} m - \lambda m, & t > 0,\ x \in \mathbb{R}, \\ (\partial_t + v^{b-1} u\partial_x)n = -(b + 1)v_x u^{b-1} n - \lambda n, & t > 0,\ x \in \mathbb{R}, \\ m(0, x) = m_0(x), \quad n(0, x) = n_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.5}$$

We present the space

$$E^s_{p,r}(T) = \begin{cases} C([0, T]; B^s_{p,r}(\mathbb{R})) \cap C^1([0, T]; B^{s-1}_{p,r}(\mathbb{R})), & 1 \leq r < \infty, \\ L^\infty([0, T]; B^s_{p,\infty}(\mathbb{R})) \cap \mathrm{Lip}([0, T]; B^{s-1}_{p,\infty}(\mathbb{R})), & r = \infty, \end{cases}$$

where $T > 0$, $s \in \mathbb{R}$, $p \in [1, \infty]$ and $r \in [1, \infty]$.

The main results of this paper are presented as follows.

Theorem 1.1. Assume $1 \leq p, r \leq \infty$ and $(u_0, v_0) \in B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R})(s > \max(\frac{5}{2}, 2 + \frac{1}{p}))$. Then, there exists a positive constant T such that problem (1.1) admits a unique solution $(u, v) \in E^s_{p,r}(T) \times E^s_{p,r}(T)$. It holds that the map $(u_0, v_0) \to (u, v)$ is continuous from a neighborhood of (u_0, v_0) in $B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R})$ into

$$C([0, T]; B^{s'}_{p,r}(\mathbb{R})) \cap C^1([0, T]; B^{s'-1}_{p,r}(\mathbb{R})) \times C([0, T]; B^{s'}_{p,r}(\mathbb{R})) \cap C^1([0, T]; B^{s'-1}_{p,r}(\mathbb{R}))$$

for all $s' < s$, $r = \infty$ and $s' = s$, $r < \infty$.

Theorem 1.2. Assume $1 \leq p, r \leq \infty$ and $(u_0, v_0) \in B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R})(\max(\frac{5}{2}, 2 + \frac{1}{p}) < s < 3)$. The positive constant T stands for the maximal existence time of solution (u, v) to problem (1.1) corresponding to initial values (u_0, v_0). Let $m = u - u_{xx}$, $n = v - v_{xx}$. Then, the corresponding solution (u, v) blows up in finite time if and only if

$$\int_0^t (1 + \lambda + \| m \|_{L^\infty} + \| n \|_{L^\infty})^b d\tau = +\infty.$$

Remark 1.1. Due to the application of Lemma 2.3, we need to restrict the regularity index s which satisfies $\max(\frac{5}{2}, 2 + \frac{1}{p}) < s < 3$ in Theorem 1.2.

Remark 1.2. The Besov space $B^s_{p,r}(\mathbb{R})$ coincides with the Sobolev space $H^s(\mathbb{R})$ in the case $p = r = 2$. Theorem 1.1 implies that under the assumption $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})(s > \frac{5}{2})$, we establish the local well-posedness for problem (1.1) and the corresponding solution $(u,v) \in C([0,T];H^s(\mathbb{R})) \cap C^1([0,T];H^{s-1}(\mathbb{R})) \times C([0,T];H^s(\mathbb{R})) \cap C^1([0,T];H^{s-1}(\mathbb{R}))(s > \frac{5}{2})$.

Throughout this paper, $\| \cdot \|_{B^s_{p,r}(\mathbb{R})}$ represents the norm in the Besov space $B^s_{p,r}(\mathbb{R})(s \in \mathbb{R})$. Since the functions in all spaces are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations if there is no ambiguity. $a \lesssim b$ means that there exists a uniform constant C, which may be different on different lines such that $a \leq Cb$. $[A;B] = AB - BA$ is the commutator between the operators A and B.

2. Preliminary

We are in a position to review some basic estimates in the Besov space.

Definition 2.1. [2] Let $s \in \mathbb{R}$, $1 \leq p,\ r \leq \infty$. The non-homogeneous Besov space is defined by $B^s_{p,r} = \{f \in S'(\mathbb{R}) | \parallel f \parallel_{B^s_{p,r}} < \infty\}$, where

$$\parallel f \parallel_{B^s_{p,r}} = \begin{cases} (\sum\limits_{j=-1}^{\infty} 2^{jrs} \parallel \Delta_j f \parallel^r_{L^p})^{\frac{1}{r}}, & r < \infty, \\ \sup\limits_{j \geq -1} 2^{js} \parallel \Delta_j f \parallel_{L^p}, & r = \infty. \end{cases}$$

Moreover, $S_j f = \sum\limits_{q=-1}^{j-1} \Delta_q f$.

Lemma 2.1. [2] Let $s \in \mathbb{R}$, $1 \leq p,\ r,\ p_j,\ r_j \leq \infty,\ j = 1,\ 2$.

1) Algebraic properties: for all $s > 0$, $B^s_{p,r} \cap L^\infty$ is an algebra. $B^s_{p,r}$ is an algebra $\Leftrightarrow B^s_{p,r} \hookrightarrow L^\infty \Leftrightarrow s > \frac{1}{p}$ or $s \geq \frac{1}{p}$ and $r = 1$.

2) Fatou's Lemma: If $(u_n)_{n \in \mathbb{N}}$ is bounded in $B^s_{p,r}$ and $u_n \to u$ in $S'(\mathbb{R})$, then it holds that $u \in B^s_{p,r}$ and

$$\parallel u \parallel_{B^s_{p,r}} \leq \lim_{n \to \infty} \inf \parallel u_n \parallel_{B^s_{p,r}} .$$

3) Complex interpolation:

$$\parallel f \parallel_{B^{\theta s_1 + (1-\theta) s_2}_{p,r}} \lesssim \parallel f \parallel^{\theta}_{B^{s_1}_{p,r}} \parallel f \parallel^{1-\theta}_{B^{s_2}_{p,r}} \quad \text{for all} \quad f \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}, \quad \theta \in [0,1].$$

4) Let $m \in \mathbb{R}$ and f is an S^m-multiplier. Then, the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.

Lemma 2.2. [2] Let $1 \leq p,\ r \leq \infty$. It holds that

(i) for $s > 0$, then

$$\parallel fg \parallel_{B^s_{p,r}} \leq C(\parallel f \parallel_{B^s_{p,r}} \parallel g \parallel_{L^\infty} + \parallel f \parallel_{L^\infty} \parallel g \parallel_{B^s_{p,r}}),$$

(ii) for $s_1 \leq \frac{1}{p}$, $s_2 > \frac{1}{p}$ ($s_2 \geq \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$, then

$$\parallel fg \parallel_{B^{s_1}_{p,r}} \leq C \parallel f \parallel_{B^{s_1}_{p,r}} \parallel g \parallel_{B^{s_2}_{p,r}},$$

where C is a positive constant independent of f and g.

We have the commutators estimates.

Lemma 2.3. [2] Assume $s > 0, 1 \leq p \leq p_1 \leq \infty, 1 \leq r \leq \infty, \frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1}$. Let v be a vector on $\mathbb{R}$. Then

$$\parallel [\Delta_j,\ v \cdot \nabla] f \parallel_{B^s_{p,r}} \leq C(\parallel \nabla v \parallel_{L^\infty} \parallel f \parallel_{B^s_{p,r}} + \parallel \nabla v \parallel_{B^{s-1}_{p_1,r}} \parallel \nabla f \parallel_{L^{p_2}}).$$

Moreover, if $0 < s < 1$, then

$$\| [\Delta_j, v\cdot\nabla]f \|_{B^s_{p,r}} \leq C \| \nabla v \|_{L^\infty} \| f \|_{B^s_{p,r}},$$

where C is a positive constant independent of f and v.

We illustrate three lemmas which are related to the transport equation

$$\begin{cases} f_t + d\cdot\nabla f = F, \\ f|_{t=0} = f_0, \end{cases} \tag{2.1}$$

where $d : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ denotes a time dependent vector field, $f_0 : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are known functions.

Lemma 2.4. [2] Let $1 \leq p \leq p_1 \leq \infty$, $1 \leq r \leq \infty$, $p' = \frac{p}{p-1}$. Assume $s > -\min(\frac{1}{p_1}, \frac{1}{p'})$ or $s > -1 - \min(\frac{1}{p_1}, \frac{1}{p'})$ if $\nabla\cdot d = 0$. Then, there exists a positive constant C_1 depending only on p, p_1, r, s such that

$$\begin{aligned} &\| f \|_{L^\infty_t([0,t];B^s_{p,r})} \\ &\leq e^{C_1\int_0^t Z(\tau)d\tau}[\| f_0 \|_{B^s_{p,r}} + \int_0^t e^{-C_1\int_0^\tau Z(\xi)d\xi} \| F(\tau) \|_{B^s_{p,r}}\, d\tau], \end{aligned} \tag{2.2}$$

where

$$Z(t) = \begin{cases} \| \nabla d(t) \|_{B^{\frac{1}{p_1}}_{p_1,\infty}\cap L^\infty}, & s < 1 + \frac{1}{p_1}, \\ \| \nabla d(t) \|_{B^{s-1}_{p_1,r}}, & s > 1 + \frac{1}{p_1} \text{ or } s = 1 + \frac{1}{p_1}, r = 1. \end{cases}$$

If $f = d$, $s > 0$ ($\nabla\cdot d = 0$, $s > -1$), $Z(t) = \| \nabla d(t) \|_{L^\infty}$, then (2.2) holds.

Lemma 2.5. [2] Let p, p_1, r, s be defined in Lemma 2.3 and $f_0 \in B^s_{p,r}$, $F \in L^1([0,T]; B^s_{p,r})$. $d \in L^\rho([0,T]; B^{-M}_{\infty,\infty})$ is a time dependent vector field for $\rho > 1$, $M > 0$, such that if $s < 1 + \frac{1}{p_1}$ then $\nabla d \in L^1([0,T]; B^{\frac{1}{p_1}}_{p_1,\infty} \cap L^\infty)$. If $s > 1 + \frac{1}{p_1}$ or $s = 1 + \frac{1}{p_1}$, $r = 1$, then $\nabla d \in L^1([0,T]; B^{s-1}_{p_1,r})$. Therefore, problem (2.1) has a unique solution $f \in L^\infty([0,T]; B^s_{p,r}) \cap (\cap_{s'<s} C([0,T]; B^{s'}_{p,1}))$ and (2.2) holds. If $r < \infty$, it holds that $f \in C([0,T]; B^s_{p,r})$.

Lemma 2.6. [27] Let $1 \leq p \leq \infty$, $1 \leq r \leq \infty$, $s > \max(\frac{1}{2}, \frac{1}{p})$, $f_0 \in B^{s-1}_{p,r}$, $F \in L^1([0,t]; B^{s-1}_{p,r})$ and the velocity function $d \in L^1([0,t]; B^{s+1}_{p,r})$. Then, the solution $f \in L^\infty([0,T]; B^{s-1}_{p,r})$ to problem (2.1) satisfies

$$\begin{aligned} &\| f \|_{L^\infty_t([0,t];B^{s-1}_{p,r})} \\ &\leq e^{C_1\int_0^t Z(\tau)d\tau}[\| f_0 \|_{B^{s-1}_{p,r}} + \int_0^t e^{-C_1\int_0^\tau Z(\xi)d\xi} \| F(\tau) \|_{B^{s-1}_{p,r}}\, d\tau], \end{aligned} \tag{2.3}$$

where $Z(t) = \int_0^t \| d(\tau) \|_{B^{s+1}_{p,r}}\, d\tau$, C_1 is a positive constant depending only on s, p, r.

3. The proof of Theorem 1.1

3.1. Existence of solutions

Taking advantage of iterative method, we construct the approximate solutions to problem (1.5) with initial values $(m_0, n_0) \in B^s_{p,r} \times B^s_{p,r}(s > \max(\frac{1}{p}, \frac{1}{2}))$.

Step 1. Set $(m^0, n^0) = (0, 0)$. We define by induction a sequence of smooth functions $(m^i, n^i)_{i\in\mathbb{N}} \in C(\mathbb{R}^+; B^\infty_{p,r}) \times C(\mathbb{R}^+; B^\infty_{p,r})$ satisfying

$$\begin{cases} (\partial_t + (u^i)^{b-1}v^i\partial_x)m^{i+1} = F_1(t,x), & t>0,\ x\in\mathbb{R}, \\ (\partial_t + (v^i)^{b-1}u^i\partial_x)n^{i+1} = F_2(t,x), & t>0,\ x\in\mathbb{R}, \\ m^{i+1}(0,x) = m_0^{i+1}(x) = S_{i+1}m_0, & x\in\mathbb{R}, \\ n^{i+1}(0,x) = n_0^{i+1}(x) = S_{i+1}n_0, & x\in\mathbb{R}, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} F_1(t,x) &= -(b+1)u^i_x(v^i)^{b-1}m^i - \lambda m^i, \\ F_2(t,x) &= -(b+1)v^i_x(u^i)^{b-1}n^i - \lambda n^i. \end{aligned} \tag{3.2}$$

It holds that all the values $(S_{i+1}m_0, S_{i+1}n_0) \in B^\infty_{p,r} \times B^\infty_{p,r}$. Employing Lemma 2.5, we deduce that for all $i \in \mathbb{N}$, the problem (3.1) admits a global solution which belongs to $C(\mathbb{R}^+; B^\infty_{p,r}) \times C(\mathbb{R}^+; B^\infty_{p,r})$.

Step 2. We intend to demonstrate that $(m^i, n^i)_{i\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T) \times E^s_{p,r}(T)$.

Making use of Lemmas 2.1 and 2.4 yields

$$\begin{aligned} \| m^{i+1}(t) \|_{B^s_{p,r}} \le{}& e^{C_1\int_0^t \|\nabla[(u^i)^{b-1}v^i](\tau)\|_{B^s_{p,r}}d\tau} \\ &\times[\| m_0 \|_{B^s_{p,r}} + \int_0^t e^{-C_1\int_0^\tau \|\nabla[(u^i)^{b-1}v^i](\xi)\|_{B^s_{p,r}}d\xi} \| F_1(\tau,\cdot) \|_{B^s_{p,r}}\, d\tau] \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \| n^{i+1}(t) \|_{B^s_{p,r}} \le{}& e^{C_1\int_0^t \|\nabla[(v^i)^{b-1}u^i](\tau)\|_{B^s_{p,r}}d\tau} \\ &\times[\| n_0 \|_{B^s_{p,r}} + \int_0^t e^{-C_1\int_0^\tau \|\nabla[(v^i)^{b-1}u^i](\xi)\|_{B^s_{p,r}}d\xi} \| F_2(\tau,\cdot) \|_{B^s_{p,r}}\, d\tau]. \end{aligned} \tag{3.4}$$

Direct calculation shows

$$\begin{aligned} \| \nabla[(u^i)^{b-1}v^i] \|_{B^s_{p,r}} &\lesssim \| (u^i)^{b-1}v^i \|_{B^{s+1}_{p,r}} \lesssim \| u^i \|^{b-1}_{B^{s+1}_{p,r}} \| v^i \|_{B^{s+1}_{p,r}} \\ &\lesssim \| m^i \|^{b-1}_{B^s_{p,r}} \| n^i \|_{B^s_{p,r}}, \\ \| \nabla[(v^i)^{b-1}u^i] \|_{B^s_{p,r}} &\lesssim \| (v^i)^{b-1}u^i \|_{B^{s+1}_{p,r}} \lesssim \| v^i \|^{b-1}_{B^{s+1}_{p,r}} \| u^i \|_{B^{s+1}_{p,r}} \\ &\lesssim \| n^i \|^{b-1}_{B^s_{p,r}} \| m^i \|_{B^s_{p,r}} . \end{aligned}$$

Applying Lemma 2.2 leads to

$$\begin{aligned} &\| F_1(t,\cdot) \|_{B^s_{p,r}} \\ \lesssim{}& \| u^i_x(v^i)^{b-1}m^i \|_{B^s_{p,r}} + \lambda \| m^i \|_{B^s_{p,r}} \\ \lesssim{}& \| u^i_x \|_{B^s_{p,r}} \| (v^i)^{b-1} \|_{B^s_{p,r}} \| m^i \|_{B^s_{p,r}} + \lambda \| m^i \|_{B^s_{p,r}} \\ \lesssim{}& \| u^i \|_{B^{s+1}_{p,r}} \| v^i \|^{b-1}_{B^s_{p,r}} \| m^i \|_{B^s_{p,r}} + \lambda \| m^i \|_{B^s_{p,r}} \\ \lesssim{}& \| m^i \|^2_{B^s_{p,r}} \| n^i \|^{b-1}_{B^s_{p,r}} + \lambda \| m^i \|_{B^s_{p,r}} \end{aligned} \tag{3.5}$$

and

$$
\begin{aligned}
&\| F_2(t,\cdot) \|_{B^s_{p,r}} \\
&\lesssim \| v^i_x (u^i)^{b-1} n^i \|_{B^s_{p,r}} + \lambda \| n^i \|_{B^s_{p,r}} \\
&\lesssim \| v^i_x \|_{B^s_{p,r}} \| (u^i)^{b-1} \|_{B^s_{p,r}} \| n^i \|_{B^s_{p,r}} + \lambda \| n^i \|_{B^s_{p,r}} \\
&\lesssim \| v^i \|_{B^{s+1}_{p,r}} \| u^i \|^{b-1}_{B^s_{p,r}} \| n^i \|_{B^s_{p,r}} + \lambda \| n^i \|_{B^s_{p,r}} \\
&\lesssim \| n^i \|^2_{B^s_{p,r}} \| m^i \|^{b-1}_{B^s_{p,r}} + \lambda \| n^i \|_{B^s_{p,r}} .
\end{aligned} \tag{3.6}
$$

Combining (3.3)–(3.6), we derive

$$
\begin{aligned}
\| m^{i+1}(t) \|_{B^s_{p,r}} \leq e^{C_1 \int_0^t \| m^i \|^{b-1}_{B^s_{p,r}} \| n^i \|_{B^s_{p,r}} d\tau} & \\
\times [\| m_0 \|_{B^s_{p,r}} + \int_0^t e^{-C_1 \int_0^\tau \| m^i \|^{b-1}_{B^s_{p,r}} \| n^i \|_{B^s_{p,r}} d\xi} & \\
\times (\| m^i \|^2_{B^s_{p,r}} \| n^i \|^{b-1}_{B^s_{p,r}} + \lambda \| m^i \|_{B^s_{p,r}}) d\tau] &
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
\| n^{i+1}(t) \|_{B^s_{p,r}} \leq e^{C_1 \int_0^t \| n^i \|^{b-1}_{B^s_{p,r}} \| m^i \|_{B^s_{p,r}} d\tau} & \\
\times [\| n_0 \|_{B^s_{p,r}} + \int_0^t e^{-C_1 \int_0^\tau \| n^i \|^{b-1}_{B^s_{p,r}} \| m^i \|_{B^s_{p,r}} d\xi} & \\
\times (\| n^i \|^2_{B^s_{p,r}} \| m^i \|^{b-1}_{B^s_{p,r}} + \lambda \| n^i \|_{B^s_{p,r}}) d\tau]. &
\end{aligned} \tag{3.8}
$$

It follows that

$$
\begin{aligned}
&\| m^{i+1}(t) \|_{B^s_{p,r}} + \| n^{i+1}(t) \|_{B^s_{p,r}} \\
&\leq C_2 \cdot e^{C_2 \int_0^t (1+\lambda+\| m^i \|_{B^s_{p,r}} + \| n^i \|_{B^s_{p,r}})^b d\tau} [(\| m_0 \|_{B^s_{p,r}} + \| n_0 \|_{B^s_{p,r}}) \\
&\quad + \int_0^t e^{-C_2 \int_0^\tau (1+\lambda+\| m^i \|_{B^s_{p,r}} + \| n^i \|_{B^s_{p,r}})^b d\xi} \\
&\quad \times (1 + \lambda + \| m^i \|_{B^s_{p,r}} + \| n^i \|_{B^s_{p,r}})^b (\| m^i \|_{B^s_{p,r}} + \| n^i \|_{B^s_{p,r}}) d\tau].
\end{aligned} \tag{3.9}
$$

We choose a positive constant T such that

$$
2bC_2^{b+1}(1 + \lambda + \| m_0 \|_{B^s_{p,r}} + \| n_0 \|_{B^s_{p,r}})^b T < 1
$$

and

$$
\begin{aligned}
&(1 + \lambda + \| m^i \|_{B^s_{p,r}} + \| n^i \|_{B^s_{p,r}})^b \\
&\leq \frac{C_2^b (1 + \lambda + \| m_0 \|_{B^s_{p,r}} + \| n_0 \|_{B^s_{p,r}})^b}{1 - 2bC_2^{b+1}(1 + \lambda + \| m_0 \|_{B^s_{p,r}} + \| n_0 \|_{B^s_{p,r}})^b t}.
\end{aligned} \tag{3.10}
$$

Inserting (3.10) into (3.9) yields

$$
(1 + \lambda + \| m^{i+1}(t) \|_{B^s_{p,r}} + \| n^{i+1}(t) \|_{B^s_{p,r}})^b
$$

$$\leq \frac{C_2^b(1+\lambda+\parallel m_0 \parallel_{B^s_{p,r}} + \parallel n_0 \parallel_{B^s_{p,r}})^b}{1-2bC_2^{b+1}(1+\lambda+\parallel m_0 \parallel_{B^s_{p,r}} + \parallel n_0 \parallel_{B^s_{p,r}})^b t}.$$

Hence, $(m^i, n^i)_{i\in\mathbb{N}}$ is uniformly bounded in $C([0,T]; B^s_{p,r}) \times C([0,T]; B^s_{p,r})$. Utilizing Lemma 2.2 gives rise to

$$\begin{aligned}
&\parallel (u^i)^{b-1} v^i \partial_x m^{i+1} \parallel_{B^{s-1}_{p,r}} \\
&\lesssim \parallel (u^i)^{b-1} v^i \parallel_{B^s_{p,r}} \parallel \partial_x m^{i+1} \parallel_{B^{s-1}_{p,r}} \\
&\lesssim \parallel u^i \parallel^{b-1}_{B^s_{p,r}} \parallel v^i \parallel_{B^s_{p,r}} \parallel m^{i+1} \parallel_{B^s_{p,r}}
\end{aligned}$$

and

$$\begin{aligned}
&\parallel (v^i)^{b-1} u^i \partial_x n^{i+1} \parallel_{B^{s-1}_{p,r}} \\
&\lesssim \parallel (v^i)^{b-1} u^i \parallel_{B^s_{p,r}} \parallel \partial_x n^{i+1} \parallel_{B^{s-1}_{p,r}} \\
&\lesssim \parallel v^i \parallel^{b-1}_{B^s_{p,r}} \parallel u^i \parallel_{B^s_{p,r}} \parallel n^{i+1} \parallel_{B^s_{p,r}} .
\end{aligned}$$

As a concequence, we obtain that $((u^i)^{b-1} v^i \partial_x m^{i+1})_{i\in\mathbb{N}}$ and $F_1(t,x)$ are uniformly bounded in $C([0,T]; B^{s-1}_{p,r})$. From (3.1), we conclude that $(\partial_t m^{i+1})_{i\in\mathbb{N}} \in C([0,T]; B^{s-1}_{p,r})$ is uniformly bounded. Hence, $(m^i)_{i\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$. Analogously, $(n^i)_{i\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$. Therefore, $(m^i, n^i)_{i\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T) \times E^s_{p,r}(T)$.

Step 3. We shall prove that $(m^i, n^i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]; B^{s-1}_{p,r}) \times C([0,T]; B^{s-1}_{p,r})$.

Applying (3.1), for all $i,\ j \in \mathbb{N}$, we arrive at

$$\begin{aligned}
&(\partial_t + (u^{i+j})^{b-1} v^{i+j} \partial_x)(m^{i+j+1} - m^{i+1}) \\
&= -[(u^{i+j})^{b-1} v^{i+j} - (u^i)^{b-1} v^i] m^{i+1}_x + F_3(t,x) - \lambda(m^{i+j} - m^i), \qquad (3.11) \\
&(\partial_t + (v^{i+j})^{b-1} u^{i+j} \partial_x)(n^{i+j+1} - n^{i+1}) \\
&= -[(v^{i+j})^{b-1} u^{i+j} - (v^i)^{b-1} u^i] n^{i+1}_x + F_4(t,x) - \lambda(n^{i+j} - n^i), \qquad (3.12)
\end{aligned}$$

where

$$\begin{aligned}
F_3(t,x) &= -(b+1)[u^{i+j}_x (v^{i+j})^{b-1} (m^{i+j} - m^i) \\
&\quad + [u^{i+j}_x (v^{i+j})^{b-1} - u^i_x (v^i)^{b-1}] m^i], \\
F_4(t,x) &= -(b+1)[v^{i+j}_x (u^{i+j})^{b-1} (n^{i+j} - n^i) \\
&\quad + [v^{i+j}_x (u^{i+j})^{b-1} - v^i_x (u^i)^{b-1}] n^i].
\end{aligned}$$

We present estimates of the terms in the right-hand side of (3.11) and (3.12). Using the relation

$$\begin{aligned}
&(u^{i+j})^{b-1} v^{i+j} - (u^i)^{b-1} v^i \\
&= [(u^{i+j})^{b-1} - (u^i)^{b-1}] v^{i+j} + (u^i)^{b-1} (v^{i+j} - v^i) \qquad (3.13)
\end{aligned}$$

and Lemma 2.2 gives rise to the estimates

$$\parallel [(u^{i+j})^{b-1} - (u^i)^{b-1}] v^{i+j} m^{i+1}_x \parallel_{B^{s-1}_{p,r}}$$

$$\lesssim \| (u^{i+j})^{b-1} - (u^i)^{b-1} \|_{B^s_{p,r}} \| v^{i+j} \|_{B^s_{p,r}} \| m_x^{i+j} \|_{B^{s-1}_{p,r}}$$
$$\lesssim \| u^{i+j} - u^i \|_{B^s_{p,r}} (\sum_{k=0}^{b-2} \| u^{i+j} \|^{b-2-k}_{B^s_{p,r}} \| u^i \|^k_{B^s_{p,r}}) \| v^{i+j} \|_{B^s_{p,r}} \| m^{i+j} \|_{B^s_{p,r}}$$
$$\lesssim \| m^{i+j} - m^i \|_{B^s_{p,r}} \tag{3.14}$$

and

$$\| (u^i)^{b-1}(v^{i+j} - v^i) m_x^{i+1} \|_{B^{s-1}_{p,r}}$$
$$\lesssim \| v^{i+j} - v^i \|_{B^s_{p,r}} \| (u^i)^{b-1} \|_{B^s_{p,r}} \| m_x^{i+j} \|_{B^{s-1}_{p,r}}$$
$$\lesssim \| v^{i+j} - v^i \|_{B^s_{p,r}} \| u^i \|^{b-1}_{B^s_{p,r}} \| m^{i+j} \|_{B^s_{p,r}}$$
$$\lesssim \| n^{i+j} - n^i \|_{B^s_{p,r}} . \tag{3.15}$$

Thus, we conclude

$$\| -[(u^{i+j})^{b-1} v^{i+j} - (u^i)^{b-1} v^i] m_x^{i+1} \|_{B^{s-1}_{p,r}}$$
$$\lesssim \| m^{i+j} - m^i \|_{B^{s-1}_{p,r}} + \| n^{i+j} - n^i \|_{B^{s-1}_{p,r}} . \tag{3.16}$$

In an analogous way, we deduce

$$\| F_3(t, x) \|_{B^{s-1}_{p,r}}$$
$$= \| -(b+1)[u_x^{i+j}(v^{i+j})^{b-1}(m^{i+j} - m^i) + [u_x^{i+j}(v^{i+j})^{b-1} - u_x^i(v^i)^{b-1}] m^i] \|_{B^{s-1}_{p,r}}$$
$$\lesssim \| u_x^{i+j} \|_{B^s_{p,r}} \| v^{i+j} \|^{b-1}_{B^s_{p,r}} \| m^{i+j} - m^i \|_{B^{s-1}_{p,r}}$$
$$+ \| [(u_x^{i+j} - u_x^i)(v^{i+j})^{b-1} + u_x^i((v^{i+j})^{b-1} - (v^i)^{b-1})] m^i \|_{B^{s-1}_{p,r}}$$
$$\lesssim \| u_x^{i+j} \|_{B^s_{p,r}} \| v^{i+j} \|^{b-1}_{B^s_{p,r}} \| m^{i+j} - m^i \|_{B^{s-1}_{p,r}}$$
$$+ \| u_x^{i+j} - u_x^i \|_{B^{s-1}_{p,r}} \| v^{i+j} \|^{b-1}_{B^s_{p,r}} \| m^i \|_{B^s_{p,r}}$$
$$+ \| u_x^i \|_{B^s_{p,r}} \| v^{i+j} - v^i \|_{B^s_{p,r}} (\sum_{k=0}^{b-2} \| v^{i+j} \|^{b-2-k}_{B^s_{p,r}} \| v^i \|^k_{B^s_{p,r}}) \| m^i \|_{B^{s-1}_{p,r}}$$
$$\lesssim \| m^{i+j} - m^i \|_{B^{s-1}_{p,r}} + \| n^{i+j} - n^i \|_{B^{s-1}_{p,r}} . \tag{3.17}$$

It follows from some calculations that

$$\| -\lambda(m^{i+j} - m^i) \|_{B^{s-1}_{p,r}} \lesssim \| m^{i+j} - m^i \|_{B^{s-1}_{p,r}},$$
$$\| (u^{i+j})^{b-1} v^{i+j} \|_{B^{s+1}_{p,r}} \lesssim \| u^{i+j} \|^{b-1}_{B^{s+1}_{p,r}} \| v^{i+j} \|_{B^{s+1}_{p,r}} \lesssim \| m^{i+j} \|^{b-1}_{B^s_{p,r}} \| n^{i+j} \|_{B^s_{p,r}},$$
$$\| (v^{i+j})^{b-1} u^{i+j} \|_{B^{s+1}_{p,r}} \lesssim \| v^{i+j} \|^{b-1}_{B^{s+1}_{p,r}} \| u^{i+j} \|_{B^{s+1}_{p,r}} \lesssim \| n^{i+j} \|^{b-1}_{B^s_{p,r}} \| m^{i+j} \|_{B^s_{p,r}} .$$

Applying Lemma 2.6, we have

$$\| m^{i+j+1}(t) - m^{i+1}(t) \|_{B^{s-1}_{p,r}} \leq e^{C_1 \int_0^t \|m^{i+j}\|^{b-1}_{B^s_{p,r}} \|n^{i+j}\|_{B^s_{p,r}} d\tau}$$
$$\times [\| m_0^{i+j+1} - m_0^{i+1} \|_{B^{s-1}_{p,r}} + \int_0^t e^{-C_1 \int_0^\tau \|m^{i+j}\|^{b-1}_{B^s_{p,r}} \|n^{i+j}\|_{B^s_{p,r}} d\xi}$$

$$\times(\| m^{i+j}-m^{i} \|_{B^{s-1}_{p,r}} + \| n^{i+j}-n^{i} \|_{B^{s-1}_{p,r}}$$
$$+\lambda \| m^{i+j}-m^{i} \|_{B^{s-1}_{p,r}})d\tau] \tag{3.18}$$

and

$$\| n^{i+j+1}(t)-n^{i+1}(t) \|_{B^{s-1}_{p,r}} \leq e^{C_1\int_0^t \|n^{i+j}\|^{b-1}_{B^s_{p,r}} \|m^{i+j}\|_{B^s_{p,r}} d\tau}$$
$$\times[\| n_0^{i+j+1}-n_0^{i+1} \|_{B^{s-1}_{p,r}} + \int_0^t e^{-C_1\int_0^\tau \|n^{i+j}\|^{b-1}_{B^s_{p,r}} \|m^{i+j}\|_{B^s_{p,r}} d\xi}$$
$$\times(\| m^{i+j}-m^{i} \|_{B^{s-1}_{p,r}} + \| n^{i+j}-n^{i} \|_{B^{s-1}_{p,r}}$$
$$+\lambda \| n^{i+j}-n^{i} \|_{B^{s-1}_{p,r}})d\tau]. \tag{3.19}$$

We note that $(m^i, n^i)_{i\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$ and

$$m_0^{i+j+1}-m_0^{i+1}=\sum_{q=i+1}^{i+j}\Delta_q m_0, \quad n_0^{i+j+1}-n_0^{i+1}=\sum_{q=i+1}^{i+j}\Delta_q n_0.$$

There exists a constant C_T which is independent of i for all $t\in[0,T]$ such that

$$\| m^{i+j+1}-m^{i+1} \|_{B^{s-1}_{p,r}} + \| n^{i+j+1}-n^{i+1} \|_{B^{s-1}_{p,r}}$$
$$\leq C_T[2^{-i}+\int_0^t (\| m^{i+j}-m^{i} \|_{B^{s-1}_{p,r}} + \| n^{i+j}-n^{i} \|_{B^{s-1}_{p,r}})d\tau].$$

Employing induction argument gives rise to

$$\| m^{i+j+1}-m^{i+1} \|_{B^{s-1}_{p,r}} + \| n^{i+j+1}-n^{i+1} \|_{B^{s-1}_{p,r}}$$
$$\leq \frac{(C_T T)^{i+1}}{(i+1)!}(\| m^{j} \|_{L^\infty([0,T];B^{s-1}_{p,r})} + \| n^{j} \|_{L^\infty([0,T];B^{s-1}_{p,r})})$$
$$+C_T\sum_{l=0}^{i}2^{-(i-l)}\frac{(C_T T)^l}{l!}.$$

Utilizing the fact that $(m^j, n^j)_{j\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0,T];B^s_{p,r})\times L^\infty([0,T];B^s_{p,r})$ independent of j, we deduce that there exists a new constant C_{T_1} such that

$$\| m^{i+j+1}-m^{i+1} \|_{L^\infty([0,T];B^{s-1}_{p,r})} + \| n^{i+j+1}-n^{i+1} \|_{L^\infty([0,T];B^{s-1}_{p,r})}\leq C_{T_1}2^{-i}.$$

Therefore, $(m^i, n^j)_{i\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})\times C([0,T];B^{s-1}_{p,r})$.
Step 4. We present the proof of existence of solutions.

Utilizing the Fatou property in Lemma 2.1 yields $(m,n)\in L^\infty([0,T];B^s_{p,r})\times L^\infty([0,T];B^s_{p,r})$. We see that $(m^i, n^i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})\times C([0,T];B^{s-1}_{p,r})$, which converges to a limit function $(m,n)\in C([0,T];B^{s-1}_{p,r})\times C([0,T];B^{s-1}_{p,r})$. Based on interpolation argument, we infer that the convergence holds in $C([0,T];B^{s'}_{p,r})\times C([0,T];B^{s'}_{p,r})$ for all $s'<s$. Sending $i\to\infty$ in (3.1) gives rise to the fact that (m,n) is a solution to (3.1). Making use of $(m,n)\in L^\infty([0,T];B^s_{p,r})\times L^\infty([0,T];B^s_{p,r})$, we derive that the right side of the first equation in (3.1) belongs to $L^\infty([0,T];B^s_{p,r})$. In the case $r<$

∞, applying Lemma 2.4 yields $m \in C([0,T]; B^{s'}_{p,r})$ for all $s' < s$. Similarly, the right side of the second equation in (3.1) belongs to $L^\infty([0,T]; B^s_{p,r})$. In the case $r < \infty$, from Lemma 2.4, we acquire $n \in C([0,T]; B^{s'}_{p,r})$ for all $s' < s$. Thus, we arrive at $(m,n) \in L^\infty([0,T]; B^s_{p,r}) \times L^\infty([0,T]; B^s_{p,r})$ and $(m,n) \in C([0,T]; B^{s'}_{p,r}) \times C([0,T]; B^{s'}_{p,r})$ for all $r < \infty$, $s' < s$.

Applying problem (1.5) yields that $(m_t, n_t) \in C([0,T]; B^{s-1}_{p,r})$ if $r < \infty$ and $(m_t, n_t) \in L^\infty([0,T]; B^{s-1}_{p,r}) \times L^\infty([0,T]; B^{s-1}_{p,r})$ otherwise. Hence, $(m,n) \in E^s_{p,r}(T) \times E^s_{p,r}(T)$. Utilizing a sequence of viscosity approximate solutions $(m_\varepsilon, n_\varepsilon)_{\varepsilon>0}$ to problem (1.5) which converges uniformly in $C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}) \times C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r})$ gives rise to the continuity of solution $(m,n) \in E^s_{p,r}(T) \times E^s_{p,r}(T)$. ■

3.2. Uniqueness and continuity with initial values

Lemma 3.2.1. Assume $1 \le p, r \le \infty$, $s > \max(\frac{1}{p}, \frac{1}{2})$. Let (m^1, n^1), (m^2, n^2) be two solutions to the Cauchy problem (1.5) with initial values (m^1_0, n^1_0), $(m^2_0, n^2_0) \in B^s_{p,r} \times B^s_{p,r}$. (m^1, n^1), $(m^2, n^2) \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; B^{s-1}_{p,r}) \times L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; B^{s-1}_{p,r})$. Assume $m^{12}_0 = m^1_0 - m^2_0$, $n^{12}_0 = n^1_0 - n^2_0$. Then, for all $t \in [0,T]$, it holds that

$$
\begin{aligned}
&\| m^1(t) - m^2(t) \|_{B^{s-1}_{p,r}} + \| n^1(t) - n^2(t) \|_{B^{s-1}_{p,r}} \\
&\le (\| m^{12}_0 \|_{B^{s-1}_{p,r}} + \| n^{12}_0 \|_{B^{s-1}_{p,r}}) e^{C_1 \int_0^t (\|m^1\|_{B^s_{p,r}} + \|n^1\|_{B^s_{p,r}})^b d\tau}.
\end{aligned}
$$

Proof. We set $m^{12} = m^1 - m^2$, $n^{12} = n^1 - n^2$. Thus, it holds that

$$
m^{12} \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; B^{s-1}_{p,r}),\ n^{12} \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; B^{s-1}_{p,r}),
$$

which implies that $(m^{12}, n^{12}) \in C([0,T]; B^{s-1}_{p,r}) \times C([0,T]; B^{s-1}_{p,r})$. We deduce

$$
\begin{aligned}
&(\partial_t + (u^1)^{b-1} v^1 \partial_x) m^{12} \\
&= -[(u^1)^{b-1} v^1 - (u^2)^{b-1} v^2] m^2_x + F_5(t,x) - \lambda m^{12}, &(3.20)\\
&(\partial_t + (v^1)^{b-1} u^1 \partial_x) n^{12} \\
&= -[(v^1)^{b-1} u^1 - (v^2)^{b-1} u^2] n^2_x + F_6(t,x) - \lambda n^{12}, &(3.21)
\end{aligned}
$$

where

$$
\begin{aligned}
F_5(t,x) &= -(b+1)[u^1_x (v^1)^{b-1} m^{12} + (u^1_x (v^1)^{b-1} - u^2_x (v^2)^{b-1}) m^2],\\
F_6(t,x) &= -(b+1)[v^1_x (u^1)^{b-1} n^{12} + (v^1_x (u^1)^{b-1} - v^2_x (u^2)^{b-1}) n^2].
\end{aligned}
$$

Using Lemma 2.6 gives rise to

$$
\begin{aligned}
\| m^{12}(t) \|_{B^{s-1}_{p,r}} \le\ & e^{C_1 \int_0^t \|m^1\|^{b-1}_{B^s_{p,r}} \|n^1\|_{B^s_{p,r}} d\tau} \\
& \times [\| m^{12}_0 \|_{B^{s-1}_{p,r}} + \int_0^t e^{-C_1 \int_0^\tau \|m^1\|^{b-1}_{B^s_{p,r}} \|n^1\|_{B^s_{p,r}} d\xi} \\
& \times (\| m^{12} \|_{B^{s-1}_{p,r}} + \| n^{12} \|_{B^{s-1}_{p,r}} + \lambda \| m^{12} \|_{B^{s-1}_{p,r}}) d\tau] \qquad (3.22)
\end{aligned}
$$

and

$$
\| n^{12}(t) \|_{B^{s-1}_{p,r}} \le e^{C_1 \int_0^t \|n^1\|^{b-1}_{B^s_{p,r}} \|m^1\|_{B^s_{p,r}} d\tau}
$$

$$\times [\| n_0^{12} \|_{B^{s-1}_{p,r}} + \int_0^t e^{-C_1 \int_0^\tau \|n^1\|^{b-1}_{B^s_{p,r}} \|m^1\|_{B^s_{p,r}} d\xi}$$
$$\times (\| m^{12} \|_{B^{s-1}_{p,r}} + \| n^{12} \|_{B^{s-1}_{p,r}} + \lambda \| n^{12} \|_{B^{s-1}_{p,r}}) d\tau]. \quad (3.23)$$

From (3.22) and (3.23), we derive

$$\begin{aligned}
&\| m^{12}(t) \|_{B^{s-1}_{p,r}} + \| n^{12}(t) \|_{B^{s-1}_{p,r}} \\
&\leq e^{C_1 \int_0^t (\|m^1\|_{B^s_{p,r}} + \|n^1\|_{B^s_{p,r}})^b d\tau} \times [(\| m_0^{12} \|_{B^{s-1}_{p,r}} + \| n_0^{12} \|_{B^{s-1}_{p,r}}) \\
&\quad + \int_0^t e^{-C_1 \int_0^\tau (\|m^1\|_{B^s_{p,r}} + \|n^1\|_{B^s_{p,r}})^b d\xi} \\
&\quad \times (\| m^{12} \|_{B^{s-1}_{p,r}} + \| n^{12} \|_{B^{s-1}_{p,r}}) d\tau].
\end{aligned} \quad (3.24)$$

Applying the Gronwall inequality yields

$$\begin{aligned}
&\| m^{12}(t) \|_{B^{s-1}_{p,r}} + \| n^{12}(t) \|_{B^{s-1}_{p,r}} \\
&\leq (\| m_0^{12} \|_{B^{s-1}_{p,r}} + \| n_0^{12} \|_{B^{s-1}_{p,r}}) e^{C_1 \int_0^t (\|m^1\|_{B^s_{p,r}} + \|n^1\|_{B^s_{p,r}})^b d\tau}.
\end{aligned}$$

This completes the proof of Lemma 3.2.1. ■

Remark 3.1. Employing the relation $\| f \|_{B^{s+2}_{p,r}(\mathbb{R})} \lesssim \| (1-\partial_x^2) f \|_{B^s_{p,r}(\mathbb{R})}$ with $s > \max(\frac{1}{p}, \frac{1}{2})$ yields $f \in B^s_{p,r}(\mathbb{R})(s > \max(2 + \frac{1}{p}, \frac{5}{2}))$.

4. The proof of Theorem 1.2

We are in a position to establish blow-up criterion of solutions to problem (1.1).

Applying the operator Δ_q to the first and the second equations in problem (1.5) respectively, we have

$$(\partial_t + u^{b-1} v \partial_x) \Delta_q m = [u^{b-1} v, \Delta_q] \partial_x m + \Delta_q F_7(t,x), \quad (4.1)$$
$$(\partial_t + v^{b-1} u \partial_x) \Delta_q n = [v^{b-1} u, \Delta_q] \partial_x n + \Delta_q F_8(t,x), \quad (4.2)$$

where

$$F_7(t,x) = -(b+1) u_x v^{b-1} m + \lambda m, \quad F_8(t,x) = -(b+1) v_x u^{b-1} n + \lambda n.$$

Utilizing Lemma 2.2, we deduce

$$\begin{aligned}
&\| F_7(t,x) \|_{B^s_{p,r}} \\
&\lesssim \| u_x v^{b-1} \|_{L^\infty} \| m \|_{B^s_{p,r}} + \| u_x v^{b-1} \|_{B^s_{p,r}} \| m \|_{L^\infty} + \lambda \| m \|_{B^s_{p,r}} \\
&\lesssim \| u_x v^{b-1} \|_{L^\infty} \| m \|_{B^s_{p,r}} + \lambda \| m \|_{B^s_{p,r}} \\
&\quad + [\| u_x \|_{L^\infty} \| v^{b-1} \|_{B^s_{p,r}} + \| u_x \|_{B^s_{p,r}} \| v^{b-1} \|_{L^\infty}] \| m \|_{L^\infty} \\
&\lesssim \| u_x \|_{L^\infty} \| v \|^{b-1}_{L^\infty} \| m \|_{B^s_{p,r}} + \lambda \| m \|_{B^s_{p,r}} \\
&\quad + [\| u_x \|_{L^\infty} \| v \|^{b-2}_{L^\infty} \| v \|_{B^s_{p,r}} + \| u_x \|_{B^s_{p,r}} \| v \|^{b-1}_{L^\infty}] \| m \|_{L^\infty}
\end{aligned}$$

$$\lesssim \| m \|_{L^\infty} \| n \|_{L^\infty}^{b-1} \| m \|_{B^s_{p,r}} + \lambda \| m \|_{B^s_{p,r}}$$
$$+ \| m \|_{L^\infty}^2 \| n \|_{L^\infty}^{b-2} \| n \|_{B^s_{p,r}} + \| m \|_{B^s_{p,r}} \| n \|_{L^\infty}^{b-1} \| m \|_{L^\infty} . \quad (4.3)$$

If $\max(\frac{1}{p}, \frac{1}{2}) < s < 1$, using Lemma 2.3 gives rise to

$$\begin{aligned} \| [u^{b-1}v, \Delta_q]\partial_x m \|_{B^s_{p,r}} &\lesssim \| \partial_x[u^{b-1}v] \|_{L^\infty} \| m \|_{B^s_{p,r}} \\ &\lesssim (\| u_x \|_{L^\infty}^{b-1} \| v \|_{L^\infty} + \| u \|_{L^\infty}^{b-1} \| v_x \|_{L^\infty}) \| m \|_{B^s_{p,r}} \\ &\lesssim \| m \|_{L^\infty}^{b-1} \| n \|_{L^\infty} \| m \|_{B^s_{p,r}} . \end{aligned} \quad (4.4)$$

Multiplying (4.1) by $(\Delta_q m)^{p-1}$ and integrating with respect to x by parts yield

$$\begin{aligned} \frac{1}{p}\frac{d}{dt} \| \Delta_q m \|_{L^p}^p &\lesssim \| \partial_x(u^{b-1}v) \|_{L^\infty} \| \Delta_q m \|_{L^p}^p \\ &+ \| [u^{b-1}v, \Delta_q]\partial_x m \|_{L^p} \| \Delta_q m \|_{L^p}^{p-1} + \| F_7(t,x) \|_{L^p} \| \Delta_q u \|_{L^p}^{p-1} . \end{aligned} \quad (4.5)$$

Consequently, we achieve

$$\begin{aligned} \frac{d}{dt} \| \Delta_q m \|_{L^p} &\lesssim \| \partial_x(u^{b-1}v) \|_{L^\infty} \| \Delta_q m \|_{L^p} \\ &+ \| [u^{b-1}v, \Delta_q]\partial_x m \|_{L^p} + \| F_7(t,x) \|_{L^p} . \end{aligned} \quad (4.6)$$

Employing Definition 2.1 leads to

$$\begin{aligned} &\| m(t) \|_{B^s_{p,r}} \\ &\lesssim \| m_0 \|_{B^s_{p,r}} + \int_0^t [(\| m \|_{L^\infty} \| n \|_{L^\infty}^{b-1} + \| m \|_{L^\infty}^{b-1} \| n \|_{L^\infty} + \lambda) \| m \|_{B^s_{p,r}} \\ &\quad + \| m \|_{L^\infty}^2 \| n \|_{L^\infty}^{b-2} \| n \|_{B^s_{p,r}}] d\tau \\ &\lesssim \| m_0 \|_{B^s_{p,r}} + \int_0^t (1 + \lambda + \| m \|_{L^\infty} + \| n \|_{L^\infty})^b \\ &\quad \times (\| m \|_{B^s_{p,r}} + \| n \|_{B^s_{p,r}}) d\tau. \end{aligned} \quad (4.7)$$

Analogously, we come to

$$\begin{aligned} \| n(t) \|_{B^s_{p,r}} &\lesssim \| n_0 \|_{B^s_{p,r}} \\ &+ \int_0^t (1 + \lambda + \| m \|_{L^\infty} + \| n \|_{L^\infty})^b (\| m \|_{B^s_{p,r}} + \| n \|_{B^s_{p,r}}) d\tau. \end{aligned} \quad (4.8)$$

Therefore, we conclude

$$\begin{aligned} \| m(t) \|_{B^s_{p,r}} + \| n(t) \|_{B^s_{p,r}} &\lesssim (\| m_0 \|_{B^s_{p,r}} + \| n_0 \|_{B^s_{p,r}}) \\ &+ \int_0^t (1 + \lambda + \| m \|_{L^\infty} + \| n \|_{L^\infty})^b (\| m \|_{B^s_{p,r}} + \| n \|_{B^s_{p,r}}) d\tau. \end{aligned} \quad (4.9)$$

It follows from the Gronwall inequality that

$$\| m(t) \|_{B^s_{p,r}} + \| n(t) \|_{B^s_{p,r}}$$

$$\lesssim (\| m_0 \|_{B^s_{p,r}} + \| n_0 \|_{B^s_{p,r}}) e^{\int_0^t (1+\lambda+\|m\|_{L^\infty}+\|n\|_{L^\infty})^b d\tau}. \tag{4.10}$$

Let T^* be the maximal existence time of solutions to problem (1.5). If $T^* < \infty$, we claim

$$\int_0^t (1 + \lambda + \| m \|_{L^\infty} + \| n \|_{L^\infty})^b d\tau = +\infty. \tag{4.11}$$

We prove the claim (4.11) by contradiction. Indeed, if (4.11) is not valid, namely

$$\int_0^t (1 + \lambda + \| m \|_{L^\infty} + \| n \|_{L^\infty})^b d\tau < +\infty, \tag{4.12}$$

we derive that $\| m(T^*) \|_{B^s_{p,r}} + \| n(T^*) \|_{B^s_{p,r}}$ is bounded by employing (4.11). This contradicts with the fact that T^* is the maximal existence time of solutions to problem (1.5). This finishes the proof of Theorem 1.2. ■

5. Conclusions

This paper is concerned with local well-posedness for the Cauchy problem of coupled system of generalized Camassa-Holm equations and blow-up dynamic of solutions in the Besov spaces $B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R})(s > \max(\frac{5}{2}, 2 + \frac{1}{p}))$. The methods employed in the proofs are based on the Littlewood-Paley theory and a priori estimate of solution to transport equation. This is different from the regularity index $s > \max(\frac{3}{2}, 1 + \frac{1}{p})$ of solutions to the Camassa-Holm equation (see [47]) and Novikov equation (see [29]). Our main new contribution is that the effects of dissipative coefficient λ and exponent b in the nonlinear terms to the solutions are analyzed. It is worth to mention that problem (1.1) contains the problem studied in [39] as special case. We extend parts of results in [29, 49] for single shallow water equation to the dissipative shallow water system. To the best of authors' knowledge, the results in Theorems 1.1 and 1.2 are new.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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