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Research article

The generalized 4-connectivity of folded Petersen cube networks

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Abstract: The generalized ℓ-connectivity $\kappa_\ell(G)$ of a graph G is a generalization of classical connectivity $\kappa(G)$ with $\kappa_2(G) = \kappa(G)$. It serves to measure the capability of connection for any ℓ vertices. The folded Petersen cube network $FPQ_{n,k}$ can be used to model the topological structure of a communication-efficient multiprocessor. This paper shows that the generalized 4-connectivity of the folded Petersen cube network $FPQ_{n,k}$ is $n + 3k - 1$. As a corollary, the generalized 3-connectivity of $FPQ_{n,k}$ also is obtained and the results on the generalized 4-connectivity of hypercube Q_n and folded Petersen graph FP_k can be verified. These conclusions provide a foundation for studying the generalized 4-connectivity of Cartesian product graphs.

Keywords: generalized 4-connectivity; folded Petersen cube networks; Cartesian product
Mathematics Subject Classification: 05C05, 05C40, 05C76, 68R10

1. Introduction

As usual, the topological structure of an interconnection network is regarded as a graph $G = (V, E)$, in which vertices correspond to processors and edges represent communication links between processors. The fault tolerance is one of the most important factors in the design and analysis of an interconnection network and it can be measured by the connectivity of a graph. If the connectivity of a network is larger, then its fault tolerance is higher. The traditional connectivity $\kappa(G)$ of a graph G is defined as the minimum number of vertices whose deletion results in a disconnected graph. An excellent theorem of Whitney [32] provided an equivalent statement about the definition of the connectivity. That is, for any 2-subset $S = \{u, v\} \subseteq V(G)$, if $\kappa(S)$ denotes the maximum number of internally disjoint paths between u and v in G, then $\kappa(G) = \min\{\kappa(S) : S \subseteq V(G), |S| = 2\}$. Clearly,

$\kappa(G)$ reflects the connectivity between any two processors. To measure the connectivity capability of more processors, Chartrand et al. [6] and Hager et al. [11] introduced independently the concept of the generalized connectivity of a graph by generalizing the equivalent definition of connectivity.

Let G be a connected graph with order n and ℓ be an integer such that $2 \leq \ell \leq n$. For $S \subseteq V(G)$, a tree T in G is called an S-tree if $S \subseteq V(T)$. Let $\kappa(S)$ denote the maximum number r of edge-disjoint S-trees $T_1, \ldots, T_r$ satisfying $V(T_i) \cap V(T_j) = S$ for any two distinct integers $i, j \in \{1, \ldots, r\}$. The generalized ℓ-connectivity $\kappa_\ell(G)$ of G is defined as $\min\{\kappa(S) \colon S \subseteq V(G), |S| = \ell\}$. Actually, $\kappa_2(G)$ is exactly $\kappa(G)$.

Though there are numerous results about the generalized ℓ-connectivity over the past years, for general integer ℓ, the exact values of $\kappa_\ell(G)$ are known for only a small class of graphs: complete graph [7], complete bipartite graph [16] and complete equipartition 3-partite graph [18]. Meanwhile, for a given graph G, any fixed integer $k \geq 2$ and a subset S of $V(G)$, the decision problem whether $\kappa(S) \geq k$ is NP-complete [19]. The upper and lower bounds of the generalized 3-connectivity of a graph [21, 25] and of Cartesian (Lexicographic) product of two graphs [13, 14, 26] were investigated, and extremal problems were studied in [17, 22]. The generalized 3-connectivity of some graph classes are known, including Cayley graphs [20,31], star graphs and bubble-sort graphs [23], alternating group graphs and (n, k)-star graphs [34], k-ary n-cubes, split-star graphs and bubble-sort star graphs [37], (n, k)-bubble sort graphs [38], etc. We refer the readers to [15, 27] for more details.

Unfortunately, the results of the generalized 4-connectivity are less known. Only hypercubes [24], dual cubes [35], exchanged hypercubes [33], (n, k)-star graphs [12], hierarchical cubic networks [36] have been studied.

The main focus of this paper is to determine the generalized 4-connectivity of the folded Petersen cube networks $FPQ_{n,k}$. The following result is obtained.

Theorem 1.1. *Let k, n be two integers. Then $\kappa_4(FPQ_{n,k}) = n + 3k - 1$.*

Theorem 1.1 implies that if $k = 0$, then $\kappa_4(FPQ_{n,0}) = n - 1$, which coincides the value of $\kappa_4(Q_n)$. The key to prove Theorem 1.1 is Theorem 1.2.

Theorem 1.2. *Let FP_k be a k-dimensional folded Petersen graph. Then $\kappa_4(FP_k) = 3k - 1$.*

For a regular graph, the following lemma is useful.

Lemma 1.1. *[24] Let G be an r-regular graph. If $\kappa_k(G) = r - 1$, then $\kappa_{k-1}(G) = r - 1$, where $k \geq 4$.*

Combining Theorem 1.1 and Lemma 1.1, the following corollary is an immediate consequence.

Corollary 1.1. *Let k, n be two integers. Then $\kappa_3(FPQ_{n,k}) = n + 3k - 1$.*

For helping readers to understand the proof process, a flow chart in Figure 1 is presented to illustrate the relationship between different lemmas, theorems and corollaries.

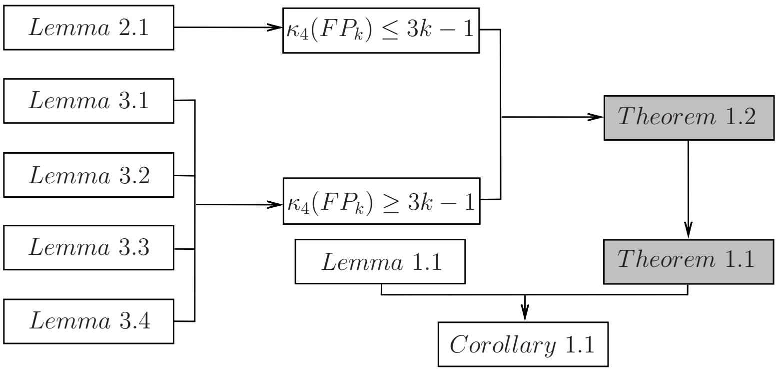


Figure 1. Flow chart.

2. Preliminaries

This section introduces some basic notations and results that will be used throughout the paper. All graphs considered in this paper are connected, simple, undirected, finite and for the notation and terminology not defined refer to [5].

2.1. *Basic notations and lemmas*

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G, we use $N_G(v)$ to denote the set of neighbors of v in G, and the degree of v is $d_G(v) = |N_G(v)|$. The minimum degree and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A graph G is r-regular if $\delta(G) = \Delta(G) = r$. For two vertices $u, v \in V(G)$, a (u, v)-path is denoted by P_{uv} and the length of a shortest (u, v)-path is called the distance between u and v, denoted by $d_G(u, v)$. A subgraph of G is a graph $H = (V', E')$ such that $V'(H) \subseteq V(G)$ and $E'(H) \subseteq E(G)$. If $V'(H) = V(G)$, then H is called a spanning subgraph of G. The subgraph of G induced by V' is denoted by $G[V']$. Let $[b] = \{1, \ldots, b\}$ for a given integer b.

Lemma 2.1. *[21] If there are two adjacent vertices of degree δ, then $\kappa_\ell(G) \leq \delta - 1$ for $3 \leq \ell \leq |V(G)|$.*

Theorem 2.1. *(Menger's theorem [5]) A graph G is r-connected if and only if any two distinct vertices of G are connected by at least r internally disjoint paths.*

Lemma 2.2. *(Fan Lemma [5]) Let G be an r-connected graph, x be a vertex of G, and let $Y \subseteq V(G)\backslash\{x\}$ be a set of at least r vertices of G. Then there exists an r-fan in G from x to Y, that is, there exists a family of r internally disjoint (x, Y)-paths whose terminal vertices are distinct in Y.*

2.2. *The folded Petersen cube networks*

The Cartesian product of graphs is an important tool to construct a bigger network. Recall that the Cartesian product of two graphs G and H, denoted by $G\square H$, is a graph with the vertex set $V(G) \times V(H)$ such that (g, h) and (g', h') are adjacent if and only if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$.

The Petersen graph $\mathbf{P}$ with a vertex set $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{9}\}$ has an outer 5-cycle and an inner 5-cycle are joined by a perfect matching (Figure 2(a) depicts $\mathbf{P}$ with decimal vertex-labeling). It is a 3-regular

3-connected graph with diameter 2. The k-dimensional folded Petersen graph FP_k is constructed by the k-th iteration of Cartesian product on Petersen graph $\mathbf{P}$, defined as $FP_k = \mathbf{P}\square\cdots\square\mathbf{P} = (V, E)$, where $V = \{x_1 \cdots x_k \colon x_i \in \{0, 1, ..., 9\}, 1 \leq i \leq k\}$ and $E = \{(x_1 \cdots x_{i-1}x_ix_{i+1} \cdots x_k, x_1 \cdots x_{i-1}yx_{i+1} \cdots x_k) \colon y \in \{0, 1, \ldots, 9\}, d_{\mathbf{P}}(y, x_i) = 1 \text{ and } 1 \leq i \leq k\}$. As depicted in Figure 1(b), FP_2 is obtained from $\mathbf{P}$ by replacing each of its vertices by $\mathbf{P}$, denoted by $\mathbf{P}^0, \mathbf{P}^1, \ldots, \mathbf{P}^9$, respectively, moreover, between $\mathbf{P}^i$ and $\mathbf{P}^j$ have a perfect matching for $d_{\mathbf{P}}(i, j) = 1$. In fact, for $k \geq 2$, FP_k is obtained from $\mathbf{P}$ by replacing each of its vertices by FP_{k-1}, and we denote the 10 connected subgraphs by $FP_k^{i:0}, FP_k^{i:1}, \ldots, FP_k^{i:9}$, respectively, where $FP_k^{i:j} = FP_k[\{x_1 \cdots x_k \in V(FP_k), x_i = j\}]$. Clearly, $FP_k^{i:j} \cong FP_{k-1}$. For convenience, FP^j denotes $FP_k^{1:j}$ for $j \in \{0, 1, \ldots, 9\}$ in the rest of this paper. It is easy to see that the subgraph induced by $V(FP^i) \cup V(FP^j)$ is isomorphic to $FP_{k-1}\square K_2$ for $d_{\mathbf{P}}(i, j) = 1$, and the edges between $V(FP^i)$ and $V(FP^j)$ are called the crossed edges.

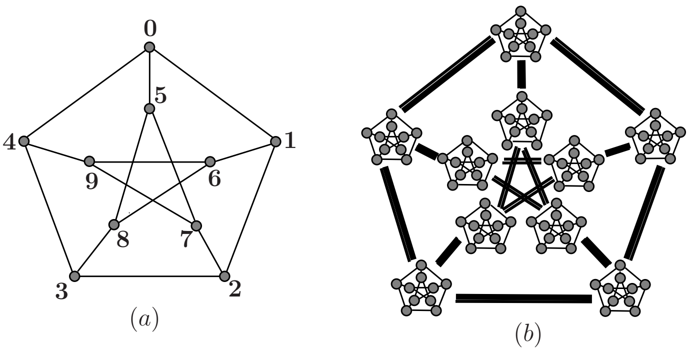


Figure 2. (a) The Petersen graph $\mathbf{P}$; (b) Scheme of FP_2.

For any $x \in V(FP^i)$, we call that x^{i_j} is a corresponding vertex of x in FP^{i_j} if x and x^{i_j} differ in exactly the first digit. In particular, a corresponding vertex x^{i_j} of x is an outside neighbor of x if $d_{\mathbf{P}}(i, i_j) = 1$. Apparently, there are exactly three outside neighbors of x. Two graphs G' and G'' are corresponding if their vertices correspond one to one and $G' \cong G''$. Furthermore, FP_k is a $3k$-regular $3k$-connected graph with diameter $2k$ and order 10^k, also vertex and edge symmetric. More details see [8, 10].

The folded Petersen cube network $FPQ_{n,k} = FP_k\square Q_n$ with $10^k \times 2^n$ vertices is introduced by Öhring and Das [28], where Q_n is an n-dimensional hypercube and it can be regarded as n-th iteration of Cartesian product on K_2. In particular, $FPQ_{0,k} = FP_k$ and $FPQ_{n,0} = Q_n$. There are a lot of topological structures like linear arrays, rings, meshes, hypercubes can be embedded into it. Some research findings on the folded Petersen cube networks have been published for the past several years, see [9, 28, 30].

The following result will be used to prove Theorem 1.2.

Lemma 2.3. *Let S be a vertex subset of the Petersen graph $\mathbf{P}$ with $|S| = 4$. Then there exist two internally disjoint S-trees in $\mathbf{P}$.*

Proof. Let $\mathbf{P}$ be a Petersen graph, $S = \{x, y, z, w\} \subseteq V(\mathbf{P})$ and $G = \mathbf{P}[S]$. Then $d_G(v) \leq 3$ for arbitrary $v \in V(G)$. Since $\mathbf{P}$ is vertex symmetric, assume that $d_G(x) = \Delta(G)$. If $d_G(x) = 3$, then $N_G(x) = \{y, z, w\}$. Two internally disjoint S-trees are demonstrated in Figure 3(a). If $d_G(x) = 2$, without loss of generality, let $N_G(x) = \{y, z\}$. It suffices to consider two cases and the required S-trees are demonstrated in Figure 3(b) and Figure 3(c), respectively. If $d_G(x) = 1$, assume that $N_G(x) = \{y\}$, it suffices to consider two cases and the required S-trees are demonstrated in Figure 3(d) and Figure 3(e), respectively. If $d_G(x) = 0$, Figure 2(f) demonstrates two internally disjoint S-trees.

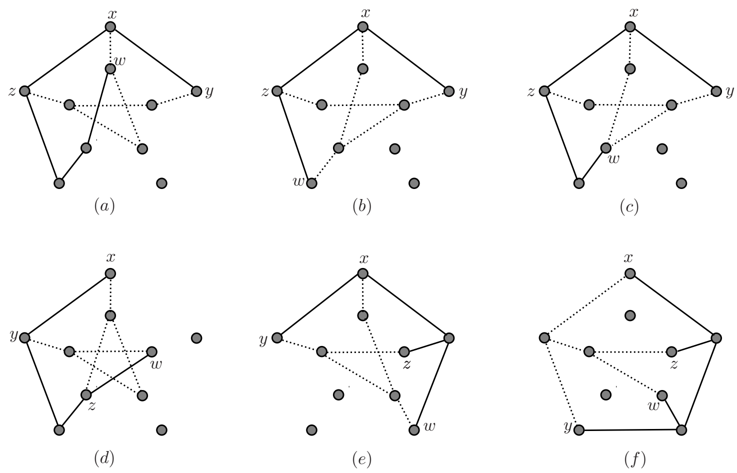


Figure 3. Illustrations of the proof of Lemma 2.3.

3. Main results

In this section, we mainly determine $\kappa_4(FP_k)$. First, propose the general idea of the proof $\kappa_4(FP_k) \geq 3k-1$. From the definition of $\kappa_4(FP_k)$, it suffices to show that $\kappa_{FP_k}(S) \geq 3k-1$ for arbitrary $S \subseteq V(FP_k)$ with $|S| = 4$. Furthermore, according to the definition of $\kappa_{FP_k}(S)$, find out $3k-1$ internally disjoint S-trees in FP_k. Let S be a set of arbitrary four distinct vertices in FP_k. Recall that FP_k can be decomposed into 10 disjoint sub-folded Petersen graphs $FP^0, FP^1, \ldots, FP^9$, each of which is isomorphic to FP_{k-1} by removing all crossed edges.

For arbitrary four vertices v_0, v_1, v_2, v_3 in $\mathbf{P}$, by Lemma 2.3, there are two internally disjoint $\{v_0, v_1, v_2, v_3\}$-trees T_1^* and T_2^* in $\mathbf{P}$. At least one of the trees T_1^* and T_2^* contains a vertex u different from v_0, v_1, v_2, v_3. Without loss of generality, let T_2^* be such a tree. Clearly, $(T_1^* \cup T_2^*)\Box FP_{k-1}$ is a subgraph of FP_k. If we can prove that $\kappa_{T_1^*\Box FP_{k-1}}(S) \geq 3k-2$ and find out an S-tree in $T_2^*\Box FP_{k-1}$, then $\kappa_{FP_k}(S) \geq \kappa_{T_1^*\Box FP_{k-1}}(S) + 1 \geq 3k-1$. In this case, the problem is converted into finding out $3k-2$ internally disjoint S-trees in $T_1^*\Box FP_{k-1}$ and an S-tree in $T_2^*\Box FP_{k-1}$ such that all of the S-trees are internally disjoint.

Observation 3.1. *For arbitrary three vertices v_0, v_1, v_2 in $\mathbf{P}$, one of the following holds.*

(i) If $\{v_0, v_1, v_2\}$ is an isolated set, then there are three internally disjoint $\{v_0, v_1, v_2\}$-trees T_1^*, T_2^*, T_3^* in $\mathbf{P}$. Moreover, there exists a vertex u_i with degree 3 different from v_0, v_1, v_2 in T_i^* for $i = 1, 2, 3$.

(ii) If $\{v_0, v_1, v_2\}$ is not an isolated set, then there are a C_5 and a $\{v_0, v_1, v_2\}$-tree T_3^* in $\mathbf{P}$ such that $v_0, v_1, v_2 \in V(C_5)$ and $V(C_5) \cap V(T_3^*) = \{v_0, v_1, v_2\}$. Moreover, there exists a vertex u with degree 3 different from v_0, v_1, v_2 in T_3^*.

For Observation 3.1(i), it is easy to see that $(\bigcup_{i=1}^{3} T_i^*)\Box FP_{k-1}$ is a subgraph of FP_k (see Figure 4). Notice that T_1^*, T_2^* and T_3^* are internally disjoint in $\mathbf{P}$, if we can prove that $\kappa_{T_1^*\Box FP_{k-1}}(S) \geq 3k-3$, find out an S-tree in $T_2^*\Box FP_{k-1}$ and another S-tree in $T_3^*\Box FP_{k-1}$, then $\kappa_{FP_k}(S) \geq \kappa_{T_1^*\Box FP_{k-1}}(S) + 2 \geq 3k-1$.

In this case, the problem is converted into finding out $3k-3$ internally disjoint S-trees in $T_1^*\Box FP_{k-1}$, an S-tree in $T_2^*\Box FP_{k-1}$ and another S-tree in $T_3^*\Box FP_{k-1}$ such that all of the S-trees are internally disjoint.

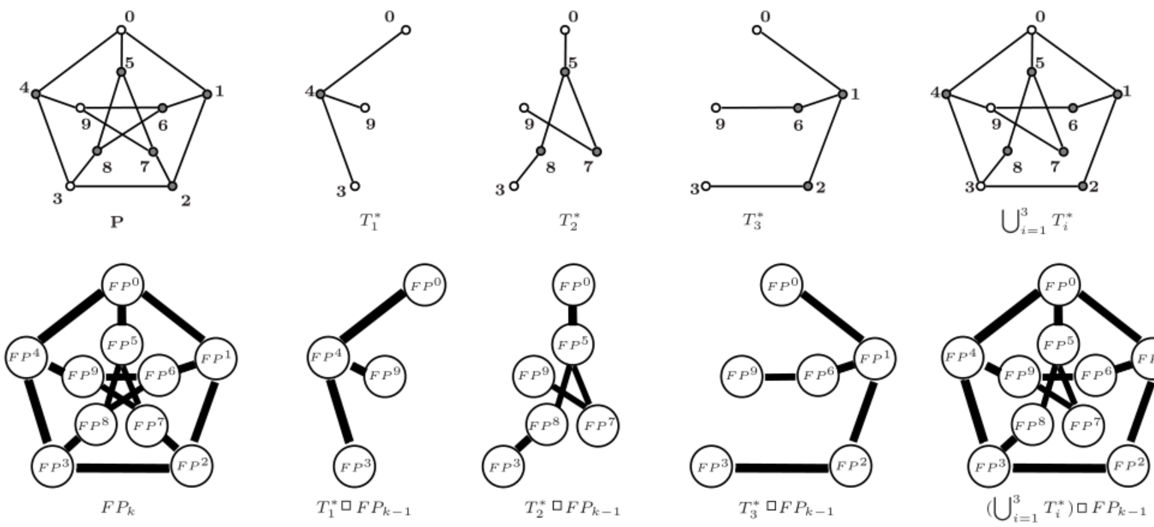


Figure 4. An example for $(\bigcup_{i=1}^3 T_i^*)\Box FP_{k-1}$ is a subgraph of FP_k.

For Observation 3.1(ii), let $C = C_5$, it can be seen that $(C \cup T_3^*)\Box FP_{k-1}$ is a subgraph of FP_k (see Figure 5). If we can prove that $\kappa_{C\Box FP_{k-1}}(S) \geq 3k-2$, and find out an S-tree in $T_3^*\Box FP_{k-1}$, then $\kappa_{FP_k}(S) \geq \kappa_{C\Box FP_{k-1}}(S) + 1 \geq 3k-1$. In this case, the problem is converted into finding out $3k-2$ internally disjoint S-trees in $C\Box FP_{k-1}$ and an S-tree in $T_3^*\Box FP_{k-1}$ such that all of the S-trees are internally disjoint.

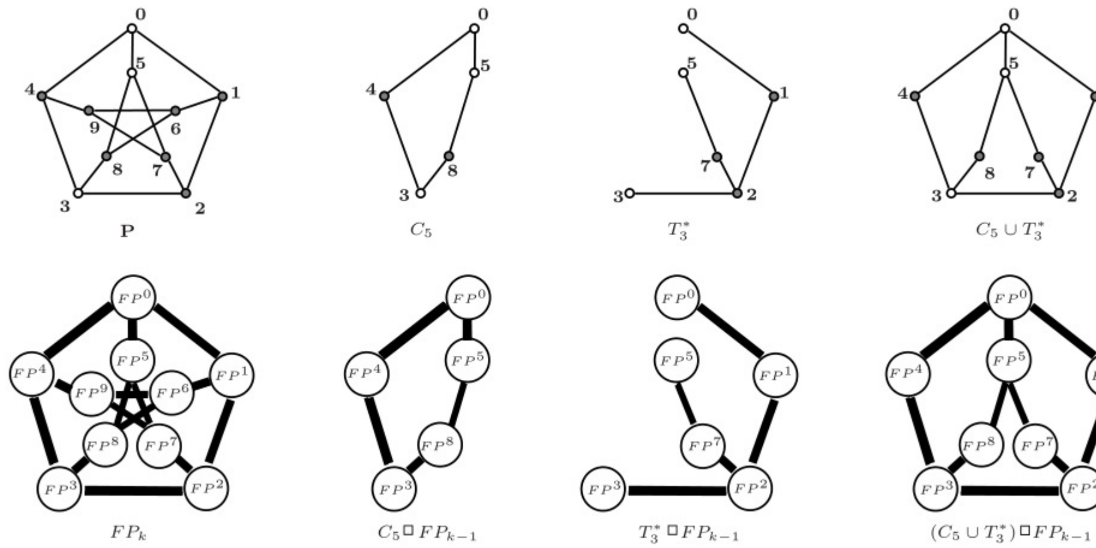


Figure 5. An example for $(C \cup T_3^*)\Box FP_{k-1}$ is a subgraph of FP_k.

For arbitrary two vertices v_0, v_1 in $\mathbf{P}$, by Theorem 2.1, there are three internally disjoint (v_0, v_1)-paths L_1^*, L_2^* and L_3^* in $\mathbf{P}$ because the Petersen graph $\mathbf{P}$ is 3-connected. Note that $\bigcup_{i=1}^3 L_i^*$ is a subgraph of $\mathbf{P}$ and $(\bigcup_{i=1}^3 L_i^*)\Box FP_{k-1}$ is a subgraph of FP_k (see Figure 6). If we can prove that $\kappa_{L_1^*\Box FP_{k-1}}(S) \geq 3k-3$, find out an S-tree in $L_2^*\Box FP_{k-1}$ and another S-tree in $L_3^*\Box FP_{k-1}$, then $\kappa_{FP_k}(S) \geq \kappa_{(\cup_{i=1}^3 L_i^*)\Box FP_{k-1}}(S) \geq \kappa_{L_1^*\Box FP_{k-1}}(S) + 2 \geq 3k-1$. In this case, the problem is converted into finding out $3k-3$ internally disjoint S-trees in $L_1^*\Box FP_{k-1}$, an S-tree in $L_2^*\Box FP_{k-1}$ and another S-tree in $L_3^*\Box FP_{k-1}$ such that all of the S-trees are internally disjoint.

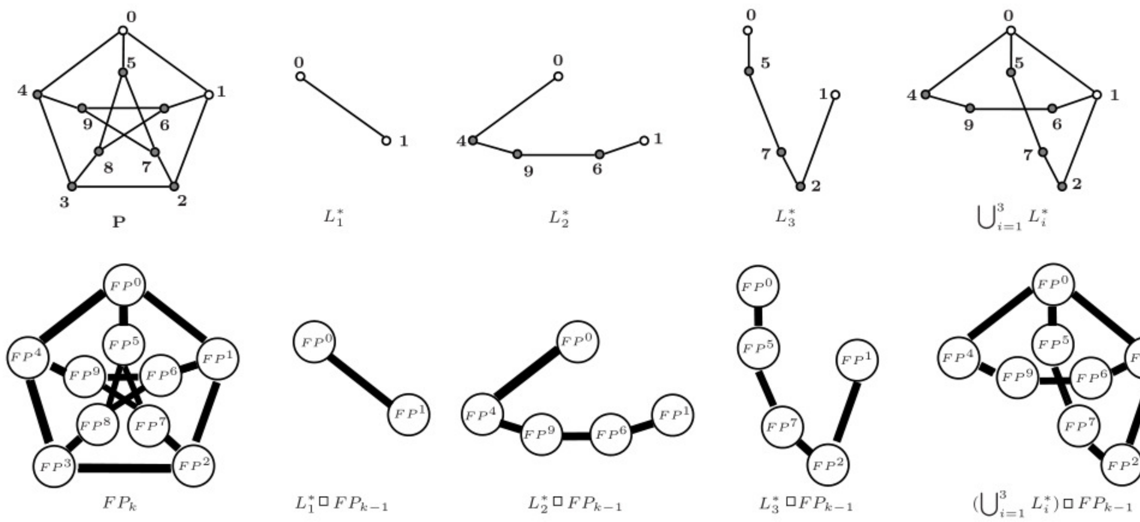


Figure 6. An example for $(\bigcup_{i=1}^3 L_i^*)\Box FP_{k-1}$ is a subgraph of FP_k.

The following four lemmas will be helpful to main result.

Lemma 3.1. *Let FP_k be a k-dimensional folded Petersen graph and $S = \{x, y, z, w\}$ be a set of arbitrary four distinct vertices in FP_k for $k \geq 2$. If the vertices in S belong to four sub-folded Petersen graphs of FP_k, then there are $3k-1$ internally disjoint S-trees in FP_k.*

Proof. Let $FP^0, FP^1, \ldots, FP^9$ be 10 disjoint sub-folded Petersen graphs of FP_k. Since the vertices in S belong to four sub-folded Petersen graphs of FP_k, there exist $v_0, v_1, v_2, v_3 \in \{0, 1, \ldots, 9\}$ such that they are mutual distinct and $x \in V(FP^{v_0})$, $y \in V(FP^{v_1})$, $z \in V(FP^{v_2})$ and $w \in V(FP^{v_3})$.

For $v_0, v_1, v_2, v_3 \in V(\mathbf{P})$, there exist two internally disjoint $\{v_0, v_1, v_2, v_3\}$-trees T_1^* and T_2^* in $\mathbf{P}$ by Lemma 2.3. At least one of the trees T_1^* and T_2^* contains a vertex c different from v_0, v_1, v_2, v_3, say T_2^* is such a tree. Since FP^c is connected, there exists an $\{x^c, y^c, z^c, w^c\}$-tree T'_{3k-2} in FP^c. There is a unique path to connect arbitrary two vertices in T_2^*. That is, we can find the corresponding path to connect arbitrary two vertices in $T_2^*\Box FP_{k-1}$. Let $T_{3k-2} = T'_{3k-2} \cup P_{xx^c} \cup P_{yy^c} \cup P_{zz^c} \cup P_{ww^c}$.

Let $x_0 = x$. Choose $3k-2$ distinct vertices $x_0, x_1, x_2, \ldots, x_{3k-3}$ from FP^{v_0} being X such that y^{v_0}, z^{v_0} and w^{v_0} belong to X and $|X| = 3k-2$. Without loss of generality, assume that $x_r = y^{v_0}, x_s = z^{v_0}, x_t = w^{v_0}$ for $r, s, t \in \{0, 1, \ldots, 3k-3\}$. Let $Y = \bigcup_{i=0}^{3k-3} y_i, Z = \bigcup_{i=0}^{3k-3} z_i$ and $W = \bigcup_{i=0}^{3k-3} w_i$ be the corresponding vertices of vertices of X in FP^{v_1}, FP^{v_2} and FP^{v_3}, respectively, where $y_i = x_i^{v_1}$, $z_i = x_i^{v_2}$ and $w_i = x_i^{v_3}$ for each $i \in \{0, 1, \ldots, 3k-3\}$. Then $|Y| = |Z| = |W| = 3k-2$. By Lemma 2.2 and $\kappa(FP_{k-1}) = 3k-3$, there are $3k-3$ internally disjoint paths $A_1, A_2, \ldots, A_{3k-3}$ from a to $A \setminus \{a\}$ in FP^{v_j} for $(a, A, j) = (x, X, 0), (y, Y, 1), (z, Z, 2), (w, W, 3)$, respectively. Since there is a unique path to connect arbitrary two vertices in T_1^*, we can find the corresponding path to connect corresponding vertices in $T_1^*\Box FP_{k-1}$. Construct $3k-2$ internally disjoint S-trees in $T_1^*\Box FP_{k-1}$ as follows: $T_i = X_i \cup P_{x_iy_i} \cup Y_i \cup P_{y_iz_i} \cup Z_i \cup P_{z_iw_i} \cup W_i$ for $i \in \{0, 1, \ldots, 3k-3\}$, where $X_0 = \{x_0\} = \{x\}$, $Y_r = \{y_r\} = \{y\}$, $Z_s = \{z_s\} = \{z\}$ and $W_t = \{w_t\} = \{w\}$. Then, $T_0, T_1, \ldots, T_{3k-2}$ are $3k-1$ internally disjoint S-trees in FP_k. See Figure 7.

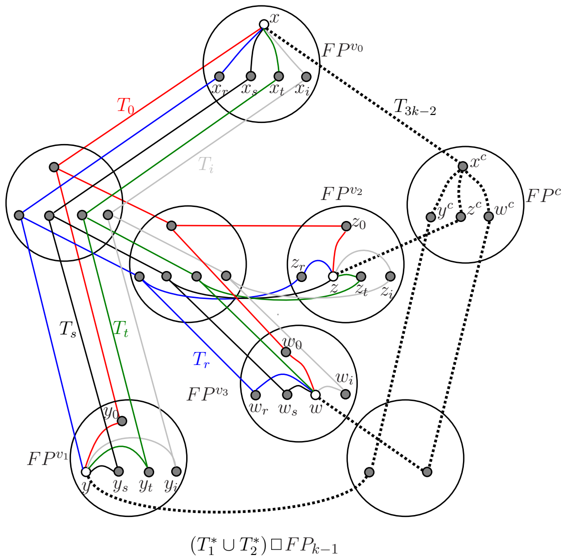


Figure 7. Illustrations of Lemma 3.1.

Lemma 3.2. *Let FP_k be a k-dimensional folded Petersen graph and $S = \{x, y, z, w\}$ be a set of arbitrary four distinct vertices in FP_k for $k \geq 2$. If the vertices in S belong to three sub-folded Petersen graphs of FP_k, then there are $3k-1$ internally disjoint S-trees in FP_k.*

Proof. Let $FP^0, FP^1, \ldots, FP^9$ be 10 disjoint sub-folded Petersen graphs of FP_k. Since the vertices in S belong to three sub-folded Petersen graphs of FP_k, there exist $v_0, v_1, v_2 \in \{0, 1, \ldots, 9\}$ such that $x, y \in V(FP^{v_0})$, $z \in V(FP^{v_1})$ and $w \in V(FP^{v_2})$ by the symmetry of FP_k.

Since FP_{k-1} is $(3k-3)$-connected, there exist $3k-3$ internally disjoint (x, y)-paths $P_1, P_2, \ldots, P_{3k-3}$ in FP^{v_0}. Choose $x_i \in V(P_i)$ such that $x_i \neq x$. Let $z_i = x_i^{v_1}$ and $w_i = x_i^{v_2}$ for $i \in [3k-3]$. By Lemma 2.2, there are $3k-3$ paths $P_{zz_1}, \ldots, P_{zz_{3k-3}}$ and $3k-3$ paths $P_{ww_1}, \ldots, P_{ww_{3k-3}}$ in FP^{v_1} and FP^{v_2}, respectively. It is possible that one of the paths P_{zz_i} (resp. P_{ww_i}) is a single vertex.

Case 1. $\{v_0, v_1, v_2\}$ is an isolated set.

By Observation 3.1(i), there exists a $\{v_0, v_1, v_2\}$-tree T_1^* which contains a vertex a with degree 3 different from v_0, v_1, v_2 in $\mathbf{P}$. Moreover, there are a unique (v_0, a)-path P_{v_0a}, a unique (v_1, a)-path P_{v_1a} and a unique (v_2, a)-path P_{v_2a} in T_1^*. It is not hard to find $3k-3$ internally disjoint S-trees in $T_1^*\Box FP_{k-1}$, that is, $T_i = P_i \cup P_{x_ix_i^a} \cup P_{zz_i} \cup P_{z_ix_i^a} \cup P_{ww_i} \cup P_{w_ix_i^a}$ for $i \in [3k-3]$, where $P_{x_ix_i^a} \cong P_{v_0a}$, $P_{z_ix_i^a} \cong P_{v_1a}$ and $P_{w_ix_i^a} \cong P_{v_2a}$.

Except T_1^*, there are two internally disjoint $\{v_0, v_1, v_2\}$-trees T_2^* and T_3^* in $\mathbf{P}$, T_2^* and T_3^* contains a vertex with degree 3 different from v_0, v_1 and v_2, denoted by b and c, respectively. Since FP^b is connected, there exists an $\{x^b, y^b, z^b, w^b\}$-tree T'_{3k-2} in FP^b (resp. $\{x^c, y^c, z^c, w^c\}$-tree T'_{3k-1} in FP^c). By the definition of FP_k, there exist the paths $P_{xx^b}, P_{yy^b}, P_{zz^b}, P_{ww^b}$ (resp. $P_{xx^c}, P_{yy^c}, P_{zz^c}, P_{ww^c}$) in FP_k such that $P_{xx^b} \cong P_{yy^b} \cong P_{v_0b}, P_{zz^b} \cong P_{v_1b}, P_{ww^b} \cong P_{v_2b}$ (resp. $P_{xx^c} \cong P_{yy^c} \cong P_{v_0c}, P_{zz^c} \cong P_{v_1c}, P_{ww^c} \cong P_{v_2c}$) where $P_{v_0b}, P_{v_1b}, P_{v_2b}$ belong to T_2^* (resp. $P_{v_0c}, P_{v_1c}, P_{v_2c}$ belong to T_3^*). Let $T_{3k-2} = T'_{3k-2} \cup P_{xx^b} \cup P_{yy^b} \cup P_{zz^b} \cup P_{ww^b}$ (resp. $T_{3k-1} = T'_{3k-1} \cup P_{xx^c} \cup P_{yy^c} \cup P_{zz^c} \cup P_{ww^c}$). Then, $T_1, T_2, \ldots, T_{3k-1}$ are $3k-1$ internally disjoint S-trees in FP_k. See Figure 8.

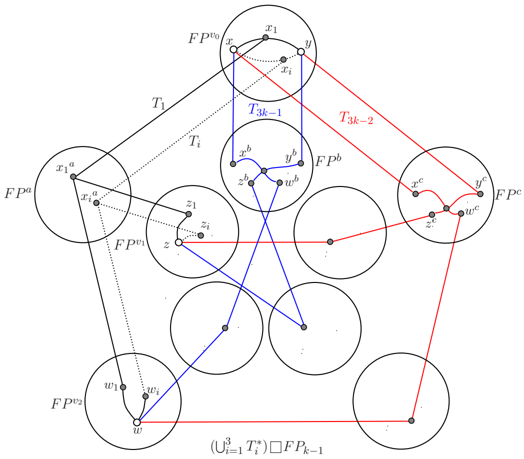
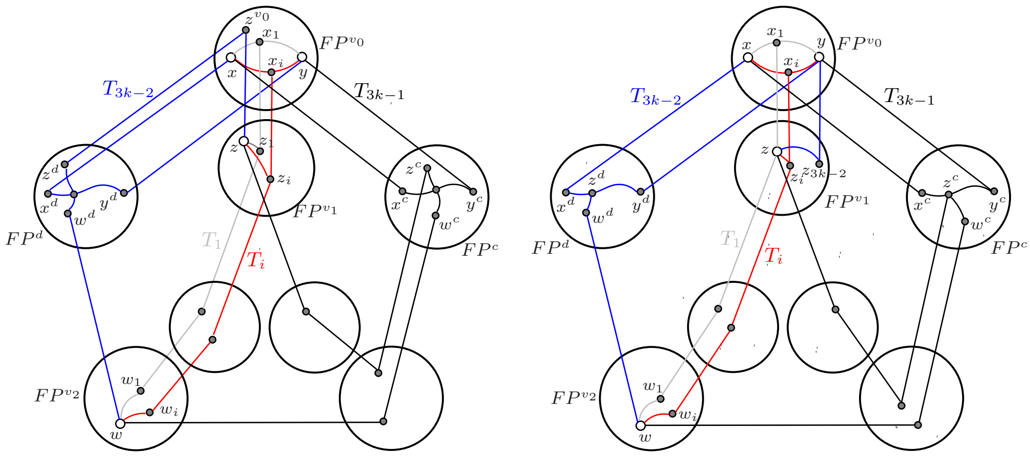


Figure 8. Illustrations of Case 1 in Lemma 3.2.

Case 2. $\{v_0, v_1, v_2\}$ is not an isolated set.

By Observation 3.1(ii), there exists a cycle C for $C = C_5$ and a $\{v_0, v_1, v_2\}$-tree T_3^* in $\mathbf{P}$ such that $V(C) \cap V(T_3^*) = \{v_0, v_1, v_2\}$. Let R be a shorter path containing vertices v_0, v_1, v_2 in C, say (v_0, v_2)-path being such a path. Then $R = P_{v_0v_1} \cup P_{v_1v_2}$. Let $T_i = P_i \cup P_{x_iz_i} \cup P_{zz_i} \cup P_{z_iw_i} \cup P_{ww_i}$ for $i \in [3k-3]$, where $P_{x_iz_i} \cong P_{v_0v_1}$ and $P_{z_iw_i} \cong P_{v_1v_2}$.



(a) $z^{v_0} \notin \bigcup_{i=1}^{3k-3} V(P_i)$ (b) $z^{v_0} \in \bigcup_{i=1}^{3k-3} V(P_i)$

Figure 9. Illustrations of Case 2 in Lemma 3.2.

Clearly, R is a $\{v_0, v_1, v_2\}$-tree with $|V(R)| \leq 4$ and there exists another (v_0, v_2)-path R' containing a non-end vertex d, then $R' = P_{v_0d} \cup P_{v_2d}$. Since FP^d is connected, there exists an $\{x^d, y^d, z^d, w^d\}$-tree T'_{3k-2} in FP^d. If $z^{v_0} \notin \bigcup_{i=1}^{3k-3} V(P_i)$ (see Figure 9(a)), then $z^{v_0} \in P_{zz^d}$, where $P_{zz^d} \cong P_{v_0v_1} \cup P_{v_0d}$. Let $T_{3k-2} = T'_{3k-2} \cup P_{xx^d} \cup P_{yy^d} \cup P_{zz^d} \cup P_{ww^d}$, where $P_{xx^d} \cong P_{yy^d} \cong P_{v_0d}$ and $P_{ww^d} \cong P_{v_2d}$. If $z^{v_0} \in \bigcup_{i=1}^{3k-3} V(P_i)$ (see Figure 9(b)), without loss of generality, assume that $z^{v_0} \in P_1$ and $x_1 = z^{v_0}$, then the path P_{zz_1} is a single vertex. Let $z_{3k-2} = y^{v_1}$. Then there exists a path $P_{zz_{3k-2}}$ in FP^{v_1}. Let $T_{3k-2} = T'_{3k-2} \cup P_{xx^d} \cup P_{yy^d} \cup P_{yz_{3k-2}} \cup P_{zz_{3k-2}} \cup P_{ww^d}$, where $P_{xx^d} \cong P_{yy^d} \cong P_{v_0d}$, $P_{yz_{3k-2}} \cong P_{v_0v_1}$ and $P_{ww^d} \cong P_{v_2d}$. The S-tree T_{3k-1} can be similarly constructed as T_{3k-1} of Case 1. Then, $T_1, T_2, \ldots, T_{3k-1}$ are $3k-1$ internally disjoint S-trees in FP_k.

Lemma 3.3. *Let FP_k be a k-dimensional folded Petersen graph and $S = \{x, y, z, w\}$ be a set of arbitrary four distinct vertices in FP_k for $k \geq 2$. If the vertices in S belong to the same sub-folded Petersen graphs of FP_k, then there are $3k-1$ internally disjoint S-trees in FP_k.*

Proof. Let $FP^0, FP^1, \ldots, FP^9$ be 10 disjoint sub-folded Petersen graphs of FP_k. The vertices in S belong to the same sub-folded Petersen graph, without loss of generality, suppose that $x, y, z, w \in V(FP^0)$. Since $FP^0 \cong FP_{k-1}$, by induction hypothesis, we have $\kappa_4(FP_{k-1}) \geq 3k-4$. Hence, there exist $3k-4$ internally disjoint S-trees $T_1, T_2, \ldots, T_{3k-4}$ in FP^0. Clearly, there exists an $\{x^1, y^1, z^1, w^1\}$-tree T'_{3k-3} in FP^1, an $\{x^4, y^4, z^4, w^4\}$-tree T'_{3k-2} in FP^4 and an $\{x^5, y^5, z^5, w^5\}$-tree T'_{3k-1} in FP^5. Let $T_{3k-3} = T'_{3k-3} \cup xx^1 \cup yy^1 \cup zz^1 \cup ww^1$, $T_{3k-2} = T'_{3k-2} \cup xx^4 \cup yy^4 \cup zz^4 \cup ww^4$ and $T_{3k-1} = T'_{3k-1} \cup xx^5 \cup yy^5 \cup zz^5 \cup ww^5$. Then, $T_1, T_2, \ldots, T_{3k-1}$ are $3k-1$ internally disjoint S-trees in FP_k.

Lemma 3.4. *Let FP_k be a k-dimensional folded Petersen graph and $S = \{x, y, z, w\}$ be a set of arbitrary four distinct vertices in FP_k for $k \geq 2$. If the vertices in S belong to two sub-folded Petersen graphs of FP_k, then there are $3k-1$ internally disjoint S-trees in FP_k.*

Proof. Let $FP^0, FP^1, \ldots, FP^9$ be 10 disjoint sub-folded Petersen graphs of FP_k. Suppose that the vertices in S belong to two distinct sub-folded Petersen graphs FP^{v_0} and FP^{v_1}, where $v_0, v_1 \in \{0, 1, \ldots, 9\}$. For $v_0, v_1 \in V(\mathbf{P})$, since the Petersen graph $\mathbf{P}$ is 3-connected, by Theorem 2.1, there are three internally disjoint (v_0, v_1)-paths L_1^*, L_2^* and L_3^* in $\mathbf{P}$. For convenience, let $|V(L_1^*)| \leq |V(L_2^*)| \leq |V(L_3^*)|$. It implies that $2 \leq |V(L_1^*)| \leq 3$. We just consider $|V(L_1^*)| = 2$ as the discussion for $|V(L_1^*)| = 3$ is similar. Without loss of generality, suppose that $v_0 = \mathbf{0}$ and $v_1 = \mathbf{1}$. By the symmetry of FP_k, the following cases be considered.

Case 1. $x, y, z \in V(FP^0)$ and $w \in V(FP^1)$.

Case 1.1. $w^0 \in \{x, y, z\}$.

Without loss of generality, suppose that $w^0 = z$. By induction hypothesis and Lemma 1.1, we can find $3k-4$ internally disjoint $\{x, y, z\}$-trees $T'_1, T'_2, \ldots, T'_{3k-4}$ in FP^0. Let $\mathcal{T}' = \bigcup_{i=1}^{3k-4} T'_i$.

Case 1.1.1. $|N_{\mathcal{T}'}(z) \cap \{x, y\}| \leq 1$.

First, construct two internally disjoint S-trees T_{3k-2} and T_{3k-1} in $L_2^* \square FP_{k-1}$ and $L_3^* \square FP_{k-1}$, respectively. Since FP^4 and FP^5 are connected, there exist an $\{x^4, y^4, z^4\}$-tree T'_{3k-2} in FP^4 and an $\{x^5, y^5, z^5\}$-tree T'_{3k-1} in FP^5, respectively. By the definition of FP_k, there exist two paths P_{wz^4} and P_{wz^5} such that $P_{wz^4} \cong L_2^* \setminus \{\mathbf{0}\}$ and $P_{wz^5} \cong L_3^* \setminus \{\mathbf{0}\}$. Let $T_{3k-2} = T'_{3k-2} \cup xx^4 \cup yy^4 \cup zz^4 \cup P_{wz^4}$ and $T_{3k-1} = T'_{3k-1} \cup xx^5 \cup yy^5 \cup zz^5 \cup P_{wz^5}$. Then T_{3k-2} and T_{3k-1} are internally disjoint S-trees in FP_k.

Next, construct $3k-3$ internally disjoint S-trees $T_1, T_2, \ldots, T_{3k-3}$ in $L_1^* \square FP_{k-1}$ such that $T_1, T_2, \ldots, T_{3k-1}$ are $3k-1$ internally disjoint S-trees in FP_k.

Since $|N_{\mathcal{T}'}(z) \cap \{x, y\}| \leq 1$, without loss of generality, assume that $|N_{T'_i}(z) \cap \{x, y\}| = 0$ and $d_{T'_i}(z) = 1$ for $i = 1, 2, \ldots, 3k-6$. Let $T_i = T'_i \cup z_i w_i \cup w_i w$, where $i \in [3k-6]$, z_i is the neighbor of z in T'_i and $w_i = z_i^1$. Note that if $|N_{\mathcal{T}'}(z) \cap \{x, y\}| = 1$, say $y \in N_{\mathcal{T}'}(z)$. By symmetry, consider the following two cases.

Case 1.1.1.1. $d_{T'_{3k-5}}(z) = d_{T'_{3k-4}}(z) = 1$.

If $y \notin N_{\mathcal{T}'}(z)$, let $T_i = T'_i \cup z_i w_i \cup w_i w$ for $i = 3k-5$ and $3k-4$, where z_i is the neighbor of z in T'_i and $w_i = z_i^1$. Otherwise, assume that $y \in N_{T'_{3k-5}}(z)$. Let $T_{3k-5} = T'_{3k-5} \cup zw$ and $T_{3k-4} = T'_{3k-4} \cup z_{3k-4} w_{3k-4} \cup w_{3k-4} w$. It is clear that $|N_{\mathcal{T}'}(z) \cap V(FP^0)| \leq 3k-4$, so there is a neighbor z_{3k-3} of z in FP^0. Let $\mathcal{T} = \bigcup_{i=1}^{3k-4} T_i$ and $W = N_{\mathcal{T}}(w)$. Then $|W \cap V(FP^1)| \leq 3k-4$. Since FP^1 is $(3k-3)$-connected, $FP^1 - W$ is still connected, thus we can find an $\{x^1, y^1, w, w_{3k-3}\}$-tree T'_{3k-3} in $FP^1 - W$, where $w_{3k-3} = z^1_{3k-3}$. Let $T_{3k-3} = T'_{3k-3} \cup xx^1 \cup yy^1 \cup zz_{3k-3} \cup z_{3k-3}w_{3k-3}$.

Case 1.1.1.2. $d_{T'_{3k-5}}(z) = 1$ and $d_{T'_{3k-4}}(z) = 2$.

If $y \notin N_{\mathcal{T}'}(z)$, let $T_i = T'_i \cup z_iw_i \cup w_iw$ for $i = 3k-5$ and $3k-4$, where z_i is one of the neighbors of z in T'_i and $w_i = z^1_i$. Let $\mathcal{T} = \bigcup_{i=1}^{3k-4} T_i$ and $W = N_{\mathcal{T}}(w)$. Then $|W \cap V(FP^1)| = 3k-4$. Since FP^1 is $(3k-3)$-connected, $FP^1 - W$ is still connected, thus there exists an $\{x^1, y^1, w\}$-tree T'_{3k-3} in $FP^1 - W$. Let $T_{3k-3} = T'_{3k-3} \cup xx^1 \cup yy^1 \cup zw$.

Suppose that $y \in N_{\mathcal{T}'}(z)$. Without loss of generality, assume that $y \in N_{T'_{3k-5}}(z)$. Clearly, T'_{3k-5} is an (x, z)-path containing y and T'_{3k-4} is an (x, y)-path containing z. That is, $T'_{3k-5} = P_{xy} \cup yz$ and $T'_{3k-4} = P_{xz} \cup P_{zy}$, where the lengths of P_{xz} and P_{zy} are least 2, see Figure 10(a). Let $T''_{3k-5} = P_{xy} \cup P_{zy}$ and $T''_{3k-4} = P_{xz} \cup yz$, see Figure 10($b$). Actually, T'_{3k-5} and T'_{3k-4} in the case $y \in N_{T'_{3k-4}}(z)$ are T''_{3k-5} and T''_{3k-4}, respectively. Let $T_i = T''_i \cup z_iw_i \cup w_iw$ for $i = 3k-5, 3k-4$, where $z_i \neq y$ and z_i is the neighbor of z in T''_i. Let $\mathcal{T} = \bigcup_{i=1}^{3k-4} T_i$ and $W = N_{\mathcal{T}}(w)$, then $|W \cap V(FP^1)| = 3k-4$ and $FP^1 - W$ is still connected, thus there exists an $\{x^1, y^1, w\}$-tree T'_{3k-3} in $FP^1 - W$. Let $T_{3k-3} = T'_{3k-3} \cup xx^1 \cup yy^1 \cup zw$, see Figure 10($c$).

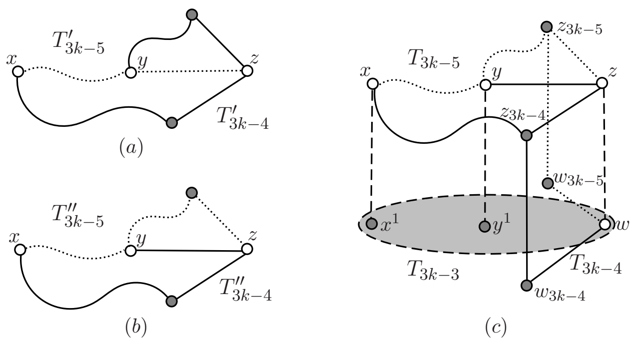


Figure 10. Illustrations of $y \in N_{\mathcal{T}'}(z)$ in Case 1.1.1.2.

Then, $T_1, T_2, \ldots, T_{3k-1}$ are $3k-1$ internally disjoint S-trees in FP_k.

Case 1.1.2. $|N_{\mathcal{T}'}(z) \cap \{x, y\}| = 2$.

Recall the decimal vertex-labeling of the FP_k. Without loss of generality, assume that $w^0 = z = 000\ldots00$. Then $w = 100\ldots00$. Notice that x, y, z are either in a subgraph of FP_k isomorphic to C_5 or C_4 when $k \geq 2$.

Case 1.1.2.1. x, y, z are in a subgraph of FP_k isomorphic to C_5.

By symmetry, assume that $x = 000\ldots01$, $y = 000\ldots04$. If $k \geq 3$, then $x, y, z, w \in FP_k^{j:0}$ for $j \in \{2, \ldots, k-1\}$ when dividing FP_k along the jth dimension. Thus, the desired trees can be found similarly to Lemma 3.3. Suppose that $k = 2$. Then $x = 01$, $y = 04$, $z = 00$ and $w = 10$. The desired trees can be found similarly to Lemma 3.2 when dividing FP_k along the 2th dimension.

Case 1.1.2.2. x, y, z are in a subgraph of FP_k isomorphic to C_4.

In this case, $k \geq 3$. Without loss of generality, assume that $x = 000\ldots01$ and $y = 000\ldots10$. If $k \geq 4$, then $x, y, z, w \in FP^{j:0}$ for $j \in \{2, \ldots, k-2\}$ when dividing FP_k along the jth dimension. Thus, we can find out desired trees similarly to Lemma 3.3. Suppose that $k = 3$. Then $x = 001$, $y = 010$, $z = 000$ and $w = 110$. The desired trees are shown in Figure. 11.

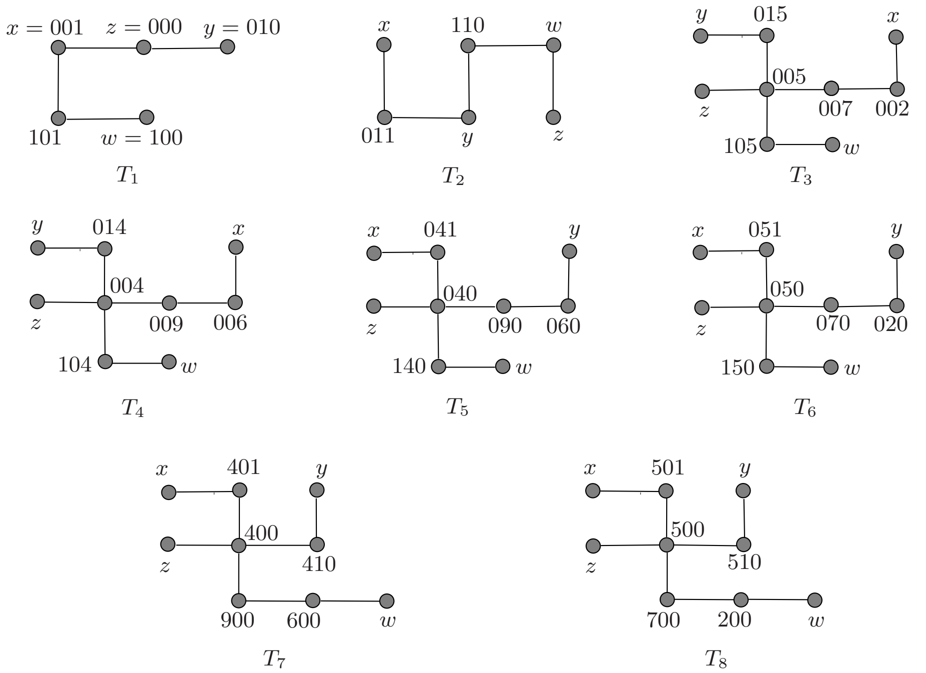


Figure 11. Eight internally disjoint S-trees of FP_3 in Case 1.1.2.2.

Case 1.2. $w^0 \notin \{x, y, z\}$.

Case 1.2.1. $|N_{FP^0}(w^0) \cap \{x, y, z\}| \leq 1$.

First, construct $3k - 3$ internally disjoint S-trees $T_1, T_2, \ldots, T_{3k-3}$ in $L_1^* \square FP_{k-1}$ such that $T_1, T_2, \ldots, T_{3k-1}$ are $3k - 1$ internally disjoint S-trees in FP_k.

Let $S' = \{x, y, z, w^0\}$. Since $FP^0 \cong FP_{k-1}$, by induction hypothesis, there exist $3k - 4$ internally disjoint S'-trees $T'_1, T'_2, \ldots, T'_{3k-4}$ in FP^0. Since $|N_{FP^0}(w^0) \cap \{x, y, z\}| \leq 1$, without loss of generality, assume that $|N_{T'_i}(w^0) \cap \{x, y, z\}| = 0$ for $i \in [3k-4]$ and $i \neq 1$. Let $T_1 = T'_1 \cup ww^0$. Since $d_{FP^0}(w^0) = 3k-3$, we have $1 \leq d_{T'_i}(w^0) \leq 2$ for $i \in [3k - 4]$. For $i = 2, \ldots, 3k - 4$, let $T_i = (T'_i \setminus w^0) \cup w_i w_i^1 \cup w_i^1 w$ if $N_{T'_i}(w^0) = \{w_i\}$ and let $T_i = (T'_i \setminus w^0) \cup w_i w_i^1 \cup w_i^1 w \cup w_j w_j^1 \cup w_j^1 w$ if $N_{T'_i}(w^0) = \{w_i, w_j\}$.

Let $\mathcal{T} = \bigcup_{i=1}^{3k-4} T_i$ and $W = N_{\mathcal{T}}(w)$. Then $|W \cap V(FP^1)| = 3k - 4$. Since FP^1 is $(3k - 3)$-connected, $FP^1 - W$ is still connected, thus there exists an $\{x^1, y^1, z^1, w\}$-tree T'_{3k-3} in $FP^1 - W$. Let $T_{3k-3} = T'_{3k-3} \cup xx^1 \cup yy^1 \cup zz^1$.

Next, construct two internally disjoint S-trees T_{3k-2} and T_{3k-1} in $L_2^* \square FP_{k-1}$ and $L_3^* \square FP_{k-1}$, respectively. Since $\mathbf{P}$ is a simple graph, there exist two non-end vertices $\mathbf{4}$ and $\mathbf{5}$ in L_2^* and L_3^*, respectively. Clearly, FP^4 is connected and there exists an $\{x^4, y^4, z^4, w^4\}$-tree T'_{3k-2} in FP^4. Let $T_{3k-2} = T'_{3k-2} \cup xx^4 \cup yy^4 \cup zz^4 \cup P_{ww^4}$, where $P_{ww^4} \cong L_2^* \setminus \{\mathbf{0}\}$. Similarly, construct $T_{3k-1} = T'_{3k-1} \cup xx^5 \cup yy^5 \cup zz^5 \cup P_{ww^5}$. Then, $T_1, T_2, \ldots, T_{3k-1}$ are $3k - 1$ internally disjoint S-trees in FP_k.

Case 1.2.2. $|N_{FP^0}(w^0) \cap \{x, y, z\}| \geq 2$.

Without loss of generality, suppose that $\{x, y\} \subseteq N_{FP^0}(w^0)$ and $w^0 = 000 \cdots 00$. Clearly, $w = 100 \cdots 00$. Note that x, y, w^0 are either in a subgraph of FP_k isomorphic to C_5 or C_4 when $k \geq 2$.

Case 1.2.2.1. x, y, w^0 are in a subgraph of FP_k isomorphic to C_5.

Without loss of generality, assume that $x = 000 \cdots 01$ and $y = 000 \cdots 04$. Let $z = 0z_2z_3 \cdots z_{k-1}z_k$. If there exists $j \in \{2, \ldots, k - 1\}$ such that $z_j = 0$, then $x, y, z, w \in FP_k^{j:0}$ when dividing FP_k along the jth dimension. Thus, we can find out desired trees similarly to Lemma 3.3. It suffices to consider the case $z_j \neq 0$ for $j = 2, \ldots, k-1$.

If $k \geq 3$, then $x, y, w \in V(FP_k^{2:0})$, $z \in V(FP_k^{2:z_2})$, $z^0 = 00z_3 \cdots z_{k-1}z_k \notin \{x, y, w\}$ and $|N(z^0) \cap \{x, y, w\}| \leq 1$ when dividing FP_k along the 2th dimension. Thus, we can find out desired trees Similarly to case 1.2.1.

Suppose that $k = 2$, there are $x = 01$, $y = 04$, $w = 10$, $z = 0z_2$ and $z_2 \notin \{0, 1, 4\}$. We can find out desired trees similarly to Lemma 3.1 when dividing FP_k along the 2th dimension.

Case 1.2.2.2. x, y, w^0 are in a subgraph of FP_k isomorphic to C_4.

In this case, $k \geq 3$. Without loss of generality, assume that $x = 000\cdots01$ and $y = 000\cdots40$. Let $z = 0z_2z_3\ldots z_{k-2}z_{k-1}z_k$. If there exists $j \in \{2, \ldots, k-2\}$ such that $z_j = 0$, then $x, y, z, w \in FP_k^{j:0}$ when dividing FP_k along the jth dimension. Thus, we can find out desired trees similarly to Lemma 3.3. It suffices to consider the case $z_2, z_3, \ldots, z_{k-2} \neq 0$.

If $k \geq 4$, we divide FP_k along the 2th dimension. Then $x, y, w \in V(FP_k^{2:0})$, $z \in V(FP_k^{2:z_2})$, $z^0 = 00z_3 \cdots z_{k-1}z_k \notin \{x, y, w\}$ and $|N(z^0) \cap \{x, y, w\}| \leq 1$ except $z^0 = 0041$. Similarly to case 1.2.1, we can find $3k-3$ internally disjoint S-trees. When $z^0 = 0041$, $x = 0001$, $y = 0040$, $w = 1000$, assume that $z = 0141$ (for $z = 0z_241$ and $z_2 \neq 0$ is similar). Clearly, $|N(z^0) \cap \{x, y, w\}| = 2$. The desired trees are shown in Figure 12.

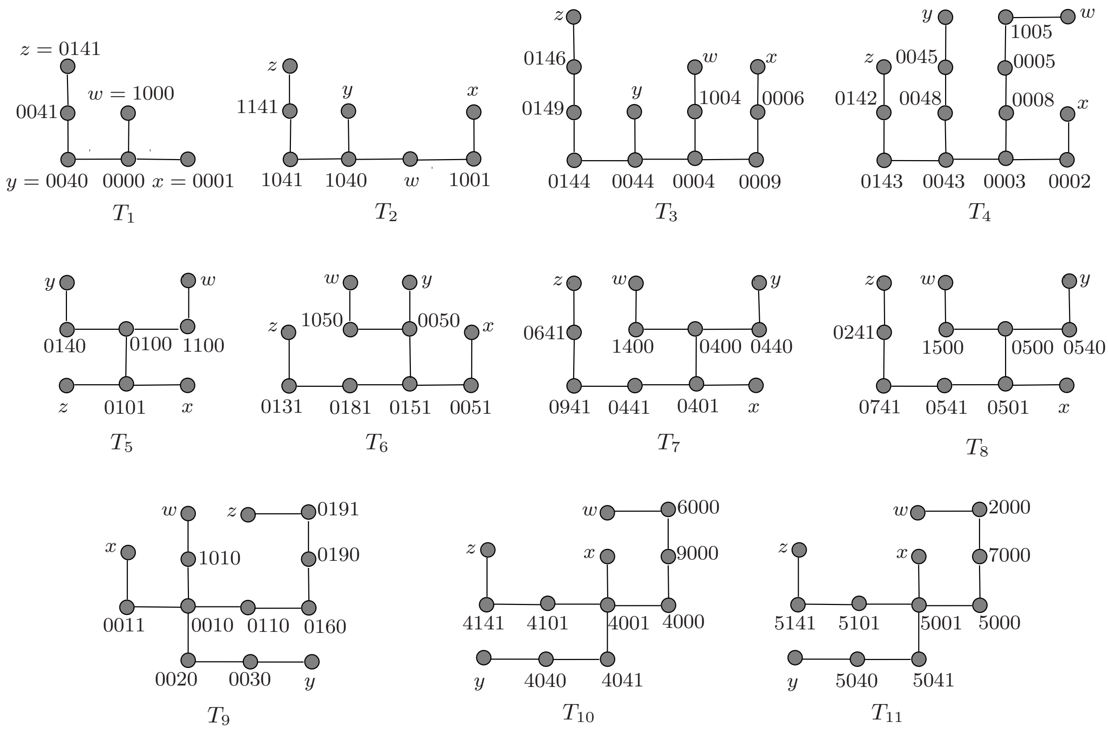


Figure 12. Eleven internally disjoint S-trees of FP_4 in Case 1.2.2.2.

Suppose that $k = 3$, we have $x = 001$, $y = 040$, $w = 100$, $z = 0z_2z_3 \notin \{x, y, w^0\}$. If $z_2 \notin \{0, 4\}$, then $x, w \in FP_3^{2:0}, y \in FP_3^{2:4}$ and $z \in FP_3^{2:z_2}$ when dividing FP_3 along the 2th dimension. Thus, we can find out desired trees similarly to Lemma 3.2. If $z_3 \notin \{0, 1\}$, then $x \in FP_3^{3:1}, y, w \in FP_3^{3:0}$ and $z \in FP_3^{3:z_3}$ when dividing FP_3 along the 3th dimension. Thus, we can find out desired trees similarly to Lemma 3.2. Suppose $z_2 \in \{0, 4\}$ and $z_3 \in \{0, 1\}$. Since $z \neq x, y, w$, then $z = 041$. The desired trees are shown in Figure 13.

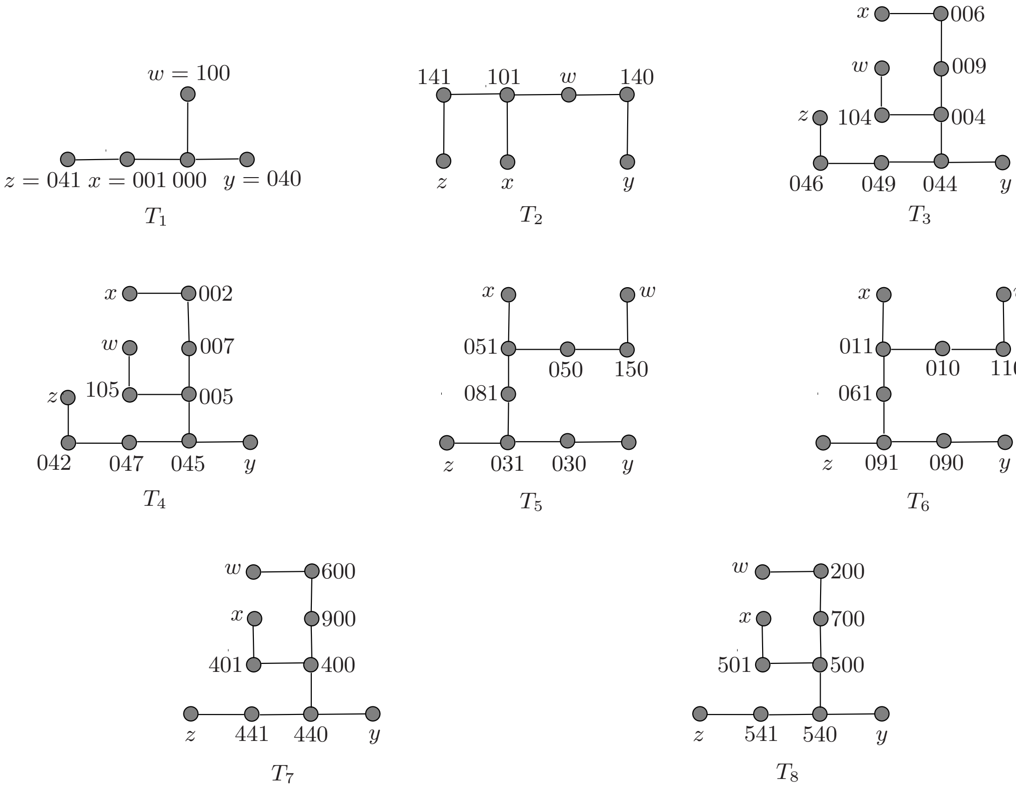


Figure 13. Eight internally disjoint S-trees of FP_3 in Case 1.2.2.2.

Case 2. $x, y \in V(FP^0)$ and $z, w \in V(FP^1)$.

Notice that $|V(L_2^*)| \geq 3$ and $|V(L_3^*)| \geq 3$. Without loss of generality, let $\mathbf{4} \in V(L_2^*)$ and $\mathbf{5} \in V(L_3^*)$. Since FP^4 is connected, there exists an $\{x^4, y^4, z^4, w^4\}$-tree T'_{3k-2} in FP^4, similarly, there exists an $\{x^5, y^5, z^5, w^5\}$-tree T'_{3k-1} in FP^5. By the definition of FP_k, there exist four paths P_{zz^4}, P_{ww^4}, P_{zz^5} and P_{ww^5} such that $P_{zz^4} \cong P_{ww^4} \cong L_2^* \setminus \{\mathbf{0}\}$ and $P_{zz^5} \cong P_{ww^5} \cong L_3^* \setminus \{\mathbf{0}\}$. Let $T_{3k-2} = T'_{3k-2} \cup xx^4 \cup yy^4 \cup P_{zz^4} \cup P_{ww^4}$ and $T_{3k-1} = T'_{3k-1} \cup xx^5 \cup yy^5 \cup P_{zz^5} \cup P_{ww^5}$. Then T_{3k-2} and T_{3k-1} are internally disjoint S-trees in FP_k. Main goal is to find out $3k-3$ internally disjoint S-trees in $L_1^* \Box FP_{k-1}$. Let S' be a set of x, y, z^0 and w^0. Without loss of generality, assume that $d(x, z^0) = \min d(u, v)$ for $u, v \in S'$.

Case 2.1. $|S'| \leq 3$.

In this case, $d(x, z^0) = 0$, that is, $z^0 = x$. Since FP_{k-1} is $(3k-3)$-connected, by Theorem 2.1, there exist $3k-3$ internally disjoint (x, y)-paths $P_1, P_2, \ldots, P_{3k-3}$ in FP^0 and $3k-3$ internally disjoint (z, w)-paths $P'_1, P'_2, \ldots, P'_{3k-3}$ in FP^1. Let $x_i \in N_{P_i}(x)$ and $z_i \in N_{P'_i}(z)$. Then there exists $z_i \in V(P'_i)$ such that z_i is a corresponding vertex of x_j in FP^1 for $i, j \in [3k-3]$, suppose that $i = j$. Hence, $x_i z_i \in E(FP_k)$. Let $T_i = P_i \cup x_i z_i \cup P'_i$ for $i \in [3k-3]$. Then, $T_1, T_2, \ldots, T_{3k-3}$ are $3k-3$ internally disjoint S-trees in $L_1^* \Box FP_{k-1}$. See Figure 14(a).

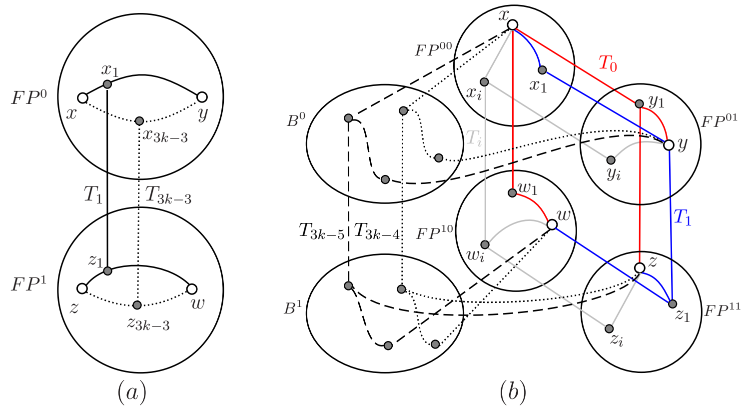


Figure 14. (a) Illustrations of Case 2.1; (b) Illustrations of Case 2.2 for $d(x, z) = 2$.

Case 2.2. $|S'| = 4$.

In this case, $d(x, z^0) \geq 1$. It implies that $d(x, z) \geq 2$. If $d(x, z) = 2$. Without loss of generality, assume that $x = 000 \cdots 00$. Then $z = 1c_2 \cdots c_k$ such that there exist an $i \in \{2, \dots, k\}$ satisfying $d_{\mathbf{P}(0,c_i)} = 1$ and others $c_i = 0$, say $z = 110 \cdots 00$. If there exist a dimension j such that $|S \cap FP^{j:i}| \neq 2$ for $j \in [k]$ and $i \in \{0, 1, \dots, 9\}$ when dividing FP_k along the jth dimension, then we can find out the desired S-trees by the above discussion. Hence, suppose that $|S \cap FP^{j:i}| = 2$ for arbitrary j when dividing FP_k along the jth dimension. Thus, $y = 01b_3 \cdots b_k$ and $w = 10b_3 \cdots b_k$, where $b_i \neq 0$ for $i \in \{3, \dots, k\}$.

Let $FP^{ij} = FP_k[d_1 d_2 d_3 \cdots d_k \in V(FP_k)\colon d_1 = i, d_2 = j]$. Then $FP^{ij} \cong FP_{k-2}$. Let x_1 be corresponding vertex of y in FP^{00}. Then $x_1 \neq x$. Choose $3k-7$ vertices $x_2, \dots, x_{3k-6}$ from $N_{FP^{00}}(x)$, denote $X = \{x_1, \dots, x_{3k-6}\}$. For $i = 1, \dots, 3k-6$, let z_i be corresponding vertices of x_i in FP^{11}, denote $Z = \{z_1, \dots, z_{3k-6}\}$. For $i = 2, \dots, 3k-6$, let y_i and w_i be corresponding vertices of x_i in FP^{01} and FP^{10}, respectively. Denote $Y = \{y_1, \dots, y_{3k-7}\}$ and $W = \{w_1, \dots, w_{3k-7}\}$, where y_1 and w_1 are be corresponding vertices of x in FP^{01} and FP^{10}, respectively. Since FP_{k-2} is $(3k-6)$-connected, there exist $3k-6$ internally disjoint (a, A)-paths $A_1, \dots, A_{3k-6}$ in FP^{ij} for $(a, A, ij) = (x, X, 00), (y, Y, 01), (z, Z, 10), (w, W, 11)$, respectively. Let $T_0 = X_1 \cup x_1 y \cup yz_1 \cup Z_1 \cup z_1 w$, $T_1 = xy_1 \cup Y_1 \cup y_1 z \cup xw_1 \cup W_1$ and $T_i = X_i \cup x_i y_i \cup Y_i \cup Z_i \cup z_i w_i \cup W_i \cup x_i w_i$ for $i = 2, \dots, 3k-6$.

Let B^0 be a subgraph induced by $V(FP^0 - FP^{00} - FP^{01})$ and B^1 be a subgraph induced by $V(FP^1 - FP^{10} - FP^{11})$. Then there exist two internally disjoint paths P_{3k-5}, P_{3k-4} to connect x and y in $FP_k[V(B^0) \cup \{x, y\}]$ and two internally disjoint paths P'_{3k-5}, P'_{3k-4} to connect z and w in $FP_k[V(B^1) \cup \{z, w\}]$. Let $T_j = P_j \cup x_j z_j \cup P'_j$ for $j = 3k-5$ and $3k-4$, where $xx_j \in E(P_j)$ and z_j is corresponding vertex of x_j in P'_i. Then, $T_0, T_1, \dots, T_{3k-4}$ are $3k-3$ internally disjoint S-trees in $L_1^* \square FP_{k-1}$. See Figure 14(b).

Suppose that $d(x, z) \geq 3$. There are $3k-4$ internally disjoint S'-trees $T'_1, T'_2, \dots, T'_{3k-4}$ in FP^0 because of the induction hypothesis. Notice that $|N_{FP_k}(z^0) \cap N_{FP_k}(w^0)| \leq 2$. Let T'_1 be a tree such that $|N_{T'_1}(z^0) \cap N_{T'_1}(w^0)| = 0$. Let $z_1 \in N_{T'_1}(z^0)$ and $w_1 \in N_{T'_1}(w^0)$. Then $z_1 \neq w_1$. Let $T_1 = T'_1 \cup \{zz^0, ww^0\}$.

Let $S'' = \{x, y, z, w^0\}$. Since $d_{FP^0}(z^0) = 3k-3$, there are $1 \leq d_{T'_i}(z^0) \leq 2$ for $i \in [3k-4]$. Construct $3k-5$ internally disjoint S''-trees as follows: $T''_i = (T'_i \setminus z^0) \cup z_i z_i^1 \cup z_i^1 z$ if $N_{T'_i}(z^0) = \{z_i\}$ and $T''_i = (T'_i \setminus z^0) \cup z_i z_i^1 \cup z_i^1 z \cup z_j z_j^1 \cup z_j^1 z$ if $N_{T'_i}(z^0) = \{z_i, z_j\}$, where $i = 2, \dots, 3k-4$. Similarly, construct $3k-5$ internally disjoint S-trees as follows: $T_i = (T'_i \setminus w^0) \cup w_i w_i^1 \cup w_i^1 w$ if $N_{T''_i}(w^0) = \{w_i\}$ and $T_i = (T''_i \setminus w^0) \cup w_i w_i^1 \cup w_i^1 w \cup w_j w_j^1 \cup w_j^1 w$ if $N_{T''_i}(w^0) = \{w_i, w_j\}$, where $i = 2, \dots, 3k-4$.

Let $\mathcal{T} = \bigcup_{i=1}^{3k-4} T_i$, $W = N_{\mathcal{T}}(z) \cup N_{\mathcal{T}}(w)$ and T_1'' be a corresponding tree of T_1' in FP^1. Then, T_1'' is a tree in $FP^1 - W$. Let $T_{3k-3} = T_1'' \cup xx^1 \cup yy^1$. Then, $T_1, T_2, \ldots, T_{3k-3}$ are $3k-3$ internally disjoint S-trees in $L_1^* \Box FP_{k-1}$.

Therefore, $T_1, T_2, \ldots, T_{3k-1}$ are $3k-1$ internally disjoint S-trees in FP_k.

Giving an algorithm to find out $3k-1$ internally disjoint S-trees in FP_k for any $S \subseteq V(FP_k)$ with $|S| = 4$, it means that $\kappa_4(FP_k) \geq 3k-1$.

Algorithm 1 Find out $3k-1$ internally disjoint S-trees in FP_k.

Input: An k-dimensional folded Petersen network FP_k and four vertices x, y, z, w of FP_k.
Output: $3k-1$ internally disjoint $\{x, y, z, w\}$-trees $\mathfrak{T}$.

Initialization: $i = 0$, $S = \{x, y, z, w\}$, $\mathfrak{T} = \emptyset$, $G_i = FP_k$
While $i < 3k-1$ and G_i is connected **do**
  construct an S-tree T_i in G_i such that $1 \leq d_{T_i}(v) \leq 2$ and $|\{v \colon d_{T_i}(v) = 1\}| \geq 2$, where $v \in S$. Moreover, $\{u \colon d_{T_i}(u) = 1\} \subseteq S$
  $\mathfrak{T} = \mathfrak{T} \cup T_i$
  $i = i + 1$
  $G_i = G_{i-1} - (V(T_{i-1}) \setminus S)$
end while
return $\mathfrak{T}$

Now we give the proof of the main result.

Proof of Theorem 1.2. Since FP_k is a $3k$-regular graph, there are $\kappa_4(FP_k) \leq 3k-1$ as Lemma 2.1. Next we will show that $\kappa_4(FP_k) \geq 3k-1$. Let $S = \{x, y, z, w\}$ be a set of arbitrary four distinct vertices in FP_k. It suffices to show that there exist $3k-1$ internally disjoint S-trees. The proof of this result by induction on k. By Lemma 2.3, the statement holds for $k = 1$. Suppose that the statement holds in FP_{k-1} for $k \geq 2$. Now consider FP_k. Decompose FP_k into 10 disjoint sub-folded Petersen graphs $FP^0, \ldots, FP^9$, each of which is isomorphic to FP_{k-1}, by removing all crossed edges. We only need to take into account the following cases because of symmetry.

Case 1. x, y, z and w belong to four distinct sub-folded Petersen graphs.

By Lemma 3.1, the desired $3k-1$ internally disjoint S-trees can be obtained in FP_k.

Case 2. x, y, z and w belong to three distinct sub-folded Petersen graphs.

By Lemma 3.2, the desired $3k-1$ internally disjoint S-trees can be obtained in FP_k.

Case 3. x, y, z and w belong to two distinct sub-folded Petersen graphs.

By Lemma 3.4, the desired $3k-1$ internally disjoint S-trees can be obtained in FP_k.

Case 4. x, y, z and w belong to the same sub-folded Petersen graph.

By Lemma 3.3, the desired $3k-1$ internally disjoint S-trees can be obtained in FP_k.

Hence, $\kappa_4(FP_k) \geq 3k-1$ and the proof is completed.

Proof of Theorem 1.1. Remember that $FPQ_{n,k}$ can be regarded as replacing every vertex of Q_n by FP_k. Take the Figure 15 as an example. Since $FPQ_{n,k}$ is $(n+3k)$-regular, we have $\kappa_4(FPQ_{n,k}) \leq n+3k-1$ by Lemma 2.1. In order to prove $\kappa_4(FPQ_{n,k}) \geq n+3k-1$, it needs to show that there are $n+3k-1$ internally disjoint S-trees in $FPQ_{n,k}$ for arbitrary $S \subseteq V(FPQ_{n,k})$ with $|S| = 4$.

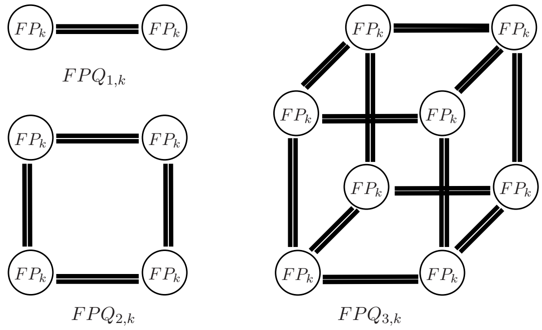


Figure 15. Scheme of $FPQ_{1,k}, FPQ_{2,k}$ and $FPQ_{3,k}$.

If $n = 1$. According to Lemma 3.3 and 3.4, it is not hard to see that there exist $3k - 3$ internally disjoint S-trees in $K_2 \square FP_{k-1}$. That is, $\kappa_4(FPQ_{1,k-1}) \geq 3k - 3$. Hence, $\kappa_4(FPQ_{1,k}) \geq 3k$.

Suppose that $n \geq 2$. When the vertices of S distribute among one copy of FP_k. Similar to Lemma 3.3, the desired $n + 3k - 1$ internally disjoint S-trees can be found in $FPQ_{n,k}$.

When the vertices of S distribute among two copies of FP_k. Since $\kappa(Q_n) = n$, there are n internally disjoint paths $L_1^*, L_2^*, \ldots, L_n^*$ connecting arbitrary two vertices of Q_n. Similar to Lemma 3.4, we can find $3k$ S-trees in $L_1^* \square FP_k$ and $n - 1$ S-trees in $(\bigcup_{i=2}^{n} L_i^*) \square FP_k$ such that these $n + 3k - 1$ S-trees are internally disjoint.

When the vertices of S distribute among three copies of FP_k. Since $\kappa_3(Q_n) = n - 1$, there are $n - 1$ internally disjoint path $T_1^*, T_2^*, \ldots, T_{n-1}^*$ connecting arbitrary three vertices of Q_n. Similar to Lemma 3.2, we can find $3k + 1$ S-trees in $T_1^* \square FP_k$ and $n - 2$ S-trees in $(\bigcup_{i=2}^{n-1} T_i^*) \square FP_k$ such that these $n + 3k - 1$ S-trees are internally disjoint.

When the vertices of S distribute among four copies of FP_k. Since $\kappa_4(Q_n) = n - 1$, there are $n - 1$ internally disjoint path $H_1^*, H_2^*, \ldots, H_{n-1}^*$ connecting arbitrary four vertices of Q_n. Similar to Lemma 3.1, we can find $3k + 1$ S-trees in $H_1^* \square FP_k$ and $n - 2$ S-trees in $(\bigcup_{i=2}^{n-1} H_i^*) \square FP_k$ such that these $n + 3k - 1$ S-trees are internally disjoint.

Therefore, $\kappa_4(FPQ_{n,k}) = n + 3k - 1$.

4. Conclusions

The generalized ℓ-connectivity is a natural generalization of the traditional connectivity and can serve for measuring the fault tolerance capability of a network. This paper centers on the generalized 4-connectivity of the folded Petersen cube network $FPQ_{n,k}$ and shows that $\kappa_4(FPQ_{n,k}) = n + 3k - 1$. As a corollary, $\kappa_3(FPQ_{n,k}) = n + 3k - 1$ is obtained easily. Furthermore, the results $\kappa_4(Q_n) = \kappa_4(FPQ_{n,0}) = n - 1$ and $\kappa_4(FP_k) = \kappa_4(FPQ_{0,k}) = 3k - 1$ can be verified. Sabidussi [29] discussed the classical connectivity of Cartesian product graphs, in the next work, we would like to research the generalized 4-connectivity of Cartesian product graphs.

Besides, fault tolerance or connectivity is mainly to provide a data to measure the reliability of a network, but in practical, when a system failure, it is worth considering how to compensate the impact of the fault and to recover the performance of the system before the failure as far as possible such that the system runs stably and reliably. Motivated by Alhasnawi et al. [1–4], it needs to design fault tolerance control to ensure steady-state operation, enhance network' fault resilience, improve network' robust and efficient operation. In future work, we would like to apply graph theory to solve practical problems.

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Conflict of interest

The authors declare that they have no competing interests.

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