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Research article

Optimal harvesting for a periodic competing system with size structure in a polluted environment

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Abstract: As a renewable resource, biological population not only has direct economic value to people's lives, but also has important ecological and environmental value. This study examines an optimal harvesting problem for a periodic, competing hybrid system of three species that is dependent on size structure in a polluted environment. The existence and uniqueness of the nonnegative solution are proved via an operator theory and fixed point theorem. The necessary optimality conditions are derived by constructing an adjoint system and using the tangent-normal cone technique. The existence of unique optimal control pair is verified by means of the Ekeland variational principle and a feedback form of the optimal policy is presented. The finite difference scheme and the chasing method are used to approximate the nonnegative T-periodic solution of the state system corresponding to a given initial datum. The objective functional represents the total profit obtained from harvesting three species. The results obtained in this work can be extended to a wide variety of fields.

Keywords: size structure; optimal harvesting; competing system; pollution; finite difference method
Mathematics Subject Classification: 49J20, 92B05

1. Introduction

In today's world of industrial pollution, toxicants are pervading the air, ecological problems have become increasingly prominent, and environmental pollution has become a major problem. When human activities expose in the wild, they may come into contact with wild animals. In the process, wild animals can easily transmit viruses they carry to humans. In fact, most new infectious diseases come from wild animals [1]. SARS, Ebola virus, AIV, H1N1 influenza, and COVID-19 are threatening the ecological balance as well as the survival of human beings and other creatures. A large amount of toxic and harmful substances are discharged into the atmosphere, and seriously affect the environmental quality. It is necessary to study the effects of toxicants on the ecosystem. Hallam et al. proposed using a dynamic methodology to examine ecotoxicology in [2–4]. They established a model of the interaction between toxicants and population, and provided sufficient conditions for the persistence and extinction of a population stressed by a toxicant. Researchers have been studying ecotoxicology since the 1980s, and a large amount of literature has been devoted to problems in the area [5–12]. However, size-structured factor has not been considered in these models. Size here refers to some continuous indices related to individuals in the given population, such as volume, maturity, diameter, length, mass, or other quantities that show its physiological or statistical characteristics.

The effects of environmental pollution on biological population, the dynamical behavioral analysis of ecosystem models, and the control problem have attracted the attention of many scholars [13–15]. For many populations, size structure is more appropriate to describe the dynamical evolution than age structure, especially for plants and fishes [16, 17]. Population models with age structure have been extensively investigated by many authors as seen in [13] and the references therein. On the other hand, the control problem with size structure has achieved remarkable results through theoretical analysis, numerical calculations, and experimental methods, such as in [14, 15, 18–26]. However, most of these studies have focused on a single species, and only a few have examined interactions among species. Among them, the optimal birth problem has also discussed in detail in [14]. In addition, Hritonenko et al. [15] have established a sized-structured forest system, where the objective function includes the net benefits from timber production and carbon sequestration. Liu et al. [18] have studied the least cost-size problem and the least cost-derivation problem for a nonlinear size-structured vermin population model with separable mortality rate, which takes fertility rate as the control variable. We also mention that Li et al. [21] have considered the optimal harvesting for a size-stage-structured population model. For other types of optimal harvesting problems, refer to [15, 22, 24, 25]. Moreover, the influence of seasonal changes and other factors, the living environment of populations often undergoes periodic changes. Research on optimal harvesting problems dependent on the model of individual size in a periodic environment has been reported in [26, 27]. In [27], Zhang et al. have discussed the optimal harvesting in a periodic food chain model by using the size structure of predators. To the best of our knowledge, few studies to date have examined optimal control problems of size-dependent population models and periodic effects in a polluted environment. Inspired by the above work, this paper discusses optimal harvesting for a periodic, competing system that is dependent on size structure in a polluted environment.

The remainder of this paper is organized as follows: In Section 2, we describe a population model with size structure in a polluted environment and its well-posedness is proved in Section 3. The optimality conditions are established in Section 4. The existence of a unique optimal control pair is obtained in Section 5. Some numerical results are presented in Section 6. At the end of this paper, some brief conclusions are provided.

2. The basic model

In [2–4], Hallam et al. proposed the following dynamic population model with toxicant effects:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = x[r_0 - r_1 C_0 - f x], \\ \frac{\mathrm{d}C_0}{\mathrm{d}t} = kC_E - gC_0 - mC_0, \\ \frac{\mathrm{d}C_E}{\mathrm{d}t} = -k_1 C_E x + g_1 C_0 x - hC_E + u, \end{cases} \tag{2.1}$$

where $x = x(t)$ is the population biomass at time t; $C_0 = C_0(t)$ is the concentration of toxicants in the organism at time t; $C_E = C_E(t)$ is the concentration of toxicants in the environment of the population at time t. The exogenous rate of input of toxicants into the environment was represented by u. They investigated the persistence and extinction of a population in a polluted environment.

Luo et al. [28] studied optimal harvesting control problem for the following age-dependent competing system of n species:

$$\begin{cases} \frac{\partial p_i}{\partial t} + \frac{\partial p_i}{\partial a} = f_i(a,t) - \mu_i(a,t)p_i - \sum\limits_{k=1,k\neq i}^{n} \lambda_{ik}(a,t)P_k(t)p_i - u_i(a,t)p_i,, \\ p_i(0,t) = \beta_i(t) \int_{a_1}^{a_2} m_i(a,t)p_i(a,t)\mathrm{d}a, \\ p_i(a,0) = p_{i0}(a), \\ P_i(t) = \int_0^{a_+} p_i(a,t)\mathrm{d}a, \quad i = 1,2,\dots,n, \ \ (a,t) \in Q, \end{cases} \tag{2.2}$$

where $Q = (0, a_+) \times (0, +\infty)$, $[a_1, a_2]$ is the fertility interval. $p_i(a,t)$ represents the density of ith population of age a at time t, and a_+ is the life expectancy of individuals; p_{i0} is the initial age distribution of ith population; $u_i(x,t)$ is the harvesting effort function, which is the control variable in the model. The existence of an optimal control, the necessary conditions of optimality for the control problem have been derived.

By combining (2.1) and (2.2), we consider the following periodic, competing system with size structure in a polluted environment:

$$\begin{cases} \frac{\partial p_i}{\partial t} + \frac{\partial (V_i(x,t)p_i)}{\partial x} = f_i(x,t) - \mu_i(x, c_{i0}(t))p_i - \sum\limits_{i,k=1,k\neq i}^{3} \lambda_{ik}(x,t)P_k(t)p_i - u_i(x,t)p_i, \\ \frac{\mathrm{d}c_{i0}}{\mathrm{d}t} = k_1 c_e(t) - g_1 c_{i0}(t) - m c_{i0}(t), \\ \frac{\mathrm{d}c_e}{\mathrm{d}t} = -k_2 c_e(t) \sum\limits_{i=1}^{3} P_i(t) + g_2 \sum\limits_{i=1}^{3} c_{i0}(t)P_i(t) - h c_e(t) + v(t), \\ V_i(0,t)p_i(0,t) = \int_0^l \beta_i(x, c_{i0}(t))p_i(x,t)\mathrm{d}x, \\ 0 \le c_{i0}(0) \le 1, \ \ 0 \le c_e(0) \le 1, \\ p_i(x,t) = p_i(x, t+T), \\ P_i(t) = \int_0^l p_i(x,t)\mathrm{d}x, \quad i = 1,2,3, \ \ (x,t) \in Q, \end{cases} \tag{2.3}$$

where $Q = (0,l) \times R_+, l \in R_+$ is the maximal size of an individual in the population, $T \in R_+$ is the period of habitat evolution of the populations. k_1, g_1, m, k_2, g_2, and h are nonnegative constants. The meaning of the variables and functional traits are as follows: $p_i(x,t)$: the density of the ith population of size x at time t; $c_{i0}(t)$: the concentration of toxicants in the ith population; $c_e(t)$: the concentration of toxicants in the environment; $V_i(x,t)$: the average rate of growth for the ith population, that is, $\frac{\mathrm{d}x}{\mathrm{d}t} = V_i(x,t)$ (see [29]); $\mu_i(x, c_{i0}(t)), \beta_i(x, c_{i0}(t))$: the mortality and fertility rates of the ith population, respectively; $v(t)$: the input rate of exogenous toxicants; $P_i(t)$: total number of individuals in the ith population; $f_i(x,t)$: the immigration rate of the ith population; $\lambda_{ik}(x,t)$: the interaction coefficient; $u_i(x,t)$: function of the harvesting efforts of the ith population of size x harvested at time t; $k_1 c_e(t)$: the organism's net uptake of toxicant from the environment; $-g_1 c_{i0}(t)$ and $-m c_{i0}(t)$: the egestion and depuration rates of the toxicant in the ith population, respectively. The units of k_1, g_1 and m are in terms of $m_e m_0^{-1} t^{-1}, t^{-1}$, and t^{-1}, respectively. $-k_2 c_e(t) \sum\limits_{i=1}^{3} P_i(t)$: the loss of the toxicant in the environment that is due to the uptake of toxicant by the total population. $g_2 \sum\limits_{i=1}^{3} c_{i0}(t)P_i(t)$: the increase in the toxicant in the environment coming from the egestion of the total population. $-hc_e(t)$: the toxicant loss from the environment itself by volatilization and so on. The unit of k_2 is in terms of $m_0^{-1}t^{-1}$; g_2 is in terms of $m_e^{-1}t^{-1}$; and h is in terms of t^{-1}, where m_e and m_0 denote the units of mass of the environment and in the ith population, respectively. The toxicant-population model with size structure is established under the condition of small toxicant capacity in the environment.

The aim of this paper is to seek the maximum of the following objective functional $J(u, v)$, that is

$$\text{Maximize}\{J(u, v) : u = (u_1(x, t), \dots, u_3(x, t)), v = v(t), (u, v) \in \Omega\}, \tag{2.4}$$

where

$$J(u, v) = \sum_{i=1}^{3} \int_0^T \int_0^l w_i(x, t)u_i(x, t)p_i(x, t)\mathrm{d}x\mathrm{d}t - \frac{1}{2}\sum_{i=1}^{3} \int_0^T \int_0^l c_i u_i^2(x, t)\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^T c_4 v^2(t)\mathrm{d}t,$$

$w_i(x, t)$ is the selling price of an individual belonging to the ith population. The positive constants c_i and c_4 are the cost factors of the ith harvested population and the curbing environmental pollution, respectively. $J(u, v)$ represents the total profit from the harvested populations during period T. The admissible control set Ω is as follows:

$$\Omega = \{(u, v) \in [L_T^\infty(Q)]^3 \times L_T^\infty(R_+) : 0 \le u_i(x, t) \le N_i \text{ a.e. } (x, t) \in Q, 0 \le v_0 \le v(t) \le v_1 \text{ a.e. } t \in R_+\},$$

where

$$L_T^\infty(Q) = \{\eta \in L^\infty(Q) : \eta(x, t) = \eta(x, t + T) \text{ a.e. } (x, t) \in Q\},$$

$$L_T^\infty(R_+) = \{\eta \in L^\infty(R_+) : \eta(t) = \eta(t + T) \text{ a.e. } t \in R_+\}.$$

This paper makes the following assumptions:

(A_1) $V_i : [0, l) \times R_+ \to R_+$ are bounded continuous functions, $V_i(x, t) > 0$ and $V_i(x, t) = V_i(x, t + T)$ for $(x, t) \in Q$, $\lim_{x\uparrow l} V_i(x, t) = 0$, and $V_i(0, t) = 1$ for $t \in R_+$. There are Lipschitz constants L_{V_i} such that

$$|V_i(x_1, t) - V_i(x_2, t)| \le L_{V_i}|x_1 - x_2| \text{ for } x_1, x_2 \in [0, l],\ t \in R_+.$$

(A_2) $0 \le \beta_i(x, c_{i0}(t)) = \beta_i(x, c_{i0}(t + T)) \le \overline{\beta}_i, \overline{\beta}_i$ are constants.

$$(A_3)\begin{cases} \mu_i(x, c_{i0}(t)) = \mu_{i0}(x) + \overline{\mu}_i(x, c_{i0}(t)) \text{ a.e. } (x, t) \in Q, \text{ where } \mu_{i0} \in L^1_{loc}([0, l)), \\ \mu_{i0}(s) \ge 0 \text{ a.e. } x \in [0, l), \int_0^l \mu_{i0}(s)\mathrm{d}s \to +\infty, \overline{\mu}_i \in L^\infty(Q), \\ \overline{\mu}_i(x, c_{i0}(t)) \ge 0, \overline{\mu}_i(x, c_{i0}(t)) = \overline{\mu}_i(x, c_{i0}(t + T)) \text{ a.e. } (x, t) \in Q. \end{cases}$$

(A_4) $f_i \in L^\infty(Q), 0 \le f_i(x, t) = f_i(x, t + T)$. $0 \le \lambda_{ik}(x, t) \le \overline{\lambda}_i, 0 \le w_i(x, t) \le w_i(x, t + T) \le \overline{w}_i, \overline{\lambda}_i$ and $\overline{w}_i$ are constants.

(A_5) There exist constants $L_\beta > 0, L_\mu > 0$ such that $|\beta_i(x, c_{i0}^1(t)) - \beta_i(x, c_{i0}^2(t))| \le L_\beta|c_{i0}^1(t) - c_{i0}^2(t)|$, $|\mu_i(x, c_{i0}^1(t)) - \mu_i(x, c_{i0}^2(t))| \le L_\mu|c_{i0}^1(t) - c_{i0}^2(t)|$.

(A_6) $g_1 \le k_1 \le g_1 + m, v_1 \le h$. (see [30])

3. Well-posedness of the state system

Definition 3.1. For $i = 1,2,3$, the unique solution $x = \varphi_i(t; t_0, x_{i0})$ of the initial value problem $x'(t) = V_i(x,t), x(t_0) = x_{i0}$ is said to be a characteristic curve of the hybrid system (2.3) through (t_0, x_{i0}). Let $z_i(t) := \varphi_i(t; 0, 0)$ denote the characteristic curve through $(0,0)$ in the $x-t$ plane.

For any point $(x,t) \in [0,l) \times [0,T]$ such that $x \le z_i(t)$, define the initial time $\tau := \tau(t,x)$, in order that $\varphi_i(t;\tau,0) = x \Leftrightarrow \varphi_i(\tau; t, x) = 0$. The solution of (2.3) is

$$p_i(x,t) = p_i(0, t - z_i^{-1}(x))\Pi_i(x;x,t) + \int_0^x \frac{f_i(r, \varphi_i^{-1}(r;t,x))}{V_i(r,\varphi_i^{-1}(r;t,x))}\frac{\Pi_i(x;x,t)}{\Pi_i(r;x,t)}\mathrm{d}r, \tag{3.1}$$

where

$$\Pi_i(s;x,t) = \exp\left\{-\int_0^s \frac{\mu_i(r, c_{i0}(\varphi_i^{-1}(r;t,x)))}{V_i(r,\varphi_i^{-1}(r;t,x))} + \frac{\sum\limits_{i,k=1,k\neq i}^{3} \lambda_{ik}(r,\varphi_i^{-1}(r;t,x))P_k(\varphi_i^{-1}(r;t,x))}{V_i(r,\varphi_i^{-1}(r;t,x))}\right.$$
$$\left. + \frac{u_i(r,\varphi_i^{-1}(r;t,x)) + V_{ix}(r,\varphi_i^{-1}(r;t,x))}{V_i(r,\varphi_i^{-1}(r;t,x))}\mathrm{d}r\right\}.$$

$$c_{i0}(t) = c_{i0}(0)\exp\{-(g_1+m)t\} + k_1\int_0^t c_e(s)\exp\{(s-t)(g_1+m)\}\mathrm{d}s. \tag{3.2}$$

$$c_e(t) = c_e(0)\exp\left\{-\int_0^t \left[k_2\sum_{i=1}^3 P_i(\tau) + h\right]\mathrm{d}\tau\right\} + \int_0^t \left[g_2\sum_{i=1}^3 c_{i0}(s)P_i(s) + v(s)\right]$$
$$\cdot \exp\left\{\int_t^s \left[k_2\sum_{i=1}^3 P_i(\tau) + h\right]\mathrm{d}\tau\right\}\mathrm{d}s. \tag{3.3}$$

By assumption (A_1), we have $V_i(0,t) = 1$. Let $b_i(t) = p_i(0,t)$. Then, by noting that $\varphi_i^{-1}(0;t,x) = \tau = t - z_i^{-1}(x)$, we have

$$b_i(t) = F_i(t) + \int_0^l K_i(t,x)b_i(t - z_i^{-1}(x))\mathrm{d}x, \tag{3.4}$$

where

$$K_i(t,x) = \beta_i(x, c_{i0}(t))\Pi_i(x;x,t), \tag{3.5}$$

$$F_i(t) = \int_0^l \beta_i(x, c_{i0}(t))\int_0^x \frac{f_i(r,\varphi_i^{-1}(r;t,x))}{V_i(r,\varphi_i^{-1}(r;t,x))}\frac{\Pi_i(x;x,t)}{\Pi_i(r;x,t)}\mathrm{d}r\mathrm{d}x. \tag{3.6}$$

Define the linear and bounded operator $\mathcal{A}_i : L_T^\infty(R_+) \to L_T^\infty(R_+)$ given by

$$(\mathcal{A}_i q)(t) = \int_0^l K_i(t,x)q_i(t - z_i^{-1}(x))\mathrm{d}x. \tag{3.7}$$

As a consequence (3.4) can be written in $L_T^\infty(R_+)$ as the following abstract equation

$$b_i = \mathcal{A}_i b_i + F_i, \tag{3.8}$$

with $F_i \in L^\infty_T(R_+)$ defined by (3.6). We denote by $r(\mathcal{A}_i)$ the spectral radius of the operator $\mathcal{A}_i$. If $r(\mathcal{A}_i) < 1$, then (3.8) has unique solution in $L^\infty_T(R_+)$.
Remark 3.1. If we denote by

$$\hat{\beta}_i(x) = ess\ sup_{t\in R_+}\beta_i(x, c_{i0}(t)) \text{ a.e. } x \in [0, l),$$

then (A_2) and (3.7) allow us to conclude that

$$r(\mathcal{A}_i) \le \int_0^l \hat{\beta}_i(x)\mathrm{d}x.$$

Theorem 3.1. Assume that $(A_1) - (A_6)$ hold. Then, the hybrid system (2.3) has a nonnegative and unique solution $(p_1(x, t), \ldots, p_3(x, t), c_{10}(t), \ldots, c_{30}(t), c_e(t))$, such that

(i) $(p_i(x, t), c_{i0}(t), c_e(t)) \in L^\infty(Q) \times L^\infty(0, T) \times L^\infty(0, T)$.

(ii) $0 \le c_{i0}(t) \le 1, 0 \le c_e(t) \le 1, \forall t \in (0, T),\ \ 0 \le p_i(x, t), \int_0^l p_i(x, t)\mathrm{d}x \le M, \forall (x, t) \in Q, i = 1, 2, 3.$
where $M = M_2 l + \|f_i(\cdot, \cdot)\|_{L^\infty(Q)}$.

Proof. Without loss of generality, we assume that $u_i(x, t) \equiv 0$. $p(x, t) = (p_1(x, t), \ldots, p_3(x, t)), c_0(t) = (c_{10}(t), \ldots, c_{30}(t))$. When t is so large that $t > z_i^{-1}(l)$, from (3.5) it follows that

$$\begin{aligned}
&|K_i^1(t, x) - K_i^2(t, x)|\\
=&|\beta_i(x, c_{i0}^1(t))\Pi_i^1(x; x, t) - \beta_i(x, c_{i0}^2(t))\Pi_i^2(x; x, t)|\\
\le&|\beta_i(x, c_{i0}^1(t)) - \beta_i(x, c_{i0}^2(t))| + |\beta_i(x, c_{i0}^2(t))||\Pi_i^1(x; x, t) - \Pi_i^2(x; x, t)|\\
\le&L_\beta|c_{i0}^1(t) - c_{i0}^2(t)| + \overline{\beta}_i\int_0^x \frac{|\mu_i(r, c_{i0}^1(\varphi_i^{-1}(r; t, x))) - \mu_i(r, c_{i0}^2(\varphi_i^{-1}(r; t, x)))|}{V_i(r, \varphi_i^{-1}(r; t, x))}\mathrm{d}r\\
&+\overline{\beta}_i\int_0^x\sum_{i,k=1,k\ne i}^{3}\lambda_{ik}(r, \varphi_i^{-1}(r; t, x))\frac{|P_k^1(\varphi_i^{-1}(r; t, x)) - P_k^2(\varphi_i^{-1}(r; t, x))|}{V_i(r, \varphi_i^{-1}(r; t, x))}\mathrm{d}r\\
\le&L_\beta|c_{i0}^1(t) - c_{i0}^2(t)| + \overline{\beta}_i\int_{\varphi_i^{-1}(0;t,x)}^t |\mu_i(\varphi_i(\sigma; t, x), c_{i0}^1(\sigma)) - \mu_i(\varphi_i(\sigma; t, x), c_{i0}^2(\sigma))|\mathrm{d}\sigma\\
&+\overline{\beta}_i\int_{\varphi_i^{-1}(0;t,x)}^t\sum_{i,k=1,k\ne i}^{3}\lambda_{ik}(\varphi_i(\sigma; t, x), \sigma)|P_k^1(\sigma) - P_k^2(\sigma)|\mathrm{d}\sigma\\
\le&L_\beta|c_{i0}^1(t) - c_{i0}^2(t)| + \overline{\beta}_iL_\mu\int_0^t|c_{i0}^1(\sigma) - c_{i0}^2(\sigma)|\mathrm{d}\sigma + \overline{\beta}_i\overline{\lambda}_{ik}\int_0^l\int_0^t\sum_{i,k=1,k\ne i}^{3}|p_k^1(x, \sigma) - p_k^2(x, \sigma)|\mathrm{d}\sigma\mathrm{d}x.
\end{aligned}$$

Let

$$M_1 = \max\left\{L_\beta, \overline{\beta}_iL_\mu, \overline{\beta}_i\overline{\lambda}_{ik}\right\},$$

$$W(t) = |c_{i0}^1(t) - c_{i0}^2(t)| + \int_0^t|c_{i0}^1(\sigma) - c_{i0}^2(\sigma)|\mathrm{d}\sigma + \int_0^l\int_0^t\sum_{i,k=1,k\ne i}^{3}|p_k^1(x, \sigma) - p_k^2(x, \sigma)|\mathrm{d}\sigma\mathrm{d}x.$$

Then, we can obtain

$$|K_i^1(t, x) - K_i^2(t, x)| \le M_1W(t). \tag{3.9}$$

By (3.6) and a similar procedure, we have

$$
\begin{aligned}
&|F_i^1(t) - F_i^2(t)| \\
=&\left|\int_0^l \beta_i(x, c_{i0}^1(t)) \int_0^x \frac{f_i(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \frac{\Pi_i^1(x; x, t)}{\Pi_i^1(r; x, t)} \mathrm{d}r\mathrm{d}x \right.\\
&\left. - \int_0^l \beta_i(x, c_{i0}^2(t)) \int_0^x \frac{f_i(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \frac{\Pi_i^2(x; x, t)}{\Pi_i^2(r; x, t)} \mathrm{d}r\mathrm{d}x \right| \\
\leq& \int_0^l |\beta_i(x, c_{i0}^1(t)) - \beta_i(x, c_{i0}^2(t))| \int_0^x \frac{f_i(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \mathrm{d}r\mathrm{d}x \\
&+ \int_0^l \beta_i(x, c_{i0}^2(t)) \int_0^x \frac{f_i(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \cdot \int_r^x \left(\frac{|\mu_i(\delta, c_{i0}^1(\varphi_i^{-1}(\delta; t, x))) - \mu_i(\delta, c_{i0}^2(\varphi_i^{-1}(\delta; t, x)))|}{V_i(\delta, \varphi_i^{-1}(\delta; t, x))} \right.\\
&\left. + \sum_{i,k=1,k\neq i}^{3} \lambda_{ik}(\delta, \varphi_i^{-1}(\delta; t, x)) \frac{|P_k^1(\varphi_i^{-1}(\delta; t, x)) - P_k^2(\varphi_i^{-1}(\delta; t, x))|}{V_i(\delta, \varphi_i^{-1}(\delta; t, x))} \right) \mathrm{d}\delta\mathrm{d}r\mathrm{d}x \\
\leq& L_\beta \int_0^l |c_{i0}^1(t) - c_{i0}^2(t)| \int_0^t f_i(\varphi_i(\sigma; t, x), \sigma)\mathrm{d}\sigma\mathrm{d}x + \overline{\beta}_i \int_0^l \int_0^t f_i(\varphi_i(\sigma; t, x), \sigma) \\
&\cdot \left(L_\mu \int_0^t |c_{i0}^1(\sigma) - c_{i0}^2(\sigma)|\mathrm{d}\sigma + \overline{\lambda}_{ik} \int_0^l \int_0^t \sum_{i,k=1,k\neq i}^{3} |p_k^1(x, \sigma) - p_k^2(x, \sigma)|\mathrm{d}\sigma\mathrm{d}x \right) \mathrm{d}\sigma\mathrm{d}x \\
\leq& \|f_i(\cdot, \cdot)\|_{L^1(Q)} \left(L_\beta |c_{i0}^1(t) - c_{i0}^2(t)| + \overline{\beta}_i L_\mu \int_0^t |c_{i0}^1(\sigma) - c_{i0}^2(\sigma)|\mathrm{d}\sigma \right.\\
&\left. + \overline{\beta}_i \overline{\lambda}_{ik} \int_0^l \int_0^t \sum_{i,k=1,k\neq i}^{3} |p_k^1(x, \sigma) - p_k^2(x, \sigma)|\mathrm{d}\sigma\mathrm{d}x \right).
\end{aligned}
$$

Consequently,

$$
|F_i^1(t) - F_i^2(t)| \leq \|f_i(\cdot, \cdot)\|_{L^1(Q)} M_1 W(t). \tag{3.10}
$$

Since

$$
\exp\left\{ - \int_0^x \frac{(V_i)_x(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \mathrm{d}r \right\} = \frac{1}{V_i(x, t)},
$$

and thanks to the periodicity of $b_i(t)$, we need only to consider the case $t \in [z_i^{-1}(l), z_i^{-1}(l) + T]$. By (3.4)–(3.6), we have

$$
\begin{aligned}
b_i(t) =& F_i(t) + \int_0^l K_i(t, x) b_i(t - z_i^{-1}(x))\mathrm{d}x \\
=& \int_0^l \beta_i(x, c_{i0}(t)) \int_0^x \frac{f_i(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \frac{\Pi_i(x; x, t)}{\Pi_i(r; x, t)} \mathrm{d}r\mathrm{d}x + \int_0^l \beta_i(x, c_{i0}(t)) \Pi_i(x; x, t) b_i(t - z_i^{-1}(x))\mathrm{d}x \\
\leq& \int_0^l \beta_i(x, c_{i0}(t)) \int_0^x \frac{f_i(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \mathrm{d}r\mathrm{d}x + \int_0^l \frac{\beta_i(x, c_{i0}(t))}{V_i(x, t)} b_i(t - z_i^{-1}(x))\mathrm{d}x \\
\leq& \int_0^l \beta_i(x, c_{i0}(t)) \int_{\varphi_i^{-1}(0; t, x)}^t f_i(\varphi_i(s; t, x), s)\mathrm{d}s\mathrm{d}x + \overline{\beta}_i \int_0^l \frac{b_i(t - z_i^{-1}(x))}{V_i(x, t)} \mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
&\le \int_0^l \beta_i(x, c_{i0}(t)) \int_0^t f_i(\varphi_i(s;t,x),s)\mathrm{d}s\mathrm{d}x + \overline{\beta}_i \int_{t-z_i^{-1}(l)}^t b(s)\mathrm{d}s\\
&\le \overline{\beta}_i \|f_i(\cdot,\cdot)\|_{L^1(Q)} + \overline{\beta}_i \int_0^t b(s)\mathrm{d}s.
\end{aligned}$$

From Bellman's lemma, we have

$$\begin{aligned}
b_i(t) \le& \overline{\beta}_i \|f(\cdot,\cdot)\|_{L^1(Q)} \exp\left\{\int_0^t \overline{\beta}_i \mathrm{d}r\right\}\\
\le& \overline{\beta}_i \|f(\cdot,\cdot)\|_{L^1(Q)} \exp\left\{\overline{\beta}_i(T + z_i^{-1}(l))\right\} =: M_2.
\end{aligned}$$

From (3.4), we get

$$\begin{aligned}
&|b_i^1(t) - b_i^2(t)|\\
\le& \left|F_i^1(t) - F_i^2(t)\right| + \int_0^l \left|K_i^1(t,x)b_i^1(t - z_i^{-1}) - K_i^2(t,x)b_i^2(t - z_i^{-1})\right|\\
\le& \left|F_i^1(t) - F_i^2(t)\right| + \int_0^l \left|K_i^1(t,x) - K_i^2(t,x)\right| b_i^1(t - z_i^{-1})\mathrm{d}x + \int_0^l K_i^2(t,x)\left|b_i^1(t - z_i^{-1}) - b_i^2(t - z_i^{-1})\right|\mathrm{d}x\\
\le& \|f_i(\cdot,\cdot)\|_{L^1(Q)} M_1 W(t) + M_2 \int_0^l M_1 W(t)\mathrm{d}x + \overline{\beta}_i \int_0^t |b_i^1(s) - b_i^2(s)|\mathrm{d}s\\
\le& M_3 W(t) + \overline{\beta}_i \int_0^t |b_i^1(s) - b_i^2(s)|\mathrm{d}s,
\end{aligned}$$

where $M_3 = M_1(\|f_i(\cdot,\cdot)\|_{L^1(Q)} + M_2 l)$. It follows from generalized Gronwall Bellman inequality that

$$\begin{aligned}
&|b_i^1(t) - b_i^2(t)|\\
\le& M_3 W(t) + \overline{\beta}_i \exp\left\{\overline{\beta}_i T\right\} M_3 \int_0^t W(s)\mathrm{d}s\\
\le& M_4 W(t),
\end{aligned}$$

where M_4 is a positive constant independent of $p_i(x,t)$.

Denote $X = [L_T^\infty(R_+, L^1(0,l))]^3 \times [L^\infty(R_+)]^4$, then we define the state space

$$Y = \left\{(p, c_0, c_e) \in X \mid p_i(x,t) \ge 0 \text{ a.e. } (x,t) \in Q, \int_0^l p_i(x,t)\mathrm{d}x \le M,\ \ 0 \le c_{i0}(t) \le 1, 0 \le c_e(t) \le 1\right\}.$$

Define a mapping

$$G : Y \to X,\ \ G(p, c_0, c_e) = (G_1(p, c_0, c_e), G_2(p, c_0, c_e), \ldots, G_7(p, c_0, c_e)),$$

where

$$G_i(p, c_0, c_e)(x,t) = p_i(0, t - z_i^{-1}(x))\Pi_i(x; x, t) + \int_0^x \frac{f_i(r, \varphi_i^{-1}(r;t,x))}{V_i(r, \varphi_i^{-1}(r;t,x))} \frac{\Pi_i(x;x,t)}{\Pi_i(r;x,t)}\mathrm{d}r,\ \ i = 1,2,3. \quad (3.11)$$

$$G_j(p, c_0, c_e)(t) = c_{j0}(0)\exp\{-(g_1 + m)t\} + k_1 \int_0^t c_e(s)\exp\{(s - t)(g_1 + m)\}\mathrm{d}s, \quad j = 4, 5, 6. \tag{3.12}$$

$$\begin{aligned} G_7(p, c_0, c_e)(t) =& c_e(0)\exp\left\{-\int_0^t \left[k_2 \sum_{i=1}^{3} P_i(\tau) + h\right]\mathrm{d}\tau\right\} + \int_0^t \left[g_2 \sum_{i=1}^{3} c_{i0}(s)P_i(s) + v(s)\right] \\ & \cdot \exp\left\{\int_t^s \left[k_2 \sum_{i=1}^{3} P_i(\tau) + h\right]\mathrm{d}\tau\right\}\mathrm{d}s. \end{aligned} \tag{3.13}$$

Then, we have

$$\begin{aligned} & \int_0^l |G_i(p, c_0, c_e)|\mathrm{d}x \\ =& \int_0^l b_i(t - z_i^{-1}(x))\Pi_i(x; x, t)\mathrm{d}x + \int_0^l \int_0^x \frac{f_i(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \frac{\Pi_i(x; x, t)}{\Pi_i(r; x, t)}\mathrm{d}r\mathrm{d}x \\ \leq& \int_0^l b_i(t - z_i^{-1}(x))\mathrm{d}x + \int_0^l \int_0^t f_i(\varphi_i(s; t, x), s)\mathrm{d}s\mathrm{d}x \\ \leq& M_2 l + \|f_i(\cdot, \cdot)\|_{L^\infty(Q)} = M. \end{aligned}$$

It is trivial to show that $G(p, c_0, c_e) \in Y$. We now discuss the compressibility of G. By (3.11), we have

$$\begin{aligned} & \int_0^l \left|G_i(p^1, c_0^1, c_e^1) - G_i(p^2, c_0^2, c_e^2)\right|\mathrm{d}x \quad (i = 1, 2, 3) \\ \leq& \int_0^l b_i^1(t - z_i^{-1}(x))\left|\Pi_i^1(x; x, t) - \Pi_i^2(x; x, t)\right|\mathrm{d}x + \int_0^l \left|b_i^1(t - z_i^{-1}(x)) - b_i^2(t - z_i^{-1}(x))\right|\Pi_i^2(x; x, t)\mathrm{d}x \\ & + \int_0^l \int_0^x \frac{f_i(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \left|\frac{\Pi_i^1(x; x, t)}{\Pi_i^1(r; x, t)} - \frac{\Pi_i^2(x; x, t)}{\Pi_i^2(r; x, t)}\right|\mathrm{d}r\mathrm{d}x \\ \leq& M_2 \int_0^l \int_0^x \left(\frac{\left|\mu_i\left(r, c_{i0}^1\left(\varphi_i^{-1}(r; t, x)\right)\right) - \mu_i\left(r, c_{i0}^2\left(\varphi_i^{-1}(r; t, x)\right)\right)\right|}{V_i(r, \varphi_i^{-1}(r; t, x))}\right. \\ & \left. + \sum_{i,k=1,k\neq i}^{3} \lambda_{ik}(r, \varphi_i^{-1}(r; t, x))\frac{\left|P_k^1(\varphi_i^{-1}(r; t, x)) - P_k^2(\varphi_i^{-1}(r; t, x))\right|}{V_i(r, \varphi_i^{-1}(r; t, x))}\right)\mathrm{d}r\mathrm{d}x \\ & + \int_0^l \frac{\left|b_i^1(t - z_i^{-1}(x)) - b_i^2(t - z_i^{-1}(x))\right|}{V_i(x, t)}\mathrm{d}x \\ & + \int_0^l \int_0^x \frac{f_i(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \int_r^x \left(\frac{|\mu_i(\delta, c_{i0}^1(\varphi_i^{-1}(\delta; t, x))) - \mu_i(\delta, c_{i0}^2(\varphi_i^{-1}(\delta; t, x)))|}{V_i(\delta, \varphi_i^{-1}(\delta; t, x))}\right. \\ & \left. + \sum_{i,k=1,k\neq i}^{3} \lambda_{ik}(\delta, \varphi_i^{-1}(\delta; t, x))\frac{|P_k^1(\varphi_i^{-1}(\delta; t, x)) - P_k^2(\varphi_i^{-1}(\delta; t, x))|}{V_i(\delta, \varphi_i^{-1}(\delta; t, x))}\right)\mathrm{d}\delta\mathrm{d}r\mathrm{d}x \\ \leq& M_2 \int_0^l \int_0^t \left(L_\mu|c_{i0}^1(s) - c_{i0}^2(s)| + \overline{\lambda}_{ik} \int_0^l \sum_{i,k=1,k\neq i}^{3} |p_k^1(x, s) - p_k^2(x, s)|\mathrm{d}x\right)\mathrm{d}s\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\|f_i(\cdot,\cdot)\|_{L^1(Q)}\int_0^t\left(L_\mu|c_{i0}^1(s)-c_{i0}^2(s)|+\overline{\lambda}_{ik}\int_0^l\sum_{i,k=1,k\neq i}^{3}|p_k^1(x,s)-p_k^2(x,s)|\mathrm{d}x\right)\mathrm{d}s\\
&+M_4\int_0^t\left(|c_{i0}^1(s)-c_{i0}^2(s)|+(z_i^{-1}(l)+T)|c_{i0}^1(s)-c_{i0}^2(s)|\right)\mathrm{d}s\\
&+M_4(z_i^{-1}(l)+T)\int_0^l\int_0^t\sum_{i,k=1,k\neq i}^{3}\left|p_k^1(x,s)-p_k^2(x,s)\right|\mathrm{d}s\mathrm{d}x\\
\leq& M_5\left(\int_0^t|c_{i0}^1(s)-c_{i0}^2(s)|\mathrm{d}s+\int_0^l\int_0^t\sum_{i,k=1,k\neq i}^{3}|p_k^1(x,s)-p_k^2(x,s)|\mathrm{d}s\mathrm{d}x\right),
\end{aligned}
$$

where $M_5=\max\{M_2lL_\mu+\|f_i(\cdot,\cdot)\|_{L^1(Q)}L_\mu+M_4(1+z_i^{-1}(l)+T),M_2\overline{\lambda}_{ik}l+\|f_i(\cdot,\cdot)\|_{L^1(Q)}\overline{\lambda}_{ik}+M_4(z_i^{-1}(l)+T)\}$.

By (3.12)–(3.13), we have

$$
\begin{aligned}
&\left|G_j(p^1,c_0^1,c_e^1)-G_j(p^2,c_0^2,c_e^2)\right|(t)\quad(j=4,5,6)\\
\leq& M_6\int_0^t|c_e^1(s)-c_e^2(s)|\mathrm{d}s,
\end{aligned}
$$

where $M_6=k_1$.

$$
\begin{aligned}
&\left|G_7(p^1,c_0^1,c_e^1)-G_7(p^2,c_0^2,c_e^2)\right|(t)\\
=&\left|c_e(0)\exp\left\{-\int_0^t\left[k_2\sum_{i=1}^{3}P_i^1(\tau)+h\right]\mathrm{d}\tau\right\}+\int_0^t\left[g_2\sum_{i=1}^{3}c_{i0}^1(s)P_i^1(s)+v(s)\right]\right.\\
&\cdot\exp\left\{\int_t^s\left[k_2\sum_{i=1}^{3}P_i^1(\tau)+h\right]\mathrm{d}\tau\right\}\mathrm{d}s-c_e(0)\exp\left\{-\int_0^t\left[k_2\sum_{i=1}^{3}P_i^2(\tau)+h\right]\mathrm{d}\tau\right\}\\
&\left.-\int_0^t\left[g_2\sum_{i=1}^{3}c_{i0}^2(s)P_i^2(s)+v(s)\right]\exp\left\{\int_t^s\left[k_2\sum_{i=1}^{3}P_i^2(\tau)+h\right]\mathrm{d}\tau\right\}\mathrm{d}s\right|\\
\leq&\int_0^t\left|g_2\sum_{i=1}^{3}c_{i0}^1(s)P_i^1(s)\exp\left\{\int_t^s k_2\sum_{i=1}^{3}P_i^1(\tau)\mathrm{d}\tau\right\}-g_2\sum_{i=1}^{3}c_{i0}^2(s)P_i^2(s)\exp\left\{\int_t^s k_2\sum_{i=1}^{3}P_i^2(\tau)\mathrm{d}\tau\right\}\right|\mathrm{d}s\\
&+\int_0^t\left|v(s)\left[\exp\left\{\int_t^s k_2\sum_{i=1}^{3}P_i^1(\tau)\mathrm{d}\tau\right\}-\exp\left\{\int_t^s k_2\sum_{i=1}^{3}P_i^2(\tau)\mathrm{d}\tau\right\}\right]\right|\mathrm{d}s\\
&+k_2\int_0^t\left|\sum_{i=1}^{3}P_i^1(\tau)-\sum_{i=1}^{3}P_i^2(\tau)\right|\mathrm{d}\tau\\
\leq&(k_2+g_2)\int_0^t\left|\sum_{i=1}^{3}P_i^1(s)-\sum_{i=1}^{3}P_i^2(s)\right|\mathrm{d}s+g_2M\int_0^t\left|\sum_{i=1}^{3}c_{i0}^1(s)-\sum_{i=1}^{3}c_{i0}^2(s)\right|\mathrm{d}s\\
&+(k_2g_2M+k_2h)\int_0^t\int_0^\tau\left|\sum_{i=1}^{3}P_i^1(s)-\sum_{i=1}^{3}P_i^2(s)\right|\mathrm{d}s\mathrm{d}\tau\\
\leq& M_7\left(\sum_{i=1}^{3}\int_0^l\int_0^t|p_i^1(x,s)-p_i^2(x,s)|\mathrm{d}s\mathrm{d}x+\sum_{i=1}^{3}\int_0^t|c_{i0}^1(s)-c_{i0}^2(s)|\mathrm{d}s\right),
\end{aligned}
$$

where $M_7 = \max\{k_2 + g_2 + k_2hT + k_2g_2MT, g_2M\}$.

We now use the Banach fixed point theorem to demonstrate that the mapping G has only one fixed point. Due to the periodicity of elements in the set Y, we consider the case $t \in [0, T]$ only. Define a new norm in $L^\infty(0, T)$ by

$$\|(p, c_0, c_e)\|_* = ess\, sup_{t\in[0,T]} e^{-\lambda t}\left\{\sum_{i=1}^{3}\int_0^l |p_i(x,t)|\mathrm{d}x + \sum_{i=1}^{3}|c_{i0}(t)| + |c_e(t)|\right\},$$

where $\lambda > 0$ is large enough. Then, we have

$$\begin{aligned}
&\|G(p^1, c_0^1, c_e^1) - G(p^2, c_0^2, c_e^2)\|_* \\
\leq &M_8 ess\, sup_{t\in[0,T]} e^{-\lambda t}\int_0^t\left\{\sum_{i=1}^{3}\int_0^l (|p_i^1(x,s) - p_i^2(x,s)|)\mathrm{d}x + \sum_{i=1}^{3}|c_{i0}^1(s) - c_{i0}^2(s)| + |c_e^1(s) - c_e^2(s)|\right\}\mathrm{d}s \\
\leq &M_8 ess\, sup_{t\in[0,T]} e^{-\lambda t}\int_0^t e^{\lambda s}\left\{e^{-\lambda s}\left[\sum_{i=1}^{3}\int_0^l (|p_i^1(x,s) - p_i^2(x,s)|)\mathrm{d}x\right.\right. \\
&\left.\left.+ \sum_{i=1}^{3}|c_{i0}^1(s) - c_{i0}^2(s)| + |c_e^1(s) - c_e^2(s)|\right]\right\}\mathrm{d}s \\
\leq &M_8\|(p^1 - p^2, c_0^1 - c_0^2, c_e^1 - c_e^2)\|_* ess\, sup_{t\in[0,T]}\left\{e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}s\right\} \\
\leq &\frac{M_8}{\lambda}\|(p^1 - p^2, c_0^1 - c_0^2, c_e^1 - c_e^2)\|_*,
\end{aligned}$$

where $M_8 = \max\{M_5, M_6, M_7\}$. Thus, choosing $\lambda > M_8$ yields that G is a strict contraction on $(Y, \|\cdot\|_*)$. The unique fixed point (p, c_0, c_e) of G must be solution to (2.3). □

Theorem 3.2. If T is small enough, then there are constants $K_j(T)$ with $\lim_{T\to 0} K_j(T) > 0, j = 1, 2,$ such that

$$\begin{aligned}
&\sum_{i=1}^{3}\|p_i^1 - p_i^2\|_{L^\infty(0,T;L^1(0,l))} + \sum_{i=1}^{3}\|c_{i0}^1 - c_{i0}^2\|_{L^\infty(0,T)} + \|c_e^1 - c_e^2\|_{L^\infty(0,T)} \\
\leq &K_1(T)T\left(\sum_{i=1}^{3}\|u_i^1 - u_i^2\|_{L^\infty(0,T;L^1(0,l))} + \|v^1 - v^2\|_{L^\infty(0,T)}\right).
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&\sum_{i=1}^{3}\|p_i^1 - p_i^2\|_{L^1(Q)} + \sum_{i=1}^{3}\|c_{i0}^1 - c_{i0}^2\|_{L^1(0,T)} + \|c_e^1 - c_e^2\|_{L^1(0,T)} \\
\leq &K_2(T)T\left(\sum_{i=1}^{3}\|u_i^1 - u_i^2\|_{L^1(Q)} + \|v^1 - v^2\|_{L^1(0,T)}\right).
\end{aligned} \tag{3.15}$$

This proof process of Theorem 3.2 is similar to that of Theorem 4.1 in [27], and is omitted here.

4. Optimality conditions

In this section, we employ tangent-normal cone techniques in nonlinear functional analysis to deduct the necessary conditions for the optimal control pair.

Theorem 4.1. If (u^*, v^*) is an optimal control pair and (p^*, c_0^*, c_e^*) is the corresponding optimal state, then

$$u_i^*(x,t) = \mathcal{F}_i\left(\frac{[w_i(x,t) - \xi_i(x,t)]p_i^*(x,t)}{c_i}\right),\ i = 1,2,3,\ \text{a.e. } (x,t) \in Q, \tag{4.1}$$

$$v^*(t) = \mathcal{F}_4\left(\frac{\xi_7(t)}{c_4}\right)\ \text{ a.e. } t \in\ (0,T), \tag{4.2}$$

in which the truncated mappings $\mathcal{F}_j$ are given by

$$\mathcal{F}_j(\eta) = \begin{cases} 0, & \eta < 0, \\ \eta, & 0 \le \eta \le N_j,\quad j = 1,2,3,4, \\ N_j, & \eta > N_j, \end{cases} \tag{4.3}$$

and $(\xi_1, \xi_2, \ldots, \xi_7)$ is the solution of the following adjoint system corresponding to (u^*, v^*):

$$\begin{cases} \frac{\partial \xi_i}{\partial t} + V_i \frac{\partial \xi_i}{\partial x} = \left[\mu_i(x, c_{i0}^*(t)) + \sum\limits_{i,k=1,k\neq i}^{3} \lambda_{ik} P_k^*(t) + u_i^*\right]\xi_i + [k_2 c_e^*(t) - g_2 c_{i0}^*(t)]\xi_7 \\ \qquad - \xi_i(0,t)\beta_i(x, c_{i0}^*(t)) + w_i u_i^*, \\ \frac{\mathrm{d}\xi_{i+3}}{\mathrm{d}t} = \int_0^l \frac{\partial \mu_i(x, c_{i0}^*(t))}{\partial c_{i0}} p_i^* \xi_i \mathrm{d}x + (g_1 + m)\xi_{i+3} - g_2 P_i^*(t)\xi_7 - \xi_i(0,t)\int_0^l \frac{\partial \beta_i(x, c_{i0}^*(t))}{\partial c_{i0}} p_i^* \mathrm{d}x, \\ \frac{\mathrm{d}\xi_7}{\mathrm{d}t} = -k_1 \sum\limits_{i=1}^{3} \xi_{i+3} + \left[k_2 \sum\limits_{i=1}^{3} P_i^*(t) + h\right]\xi_7, \\ \xi_i(l,t) = 0,\ \ \xi_i(x,t) = \xi_i(x, t+T),\ \ i = 1,2,3, \\ \xi_j(T) = 0,\ \ j = 4, \ldots, 7. \end{cases} \tag{4.4}$$

Proof. The existence of a unique, bounded solution to the adjoint system (4.4) can be treated in the same manner as the state system (2.3). For any given $(v_1, v_2) \in \mathcal{T}_\Omega(u^*, v^*)$ (the tangent cone of Ω at (u^*, v^*)), $u^* = (u_1^*, \ldots, u_3^*), v_1 = (v_{11}, \ldots, v_{31})$, $(u^* + \varepsilon v_1, v^* + \varepsilon v_2) \in \Omega$ provided that ε is small enough. Then, from $J(u^* + \varepsilon v_1, v^* + \varepsilon v_2) \le J(u^*, v^*)$, we derive

$$\begin{aligned} &\sum_{i=1}^{3}\int_0^T\int_0^l w_i(u_i^* + \varepsilon v_{i1})p_i^\varepsilon \mathrm{d}x\mathrm{d}t - \frac{1}{2}\sum_{i=1}^{3}\int_0^T\int_0^l c_i(u_i^* + \varepsilon v_{i1})^2 \mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^T c_4(v^* + \varepsilon v_2)^2 \mathrm{d}t \\ \le &\sum_{i=1}^{3}\int_0^T\int_0^l w_i u_i^* p_i^* \mathrm{d}x\mathrm{d}t - \frac{1}{2}\sum_{i=1}^{3}\int_0^T\int_0^l c_i {u_i^*}^2 \mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^T c_4 {v^*}^2 \mathrm{d}t, \end{aligned}$$

and then deduce that

$$\sum_{i=1}^{3}\int_0^T\int_0^l w_i(u_i^* z_i + v_{i1}p_i^*)\mathrm{d}x\mathrm{d}t - \sum_{i=1}^{3}\int_0^T\int_0^l c_i u_i^* v_{i1}\mathrm{d}x\mathrm{d}t - \int_0^T c_4 v^* v_2 \mathrm{d}t \le 0, \tag{4.5}$$

where $\frac{1}{\varepsilon}(p_i^\varepsilon - p_i^*) \to z_i, \frac{1}{\varepsilon}(c_{i0}^\varepsilon - c_{i0}^*) \to z_{i+3}, \frac{1}{\varepsilon}(c_e^\varepsilon - c_e^*) \to z_7,$ as $\varepsilon \to 0$. By Theorem 3.2, we get the existence of $z_1, z_2, \ldots, z_7$. $(p^\varepsilon, c_0^\varepsilon, c_e^\varepsilon)$ is the state corresponding to $(u^* + \varepsilon v_1, v^* + \varepsilon v_2)$. It follows from the state system (2.3) that $(z_1, z_2, \dots, z_7)$ satisfies

$$\left\{\begin{array}{l}\frac{\partial z_i}{\partial t} + V_i \frac{\partial z_i}{\partial x} = -\left[\mu_i(x, c_{i0}^*(t)) + \sum\limits_{i,k=1,k\neq i}^{3} \lambda_{ik} P_k^*(t) + V_{ix} + u_i^*\right] z_i - \sum\limits_{i,k=1,k\neq i}^{3} \lambda_{ik} Z_k(t) p_i^* \\ \qquad\qquad - \frac{\partial \mu_i(x, c_{i0}^*(t))}{\partial c_{i0}} p_i^* z_{i+3} - v_{i1} p_i^*, \\ \frac{\mathrm{d} z_{i+3}}{\mathrm{d}t} = k_1 z_7 - g_1 z_{i+3} - m z_{i+3}, \\ \frac{\mathrm{d} z_7}{\mathrm{d}t} = -k_2 c_e^*(t) \sum\limits_{i=1}^{3} Z_i(t) + g_2 \sum\limits_{i=1}^{3} \left[c_{i0}^* Z_i(t) + z_{i+3} P_i^*(t)\right] - \left[k_2 \sum\limits_{i=1}^{3} P_i^*(t) + h_1\right] z_7 + v_2, \\ V_i(0,t) z_i(0,t) = \int_0^l \beta_i(x, c_{i0}^*(t)) z_i \mathrm{d}x + \int_0^l \frac{\partial \beta_i(a, c_{i0}^*(t))}{\partial c_{i0}} p_i^* z_{i+3} \mathrm{d}x, \\ z_i(x,t) = z_i(x, t+T), \\ z_{i+3}(0) = z_7(0) = 0, \\ P_i^*(t) = \int_0^l p_i^*(x,t) \mathrm{d}x, \ \ Z_i(t) = \int_0^l z_i(x,t) \mathrm{d}x, \ \ i = 1,2,3. \end{array}\right. \tag{4.6}$$

We multiply the first three equations in (4.6) by $\xi_1, \xi_2, \dots, \xi_7$, respectively, and integrate on Q and $(0, T)$. By using (4.4), we have

$$\sum_{i=1}^{3} \int_0^T \int_0^l w_i u_i^* z_i \mathrm{d}x\mathrm{d}t = -\sum_{i=1}^{3} \int_0^T \int_0^l v_{i1} \xi_i p_i^* \mathrm{d}x\mathrm{d}t + \int_0^T v_2 \xi_7 \mathrm{d}t. \tag{4.7}$$

Substituting (4.7) into (4.5) gives

$$\sum_{i=1}^{3} \int_0^T \int_0^l [(w_i - \xi_i) p_i^* - c_i u_i^*] v_{i1} \mathrm{d}x\mathrm{d}t + \int_0^T (-c_4 v^* + \xi_7) v_2 \mathrm{d}t \leq 0,$$

for any $(v_1, v_2) \in \mathcal{T}_\Omega(u^*, v^*)$. Consequently, the structure of normal cone tells us that $((w_i - \xi_i)p_i^* - c_i u_i^*, -c_4 v^* + \xi_7) \in \mathcal{N}_\Omega(u^*, v^*)$ (the normal cone of Ω at (u^*, v^*)), which gives the desired result. □

Theorem 4.2. If T is small enough, then there is a constant K_3, such that

$$\begin{aligned}&\sum_{i=1}^{3} \|\xi_i^1 - \xi_i^2\|_{L^\infty(Q)} + \sum_{i=1}^{3} \|\xi_{i+3}^1 - \xi_{i+3}^2\|_{L^\infty(0,T)} + \|\xi_7^1 - \xi_7^2\|_{L^\infty(0,T)} \\ \leq & K_3 T \left(\sum_{i=1}^{3} \|u_i^1 - u_i^2\|_{L^\infty(Q)} + \|v^1 - v^2\|_{L^\infty(0,T)} \right).\end{aligned} \tag{4.8}$$

The proof process of Theorem 4.2 is similar to that of Theorem 3.2, and is omitted here.

5. Existence of optimal control pair

In order to show that there exists a unique optimal control pair by means of the Ekeland variational principle, we embed the functional $\widetilde{J}(u, v)$ into $[L^1(Q)]^3 \times L^1(0, T)$. We define

$$\widetilde{J}(u,v) = \left\{ \begin{array}{ll} J(u,v), & (u,v) \in \Omega, \\ -\infty, & \text{otherwise.} \end{array}\right.$$

Lemma 5.1. $\widetilde{J}(u, v)$ is upper semi-continuous with respect to (u, v) in $[L^1(Q)]^3 \times L^1(0, T)$.

Proof. Let $(u^n, v^n) \to (u, v)$ as $n \to \infty$, (p^n, c_0^n, c_e^n) and (p, c_0, c_e) be the states of (2.3) corresponding to (u^n, v^n) and (u, v), respectively. By Riesz theorem, there is a subsequence, denoted still by (u^n, v^n) such that

$$[u^n(x,t)]^2 \to u^2(x,t) \text{ a.e. } (x,t) \in Q, \quad [v^n(t)]^2 \to v^2(t) \text{ a.e. } t \in (0,T), \text{ as } n \to \infty.$$

Thus, from the Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty} \int_0^T \int_0^l [u_i^n(x,t)]^2 \mathrm{d}x\mathrm{d}t = \int_0^T \int_0^l u_i^2(x,t)\mathrm{d}x\mathrm{d}t, \quad \lim_{n\to\infty} \int_0^T [v^n(t)]^2 \mathrm{d}t = \int_0^T v^2(t)\mathrm{d}t.$$

On the other hand, it follows from (3.15) that

$$\begin{aligned}
&\left| \int_0^T \int_0^l w_i(x,t)u_i^n(x,t)p_i^n(x,t)\mathrm{d}x\mathrm{d}t - \int_0^T \int_0^l w_i(x,t)u_i(x,t)p_i(x,t)\mathrm{d}x\mathrm{d}t \right| \\
\le& \int_0^T \int_0^l w_i(x,t)p_i^n(x,t)|u_i^n(x,t) - u_i(x,t)|\mathrm{d}x\mathrm{d}t + \int_0^T \int_0^l w_i(x,t)u_i(x,t)|p_i^n(x,t) - p_i(x,t)|\mathrm{d}x\mathrm{d}t \\
\le& M\overline{w}_i\|u_i^n - u_i\|_{L^1(Q)} + N_i\overline{w}_i\|p_i^n - p_i\|_{L^1(Q)} \\
\le& M\overline{w}_i\|u_i^n - u_i\|_{L^1(Q)} + N_i\overline{w}_i K_2(T)T(\|u_i^n - u_i\|_{L^1(Q)} + \|v^1 - v^2\|_{L^1(0,T)}).
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \int_0^T \int_0^l w_i(x,t)u_i^n(x,t)p_i^n(x,t)\mathrm{d}x\mathrm{d}t = \int_0^T \int_0^l w_i(x,t)u_i(x,t)p_i(x,t)\mathrm{d}x\mathrm{d}t.$$

In a word, we have proved that $\limsup_{n\to\infty} \widetilde{J}(u^n, v^n) \le \widetilde{J}(u, v)$. □

Theorem 5.1. If T is sufficiently small, there exists one and only one optimal control pair (u^*, v^*), which is in feedback and is determined by (4.1)–(4.4) and (2.3), where C_1 and C_2 are the supremum of $|p_i|$ and $|\xi_j|$, $i = 1, 2, 3$, $j = 1, 2, \ldots, 7$, respectively.

Proof. Define the mapping $\mathcal{L} : \Omega \to \Omega$ as follows:

$$\begin{aligned}
\mathcal{L}(u, v) &= \mathcal{F}\left(\frac{(w_1 - \xi_1)p_1}{c_1}, \ldots, \frac{(w_3 - \xi_3)p_3}{c_3}, \frac{\xi_7}{c_4}\right) \\
&= \left(\mathcal{F}_1\left(\frac{(w_1 - \xi_1)p_1}{c_1}\right), \ldots, \mathcal{F}_3\left(\frac{(w_3 - \xi_3)p_3}{c_3}\right), \mathcal{F}_4\left(\frac{\xi_7}{c_4}\right)\right),
\end{aligned}$$

where (p, c_0, c_e) and $(\xi_1, \xi_1, \ldots, \xi_7)$ are the state and adjoint state, respectively, corresponding to the control (u, v). We show that $\mathcal{L}$ admits a unique fixed point, which maximizes the functional $\mathcal{L}$.

From Lemma 5.1 and the Ekeland variational principle, for any given $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \Omega$ such that

$$\widetilde{J}(u^\varepsilon, v^\varepsilon) \ge \sup_{(u,v)\in\Omega} \widetilde{J}(u, v) - \varepsilon, \tag{5.1}$$

$$\widetilde{J}(u^\varepsilon, v^\varepsilon) \ge \sup_{(u,v)\in\Omega} \left\{ \widetilde{J}(u, v) - \sqrt{\varepsilon}\left(\sum_{i=1}^{3} \|u_i^\varepsilon - u_i\|_{L^1(Q)} + \|v^\varepsilon - v\|_{L^1(0,T)} \right) \right\}. \tag{5.2}$$

Thus, the perturbed functional

$$\widetilde{J}_\varepsilon(u,v)=\widetilde{J}(u,v)-\sqrt{\varepsilon}\left(\sum_{i=1}^{3}\|u_i^\varepsilon-u_i\|_{L^1(Q)}+\|v^\varepsilon-v\|_{L^1(0,T)}\right),$$

attains its supremum at $(u^\varepsilon,v^\varepsilon)$. Then, we argue as in Theorem 4.1:

$$\begin{aligned}&(u^\varepsilon,v^\varepsilon)=\mathcal{L}(u^\varepsilon,v^\varepsilon)\\&=\left(\mathcal{F}_1\left(\frac{(w_1-\xi_1^\varepsilon)p_1^\varepsilon+\sqrt{\varepsilon}\theta_1^\varepsilon}{c_1}\right),\dots,\mathcal{F}_3\left(\frac{(w_3-\xi_3^\varepsilon)p_3^\varepsilon+\sqrt{\varepsilon}\theta_3^\varepsilon}{c_3}\right),\mathcal{F}_4\left(\frac{\xi_7^\varepsilon+\sqrt{\varepsilon}\theta_4^\varepsilon}{c_4}\right)\right),\end{aligned}\tag{5.3}$$

where $(p^\varepsilon,c_0^\varepsilon,c_e^\varepsilon)$ and $(\xi_1^\varepsilon,\xi_2^\varepsilon,\dots,\xi_7^\varepsilon)$ are the state and adjoint state, respectively, corresponding to the control $(u^\varepsilon,v^\varepsilon)$, $\theta_1^\varepsilon,\dots,\theta_3^\varepsilon\in L^\infty(Q),\theta_4^\varepsilon\in L^\infty(0,T)$, and with $|\theta_i^\varepsilon|\le 1, i=1,2,3,4$.

First, we show that $\mathcal{L}$ has only one fixed point. Let (p^j,c_0^j,c_e^j) and $(\xi_1^j,\xi_2^j,\dots,\xi_7^j)$ be the state and adjoint state corresponding to the control (u^j,v^j), $j=1,2$. By (3.14) and (4.8), we have

$$\begin{aligned}&\|\mathcal{L}(u^1,v^1)-\mathcal{L}(u^2,v^2)\|_\infty\\&=\sum_{i=1}^{3}\left\|\mathcal{F}_i\left(\frac{(w_i-\xi_i^1)p_i^1}{c_i}\right)-\mathcal{F}_i\left(\frac{(w_i-\xi_i^2)p_i^2}{c_i}\right)\right\|_{L^\infty(Q)}+\left\|\mathcal{F}_4\left(\frac{\xi_7^1}{c_4}\right)-\mathcal{F}_4\left(\frac{\xi_7^2}{c_4}\right)\right\|_{L^\infty(0,T)}\\&\le\sum_{i=1}^{3}\left\|\frac{(w_i-\xi_i^1)p_i^1}{c_i}-\frac{(w_i-\xi_i^2)p_i^2}{c_i}\right\|_{L^\infty(Q)}+\left\|\frac{\xi_7^1}{c_4}-\frac{\xi_7^2}{c_4}\right\|_{L^\infty(0,T)}\\&\le\sum_{i=1}^{3}\left\|\frac{w_i(p_i^1-p_i^2)}{c_i}+\frac{|\xi_i^1|(p_i^1-p_i^2)}{c_i}+\frac{|p_i^2|(\xi_i^1-\xi_i^2)}{c_i}\right\|_{L^\infty(Q)}+\left\|\frac{\xi_7^1-\xi_7^2}{c_4}\right\|_{L^\infty(0,T)}\\&\le T\left(\sum_{i=1}^{3}\frac{1}{c_i}(\overline{w}_iK_1+C_2K_1+C_1K_3)+K_3\right)\cdot\left(\sum_{i=1}^{3}\|u_i^1-u_i^2\|_{L^\infty(Q)}+\|v^1-v^2\|_{L^\infty(0,T)}\right).\end{aligned}$$

Clearly, $\mathcal{L}$ is a contraction if T is sufficiently small. Hence, $\mathcal{L}$ has a unique fixed point (u^*,v^*).

Next, we prove $(u^\varepsilon,v^\varepsilon)\to(u^*,v^*)$ as $\varepsilon\to0^+$. The relations (4.1), (4.2) and (5.3) lead to

$$\begin{aligned}&\|\mathcal{L}(u^\varepsilon,v^\varepsilon)-(u^\varepsilon,v^\varepsilon)\|_\infty\\&=\Bigg\|\left(\mathcal{F}_1\left(\frac{(w_1-\xi_1^\varepsilon)p_1^\varepsilon}{c_1}\right),\dots,\mathcal{F}_3\left(\frac{(w_3-\xi_3^\varepsilon)p_3^\varepsilon}{c_3}\right),\mathcal{F}_4\left(\frac{\xi_7^\varepsilon}{c_4}\right)\right)\\&\quad-\left(\mathcal{F}_1\left(\frac{(w_1-\xi_1^\varepsilon)p_1^\varepsilon}{c_1}+\frac{\sqrt{\varepsilon}\theta_1^\varepsilon}{c_1}\right),\dots,\mathcal{F}_3\left(\frac{(w_3-\xi_3^\varepsilon)p_3^\varepsilon}{c_3}+\frac{\sqrt{\varepsilon}\theta_3^\varepsilon}{c_3}\right),\mathcal{F}_4\left(\frac{\xi_7^\varepsilon}{c_4}+\frac{\sqrt{\varepsilon}\theta_4^\varepsilon}{c_4}\right)\right)\Bigg\|_\infty\\&\le\sum_{i=1}^{3}\left\|\mathcal{F}_i\left(\frac{(w_i-\xi_i^\varepsilon)p_i^\varepsilon}{c_i}\right)-\mathcal{F}_i\left(\frac{(w_i-\xi_i^\varepsilon)p_i^\varepsilon}{c_i}+\frac{\sqrt{\varepsilon}\theta_i^\varepsilon}{c_i}\right)\right\|_{L^\infty(Q)}+\left\|\mathcal{F}_4\left(\frac{\xi_7^\varepsilon}{c_4}\right)-\mathcal{F}_4\left(\frac{\xi_7^\varepsilon}{c_4}+\frac{\sqrt{\varepsilon}\theta_4^\varepsilon}{c_4}\right)\right\|_{L^\infty(0,T)}\\&=\sum_{i=1}^{3}\left\|\frac{(w_i-\xi_i^\varepsilon)p_i^\varepsilon}{c_i}-\frac{(w_i-\xi_i^\varepsilon)p_i^\varepsilon}{c_i}-\frac{\sqrt{\varepsilon}\theta_i^\varepsilon}{c_i}\right\|_{L^\infty(Q)}+\left\|\frac{\xi_7^\varepsilon}{c_4}-\frac{\xi_7^\varepsilon}{c_4}-\frac{\sqrt{\varepsilon}\theta_4^\varepsilon}{c_4}\right\|_{L^\infty(0,T)}\\&\le\sqrt{\varepsilon}\sum_{i=1}^{3}\frac{\|\theta_i^\varepsilon(x,t)\|_{L^\infty(Q)}}{c_i}+\sqrt{\varepsilon}\frac{\|\theta_4^\varepsilon(t)\|_{L^\infty(0,T)}}{c_4}\end{aligned}$$

$$\leq \sqrt{\varepsilon}\sum_{i=1}^{4}\frac{1}{c_i},$$

it is easy to derive that

$$\begin{aligned}
&\|(u^*,v^*)-(u^\varepsilon,v^\varepsilon)\|_\infty\\
\leq&\|\mathcal{L}(u^*,v^*)-\mathcal{L}(u^\varepsilon,v^\varepsilon)\|_\infty+\|\mathcal{L}(u^\varepsilon,v^\varepsilon)-(u^\varepsilon,v^\varepsilon)\|_\infty\\
\leq&T\left(\sum_{i=1}^{3}\frac{1}{c_i}(\overline{w}_iK_1+C_2K_1+C_1K_3)+K_3\right)\cdot\left(\sum_{i=1}^{3}\|u_i^*-u_i^\varepsilon\|_{L^\infty(Q)}+\|v^*-v^\varepsilon\|_{L^\infty(0,T)}\right)+\sqrt{\varepsilon}\sum_{i=1}^{4}\frac{1}{c_i}.
\end{aligned}$$

So, if T is small enough, the following result holds:

$$\sum_{i=1}^{3}\|u_i^*-u_i^\varepsilon\|_{L^\infty(Q)}+\|v^*-v^\varepsilon\|_{L^\infty(0,T)}\leq\frac{\sqrt{\varepsilon}\sum\limits_{i=1}^{4}\frac{1}{c_i}}{1-T\left(\sum\limits_{i=1}^{3}\frac{1}{c_i}(\overline{w}_iK_1+C_2K_1+C_1K_3)+K_3\right)},$$

which gives the desired result.

Finally, passing to the limit $\varepsilon\to0^+$ in the inequality of (5.2) and using Lemma 5.1 yield $\widetilde{J}(u^*,v^*)\geq\limsup_{(u,v)\in\Omega}\widetilde{J}(u,v)$, which finishes the proof. □

6. Numerical approximation

In this section, our goal is to obtain a numerical approximation for the nonnegative T-periodic solution of the system (2.3). We numerically study the evolution of a single species in a polluted environment as a simplification of the complete model (2.3). If the harvest effort term and the summation term are considered, it will be transformed into the optimization problem (2.3)–(2.4), which is complex.

Suppose the computational domain $\tilde{Q}=[0,l]\times[0,\tilde{T}]$ is divided into an $J\times N$ mesh with the spacial step size $h=\frac{l}{J}=0.01$ in the x direction and time step size $\tau=\frac{\tilde{T}}{N}=0.02$. The grid points (x_j,t_n) are defined by

$$\begin{aligned}
x_j&=jh,\quad j=0,1,2,\ldots,J;\\
t_n&=n\tau,\quad n=0,1,2,\ldots,N,
\end{aligned}$$

where J and N are two integers. The p_j^n and f_j^n terms denote the solution $p(jh,n\tau)$ and source term $f(jh,n\tau)$ of the finite difference equation, respectively.

Based on the state system (2.3), the finite difference scheme can be written as follows:

$$\frac{p_j^n-p_j^{n-1}}{\tau}+V\frac{p_j^n-p_{j-1}^n}{h}+V_xp_j^n+\mu p_j^n-f_j^n=0,\tag{6.1}$$

where $j=1,2,\ldots,J;n=1,2,\ldots,N$. It follows from (6.1) that

$$-dVp_{j-1}^n+[1+dV+\tau(V_x+\mu)]p_j^n=p_j^{n-1}+\tau f_j^n,\tag{6.2}$$

where $d=\frac{\tau}{h}$.

Since $V(0,t) = 1$, then the boundary condition $p(0,t) = \int_0^l \beta(x, c_0(t))p(x,t)\mathrm{d}x$ and initial condition $p(x,0) = p_0(x)$ can be discretized as

$$\begin{cases} p_j^0 = p_{0j}, \\ p_0^n = \sum\limits_{j=1}^{J} \beta_j p_j^n h. \end{cases} \tag{6.3}$$

From (6.2) and (6.3), we have the matrix associated with the system of linear equations of the finite difference method

$$\mathbf{A}\mathbf{P}^n = \mathbf{P}^{n-1} + \tau\mathbf{F}, \tag{6.4}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 + dV + \tau(V_x + \mu) - dV\beta h & -dV\beta h & \dots & -dV\beta h & -dV\beta h \\ -dV & 1 + dV + \tau(V_x + \mu) & \dots & 0 & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & \dots & -dV & 1 + dV + \tau(V_x + \mu) \\ 0 & 0 & \dots & 0 & -dV \end{bmatrix},$$

$$\mathbf{P}^n = (p_1^n, p_2^n, \dots, p_J^n)^{\mathrm{T}}, \quad \mathbf{F} = (f_1^n, f_2^n, \dots, f_J^n)^{\mathrm{T}}.$$

Note that $\mathbf{A}$ is an upper triangular matrix, so the nonlinear algebraic equations (6.4) have solutions. In this paper, we choose the following parameters:

$$\begin{cases} \beta(x, c_0(t)) = 100x^2(1-x)(1+\sin(\pi x))\left|\sin\frac{2\pi c_0(t)}{T}\right|, \\ \mu(x, c_0(t)) = e^{-4x}(1-x)^{-1.4}(2 + \cos\frac{2\pi c_0(t)}{T}), \\ V(x,t) = 1 - x, \ f(x,t) = 2 + (1+x)\sin(\frac{2\pi t}{T}), \\ p_0(x) = e^x, \ u(x,t) = 0, \ x = 1, \ T = \frac{1}{3}, \ \tilde{T} = 6T. \end{cases}$$

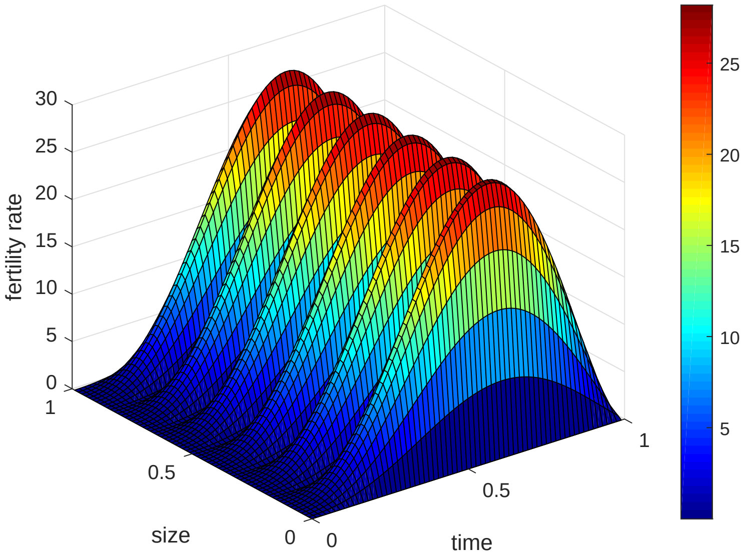

Figure 1. Fertility rate of the population.

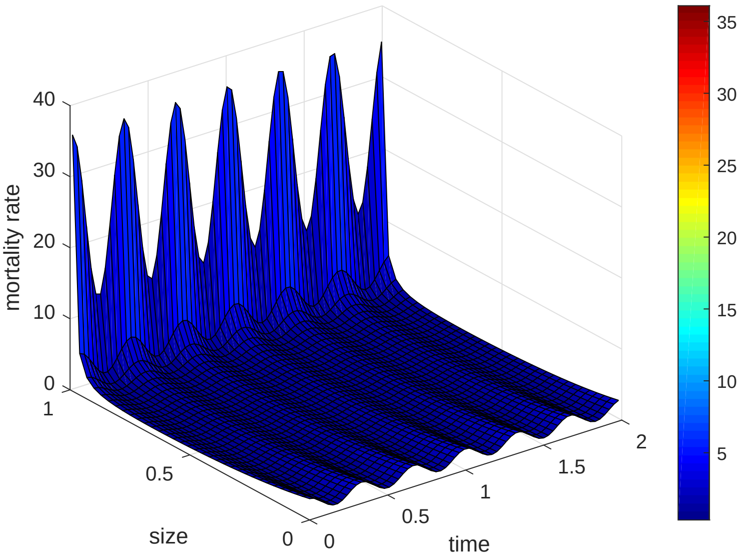

Figure 2. Mortality rate of the population.

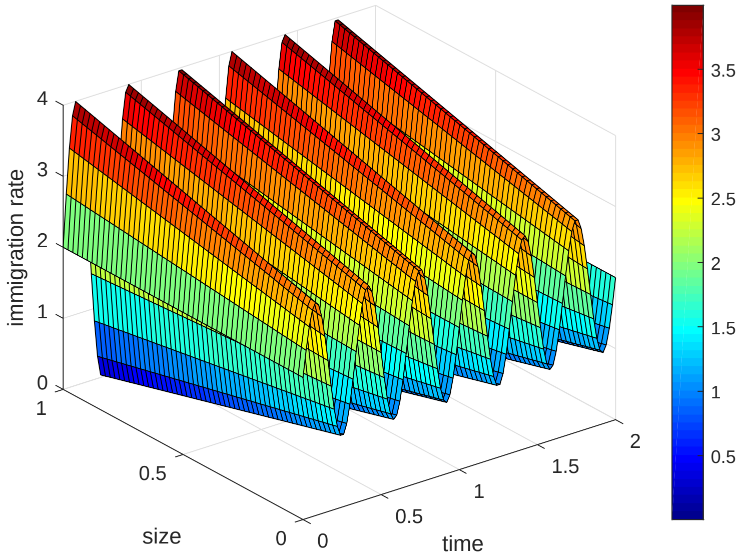

Figure 3. Immigration rate of the population.

In this paper, we used the backward difference scheme and chasing method, and (6.4) was solved through programming. The fertility rate, mortality rate, and immigration rate were T-periodic and were all greater than zero, which is consistent with the assumptions. We considered $T = \frac{1}{3}$. Their graphs are given in Figures 1–3, respectively. The fertility rate was the highest when the size was half and the mortality rate was the highest when the size was the maximum, which conformed to the empirical situation. Therefore, the selection of parameters β, μ, and f was reasonable.

The graphic of the numerical solution p is given in Figure 4. Over time, solution p showed T-periodic changes. We take the numerical solution of (2.3), corresponding to an arbitrary positive initial datum p_0, on some interval $[kT, (k+1)T]$, where k is large enough. On such an interval, the solution p was already stable. We can then get the periodic solution of (2.3) by extending the numerical solution p. During computation we found that any positive initial datum p_0 was appropriate for use.

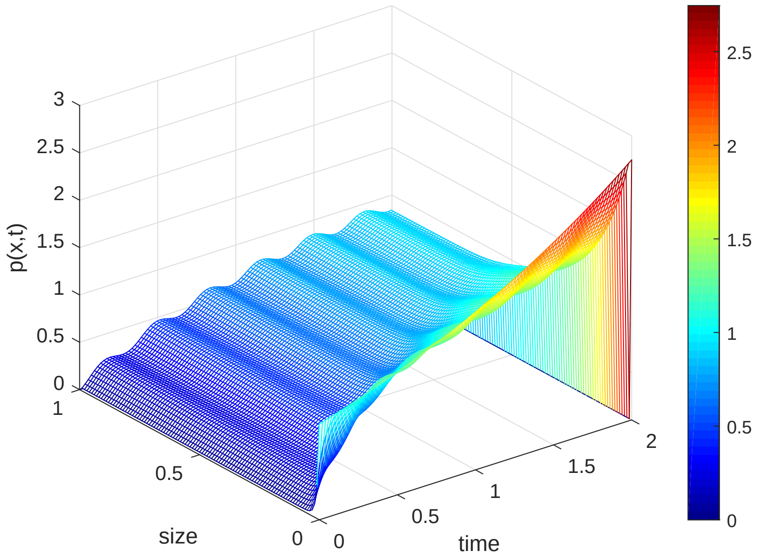

Figure 4. Numerical solution of the system.

7. Conclusions

The study of time periodic models is of great importance due to the fact that the vital rates and the inflow are often time periodic. In the foregoing, we have established the existence and uniqueness of a nonnegative solution of the hybrid system (2.3). The necessary conditions for optimal controls were provided. The existence of the unique optimal control pair was investigated. Some numerical results were finally presented. The results implied that the solution of (2.3) always maintains the pattern of increasing periodically, and any positive initial datum p_0 is appropriate. Over time, the density of the population increased first and then decreased in a cycle. The bang-bang structure of solutions is much more common in optimal population management.

Furthermore, if $V_i(x,t) = 1$ for $Q = (0,l) \times R_+, i = 1,2,3$, the state system degenerates into an age-structured model, and our results cover the corresponding results [5–7]. Note that the individual price factor $w_i(x,t)$ plays an important role in the structure of the optimal controller (4.1). However, as we do not have a clear biological meaning for the solutions $\xi_i (i = 1, 2, \dots, 7)$ of the adjoint system (4.4), it is difficult to give a precise explanation of the threshold conditions (4.1) and (4.2). In specific applications, the optimal population density and optimal policy are calculated by combining the state system and the adjoint system. This is a challenging problem, and future work in the area should address it.

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Conflict of interest

None of the authors has a conflict of interest in the publication of this paper.

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